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COMPETITIVE EXCLUSION AND COEXISTENCE IN A TWO-STRAIN PATHOGEN MODEL WITH DIFFUSION

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Abstract. We consider a two-strain pathogen model described by a system of reaction-diffusion equations. We define a basic reproduction number R_0 and show that when the model parameters are constant (spatially homogeneous), if $R_0 > 1$ then one strain will outcompete the other strain and drive it to extinction, but if $R_0 \leq 1$ then the disease-free equilibrium is globally attractive. When we assume that the diffusion rates are equal while the transmission and recovery rates are heterogeneous, then there are two possible outcomes under the condition $R_0 > 1$: 1) Competitive exclusion where one strain dies out. 2) Coexistence between the two strains. Thus, spatial heterogeneity promotes coexistence.

1. **Introduction.**

Numerous studies have been conducted to understand the dynamics of epidemic differential equation models with multiple-pathogen strains (e.g., [1, 2, 3, 9, 11, 27]). Such models are important as they provide insights into the evolutions, persistence and treatment of diseases such as infuenza, hantavirus, HIV-AIDS, and other sexually transmitted diseases (e.g., [5, 8, 10, 14, 15]). The focus of many such studies is on understanding when coexistence between strains is possible and when competitive exclusion is the outcome. In [11] by assuming that the birth rate is a function of the total population, it was shown that coexistence is not possible, and competitive exclusion is the only outcome with the winner being the strain that has the largest basic reproduction number. In [1, 2, 9] the authors showed that by allowing the mortality to be a function of the total population, then coexistence between strains is possible as well as competitive exclusion. The authors of the papers [27] and [3] extended such results to periodic and general nonautonomous environments, respectively.

Recently, researchers focused their attention on understanding the effect of diffusion on the dynamics of disease transmission models (e.g., [7, 16, 21, 29, 30, 31, 32]).

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In these papers, the models focused on one pathogen strain and considered standard incidence to model transmission of the disease. For example, the authors in [7] presented the following reaction-diffusion model to describe disease transmission:

$$\begin{aligned} \frac{\partial S}{\partial t} &= d_S \Delta S - \beta(x)\frac{SI}{S+I} + \gamma(x) I, \qquad x \in \Omega, \quad t > 0, \\ \frac{\partial I}{\partial t} &= d_I \Delta I + \beta(x)\frac{SI}{S+I} - \gamma(x) I, \qquad x \in \Omega, \quad t > 0 \end{aligned} \tag{1.1}$$

with no-flux boundary condition

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{1.2}$$

Here, $S(x,t)$ and $I(x,t)$ represent the densities of susceptible and infected individuals at location x and time t, respectively. The parameters d_S and d_I are positive diffusion coefficients for the susceptible and infected populations, respectively. The functions $\beta(x)$ and $\gamma(x)$ denote the disease transmission rate and the recovery rate from the infected class, respectively. In this paper a basic reproduction number R_0 was defined and it was shown that if $R_0 < 1$ then the disease-free equilibrium is globally attractive. Furthermore, when $R_0 > 1$ then there exists a unique endemic equilibrium and this endemic equilibrium tends to a spatially inhomogeneous disease-free equilibrium as the mobility of susceptible individuals tends to zero, i.e., as $d_S \to 0$.

In [29] the stability of the endemic equilibrium for the model (1.1)-(1.2) was studied. Therein, the global stability of this equilibrium was shown under the case $R_0 > 1$ and some additional restrictions, namely when the diffusion coefficients are equal or $\beta/\gamma = a =$ constant. In [21] the authors extended such investigation to the model in [7] with Dirichlet boundary condition which described a hostile environment at the boundary. They defined a basic reproduction number R_0 and showed that the disease-free equilibrium is globally attractive when $R_0 < 1$, that is, the disease dies out. When $R_0 > 1$, they showed the existence of an endemic equilibrium and established partial results on the global stability of the endemic equilibrium. In [31] the authors extended their investigation to the model in [7] with time-periodic parameters β and γ. They defined a basic reproduction number R_0 of this periodic model and showed that a combination of spatial heterogeneity and temporal periodicity tends to enhance the persistence of the disease.

It is worth mentioning that more recently in [33], a two-strain SIS epidemic reaction-diffusion model with homogeneous Neumann boundary conditions has been considered:

$$\begin{aligned} \frac{\partial S}{\partial t} &= d_S \Delta S - (\beta_1(x) I_1 + \beta_2(x) I_2)\frac{S}{S+I_1+I_2} \\ &\quad + \gamma_1(x) I_1 + \gamma_2(x) I_2, \qquad x \in \Omega, \quad t > 0, \\ \frac{\partial I_1}{\partial t} &= d_1 \Delta I_1 + \beta_1(x)\frac{SI_1}{S+I_1+I_2} - \gamma_1(x) I_1, \qquad x \in \Omega, \quad t > 0, \\ \frac{\partial I_2}{\partial t} &= d_2 \Delta I_2 + \beta_2(x)\frac{SI_2}{S+I_1+I_2} - \gamma_2(x) I_2, \qquad x \in \Omega, \quad t > 0, \end{aligned} \tag{1.3}$$

under the condition

$$\int_\Omega (I_1(x,0) + I_2(x,0))\, dx > 0 \text{ with } S(x,0) > 0, I_1(x,0) > 0, I_2(x,0) > 0 \text{ for } x \in \Omega.$$

In this paper a basic reproduction number R_0 was defined and it was shown that if $R_0 < 1$ then the disease-free equilibrium is globally attractive. The local stability of the semi-trivial equilibria was also studied. More importantly, using the monotonicity of the equilibrial system, the existence of coexistence equilibrium was established. Numerical investigations were conducted as well.

The purpose of our paper is two fold: 1) to investigate a multiple-strain pathogen model with diffusion and 2) to use a mass action incidence term (e.g., see [1] for models without diffusion) to model the transmission of the disease. Even though our model bears a resemblance to model (1.3), there is a main difference between our model and model (1.3): Our model assumes a bilinear disease transmission term, whereas model (1.3) assumes that infection is frequency dependent. Thus, the two models are fundamentally different and require different methods to analyze. Furthermore, we study the global attractivity of the endemic equilibrium and present conditions for competitive exclusion or coexistence between the strains to occur.

The paper is organized as follows: In section 2 we present a two-strain reaction-diffusion model. In section 3 we study the model under the assumptions that all the parameters are constant. We define a basic reproduction number R_0 and study the long-time behavior of the solution. In particular, we show that if $R_0 \leq 1$ the disease-free equilibrium is globally attractive and if $R_0 > 1$ competitive exclusion is the only outcome between the strains. In section 4 we consider the case where the diffusion rates of susceptible and infected individuals are equal while the transmission and recovery parameters are heterogeneous. We study the long-time behavior of the solution as it relates to the basic reproduction number R_0. In particular, we provide conditions under which competitive exclusion between the strains occurs and conditions under which coexistence between the strains is the outcome. In section 5 we verify the validity of such conditions. In section 6 we make concluding remarks.

2. **The model.**

In this section we present a two-strain epidemic model with diffusion. In particular, let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Let $S(x,t)$ be the density of susceptible individuals at location x and time t, and let $I_i(x,t)$ be the density of individuals infected with strain i $(i = 1, 2)$ at location x and time t. We assume that the individuals randomly move in the domain Ω with diffusion rates d_S and d_i $(i = 1, 2)$ for susceptible and infected individuals, respectively. If any individual cannot be infected with two diseases at the same time, and recovered individuals become susceptible immediately, an SIS epidemic reaction-diffusion model can be formulated as follows:

$$\begin{aligned} \tfrac{\partial S}{\partial t} &= d_S \Delta S - (\beta_1(x) I_1 + \beta_2(x) I_2) S + \gamma_1(x) I_1 + \gamma_2(x) I_2, && x \in \Omega, \quad t > 0, \\ \tfrac{\partial I_1}{\partial t} &= d_1 \Delta I_1 + \beta_1(x) S I_1 - \gamma_1(x) I_1, && x \in \Omega, \quad t > 0, \\ \tfrac{\partial I_2}{\partial t} &= d_2 \Delta I_2 + \beta_2(x) S I_2 - \gamma_2(x) I_2, && x \in \Omega, \quad t > 0, \end{aligned} \tag{2.1}$$

where the disease transmission rate function $\beta_i(x)$ describes the effective interaction between the susceptible individuals and the individuals infected with disease i at location x, and the function $\gamma_i(x)$ represents the recovery rate of the individuals infected with disease i at location x. All β_i and γ_i are positive Hölder continuous functions in $\overline{\Omega}$. Furthermore, we assume that there is no flux on the boundary $\partial\Omega$, that is,

$$\frac{\partial S}{\partial n} = \frac{\partial I_i}{\partial n} = 0, \quad x \in \partial\Omega, \ t > 0, \tag{2.2}$$

where $\partial/\partial n$ is the outward normal derivative to $\partial\Omega$. We also assume that the initial data satisfy

(H1) $S(x,0)$ and $I_i(x,0)$ are nonnegative continuous functions in $\overline{\Omega}$, and initially the number of individuals infected with disease i is positive, i.e., $\int_\Omega I_i(x,0)dx > 0$.

Let

$$\int_\Omega (S(x,0) + I_1(x,0) + I_2(x,0))dx \equiv N$$

be the total number of individuals at $t = 0$. Adding up the three equations in (2.1) and then integrating over the domain Ω, we find

$$\frac{\partial}{\partial t}\int_\Omega (S + I_1 + I_2)dx = 0, \quad t > 0,$$

which implies that the total population size is a constant given by

$$\int_\Omega (S(x,t) + I_1(x,t) + I_2(x,t))dx = N. \tag{2.3}$$

We then establish the global existence and boundedness results for the model.

Theorem 2.1. *Suppose that hypothesis (H1) holds. Then the solution* $(S(x,t), I_1(x,t), I_2(x,t))$ *of problem* (2.1)-(2.2) *exists uniquely and globally. Moreover, there exists a positive constant* M *depending on the initial data and* $\max_{x\in\overline{\Omega}}\{\gamma_i(x)/\beta_i(x)\}$ *(*$i = 1,2$*) such that*

$$0 < S(x,t), I_1(x,t), I_2(x,t) \le M \quad \text{for } x \in \overline{\Omega},\ t \in (0,\infty). \tag{2.4}$$

Proof. Let $((\hat{S}(x,t), \hat{I}_1(x,t), \hat{I}_2(x,t))$ be the local solution of the following problem:

$$\begin{array}{lll}
\frac{\partial \hat{S}}{\partial t} = d_S\Delta\hat{S} + \gamma_1(x)\hat{I}_1 + \gamma_2(x)\hat{I}_2, & x \in \Omega, & t > 0,\\
\frac{\partial \hat{I}_1}{\partial t} = d_1\Delta\hat{I}_1 + \beta_1(x)\hat{S}\hat{I}_1 - \gamma_1(x)\hat{I}_1, & x \in \Omega, & t > 0,\\
\frac{\partial \hat{I}_2}{\partial t} = d_2\Delta\hat{I}_2 + \beta_2(x)\hat{S}\hat{I}_2 - \gamma_2(x)\hat{I}_2, & x \in \Omega, & t > 0,\\
\frac{\partial \hat{S}}{\partial n} = \frac{\partial \hat{I}_1}{\partial n} = \frac{\partial \hat{I}_2}{\partial n} = 0, & x \in \partial\Omega, & t > 0,\\
\hat{S}(x,0) = S(x,0), \quad \hat{I}_1(x,0) = I_1(x,0), \quad \hat{I}_2(x,0) = I_2(x,0) & x \in \overline{\Omega}. &
\end{array} \tag{2.5}$$

Then $((\hat{S}(x,t), \hat{I}_1(x,t), \hat{I}_2(x,t))$ and $(0,0,0)$ are a pair of coupled upper and lower solutions of problem (2.1)-(2.2), and it follows that there exists a unique solution $((S(x,t), I_1(x,t), I_2(x,t))$ of (2.1)-(2.2) for $x \in \overline{\Omega}$ and $t \in [0, T_{\max})$, where $T_{\max}$ is the maximal existence time. Moreover, by the maximum principle, the solution is positive in $\overline{\Omega} \times (0, T_{\max})$ (cf. [28]). We now consider the problem for $S(x,t)$ in $\Omega \times (0, T_{\max})$:

$$\begin{array}{lll}
S_t = d_S\Delta S + (\gamma_1(x) - \beta_1(x)S)I_1 & & \\
\qquad +(\gamma_2(x) - \beta_2(x)S)I_2, & x \in \Omega, & t \in (0, T_{\max}),\\
\frac{\partial S}{\partial n} = 0, & x \in \partial\Omega, & t \in (0, T_{\max}).
\end{array} \tag{2.6}$$

Choose $M_0 = \max\{\max_{x\in\overline{\Omega}} S(x,0), \max_{x\in\overline{\Omega}}\{\gamma_1(x)/\beta_1(x)\}, \max_{x\in\overline{\Omega}}\{\gamma_2(x)/\beta_2(x)\}\}$. Then for any nonnegative functions $I_1(x,t)$ and $I_2(x,t)$, M_0 and 0 are a pair of upper and lower solutions of problem (2.6). By the comparison principle, one can see that $S(x,t) \le M_0$ in $\overline{\Omega} \times [0, T_{\max})$. Since $\int_\Omega I_i(x,t)dx \le N$, in view of Theorem 3.1 in [4], there exists a positive constant M_i depending on $I_i(x,0)$ such that $I_i(x,t) \le M_i$ ($i = 1,2$) in $\overline{\Omega} \times [0, T_{\max})$. Hence, it follows from the standard theory for semilinear parabolic systems that $T_{\max} = \infty$. ∎

3. **The case of constant coefficients.** In this section, we consider the case that all coefficients are spatially homogeneous, i.e., β_i and γ_i $(i = 1, 2)$ are positive constants. We first consider the case $\gamma_1/\beta_1 \neq \gamma_2/\beta_2$. To this end, we study the existence of the equilibria and define a basic reproduction number for problem (2.1)-(2.3). We then briefly discuss the case $\gamma_1/\beta_1 = \gamma_2/\beta_2$.

3.1. **Equilibria.** The global attractivity result to be established in Section 3.2 will show that every solution of problem (2.1)-(2.3) converges to a nonnegative constant equilibrium, which implies the nonexistence of non-constant equilibria. Thus we only need to consider the following equilibrium system:

$$\begin{aligned}&(\beta_1\bar{S} - \gamma_1)\bar{I}_1 = 0,\\ &(\beta_2\bar{S} - \gamma_2)\bar{I}_2 = 0,\\ &\bar{S} + \bar{I}_1 + \bar{I}_2 = \tfrac{N}{|\Omega|}.\end{aligned} \tag{3.1}$$

As considered in other epidemic models, we will focus on the *disease-free equilibrium* (DFE) and the *endemic equilibrium* (EE). A DFE is a solution of (3.1) with $\bar{I}_1 = \bar{I}_2 = 0$ and $\bar{S}$ a positive constant, while an EE is a nonnegative solution of (3.1) with $\bar{I}_1 > 0$ or $\bar{I}_2 > 0$. Since (3.1) is an algebraic system, one can easily find the following.

Proposition 1. *There are three cases:*

(a) If $\gamma_1/\beta_1, \gamma_2/\beta_2 \geq N/|\Omega|$, there exists a unique DFE $(N/|\Omega|, 0, 0)$ and no EEs;
(b) If $\gamma_2/\beta_2 \geq N/|\Omega| > \gamma_1/\beta_1$ or $\gamma_1/\beta_1 \geq N/|\Omega| > \gamma_2/\beta_2$, there exists a unique DFE $(N/|\Omega|, 0, 0)$ and an EE $(\gamma_1/\beta_1, N/|\Omega| - \gamma_1/\beta_1, 0)$ or $(\gamma_2/\beta_2, 0, N/|\Omega| - \gamma_2/\beta_2)$;
(c) If $N/|\Omega| > \gamma_1/\beta_1, \gamma_2/\beta_2$, there exists a unique DFE $(N/|\Omega|, 0, 0)$ and two EEs $(\gamma_1/\beta_1,\ N/|\Omega| - \gamma_1/\beta_1, 0)$ and $(\gamma_2/\beta_2, 0, N/|\Omega| - \gamma_2/\beta_2)$.

We then introduce a basic reproduction number $\mathcal{R}_0$ in order to analyze the model's dynamics. By virtue of Proposition 1, we define a basic reproduction number as follows:

$$\mathcal{R}_0 = \max\left\{\frac{N\beta_1}{|\Omega|\gamma_1}, \frac{N\beta_2}{|\Omega|\gamma_2}\right\}.$$

For convenience, we also define a related number $\mathcal{R}_1$ by

$$\mathcal{R}_1 = \min\left\{\frac{N\beta_1}{|\Omega|\gamma_1}, \frac{N\beta_2}{|\Omega|\gamma_2}\right\}.$$

Note that the assumption $\gamma_1/\beta_1 \neq \gamma_2/\beta_2$ implies $\mathcal{R}_0 > \mathcal{R}_1$. In this section, from now on, we always denote the unique DFE by E_0 and let $E_1 = (\gamma_1/\beta_1, N/|\Omega| - \gamma_1/\beta_1, 0)$ and $E_2 = (\gamma_2/\beta_2, 0, N/|\Omega| - \gamma_2/\beta_2)$. Then Proposition 1 is equivalent to the following.

Proposition 2. *There are three cases:*

(a) If $\mathcal{R}_0 \leq 1$, there exists a unique DFE E_0 and no EEs;
(b) If $\mathcal{R}_0 > 1 \geq \mathcal{R}_1$, there exists a unique DFE E_0 and an EE E_1 or E_2;
(c) If $\mathcal{R}_1 > 1$, there exists a unique DFE E_0 and two EEs E_1 and E_2.

3.2. **Global attractivity.** To conduct our discussion on global attractivity, we mainly rely on the LaSalle Invariance principle for nonlinear dynamical systems (see [19]). Let $X = L^p(\Omega)$ with $p > m$. We define a closed linear operator A with dense domain $D(A)$ given by

$$Au = -\Delta u, \quad D(A) = \left\{ u \in W^{2,p}(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

Then $-A$ generates an analytic semigroup e^{-tA} on X. Let X_α $(0 \leq \alpha \leq 1)$ be the fractional power space of X with respect to A. Since the embedding $X_\alpha \subset C^{1,\mu}(\overline{\Omega})$ is compact if $1 + \mu < 2\alpha - m/p$, we choose α close to 1 and p large such that X_α is compactly embedded into $C^{1,\mu}(\overline{\Omega})$. Let $P \subset X_\alpha$ be the cone of all nonnegative functions of X_α with nonempty interior. We introduce

$$D = \left\{ (u, v, w) \in X_\alpha \times X_\alpha \times X_\alpha : \int_\Omega (u + v + w) dx = N \text{ and } u, v, w \in P \right\}.$$

Then D is a closed subset of $X_\alpha \times X_\alpha \times X_\alpha$, and the solution (S, I_1, I_2) of problem (2.1)-(2.3) induces a nonlinear dynamical system $\{\Phi(t), \ \ t \in \mathbb{R}^+\}$ on D given by

$$\Phi(t)(S(0), I_1(0), I_2(0)) := (S(t), I_1(t), I_2(t)), \quad t \in \mathbb{R}^+,$$

where (S, I_1, I_2) is the solution with the initial condition $(S(0), I_1(0), I_2(0)) \in D$.

We then recall a well-known result [24].

Lemma 3.1. *Let $u(x,t)$ be the solution of the problem:*

$$\begin{aligned} u_t &= d\Delta u \qquad && x \in \Omega, \quad t > 0, \\ \tfrac{\partial u}{\partial n} &= 0 && x \in \partial\Omega, \quad t > 0 \end{aligned}$$

with $u(x,0) \geq 0$. Then $u(x,t)$ converges to $\int_\Omega u(x,0)dx/|\Omega|$ uniformly in $x \in \overline{\Omega}$ as $t \to \infty$.

We are now in a position to establish the main result.

Theorem 3.2. *The following statements hold.*

(a) If $\mathcal{R}_0 \leq 1$, then E_0 is globally attractive;
(b) If $\mathcal{R}_0 > 1$ and if $\gamma_2/\beta_2 > \gamma_1/\beta_1$ or $\gamma_1/\beta_1 > \gamma_2/\beta_2$, then E_1 or E_2 is globally attractive.

Proof. We first consider the case that $\mathcal{R}_0 \leq 1$. Define a continuously differentiable real valued function $V : D \to \mathbb{R}$ by

$$V(S, I_1, I_2) = \frac{1}{2} \int_\Omega \left(S - \frac{N}{|\Omega|} \right)^2 dx + B_1 \int_\Omega I_1 dx + B_2 \int_\Omega I_2 dx$$

for all $(S, I_1, I_2) \in D$ with B_1, B_2 constants to be determined. We can check that for all $(S, I_1, I_2) \in D \cap (D(A) \times D(A) \times D(A))$,

$$\begin{aligned} \dot{V}(S, I_1, I_2) &= \limsup_{t \to 0^+} \frac{V(\Phi(t)(S, I_1, I_2)) - V(S, I_1, I_2)}{t} \\ &= \int_\Omega \left(S - \frac{N}{|\Omega|} \right) (d_S \Delta S - (\beta_1 I_1 + \beta_2 I_2)S + \gamma_1 I_1 + \gamma_2 I_2) dx \end{aligned}$$

$$
\begin{aligned}
&\quad + B_1 \int_\Omega (d_1 \Delta I_1 + I_1(\beta_1 S - \gamma_1))dx \\
&\quad + B_2 \int_\Omega (d_2 \Delta I_2 + I_2(\beta_2 S - \gamma_2))dx \\
&= -d_S \int_\Omega |\nabla S|^2 dx - \int_\Omega I_1(\beta_1 S - \gamma_1)\left(S - \frac{N}{|\Omega|} - B_1\right)dx \\
&\quad - \int_\Omega I_2(\beta_2 S - \gamma_2)\left(S - \frac{N}{|\Omega|} - B_2\right)dx \\
&= -d_S \int_\Omega |\nabla S|^2 dx - \beta_1 \int_\Omega I_1 \left(S - \frac{\gamma_1}{\beta_1}\right)^2 dx - \beta_2 \int_\Omega I_2 \left(S - \frac{\gamma_2}{\beta_2}\right)^2 dx \\
&\le 0,
\end{aligned}
$$

where

$$
B_1 = \frac{\gamma_1}{\beta_1} - \frac{N}{|\Omega|} \quad \text{and} \quad B_2 = \frac{\gamma_2}{\beta_2} - \frac{N}{|\Omega|}.
$$

Since V is continuously differentiable and $D \cap (D(A) \times D(A) \times D(A))$ is dense in D, we find that $\dot{V}(S, I_1, I_2) \le 0$ for all $(S, I_1, I_2) \in D$. Thus V is a Lyapunov functional on D.

Let $E := \{(S, I_1, I_2) \in D : \dot{V}(S, I_1, I_2) = 0\}$ and M be the largest positively invariant subset of E. It follows from Theorem 2.1 and some standard arguments (see [18, 19]) that the orbit $\{(S(t), I_1(t), I_2(t)), t > 0\}$ is pre-compact in D. So by the LaSalle Invariance principle, we have that $\lim_{t\to\infty} dist(\Phi(t)(S(0), I_1(0), I_2(0)), M) = 0$. In view of $\dot{V}$, $\int_\Omega |\nabla S|^2 dx = 0$ implies that S is a constant, and $\int_\Omega I_i(S - \gamma_i/\beta_i)^2 dx = 0$ implies that either $I_i = 0$ or $S = \gamma_i/\beta_i$, $i = 1, 2$. If $S = \gamma_i/\beta_i$, then $\int_\Omega I_i dx = N - |\Omega|\gamma_i/\beta_i \le 0$ which yields $I_i = 0$. Thus, we must have $I_i = 0$, and so $E = \{E_0\}$. Hence, $M = \{E_0\}$, and it follows that E_0 is globally attractive.

We then consider the case that $\mathcal{R}_0 > 1 \ge \mathcal{R}_1$ with $\gamma_2/\beta_2 > \gamma_1/\beta_1$, that is, $\gamma_2/\beta_2 \ge N/|\Omega| > \gamma_1/\beta_1$. In this case, there exist two equilibria E_0 and E_1. Using the same Lyapunov functional V, we find

$$
E = \left\{ \left(\frac{\gamma_1}{\beta_1}, w, 0\right) : \int_\Omega w dx = N - \frac{\gamma_1}{\beta_1}|\Omega| \right\} \cup \{E_0\}.
$$

We claim that M consists of E_0 and E_1 at most. To see this, we make use of the invariant property of M, i.e., $\Phi(t)M = M$ for all $t > 0$ (see [18, 19]). Let $\epsilon > 0$ be given and B_ϵ be the open ball in D centered at E_1 with radius ϵ. Suppose that $(\gamma_1/\beta_1, w, 0) \in M$ with $\int_\Omega w dx = N - \gamma_1|\Omega|/\beta_1$. By Lemma 3.1, we have that

$$
\Phi(t)\left(\frac{\gamma_1}{\beta_1}, w, 0\right) \to E_1 \quad \text{as } t \to \infty.
$$

So there exists $T > 0$ such that $\Phi(t)(\gamma_1/\beta_1, w, 0) \in B_\epsilon$ for $t > T$. It follows from the invariance property that $M \subseteq B_\epsilon \cup \{E_0\}$. Since $\epsilon > 0$ is arbitrary, the claim is valid.

The LaSalle Invariance principle implies that $(S(t), I_1(t), I_2(t)) \to M$. It then follows that either the trajectory converges to E_0 or it converges to E_1. Assume that $\lim_{t\to\infty} \Phi(t)(S(0), I_1(0), I_2(0)) = E_0$. Since $\beta_1 N/|\Omega| > \gamma_1$, we can choose $\epsilon > 0$ so small that $\beta_1(N/|\Omega| - \epsilon) > \gamma_1$. For this ϵ, there exists a $T > 0$ such that $S(x, t) > N/|\Omega| - \epsilon$ for all $(x, t) \in \overline{\Omega} \times [T, \infty)$. By $(2.1)_2$, the following inequality holds

$$\frac{\partial I_1}{\partial t} - d_1 \Delta I_1 \geq \beta_1 \left(\frac{N}{|\Omega|} - \epsilon \right) I_1 - \gamma_1 I_1 \quad \text{for } (x,t) \in \Omega \times [T, \infty). \tag{3.2}$$

We then consider a related problem:

$$\begin{array}{lll} J_t = d_1 \Delta J + \beta_1 \left(\frac{N}{|\Omega|} - \epsilon \right) J - \gamma_1 J, & x \in \Omega, & t \in (T, \infty), \\ \frac{\partial J}{\partial n} = 0, & x \in \partial\Omega, & t \in (T, \infty), \\ J(x,T) = \min_{x \in \overline{\Omega}} I_1(x,T) \equiv I_{1m}(T), & x \in \overline{\Omega}. & \end{array} \tag{3.3}$$

The comparison principle yields that $I_1 \geq J$ on $\overline{\Omega} \times [T, \infty)$, and it is easy to check that $J = I_{1m}(T) e^{(\beta_1 N/|\Omega| - \epsilon\beta_1 - \gamma_1)t}$. Since $I_1 \geq J$ and $J \to \infty$ as $t \to \infty$, this contradicts the fact that I_1 is uniformly bounded. Hence, $M = \{E_1\}$, and E_1 is globally attractive. The case that $\mathcal{R}_0 > 1 \geq \mathcal{R}_1$ with $\gamma_1/\beta_1 > \gamma_2/\beta_2$ can be discussed analogously.

We now consider the case that $\mathcal{R}_1 > 1$ with $\gamma_2/\beta_2 > \gamma_1/\beta_1$, i.e., $N/|\Omega| > \gamma_2/\beta_2 > \gamma_1/\beta_1$. Using the same functional V, we obtain

$$\begin{aligned} E &= \left\{ \left(\frac{\gamma_1}{\beta_1}, w, 0 \right) : \int_\Omega w dx = N - \frac{\gamma_1}{\beta_1} |\Omega| \right\} \\ &\quad \cup \left\{ \left(\frac{\gamma_2}{\beta_2}, 0, w \right) : \int_\Omega w dx = N - \frac{\gamma_2}{\beta_2} |\Omega| \right\} \cup \{E_0\}. \end{aligned}$$

Proceeding as before, we can claim that M consists of E_0, E_1, and E_2 at most. Then the Lasalle invariance principle implies that the trajectory converges to E_0, E_1 or E_2. Assume that $\lim_{t\to\infty} \Phi(t)(S(0), I_1(0), I_2(0)) = E_2$. Since $\gamma_2/\beta_2 > \gamma_1/\beta_1$, we can choose $\epsilon > 0$ so small that $\beta_1 \gamma_2/\beta_2 - \beta_1 \epsilon - \gamma_1 > 0$. For this ϵ, there exists a $T > 0$ such that $S(x,t) > \gamma_2/\beta_2 - \epsilon$ for all $(x,t) \in \overline{\Omega} \times [T, \infty)$. By $(2.1)_2$, the following inequality holds

$$\frac{\partial I_1}{\partial t} - d_1 \Delta I_1 \geq \beta_1 \left(\frac{\gamma_2}{\beta_2} - \epsilon \right) I_1 - \gamma_1 I_1 \quad \text{for } (x,t) \in \overline{\Omega} \times [T, \infty). \tag{3.4}$$

We then consider a related problem:

$$\begin{array}{lll} J_t = d_1 \Delta J + \beta_1 \left(\frac{\gamma_2}{\beta_2} - \epsilon \right) J - \gamma_1 J, & x \in \Omega, & t \in (T, \infty), \\ \frac{\partial J}{\partial n} = 0, & x \in \partial\Omega, & t \in (T, \infty), \\ J(x,T) = \min_{x \in \overline{\Omega}} I_1(x,T) \equiv I_{1m}(T), & x \in \overline{\Omega}. & \end{array} \tag{3.5}$$

The comparison principle yields that $I_1 \geq J$ on $\overline{\Omega} \times [T, \infty)$, and it is easy to check that $J = I_{1m}(T) e^{(\beta_1 \gamma_2/\beta_2 - \beta_1 \epsilon - \gamma_1)t}$. Since $I_1 \geq J$ and $J \to \infty$ as $t \to \infty$, this contradicts the fact that I_1 is uniformly bounded. So the trajectory cannot converge to $\{E_2\}$. Similarly, one can see that it cannot converge to $\{E_0\}$. Hence, E_1 is globally attractive. The case that $\mathcal{R}_1 > 1$ with $\gamma_1/\beta_1 > \gamma_2/\beta_2$ can be analyzed similarly. □

We then briefly discuss the case that $\gamma_1/\beta_1 = \gamma_2/\beta_2$. It is easy to see that if $\mathcal{R}_0 > 1$ there are infinitely many endemic equilibria contained in the set:

$$L \equiv \left\{ (\gamma_1/\beta_1, \bar{I}_1, \bar{I}_2) : \bar{I}_1 + \bar{I}_2 = \frac{N}{|\Omega|} - \frac{\gamma_1}{\beta_1} \text{ and } \bar{I}_1, \bar{I}_2 \in \mathbb{R}^+ \right\}.$$

We have the following result.

Theorem 3.3. *If $\gamma_1/\beta_1 = \gamma_2/\beta_2$, then the following statements hold.*

(a) If $\mathcal{R}_0 \le 1$, then E_0 is globally attractive;
(b) If $\mathcal{R}_0 > 1$, then L is a global attractor, i.e.,

$$\lim_{t\to\infty} dist((S(x,t), I_1(x,t), I_2(x,t)), L) = 0$$

uniformly for $x \in \overline{\Omega}$.

Proof. We use the same Lyapunov functional V as introduced in the proof of Theorem 3.2. The case that $\mathcal{R}_0 \le 1$ can be proved exactly the same. For the case that $\mathcal{R}_0 > 1$, the total derivative $\dot{V} = 0$ leads to the set

$$E = \left\{ \left(\frac{\gamma_1}{\beta_1}, w_1, w_2 \right) : \int_\Omega (w_1 + w_2) dx = N - \frac{\gamma_1}{\beta_1} |\Omega| \right\} \cup \{E_0\}.$$

Let M be the largest positively invariant subset of E, and we can show analogously as for Theorem 3.2 that M only consists of spatially homogeneous equilibria. Then by the Lasalle invariance principle, M attracts every solution of (2.1)-(2.3). And it then suffices to show that the solution with initial data satisfying (H1) does not converge to E_0. The proof is similar to that for Theorem 3.2, and hence is omitted. □

4. **The case of equal diffusion rates.** If all the coefficients are homogeneous, Theorem 3.2 indicates that one disease will drive the other one to extinction in the long run if $\mathcal{R}_0 > 1$, and this suggests that there may exist some competition between the two diseases. Moreover, if $\gamma_1/\beta_1 \ne \gamma_2/\beta_2$, since the only endemic equilibria are E_1 and E_2, there does not exist a coexistence equilibrium. Then one can naturally ask that whether it is still the case when the coefficients are spatially inhomogeneous. To answer such a question, in the sequel we let $d \equiv d_S = d_1 = d_2$ and assume that the coefficients β_i, γ_i $(i = 1, 2)$ are spatially inhomogeneous. We will show that it is possible for a coexistence equilibrium to exist, and thus the model with spatially heterogeneous coefficients may induce more complicated dynamical behavior.

4.1. **Basic reproduction number.** Adding up the three equations in (2.1), we have that

$$\frac{\partial(S + I_1 + I_2)}{\partial t} - d\Delta(S + I_1 + I_2) = 0 \quad \text{in } \Omega \times (0, \infty),$$

and the boundary condition (2.2) implies that

$$\frac{\partial(S + I_1 + I_2)}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

It then follows from Lemma 3.1 that $S + I_1 + I_2 \to N/|\Omega|$ as $t \to \infty$. This suggests that we may consider the Lotka-Volterra competition model:

$$\begin{aligned} \tfrac{\partial I_1}{\partial t} &= d\Delta I_1 + \left(\tfrac{\beta_1 N}{|\Omega|} - \gamma_1 - \beta_1 I_1 - \beta_1 I_2 \right) I_1, \qquad x \in \Omega, \quad t > 0, \\ \tfrac{\partial I_2}{\partial t} &= d\Delta I_2 + \left(\tfrac{\beta_2 N}{|\Omega|} - \gamma_2 - \beta_2 I_1 - \beta_2 I_2 \right) I_2, \qquad x \in \Omega, \quad t > 0. \end{aligned} \tag{4.1}$$

However, the dynamics of the above model can be quite complicated. For certain results about (4.1), the readers are referred to [13, 17, 20, 22, 23, 25, 26] and the references therein.

For a Hölder continuous function $\alpha(x)$ on $\overline{\Omega}$, let $\lambda^*(\alpha)$ be the principal eigenvalue of the following problem

$$\begin{aligned} &d\Delta\varphi + \alpha(x)\varphi + \lambda\varphi = 0, \quad x \in \Omega, \\ &\tfrac{\partial \varphi}{\partial n} = 0, \qquad\qquad\qquad\quad\; x \in \partial\Omega. \end{aligned}$$

Then $\lambda^*(\alpha)$ is given by the variational formula:

$$\lambda^*(\alpha) = \inf\left\{ \int_\Omega \left(d|\nabla\varphi|^2 - \alpha\varphi^2\right) dx : \quad \varphi \in H^1(\Omega) \text{ and } \int_\Omega \varphi^2 dx = 1 \right\}. \tag{4.2}$$

In view of (4.1) and the variational formula, for $i = 1, 2$ we define

$$r_i = \sup\left\{ \frac{\frac{N}{|\Omega|}\int_\Omega \beta_i \varphi^2 dx}{\int_\Omega (d|\nabla\varphi|^2 + \gamma_i\varphi^2) dx} : \quad \varphi \in H^1(\Omega) \text{ and } \varphi \neq 0 \right\}.$$

Remark 1. There is a simple relation between r_i and $\lambda^*(N\beta_i/|\Omega| - \gamma_i)$, $i = 1, 2$ (see [7]):

(i) $r_i < 1$ if and only $\lambda^*(N\beta_i/|\Omega| - \gamma_i) > 0$;
(ii) $r_i = 1$ if and only $\lambda^*(N\beta_i/|\Omega| - \gamma_i) = 0$;
(iii) $r_i > 1$ if and only $\lambda^*(N\beta_i/|\Omega| - \gamma_i) < 0$.

We then define a basic reproduction number $\mathcal{R}_0$ and a related number $\mathcal{R}_1$ as follows:

$$\mathcal{R}_0 = \max\{r_1, r_2\}, \qquad \mathcal{R}_1 = \min\{r_1, r_2\}.$$

Clearly, this definition is consistent with the previous homogenous case.

The following result is well known (see [13]).

Lemma 4.1. *Suppose that α and β are Hölder continuous function on $\overline{\Omega}$ with $\alpha(x_0) > 0$ for some $x_0 \in \overline{\Omega}$ and $\beta(x) > 0$ for $x \in \overline{\Omega}$, and $u_0 \in C(\overline{\Omega})$ is nonnegative. Consider the problem:*

$$\begin{aligned} &u_t = d\Delta u + (\alpha(x) - \beta(x)u)u && x \in \Omega,\ t > 0 \\ &\tfrac{\partial u}{\partial n} = 0 && x \in \partial\Omega,\ t > 0 \\ &u(x, 0) = u_0(x) && x \in \overline{\Omega}. \end{aligned} \tag{4.3}$$

Then the following statements hold.

(a) If $\lambda^*(\alpha) \geq 0$, then all solutions of (4.3) converge to zero uniformly;
(b) If $\lambda^*(\alpha) < 0$, then for each nontrivial u_0, the solution $u(x,t)$ of (4.3) converges to $\bar{u}(x)$ uniformly, where $\bar{u}(x)$ is the unique positive steady state of (4.3).

4.2. **Disease-free equilibrium.** We first show that the disease-free equilibrium exists uniquely.

Proposition 3. *Problem (2.1)-(2.3) has a unique DFE given by $E_0 \equiv (\tilde{S}, 0, 0) = (N/|\Omega|,\ 0, 0)$.*

Proof. Clearly, $(N/|\Omega|, 0, 0)$ is a DFE. Now for any DFE $(\tilde{S}, 0, 0)$, by (2.1), we have that $\Delta\tilde{S} = 0$. Then by the maximum principle and the boundary condition $\partial\tilde{S}/\partial n = 0$, $\tilde{S}$ must be a constant in $\overline{\Omega}$. It then follows from (2.3) that $\tilde{S} = N/|\Omega|$. □

Similar to [7, 16], we now linearize (2.1) around the DFE. Let $\eta(x,t) = S(x,t) - N/|\Omega|$, $\xi_1(x,t) = I_1(x,t)$, and $\xi_2(x,t) = I_2(x,t)$. Using (2.1) and dropping high order terms, we obtain the following system:

$$\begin{aligned}
\tfrac{\partial \eta}{\partial t} &= d\Delta\eta - \left(\tfrac{N}{|\Omega|}\beta_1 - \gamma_1\right)\xi_1 - \left(\tfrac{N}{|\Omega|}\beta_2 - \gamma_2\right)\xi_2, && x\in\Omega, \quad t>0,\\
\tfrac{\partial \xi_1}{\partial t} &= d\Delta\xi_1 + \left(\tfrac{N}{|\Omega|}\beta_1 - \gamma_1\right)\xi_1, && x\in\Omega, \quad t>0,\\
\tfrac{\partial \xi_2}{\partial t} &= d\Delta\xi_2 + \left(\tfrac{N}{|\Omega|}\beta_2 - \gamma_2\right)\xi_2, && x\in\Omega, \quad t>0.
\end{aligned}$$

Let $(\eta(x,t), \xi_1(x,t), \xi_2(x,t)) = (e^{-\lambda t}\phi(x), e^{-\lambda t}\psi_1(x), e^{-\lambda t}\psi_2(x))$. We then derive an eigenvalue problem:

$$\begin{aligned}
& d\Delta\phi - \left(\tfrac{N}{|\Omega|}\beta_1 - \gamma_1\right)\psi_1 - \left(\tfrac{N}{|\Omega|}\beta_2 - \gamma_2\right)\psi_2 + \lambda\phi = 0, && x\in\Omega,\\
& d\Delta\psi_1 + \left(\tfrac{N}{|\Omega|}\beta_1 - \gamma_1\right)\psi_1 + \lambda\psi_1 = 0, && x\in\Omega\\
& d\Delta\psi_2 + \left(\tfrac{N}{|\Omega|}\beta_2 - \gamma_2\right)\psi_2 + \lambda\psi_2 = 0, && x\in\Omega
\end{aligned} \tag{4.4}$$

with boundary conditions

$$\frac{\partial\phi}{\partial n} = \frac{\partial\psi_1}{\partial n} = \frac{\partial\psi_2}{\partial n} = 0, \quad x\in\partial\Omega. \tag{4.5}$$

In view of (2.3) and Proposition 3, we impose an additional condition

$$\int_\Omega (\phi + \psi_1 + \psi_2)dx = 0. \tag{4.6}$$

The following lemma shows that the stability of the DFE relies on the magnitude of $\mathcal{R}_0$.

Proposition 4. *The DFE is stable if $\mathcal{R}_0 < 1$, and it is unstable if $\mathcal{R}_0 > 1$.*

Proof. Suppose that $\mathcal{R}_0 < 1$, i.e., $r_1, r_2 < 1$. By Remark 1, we have $\lambda^*(N\beta_i/|\Omega| - \gamma_i) > 0$, $i = 1,2$. It suffices to show that if $(\lambda, \phi, \psi_1, \psi_2)$ solves the eigenvalue problem (4.4)-(4.6) with at least one of ϕ, ψ_1, or ψ_2 nontrivial, then $Re(\lambda)$ is positive. Assume that $\psi_1 = \psi_2 = 0$, then $d\Delta\phi + \lambda\phi = 0$ in Ω and $\partial\phi/\partial n = 0$ on $\partial\Omega$, which implies that λ is positive as long as ϕ is not a constant. If ϕ is a constant, by (4.6), we have $\phi \equiv 0$, which is a contradiction. We then assume that at least one of ψ_1 and ψ_2 is nontrivial. Without loss of generality, we may assume that ψ_1 is nontrivial. It then follows that λ is an eigenvalue of the problem

$$d\Delta\psi_1 + \left(\frac{N}{|\Omega|}\beta_1 - \gamma_1\right)\psi_1 + \lambda\psi_1 = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\psi_1}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{4.7}$$

and so $Re(\lambda) \geq \lambda^*(N\beta_1/|\Omega| - \gamma_1) > 0$.

Now suppose that $\mathcal{R}_0 > 1$, i.e., $r_1 > 1$ or $r_2 > 1$. Without loss of generality, we may assume that $r_1 > 1$, and it follows from Remark 1 that $\lambda^*(N\beta_1/|\Omega| - \gamma_1) < 0$. Let $\lambda = \lambda^*(N\beta_1/|\Omega| - \gamma_1)$ and ψ_1 be a positive eigenvector of problem (4.7) corresponding to the principal eigenvalue $\lambda^*(N\beta_1/|\Omega| - \gamma_1)$. Then let ϕ be the solution of

$$d\Delta\phi - \left(\frac{N}{|\Omega|}\beta_1 - \gamma_1\right)\psi_1 + \lambda\phi = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

One can see that such $(\lambda, \phi, \psi_1, 0)$ is a solution of the eigenvalue problem (4.4)-(4.6) with $\lambda < 0$ and $\psi_1 > 0$. So the DFE is unstable. □

We then study the global attractivity of the DFE.

Theorem 4.2. *If $\mathcal{R}_0 \le 1$, then the DFE is globally attractive.*

Proof. We first consider the case $\mathcal{R}_0 < 1$. Let $\epsilon > 0$ be given. Since $S(x,t) + I_1(x,t) + I_2(x,t) \to N/|\Omega|$ as $t \to \infty$, there exists a $T > 0$ such that $S(x,t) \le N/|\Omega| + \epsilon - I_1(x,t) - I_2(x,t)$ for $(x,t) \in \overline{\Omega} \times [T,\infty)$. Then by (2.1)-(2.2), we have the following

$$\begin{aligned}
&\tfrac{\partial I_1}{\partial t} - d\Delta I_1 \le I_1\left(\left(\tfrac{N}{|\Omega|}+\epsilon\right)\beta_1 - \gamma_1 - \beta_1 I_1\right), && x\in\Omega, && t\in(T,\infty),\\
&\tfrac{\partial I_2}{\partial t} - d\Delta I_2 \le I_2\left(\left(\tfrac{N}{|\Omega|}+\epsilon\right)\beta_2 - \gamma_2 - \beta_2 I_2\right), && x\in\Omega, && t\in(T,\infty),\\
&\tfrac{\partial I_1}{\partial n} = \tfrac{\partial I_2}{\partial n} = 0, && x\in\partial\Omega, && t\in(T,\infty).
\end{aligned} \tag{4.8}$$

Let $\hat{I}_i$, $i = 1, 2$, be the solution of a related problem:

$$\begin{aligned}
&\tfrac{\partial \hat{I}_i}{\partial t} = d\Delta \hat{I}_i + \hat{I}_i\left(\left(\tfrac{N}{|\Omega|}+\epsilon\right)\beta_i - \gamma_i - \beta_i \hat{I}_i\right), && x\in\Omega, && t\in(T,\infty),\\
&\tfrac{\partial \hat{I}_i}{\partial n} = 0, && x\in\partial\Omega, && t\in(T,\infty),\\
&\hat{I}_i(x,T) = I_i(x,T), && x\in\overline{\Omega}.
\end{aligned} \tag{4.9}$$

The comparison principle yields that $I_i(x,t) \le \hat{I}_i(x,t)$ on $\overline{\Omega}\times[T,\infty)$. Furthermore, it follows from $\mathcal{R}_0 < 1$ and Remark 1 that $\lambda^*(N\beta_i/|\Omega| - \gamma_i) > 0$. We then choose ϵ so small that $\lambda^*((N/|\Omega| + \epsilon)\beta_i - \gamma_i) > 0$. Hence, by Lemma 4.1, $\hat{I}_i(x,t) \to 0$ as $t \to \infty$ uniformly for $x \in \overline{\Omega}$, and so does $I_i(x,t)$, $i = 1, 2$. Then, it follows from $S(x,t) + I_1(x,t) + I_2(x,t) \to N/|\Omega|$ that $S(x,t) \to N/|\Omega|$ as $t \to \infty$ uniformly for $x \in \overline{\Omega}$.

We then consider the case $\mathcal{R}_0 = 1$, i.e, at least one of $\lambda^*(N\beta_i/|\Omega| - \gamma_i)$, $i = 1, 2$, is zero. Without loss of generality, we may assume that $\lambda^*(N\beta_1/|\Omega| - \gamma_1) = 0$ and $\lambda^*(N\beta_2/|\Omega| - \gamma_2) > 0$. We then have that for small ϵ, $\hat{I}_1(x,t) \to \hat{I}_1^*(x)$, where $\hat{I}_1^*(x)$ is the corresponding positive equilibrium, and $\hat{I}_2(x,t) \to 0$. On the other hand, if $\epsilon \to 0$, then $\hat{I}_1^*(x) \to 0$. Hence, $I_i(x,t) \to 0$, $i = 1, 2$, as $t \to \infty$ uniformly for $x \in \overline{\Omega}$. □

4.3. **Endemic equilibrium.** Let $(\bar{S}, \bar{I}_1, \bar{I}_2)$ be an equilibrium of (2.1)-(2.3). It then satisfies the following system:

$$\begin{aligned}
&d\Delta\bar{S} - (\beta_1\bar{I}_1 + \beta_2\bar{I}_2)\bar{S} + \gamma_1\bar{I}_1 + \gamma_2\bar{I}_2 = 0, && x\in\Omega,\\
&d\Delta\bar{I}_1 + \beta_1\bar{S}\bar{I}_1 - \gamma_1\bar{I}_1 = 0, && x\in\Omega,\\
&d\Delta\bar{I}_2 + \beta_2\bar{S}\bar{I}_2 - \gamma_2\bar{I}_2 = 0, && x\in\Omega.
\end{aligned} \tag{4.10}$$

Adding up the three equations in (4.10) yields $d\Delta(\bar{S} + \bar{I}_1 + \bar{I}_2) = 0$. In view of (2.2)-(2.3) and the maximum principle, we have that $\bar{S} = N/|\Omega| - \bar{I}_1 - \bar{I}_2$. Substituting this into (4.10), we arrive at

$$\begin{aligned}
&d\Delta\bar{I}_1 + \left(\tfrac{N\beta_1}{|\Omega|} - \gamma_1 - \beta_1\bar{I}_1 - \beta_1\bar{I}_2\right)\bar{I}_1 = 0, && x\in\Omega,\\
&d\Delta\bar{I}_2 + \left(\tfrac{N\beta_2}{|\Omega|} - \gamma_2 - \beta_2\bar{I}_1 - \beta_2\bar{I}_2\right)\bar{I}_2 = 0, && x\in\Omega.
\end{aligned} \tag{4.11}$$

Thus, to study the existence of the endemic equilibria, it suffices to consider the nonnegative solutions of system (4.11) subject to the Neumann boundary conditions:

$$\frac{\partial\bar{I}_1}{\partial n} = \frac{\partial\bar{I}_2}{\partial n} = 0, \quad x\in\overline{\Omega}. \tag{4.12}$$

By Lemma 4.1 and Remark 1, problem (4.11)-(4.12) has a nonnegative solution $(U, 0)$ or $(0, V)$ with U or V strictly positive on $\overline{\Omega}$ if and only if $r_1 > 1$ or $r_2 > 1$. We then have the following result.

Theorem 4.3. *If $\mathcal{R}_0 > 1 \geq \mathcal{R}_1$, then there exists an EE $E_1 = (N/|\Omega| - U, U, 0)$ or $E_2 = (N/|\Omega| - V, 0, V)$ of model (2.1)-(2.3), which is globally attractive.*

Proof. Since $\mathcal{R}_0 > 1 \geq \mathcal{R}_1$, one of r_i, $i = 1, 2$, is greater than 1 and the other one is less than or equal to 1. Without loss of generality, we may assume that $r_1 > 1$ and $r_2 \leq 1$. Then problem (4.11)-(4.12) has a nonnegative solution $(U, 0)$. And it is easy to check that $E_1 = (N/|\Omega| - U, U, 0)$ is an EE.

We then show the global attractivity of E_1. Since $r_2 \leq 1$, using similar arguments as in the proof of Theorem 4.2, one can see that $I_2(x,t) \to 0$ as $t \to \infty$. Let $\epsilon > 0$ be given. Since $S + I_1 + I_2 \to N/|\overline{\Omega}|$ and $I_2 \to 0$ as $t \to \infty$, there exists a $T > 0$ such that $N/|\Omega| - I_1(x,t) - \epsilon \leq S(x,t) \leq N/|\Omega| - I_1(x,t) + \epsilon$ for $(x,t) \in \overline{\Omega} \times [T, \infty)$. So I_1 satisfies the following inequality

$$I_1\left(\left(\frac{N}{|\Omega|} - \epsilon\right)\beta_1 - \gamma_1 - \beta_1 I_1\right) \leq \frac{\partial I_1}{\partial t} - d\Delta I_1 \leq I_1\left(\left(\frac{N}{|\Omega|} + \epsilon\right)\beta_1 - \gamma_1 - \beta_1 I_1\right) \tag{4.13}$$

for $(x,t) \in \Omega \times (T, \infty)$. Let $\check{I}$ and $\hat{I}$ solve two related problems, respectively:

$$\begin{cases} \check{I}_t = d\Delta\check{I} + \check{I}\left(\left(\frac{N}{|\Omega|} - \epsilon\right)\beta_1 - \gamma_1 - \beta_1\check{I}\right), & x \in \Omega, \quad t \in (T, \infty), \\ \frac{\partial \check{I}}{\partial n} = 0, & x \in \partial\Omega, \quad t \in (T, \infty), \\ \check{I}(x, T) = I_1(x, T), & x \in \overline{\Omega} \end{cases} \tag{4.14}$$

and

$$\begin{cases} \hat{I}_t = d\Delta\hat{I} + \hat{I}\left(\left(\frac{N}{|\Omega|} + \epsilon\right)\beta_1 - \gamma_1 - \beta_1\hat{I}\right), & x \in \Omega, \quad t \in (T, \infty), \\ \frac{\partial \hat{I}}{\partial n} = 0, & x \in \partial\Omega, \quad t \in (T, \infty), \\ \hat{I}(x, T) = I_1(x, T), & x \in \overline{\Omega}. \end{cases} \tag{4.15}$$

By the comparison principle, we find that $\check{I}(x,t) \leq I_1(x,t) \leq \hat{I}(x,t)$ for $(x,t) \in \overline{\Omega} \times [T, \infty)$. If ϵ is small, we have that $\lambda^*((N/|\Omega| \pm \epsilon)\beta_1 - \gamma_1) < 0$, and it then follows that $\check{I}(x,t) \to \check{I}^*_\epsilon(x)$ and $\hat{I}(x,t) \to \hat{I}^*_\epsilon(x)$, where $\check{I}^*_\epsilon$ and $\hat{I}^*_\epsilon$ are the corresponding positive equilibria, respectively. Taking $\epsilon \to 0$, we further have that $\check{I}^*_\epsilon \to U(x)$ and $\hat{I}^*_\epsilon \to U(x)$. Hence, $I_1(x,t) \to U(x)$, and consequently, $S \to N/|\Omega| - U$, as $t \to \infty$ uniformly for $x \in \overline{\Omega}$. □

We now discuss the case $\mathcal{R}_1 > 1$. By Lemma 4.1, the endemic equilibria E_1 and E_2 both exist. In the case of homogeneous coefficients, we observed the competitive exclusion behavior: either E_1 drives E_2 to extinction or vice versa. We may ask whether this is still the case when the coefficients are heterogeneous. To answer such a question, we first introduce two auxiliary functions $\check{U}$ and $\check{V}$ as follows. If $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1 V) < 0$, we let $\check{U}$ be the unique positive solution of the problem:

$$\begin{aligned} & d\Delta\check{U} + \check{U}\left(\frac{N\beta_1}{|\Omega|} - \gamma_1 - \beta_1\check{U} - \beta_1 V\right) = 0, && x \in \Omega, \\ & \frac{\partial \check{U}}{\partial n} = 0, && x \in \partial\Omega, \end{aligned}$$

and if $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2 U) < 0$, we let $\check{V}$ be the unique positive solution of the problem:

$$\begin{aligned} & d\Delta\check{V} + \check{V}\left(\tfrac{N\beta_2}{|\Omega|} - \gamma_2 - \beta_2\check{V} - \beta_2 U\right) = 0, && x \in \Omega, \\ & \tfrac{\partial\check{V}}{\partial n} = 0, && x \in \partial\Omega. \end{aligned}$$

Biologically, $\check{U}$ is the lowest possible asymptotic density for the first disease and $\check{V}$ is the lowest possible asymptotic density for the second disease (see p. 289 of [13]). We then have the following competitive exclusion result.

Theorem 4.4. *Suppose that $\mathcal{R}_1 > 1$.*

(a) If $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1 V) < 0$ and $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2\check{U}) \geq 0$, then E_1 is globally attractive;

(b) If $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2 U) < 0$ and $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1\check{V}) \geq 0$, then E_2 is globally attractive.

Proof. Suppose that $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1 V) < 0$ and $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2\check{U}) \geq 0$. Since the solution of the competition model (4.1) always converges, we can proceed analogously to show that $(I_1(x,t), I_2(x,t)) \to (I_1^*(x), I_2^*(x))$ as $t \to \infty$ uniformly for $x \in \overline{\Omega}$, where $(I_1^*(x), I_2^*(x))$ is a solution of (4.11)-(4.12). It is easy to see that $I_2^* \leq V$. Let $\epsilon > 0$ be given. Since $S + I_1 + I_2 \to N/|\Omega|$ and $I_2 \to I_2^*$ as $t \to \infty$, there exists $T_1 > 0$ such that

$$\frac{\partial I_1}{\partial t} - d\Delta I_1 \geq I_1\left(\frac{N}{|\Omega|}\beta_1 - \gamma_1 - \beta_1 I_1 - \beta_1(I_2^* + \epsilon)\right) \quad \text{for } (x,t) \in \Omega \times (T_1, \infty).$$

Let $\check{I}$ be the solution of the following problem:

$$\begin{aligned} & \tfrac{\partial\check{I}}{\partial t} = d\Delta\check{I} + \check{I}\left(\tfrac{N}{|\Omega|}\beta_1 - \gamma_1 - \beta_1\check{I} - \beta_1(V+\epsilon)\right), && x \in \Omega, \quad t \in (T_1, \infty), \\ & \tfrac{\partial\check{I}}{\partial n} = 0, && x \in \partial\Omega, \quad t \in (T_1, \infty), \\ & \check{I}(x,T) = I_1(x,T_1), && x \in \overline{\Omega}. \end{aligned} \tag{4.16}$$

The comparison principle yields that $I_1 \geq \check{I}$ on $\overline{\Omega} \times [T_1, \infty)$. If $\epsilon > 0$ is small enough, then $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1(V+\epsilon)) < 0$, and it follows from Lemma 4.1 that $\check{I}(x,t)$ converges to a positive equilibrium $\check{I}_\epsilon^*(x)$. Moreover, $\check{I}_\epsilon^*(x) \to \check{U}(x)$ as $\epsilon \to 0$. Let $\delta > 0$ be given. We then have that $I_1(x,t) \geq \check{U}(x) - \delta/2$ for large t. Noticing the fact that $S + I_1 + I_2 \to N/|\Omega|$ as $t \to \infty$, there exists $T_2 > T_1$ such that

$$\frac{\partial I_2}{\partial t} - d\Delta I_2 \leq I_2\left(\frac{N}{|\Omega|}\beta_2 - \gamma_2 - \beta_2(\check{U} - \delta) - \beta_2 I_2\right) \quad \text{for } (x,t) \in \Omega \times (T_2, \infty).$$

Let $\hat{I}$ be the solution of the following problem:

$$\begin{aligned} & \tfrac{\partial\hat{I}}{\partial t} = d\Delta\hat{I} + \hat{I}\left(\tfrac{N}{|\Omega|}\beta_2 - \gamma_2 - \beta_2(\check{U} - \delta) - \beta_2\hat{I}\right), && x \in \Omega, \quad t \in (T_2, \infty), \\ & \tfrac{\partial\hat{I}}{\partial n} = 0, && x \in \partial\Omega, \quad t \in (T_2, \infty), \\ & \hat{I}(x,T) = I_2(x,T_2), && x \in \overline{\Omega}. \end{aligned} \tag{4.17}$$

The comparison principle then implies that $I_2 \leq \hat{I}$ on $\overline{\Omega} \times [T_2, \infty)$. Since $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2\check{U}) \geq 0$, it follows from Lemma 4.1 that $\hat{I} \to \hat{I}_\delta^*$ (≥ 0) as $t \to \infty$. Taking $\delta \to 0$ then gives $\hat{I}_\delta^* \to 0$, which leads to that $I_2 \to 0$ as $t \to \infty$. Consequently, $I_1 \to U$ as $t \to \infty$, and so E_1 is globally attractive. Similarly, we can prove the other case that $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2 U) < 0$ and $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1\check{V}) \geq 0$. $\square$

An endemic equilibrium $(\bar{S}, \bar{I}_1, \bar{I}_2)$ is called a coexistence equilibrium if both $\bar{I}_1$ and $\bar{I}_2$ are nontrivial. If such an equilibrium exists, then we certainly do not have the competitive exclusion.

Theorem 4.5. *Suppose that $\mathcal{R}_1 > 1$. We assume that both $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1 V)$ and $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2 U)$ are negative. Then there exists at least one coexistence equilibrium of (2.1)-(2.3). Moreover, the model (2.1)-(2.3) is permanent in the sense that there exist positive constants m_0 and M_0 such that*

$$m_0 \le S(x,t), I_1(x,t), I_2(x,t) \le M_0$$

for all $x \in \overline{\Omega}$ and $t \ge T_0$, where T_0 is dependent on the initial condition.

Proof. As in the proof of Theorem 4.4, we can see that $(I_1(x,t), I_2(x,t)) \to (I_1^*(x), I_2^*(x))$ as $t \to \infty$ uniformly for $x \in \overline{\Omega}$, where $(I_1^*(x), I_2^*(x))$ is a solution of (4.11)-(4.12). Moreover, the assumption on the principal eigenvalues guarantees that $(I_1^*(x), I_2^*(x))$ is strictly componentwise positive (see p. 307 of [13]). In addition, by the comparison principle, we have $I_1^* + I_2^* < N/|\Omega|$. Hence, noticing the fact that $S + I_1 + I_2 \to N/|\overline{\Omega}|$, we conclude that $(S, I_1, I_2) \to (S^*, I_1^*, I_2^*)$ as $t \to \infty$ with $S^* = N/|\overline{\Omega}| - I_1^* - I_2^*$ strictly positive on $\overline{\Omega}$.

It then suffices to prove that there does not exist a sequence $\{(I_{1,n}^*, I_{2,n}^*)\}$ of equilibria of (4.11)-(4.12) which converges to $(U, 0)$ or $(0, V)$ or satisfies $I_{1,n}^* + I_{2,n}^* \to N/|\Omega|$. The nonexistence of such a sequence converging to $(U, 0)$ or $(0, V)$ has been proved in [13] (see p. 290). So we only need to assume that $\{(I_{1,n}^*, I_{2,n}^*)\}$ exists and satisfies $I_{1,n}^* + I_{2,n}^* \to N/|\Omega|$. Noticing (4.11) and the uniform boundededness of $\{(I_{1,n}^*, I_{2,n}^*)\}$, it follows from a standard compact argument that there exists a subsequence $\{(I_{1,n_j}^*, I_{2,n_j}^*)\}$ such that $\{(I_{1,n_j}^*, I_{2,n_j}^*)\} \to (\tilde{I}_1^*, \tilde{I}_2^*)$, where $\tilde{I}_i^*$, $i = 1, 2$, satisfies that $d\Delta \tilde{I}_i^* - \gamma_i \tilde{I}_i^* = 0$. It then follows from the maximum principle that $\tilde{I}_i^* \equiv 0$, which contradicts $\tilde{I}_1^* + \tilde{I}_2^* = N/|\Omega|$. □

5. Conditions for competitive exclusion and coexistence.

In this section, we verify the validity of conditions in Theorems 4.4 and 4.5. We first consider Theorem 4.4. It suffices to verify the validity of conditions in part (a). In view of (4.2) and Remark 1, we choose $N\beta_2/|\Omega| > \gamma_2$ in $\overline{\Omega}$. It then follows from Lemma 4.1 that $(0, V)$ exists. Moreover, $V \le N/|\Omega| - \min_{\overline{\Omega}}\{\gamma_2/\beta_2\}$ in $\overline{\Omega}$. Since β_1 does not depend on β_2 and γ_2, we can choose β_1 so large that $\min_{\overline{\Omega}}\{\gamma_2/\beta_2\} > \max_{\overline{\Omega}}\{\gamma_1/\beta_1\}$, which implies that $N\beta_1/|\Omega| - \gamma_1 - \beta_1 V > 0$ in $\overline{\Omega}$. Then by (4.2), $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1 V) < 0$, which in conjunction with Lemma 4.1 guarantees the existence of $\check{U}$. Clearly, $\check{U} \ge N/|\Omega| - \max_{\overline{\Omega}}\{\gamma_1/\beta_1\} - V \ge \min_{\overline{\Omega}}\{\gamma_2/\beta_2\} - \max_{\overline{\Omega}}\{\gamma_1/\beta_1\}$ in $\overline{\Omega}$. For simplicity, we now let γ_2 take the form $\gamma_2 = r_2\beta_2$, where r_2 is a positive constant close to $N/|\Omega|$ such that $(N/|\Omega| + \max_{\overline{\Omega}}\{\gamma_1/\beta_1\})/2 \le r_2 < N/|\Omega|$. This implies that $N\beta_2/|\Omega| - \gamma_2 - \beta_2\check{U} \le 0$ in $\overline{\Omega}$. It then follows from (4.2) that $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2\check{U}) \ge 0$.

We now consider Theorem 4.5. For any $\alpha \in C(\overline{\Omega})$, it is well known that $\lambda^*(\alpha) \to \min_{\overline{\Omega}}\{-\alpha(x)\}$ as $d \to 0$ [7]. Let $\alpha_+(x) = \max\{\alpha(x), 0\}$ for any $x \in \overline{\Omega}$. Suppose that $\alpha(x) > 0$ for some $x \in \overline{\Omega}$, then there exists $d^* > 0$ such that the unique positive solution u of the problem

$$\begin{aligned} &d\Delta u + (\alpha(x) - u)u = 0, && x \in \Omega, \\ &\frac{\partial u}{\partial n} = 0, && x \in \partial\Omega \end{aligned} \tag{5.1}$$

exists for all $d < d^*$, and $u \to \alpha_+$ as $d \to 0$.

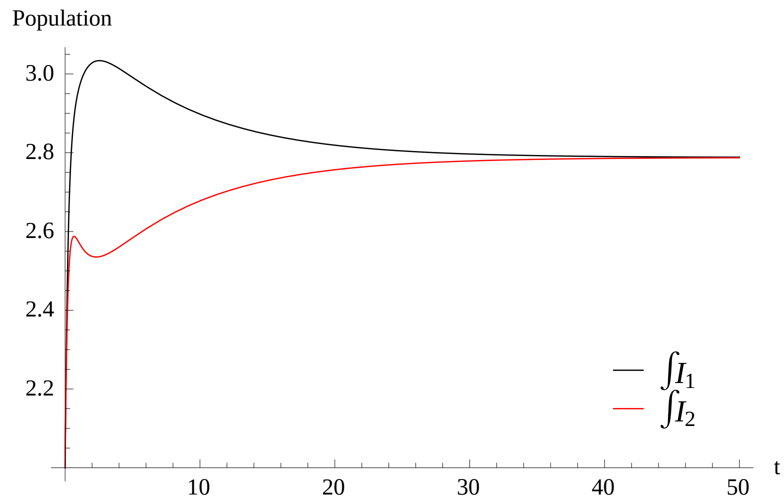

FIGURE 1. Population of infected individuals

For simplicity, let Ω be the one dimensional open interval $(0,1)$ and $\beta_1 = \beta_2 = 1$. Suppose that $N/|\Omega| > \gamma_i$ for $i = 1, 2$ so that $\mathcal{R}_1 > 1$. Then we have that $U \to N/|\Omega| - \gamma_1$ and $V \to N/|\Omega| - \gamma_2$ as $d \to 0$. Moreover, one can show that

$$\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1 V) \to \min_{\overline{\Omega}}\{\gamma_1(x) - \gamma_2(x)\} \tag{5.2}$$

and

$$\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2 U) \to \min_{\overline{\Omega}}\{\gamma_2(x) - \gamma_1(x)\} \tag{5.3}$$

as $d \to 0$.

Let $\gamma_1 = 1 + x$ and $\gamma_2 = 2 - x$. Choose the initial data as

$$\begin{aligned} S(x,0) &= 3 + \cos \pi x, \\ I_1(x,0) &= I_2(x,0) = 2 + \cos \pi x \end{aligned}$$

such that $N/|\Omega| > \gamma_i$ holds for $i = 1, 2$. Then it follows from $\gamma_1 - \gamma_2 = 2x - 1$ and (5.2)-(5.3) that both $\lambda^*(N\beta_1/|\Omega| - \gamma_1 - \beta_1 V)$ and $\lambda^*(N\beta_2/|\Omega| - \gamma_2 - \beta_2 U)$ are negative for small diffusion rate d. Therefore the conditions in Theorem 4.5 hold, and we expect the coexistence between the two strains for small diffusion rates here. This is confirmed by the numerical simulation. In Figure 1, we can see that the population of infected individuals converges to a nonzero value. Actually, (S, I_1, I_2) converges to the endemic equilibrium (the steady states of the two strains are shown in Figure 2).

6. **Conclusion.** We have established competitive exclusion and coexistence results for a multi-strain pathogen model with diffusion and with mass action incidence term. We showed that for a spatially homogenous environment competitive exclusion is the only outcome when the basic reproduction number is greater than one and when it is less than one then the disease-free equilibrium is globally attractive. These results are analogous to many differential equation models, as to be expected since the transmission and recovery rates are independent of space. We also showed that when the transmission and recovery rates are non-homogeneous then coexistence is possible under additional conditions on the model parameters.

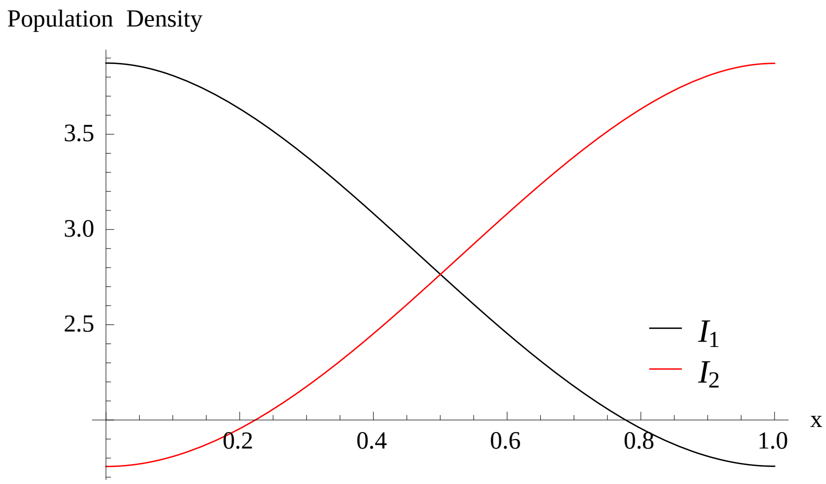

FIGURE 2. Infected individuals at t=50

One of the key properties in our model that facilitate the analysis is the conservation of the total population given in equation (2.3). When such assumption is dropped in ordinary differential equation models to allow for modeling disease added mortality (e.g., [1, 2, 3]), it results in the added difficulty that the total population can no longer be conserved and its dynamics can only be described by a differential inequality. In the future we hope to extend the current study to accommodate such important property of certain diseases that lead to mortality.

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