

## HETEROCLINIC BIFURCATION FOR A GENERAL PREDATOR-PREY MODEL WITH ALLEE EFFECT AND STATE FEEDBACK IMPULSIVE CONTROL STRATEGY

QIZHEN XIAO AND BINXIANG DAI

School of Mathematics and Statistics  
Central South University  
Changsha, 410083, China

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**ABSTRACT.** In this paper, we analyze a general predator-prey model with state feedback impulsive harvesting strategies in which the prey species displays a strong Allee effect. We firstly show the existence of order-1 heteroclinic cycle and order-1 positive periodic solutions by using the geometric theory of differential equations for the unperturbed system. Based on the theory of rotated vector fields, the order-1 positive periodic solutions and heteroclinic bifurcation are studied for the perturbed system. Finally, some numerical simulations are provided to illustrate our main results. All the results indicate that the harvesting rate should be maintained at a reasonable range to keep the sustainable development of ecological systems.

**1. Introduction.** It is known that many evolution processes are characterized by the fact that at certain moments of time some species experience abrupt changes of states. It is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. Moreover, in some real world problems such as pest integrated management, fish harvesting and wildlife management, the corresponding control strategies which cause the perturbations will only be implemented if the density of a population reaches a threshold value rather than impulses occurring at certain fixed moments of time. This leads to impulsive state feedback control strategies (see [6, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 23] and references therein). What's more, the predator-prey interaction is ubiquitous in nature and it has always been a key issue in mathematical modeling of ecological processes since the pioneering work of Lotka and Volterra.

Through experimental studies, Warder Clyde Allee, an American ecologist, demonstrated that goldfish grow more rapidly when there are more individuals within the tank [1]. This led him to conclude that aggregation can improve the survival rate of individuals and that cooperation may be crucial in the overall evolution of social structures. Such a biological phenomenon is referred to as an Allee effect. Generally speaking, a population is said to have an Allee effect if the growth rate

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per capita is initially an increasing function for the low density. Moreover it is called a strong Allee effect if the per capita growth rate in the limit of low density is negative, and a weak Allee effect if the per capita growth rate is positive at zero density. Allee effect may occur under several mechanisms, such as difficulties in finding mates when the population density is low [8, 13], social dysfunction at small population sizes and increased predation risk due to failing flocking or schooling behavior [4]. What's more, analysis of systems involving Allee effect has gained lots of concerns in many fields such as conservation biology [5], sustainable harvesting [14], biological invasions [3], meta population dynamics [25] and interacting species [24].

Here, we introduce a general impulsive harvesting predator-prey system by incorporating the influence of the relative speed of the predator population growth to the growth rate of its prey, where the prey species displays a strong Allee effect. Let  $x(t)$  and  $y(t)$  denote the population densities of the prey and predator species, respectively;  $K > 0$  is the carrying capacity of the prey and  $d > 0$  is the death rate of the predator. The species interactions with the state feedback control strategy are modeled by the following predator-prey system:

$$\left\{ \begin{array}{l} x'(t) = g(x)(f(x) - y) + \delta y(g(x) - d), \\ y'(t) = y(g(x) - d). \end{array} \right\} \quad \begin{array}{l} y < \tau, \\ y = \tau, \end{array} \quad (1)$$

$$\left\{ \begin{array}{l} \Delta x = -px, \\ \Delta y = -qy. \end{array} \right\}$$

$$x(0^+) = x_0, y(0^+) = y_0 < \tau,$$

where  $x(0^+)$  and  $y(0^+)$  denote the initial densities of the prey and predator populations, respectively. The initial condition of system (1) can be any point in the non-negative plane  $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ . Throughout this paper we assume that the initial density of the predator population is always less than  $\tau$ . Otherwise, the initial values are taken after an impulsive harvesting application. What's more, we assume that  $f, g$  and  $\tau$  satisfy

- (A1):**  $f \in C^1(\mathbb{R}_+, \mathbb{R})$ ,  $f(K) = 0$ ; there exists an  $m \in (0, K)$  such that  $f(m) = 0$ ;  $f(x) > 0$  for  $m < x < K$  and  $f(x) < 0$  for  $x < m$ ; there exists an  $n \in (m, K)$  such that  $f'(x) > 0$  on  $[m, n)$  and  $f'(x) < 0$  on  $(n, K]$ ;
- (A2):**  $g \in C^1(\mathbb{R}_+, \mathbb{R})$ ,  $g(0) = 0$ ,  $g(x) > 0$ ,  $g'(x) > 0$  for  $x \in (0, K]$ ; there exists a  $\lambda \in (m, K)$  such that  $g(\lambda) = d$ ;
- (A3):**  $0 < \tau < f(\lambda)$ .

In system (1),  $g(x)$  is the so called predator functional response, and  $g(x)f(x)$  is the net growth rate of the prey when  $\delta = 0$ .  $\delta > 0$  represents the influence of the relative speed of the predator population growth to the growth rate of the prey.  $\Delta x$  and  $\Delta y$  are discontinuous changes in, respectively,  $x$  and  $y$  as a consequence of the impulsive harvesting, and  $p, q \in [0, 1)$  are the harvesting rates for species  $x$  and  $y$ , respectively. Here, we assume that the density of the predator species is observable. In the absence of predation, the prey has a strong Allee effect which can be gained from the assumption (A1) and  $m$  is the Allee effect threshold value. The condition (A2) exhibits the commonly used functional responses: Holling type I, Holling type II, Holling type III, Ivlev type and some other equivalent forms. From the point of view of biology, we only restrict our attention to system (1) in  $\mathbb{R}_+^2$ . For system (1), in the absence of the impulsive harvesting and  $\delta = 0$ , authors of [18] gave a global bifurcation analysis and proved the existence/uniqueness and the nonexistence of limit cycles for appropriate ranges of parameters.

The remaining of this paper is organized as follows. Firstly, some preliminaries are given in the coming section. Then in Section 3, the existence of order-1 positive periodic solutions and heteroclinic bifurcation with respect to  $p$  are proved for the unperturbed system (i.e., system (1) with  $\delta = 0$ ). Followed in Section 4, when  $p$  is fixed and  $\delta > 0$ , the order-1 positive periodic solutions and heteroclinic bifurcation about  $\delta$  associated with system (1) are presented. The results are summarized and supported with numerical simulations in Section 5, followed by some conclusions.

**2. Preliminaries.** For the convenience of statements in this paper, we introduce the following definitions and lemmas.

**Definition 2.1.** [2] Consider the state-dependent impulsive differential equations

$$\begin{cases} x'(t) = \hat{P}(x, y), & y'(t) = \hat{Q}(x, y), & (x, y) \notin M\{x, y\}, \\ \Delta x = \alpha(x, y), & \Delta y = \beta(x, y), & (x, y) \in M\{x, y\}. \end{cases} \quad (2)$$

We define the dynamical system consisting of the solution mapping of system (2) as a semicontinuous dynamical system, denoted as  $(\Omega, \hat{f}, \varphi, M)$ . We require that the initial point  $P$  of the system (2) should not be in the impulsive set  $M\{x, y\}$ , that is  $P \in \Omega = \mathbb{R}_+^2 \setminus M\{x, y\}$ . For any  $P \in \Omega$ , the function  $\hat{f}_P : \mathbb{R}_+ \rightarrow \Omega$  defined by  $\hat{f}_P(t) = \hat{f}(P, t)$  is continuous and we call  $\hat{f}_P(t)$  the trajectory of  $P$ . Here  $\varphi$  is a continuous mapping satisfying  $\varphi(M) = N$  and we call  $\varphi$  the impulsive mapping and  $N\{x, y\}$  the impulsive range.  $M\{x, y\}$  and  $N\{x, y\}$  represent the straight lines or curves in the plane  $\mathbb{R}_+^2$ .

**Definition 2.2.** [2] A trajectory  $\hat{f}(P, t)$  is called an order-1 periodic solution with period  $T$  if there exist  $P \in N$  and  $T \in (0, \infty)$  such that  $\hat{f}(P, T) = Q \in M$  with  $\varphi(Q) = P$  and  $\hat{f}(P, t) \notin M$  for  $0 < t < T$ . The trajectory  $\{\hat{f}(P, t) | t \in (0, T]\}$  together with the impulsive line  $\overline{QP}$  is called an order-1 cycle. If the order-1 cycle has a singularity, then it is called an order-1 singular cycle. Furthermore, if the order-1 cycle has only one singularity and it is a saddle, then the order-1 singular cycle is called an order-1 homoclinic cycle; if the order-1 cycle has two singularities and they are saddles, then the order-1 singular cycle is called an order-1 heteroclinic cycle.

Notice that when the impulsive set  $M$  and the impulsive range  $N$  of system (2) are straight lines, a coordinate system can be well defined in the impulsive range  $N$ . Let  $A \in N$  be a point, and let its coordinate be  $a$ . Assume that the trajectory from the point  $A$  intersects the impulse set  $M$  at a point  $A'$ , and, after the impulsive effect, the point  $A'$  is mapped to the point  $A_1 \in N$  with the coordinate  $a_1$ . Then we define the successor point and the successor function as follows.

**Definition 2.3.** [2] The point  $A_1$  is called the successor point of  $A$ , and the function  $\hat{g}(A) = a_1 - a$  is called the successor function of the point  $A$ .

**Lemma 2.4.** [2] *The successor function  $\hat{g}(A)$  is continuous.*

**Lemma 2.5.** [2] *For system (2), if there exist  $\tilde{A}, \bar{A} \in N$  satisfying  $\hat{g}(\tilde{A})\hat{g}(\bar{A}) < 0$ , then the system possesses at least one order-1 periodic solution.*

Notice that system (1) can be considered as a perturbed system with parameter  $\delta$  changing from  $\delta = 0$  to  $\delta > 0$ . Therefore, we give some properties about the

general perturbed system of system (2), which is given by

$$\begin{cases} x'(t) = \tilde{P}(x, y, \delta), & y'(t) = \tilde{Q}(x, y, \delta), & (x, y) \notin M\{x, y\}, \\ \Delta x = \alpha(x, y), & \Delta y = \beta(x, y), & (x, y) \in M\{x, y\}. \end{cases} \tag{3}$$

Let  $\Delta = \begin{vmatrix} \tilde{P} & \tilde{Q} \\ \frac{\partial \tilde{P}}{\partial \delta} & \frac{\partial \tilde{Q}}{\partial \delta} \end{vmatrix}$ . We have the following definition and lemma.

**Definition 2.6.** [22] For any often point on the trajectory of system (3), if  $\Delta > 0$ , then system (3) constitutes positive rotated vector fields concerning the parameter  $\delta$ ; if  $\Delta < 0$ , then system (3) constitutes negative rotated vector fields.

**Lemma 2.7.** [22] *In the positive (negative) rotated vector fields of system (3), the rotated direction of vector fields is counterclockwise (clockwise) when parameter  $\delta$  changes from  $\delta = 0$  to  $\delta > 0$ .*

**3. The existence of order-1 positive periodic solutions and heteroclinic bifurcation for system (1) with  $\delta = 0$ .** When  $\delta = 0$ , system (1) degenerates into the following system:

$$\begin{cases} \left. \begin{aligned} x'(t) &= g(x)(f(x) - y), \\ y'(t) &= y(g(x) - d). \end{aligned} \right\} & y < \tau, \\ \left. \begin{aligned} \Delta x &= -px, \\ \Delta y &= -qy. \end{aligned} \right\} & y = \tau, \\ x(0^+) &= x_0, y(0^+) = y_0 < \tau, \end{cases} \tag{4}$$

where the assumptions (A1)-(A3) are also satisfied. We call system (4) as the unperturbed system.

It is typical for predator-prey systems that the  $x$ - and  $y$ -axes and the interior of the first quadrant are all invariant under system (4) with the assumptions (A1) and (A2) satisfied and thus the solutions with positive initial conditions are positive for all time  $t$ . In the following we investigate the long-term behavior of system (4).

**Theorem 3.1.** *All the solutions of system (4) with the positive initial condition  $(x_0, y_0)$  are uniformly ultimately bounded by a region  $\hat{\Omega}$ , where*

$$\hat{\Omega} = \{(x, y) : 0 < x \leq K, y \geq 0, x + y \leq \frac{f(n)g(K)}{d} + K\}.$$

*Proof.* Denote  $l_1 : x = K$ . Calculating the time derivative of  $l_1$  along the trajectories of system (4) yields that

$$\frac{dl_1}{dt} = \frac{dx}{dt} \Big|_{x=K} = g(K)(f(K) - y) = -g(K)y < 0.$$

Thus, the trajectory of system (4) is from the right of  $l_1$  through  $l_1$  into the left.

Denote  $l_2 : V(x, y) = x + y - \left(\frac{f(n)g(K)}{d} + K\right) = 0$ . By a direct calculation, we have

$$\frac{dl_2}{dt} = \left(\frac{dx}{dt} + \frac{dy}{dt}\right) \Big|_{V(x,y)=0} = f(x)g(x) - f(n)g(K) + d(x - K) < 0.$$

Thus,  $l_2 : V = 0$  is non-tangent and the trajectory of system (4) is from the upper right of  $l_2$  through  $l_2$  into the lower left. Hence, system (4) is uniformly ultimately bounded by  $\{(x, y) : 0 < x \leq K, y \geq 0, x + y \leq \frac{f(n)g(K)}{d} + K\}$ . This leads to the conclusion of the theorem.  $\square$

**Theorem 3.2.** For system (4), there always exist one trivial equilibrium  $S_0 = (0, 0)$ , two boundary equilibria  $S_1 = (m, 0)$  and  $S_2 = (K, 0)$  and a positive equilibrium  $S_+ = (\lambda, f(\lambda))$ . The trivial equilibrium  $S_0$  is a stable node;  $S_1$  is a saddle point with the stable manifold entering  $S_1$  from the region above the prey nullcline  $y = f(x)$  and its unstable manifold given by  $\{(x, 0) : x > 0\}$ ;  $S_2$  is a saddle point with the stable manifold given by  $\{(x, 0) : x > 0\}$  and its unstable manifold entering the region  $\{(x, y) \in \mathbb{R}_+^2 : y > f(x)\}$  from  $S_2$ ;  $S_+ = (\lambda, f(\lambda))$  is unstable for  $\lambda \in (m, n)$  and stable otherwise.

*Proof.* Followed by the assumptions (A1) and (A2), it is easy to see that system (4) has four fixed points given by

$$(x, y) = (0, 0), (m, 0), (K, 0), (\lambda, f(\lambda)),$$

where  $S_0 = (0, 0)$  is a trivial equilibrium,  $S_1 = (m, 0)$  and  $S_2 = (K, 0)$  are boundary equilibria and  $S_+ = (\lambda, f(\lambda))$  is the positive equilibrium. The stability of the equilibrium state is determined by the nature of eigenvalues of the Jacobian matrix

$$J(x, y) = \begin{pmatrix} g'(x)f(x) + g(x)f'(x) - g'(x)y & -g(x) \\ yg'(x) & g(x) - d \end{pmatrix}.$$

The Jacobian matrix  $J(x, y)$  evaluated at the trivial equilibrium point  $S_0$  takes the form

$$J(x, y)|_{S_0} = \begin{pmatrix} g'(0)f(0) & 0 \\ 0 & g(0) - d \end{pmatrix}.$$

Using (A1) and (A2) we can conclude that the eigenvalues  $\lambda_1 = g'(0)f(0) < 0$  and  $\lambda_2 = g(0) - d < 0$ . Hence,  $(0, 0)$  is a stable node.

At the boundary equilibrium point  $S_1$  Jacobian matrix  $J(x, y)$  is given by

$$J(x, y)|_{S_1} = \begin{pmatrix} g(m)f'(m) & -g(m) \\ 0 & g(m) - d \end{pmatrix}.$$

Then the eigenvalues are  $\lambda_1 = g(m)f'(m) > 0$  and  $\lambda_2 = g(m) - d < g(\lambda) - d = 0$ . So  $S_1$  is a saddle with the unstable manifold  $\{(x, 0) : x > 0\}$ . Now, we show that the stable manifold of the fixed point  $S_1$  (denoted by  $W^s(S_1)$ ) approaches  $S_1$  asymptotically as  $t \rightarrow +\infty$  from the region  $\{(x, y) \in \mathbb{R}_+^2 : y > f(x)\}$ . Obviously, an eigenvector of  $J(x, y)|_{S_1}$  corresponding to the eigenvalue  $\lambda_2$  is

$$(x_{S_1}, y_{S_1})^T = \left( \frac{g(m)}{g(m)f'(m) - (g(m) - d)}, 1 \right)^T.$$

Hence the tangential direction of  $W^s(S_1)$  at  $S_1$  is  $k_{S_1} = \frac{g(m)f'(m) - (g(m) - d)}{g(m)} = f'(m) - \frac{g(m) - d}{g(m)} = f'(m) - \frac{\lambda_2}{g(m)} > f'(m)$ , which implies that  $W^s(S_1)$  is above the prey-nullcline  $y = f(x)$ .

The Jacobian matrix  $J(x, y)$  at the equilibrium  $S_2$  is given by

$$J(x, y)|_{S_2} = \begin{pmatrix} g(K)f'(K) & -g(K) \\ 0 & g(K) - d \end{pmatrix}.$$

It follows that the eigenvalues  $\lambda_1 = g(K)f'(K) < 0$  and  $\lambda_2 = g(K) - d > g(\lambda) - d = 0$ . Hence,  $S_2$  is a saddle with the stable manifold  $\{(x, 0) : x > 0\}$ . An eigenvector corresponding to the eigenvalue  $\lambda_2$  is

$$(x_{S_2}, y_{S_2})^T = \left( \frac{g(K)}{g(K)f'(K) - (g(K) - d)}, 1 \right)^T.$$

Thus, the tangential direction of the unstable manifold of  $S_2$  (denoted by  $W^u(S_2)$ ) at  $S_2$  is  $k_{S_2} = \frac{g(K)f'(K)-(g(K)-d)}{g(K)} = f'(K) - \frac{g(K)-d}{g(K)} = f'(K) - \frac{\lambda_2}{g(K)} < f'(K)$ , which implies that  $W^u(S_2)$  is above the prey-nullcline  $y = f(x)$ .

We can calculate the Jacobian matrix at the positive equilibrium  $S_+$  as follows

$$J(x, y)|_{S_+} = \begin{pmatrix} g(\lambda)f'(\lambda) & -g(\lambda) \\ f(\lambda)g'(\lambda) & 0 \end{pmatrix}.$$

Then the eigenvalues  $\lambda_1$  and  $\lambda_2$  satisfy  $\lambda_1 + \lambda_2 = g(\lambda)f'(\lambda)$  and  $\lambda_1\lambda_2 = f(\lambda)g(\lambda)g'(\lambda) > 0$  and thus  $S_+$  is stable if  $\lambda \in (n, K)$  and unstable if  $\lambda \in (m, n)$ .  $\square$

Now, we mainly discuss the existence of positive periodic solutions and heteroclinic cycles for system (4) via the geometric theory of differential equations. We always assume that  $\lambda \in (n, K)$  holds in the following. For system (4),  $M\{x, y\} = \{(x, y) : x \in \mathbb{R}_+, y = \tau\}$ ,  $N\{x, y\} = \{(x, y) : x \in \mathbb{R}_+, y = (1 - q)\tau\}$ .

Suppose that the stable manifold  $W^s(S_1)$  intersects  $y = \tau$  and  $y = (1 - q)\tau$  respectively at points  $A$  and  $B$ , the unstable manifold  $W^u(S_2)$  intersects  $y = \tau$  and  $y = (1 - q)\tau$  respectively at points  $A_1$  and  $B_1$  and the predator-nullcline  $x = \lambda$  intersects  $y = (1 - q)\tau$  at the point  $C$ . Obviously, the impulsive function  $\varphi(x) = (1 - p)x$ ,  $x \in (0, K]$ . Let

$$p^* = 1 - \frac{x_B}{x_{A_1}}, \quad p^0 = 1 - \frac{x_C}{x_{A_1}}. \tag{5}$$

(i) if  $p^0 < p < p^*$ , then  $x_B < \varphi(x_{A_1}) < x_C$ . We claim that there exists a unique order-1 positive periodic solution for system (4). In fact, for given  $p \in (p^0, p^*)$ , assume that  $A_1$  is mapped into  $D$  due to the impulsive effect, then  $D$  is between  $B$  and  $C$ . Suppose that the trajectory starting from the point  $D$  intersects  $y = \tau$  at the point  $D_1$ . Note that any two trajectories cannot intersect. Hence,  $x_{D_1} < x_{A_1}$ . Based on the impulsive effect,  $D_1$  jumps to the point  $D_1^+$ . Then

$$x_{D_1^+} = \varphi(x_{D_1}) = (1 - p)x_{D_1} < (1 - p)x_{A_1} = \varphi(x_{A_1}) = x_D,$$

which implies that the successor function satisfies

$$\hat{g}(D) = x_{D_1^+} - x_D < 0.$$

On the other hand, take a point  $B_\epsilon$  such that  $x_B < x_{B_\epsilon} < x_{D_1^+}$ . Assume that the trajectory with the initial point  $B_\epsilon$  intersects  $y = \tau$  at the point  $B_{\epsilon 1}$ , then  $x_{D_1} < x_{B_{\epsilon 1}} < x_{A_1}$ . Due to the impulsive effect,  $B_{\epsilon 1}$  is mapped into the point  $B_{\epsilon 1}^+$ . Then

$$x_{B_{\epsilon 1}^+} = (1 - p)x_{B_{\epsilon 1}} > (1 - p)x_{D_1} = x_{D_1^+}.$$

Thus

$$\hat{g}(B_\epsilon) = x_{B_{\epsilon 1}^+} - x_{B_\epsilon} > x_{D_1^+} - x_{B_\epsilon} > 0.$$

It follows from Lemma 2.4 and Lemma 2.5 that there exists at least one order-1 positive periodic solution with the initial point between  $B_\epsilon$  and  $D$ . See Fig. 1 for a graph illustration.

Next we discuss the uniqueness of the order-1 positive periodic solution. Let  $E$  and  $F$  be any two points on the line segment  $\overline{B_\epsilon D}$  with  $x_E < x_F$  and the corresponding trajectories be

$$\begin{aligned} E &\rightarrow E_1 \rightarrow E_1^+; \\ F &\rightarrow F_1 \rightarrow F_1^+. \end{aligned}$$

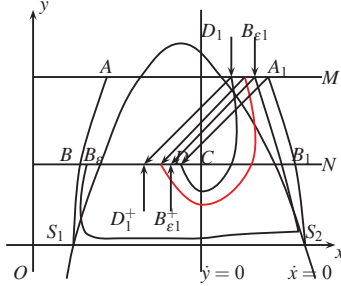


FIGURE 1. Illustration of the order-1 positive periodic solution in (4).

If  $x_E < x_F$ , then  $x_{F_1} < x_{E_1}$ . Thus,

$$x_{F_1^+} = (1 - p)x_{F_1} < (1 - p)x_{E_1} = x_{E_1^+}.$$

It follows that

$$\hat{g}(E) - \hat{g}(F) = x_{E_1^+} - x_E - (x_{F_1^+} - x_F) = (x_F - x_E) + (x_{E_1^+} - x_{F_1^+}) > 0,$$

which means that  $\hat{g}(\cdot)$  is a monotone decreasing function for any point in  $\overline{B_\varepsilon D}$ . Therefore, there exists a unique  $H$  such that  $\hat{g}(H) = 0$ . Fig. 2 gives a geometrical interpretation of the monotonicity of the successor function.

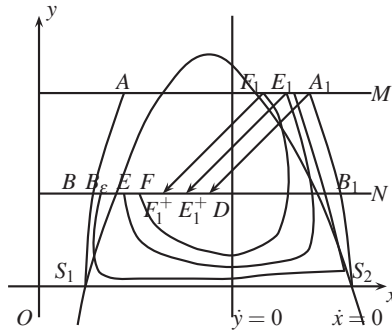


FIGURE 2. Geometry associated with the monotonicity of the successor function.

(ii) if  $p = p^*$ , then  $\varphi(x_{A_1}) = x_B$ . Thus,  $\widetilde{BS_1}$ ,  $\widetilde{S_1S_2}$ ,  $\widetilde{S_2A_1}$  together with the impulsive line  $\overline{A_1B}$  constitute a heteroclinic cycle for system (4) (see Fig. 3).

(iii) if  $p > p^*$ , then  $\varphi(x_{A_1}) < x_B$ . Notice that  $y_0 < \tau < f(\lambda)$ . Therefore, the trajectory with the initial condition  $(x_0, y_0)$  associated with system (4) will ultimately approach the origin, which implies that both the prey and predator populations will go extinct.

Summarizing the above, we have the following theorem:

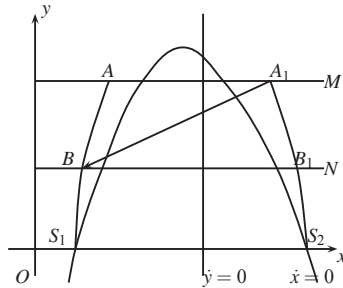


FIGURE 3. The illustration of the order-1 heteroclinic cycle in (4)

**Theorem 3.3.** *For system (4), there exist  $p^*, p^0$  satisfying (5), such that for any  $p \in (p^0, p^*)$ , system (4) has a unique order-1 positive periodic solution; if  $p = p^*$ , then the positive periodic solution disappears and bifurcates an order-1 heteroclinic cycle; if  $p > p^*$ , then system (4) has no periodic solutions and the trajectories will ultimately approach the origin.*

**4. Heteroclinic bifurcation for system (1) with  $\delta > 0$ .** In practical ecology, people always think that the growth rate of the prey is affected not only by the predator population density but also the relative speed of the predator population growth to a certain extent [7]. Therefore, in this section we consider the perturbed system (1).

In the following, we always suppose  $p = p^*$ . Choosing  $\delta$  as a control parameter, we discuss the heteroclinic bifurcation for system (1) by using the theory of rotated vector fields. It must be borne in mind that the location and the index of the singular points of system (1) remain unchanged and  $M\{x, y\}$  and  $N\{x, y\}$  have same forms as the system (4).

For system (1), we have  $\tilde{P}(x, y, \delta) = g(x)(f(x) - y) + \delta y(g(x) - d)$ ,  $\tilde{Q}(x, y, \delta) = y(g(x) - d)$ . Then

$$\Delta = \begin{vmatrix} \tilde{P} & \tilde{Q} \\ \frac{\partial \tilde{P}}{\partial \delta} & \frac{\partial \tilde{Q}}{\partial \delta} \end{vmatrix} = -y^2(g(x) - d)^2 < 0.$$

It follows from Lemma 2.7 that the rotated direction of vector fields of system (1) is clockwise.

For the perturbed system (1), suppose that the stable manifold of  $S_1$  intersects  $y = \tau$  and  $y = (1 - q)\tau$  respectively at points  $A_\delta$  and  $B_\delta$ , and the unstable manifold of  $S_2$  intersects  $y = \tau$  and  $y = (1 - q)\tau$  respectively at points  $A_{1\delta}$  and  $B_{1\delta}$ . In view of the rotated direction of vector fields of system (1), it is easy to see that  $x_{B_\delta} > x_B$  and  $x_{A_{1\delta}} > x_{A_1}$  when parameter  $\delta$  changes from  $\delta = 0$  to  $\delta > 0$ . Assume that  $A_{1\delta}$  is mapped into  $A_{1\delta}^+$ , we have the following three situations:

- if  $x_{A_{1\delta}^+} < x_{B_\delta}$ , then system (1) doesn't possess periodic solutions and both species become extinct;
- if  $x_{A_{1\delta}^+} = x_{B_\delta}$ , then system (1) has an order-1 heterocline cycle which is formed by  $\widetilde{B_\delta S_1}$ ,  $\widetilde{S_1 S_2}$ ,  $\widetilde{S_2 A_{1\delta}}$  together with the impulsive line  $\overline{A_{1\delta} B_\delta}$ ;



- if  $x_{A_{1\delta}^+} > x_{B_\delta}$ , we claim that there exists a unique order-1 positive periodic solution for the system (1). In fact, suppose that the trajectory starting from the point  $A_{1\delta}^+$  firstly intersects  $y = \tau$  at the point  $A_{2\delta}$  and then jumps into  $A_{2\delta}^+$  due to the impulsive effect. Since different trajectories cannot intersect, we get  $x_{A_{2\delta}^+} < x_{A_{1\delta}^+}$ . Therefore, the successor function satisfies

$$\hat{g}(A_{1\delta}^+) = x_{A_{2\delta}^+} - x_{A_{1\delta}^+} < 0. \tag{6}$$

On the other hand, take a point  $B_{\delta\eta}$  such that  $x_{B_\delta} < x_{B_{\delta\eta}} < x_{A_{2\delta}^+}$  and assume that  $B_{\delta\eta} \rightarrow B_{\delta\eta 1} \rightarrow B_{\delta\eta 1}^+$ . By the same arguments as in Section 3, we get  $B_{\delta\eta 1}$  is between  $A_{1\delta}$  and  $A_{2\delta}$  and  $x_{A_{2\delta}^+} < x_{B_{\delta\eta 1}^+} < x_{A_{1\delta}^+}$ . Therefore

$$\hat{g}(B_{\delta\eta}) = x_{B_{\delta\eta 1}^+} - x_{B_{\delta\eta}} > x_{A_{2\delta}^+} - x_{B_{\delta\eta}} > 0. \tag{7}$$

Using the inequalities (6) and (7), we can conclude that there exists an order-1 positive periodic solution with the initial point between  $B_{\delta\eta}$  and  $A_{1\delta}^+$ . Fig. 4 is referred to as a geometrical interpretation of the existence of the positive periodic solution. Similarly as the before, we can prove that the order-1 positive periodic solution is unique.

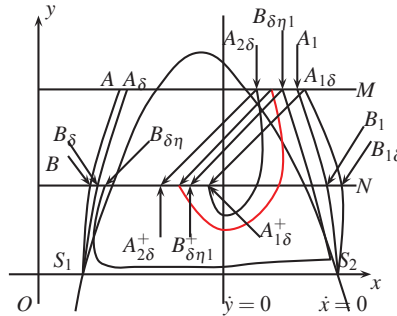


FIGURE 4. Geometry associated with the existence of the order-1 positive periodic solution for the perturbed system (1).

Summarizing the above, we have the following theorem.

**Theorem 4.1.** For  $p = p^*$ , if  $x_{A_{1\delta}^+} > x_{B_\delta}$ , then system (1) possesses a unique order-1 positive periodic solution; if  $x_{A_{1\delta}^+} = x_{B_\delta}$ , then the order-1 positive periodic solution breaks and system (1) undergoes an order-1 heteroclinic bifurcation; if  $x_{A_{1\delta}^+} < x_{B_\delta}$ , then the trajectories of system (1) will ultimately approach the origin.

**Remark 1.** Theorem 3.3 shows that the unperturbed system (4) (i.e., system (1) with  $\delta = 0$ ) possesses an order-1 heteroclinic cycle when  $p = p^*$ . However, Theorem 4.1 implies that the dynamical behaviors of the original system (4) will be changed if the perturbation appears (i.e.,  $\delta > 0$  in system (1)).

**5. Summary and discussion.** Based on the geometric theory of differential equations and the method of the successor function, we discuss a general predator-prey system with the state feedback control strategy in which the prey species displays

a strong Allee effect. The relation between the harvest rate  $p$  and the dynamics of system (4) is presented (see Theorem 3.3). If the harvest rate  $p$  satisfies  $p^0 < p < p^*$ , then a unique order-1 positive periodic solution to system (4) exists. When  $p$  increases and attains  $p^*$ , the positive periodic solution breaks and bifurcates an order-1 heteroclinic cycle. If  $p$  keeps on increasing, then the order-1 heteroclinic cycle disappears and both species tend to become extinct, in which case the system will collapse. Considering the influence of the relative speed of the predator population growth to the growth rate of the prey and assuming  $p = p^*$ , we study the existence of order-1 positive periodic solutions and heteroclinic bifurcation for the perturbed system (1). Using the theory of rotated vector fields, we get that system (1) possesses a unique order-1 positive periodic solution if  $x_{A_{1\delta}^+} > x_{B_\delta}$ . In the case of  $x_{A_{1\delta}^+} = x_{B_\delta}$ , system (1) undergoes an order-1 heteroclinic bifurcation. Otherwise, system (1) will collapse.

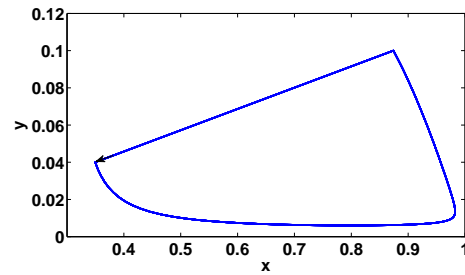
To testify the effectiveness of these theoretical results, concrete examples are performed. In these numerical tests, we take the functional response as Holling type I ( $g(x) = x$  ( $0 < x \leq K$ )), Holling type II ( $g(x) = \frac{2x}{1+x}$ ), Holling type III ( $g(x) = \frac{2x^2}{1+x^2}$ ), and Ivlev type ( $g(x) = 1 - e^{-2x}$ ), respectively.  $f(x)$  is chosen to satisfy  $g(x)f(x) = x(1-x)(x-0.2)$ , that is, both the intrinsic growth rate and the carrying capacity for the prey species are normalized to 1 and the strong Allee threshold value  $m$  equals to 0.2. Other parameters are  $d = 0.8, q = 0.6, \tau = 0.1$ . Obviously, the saddle points are  $(0.2, 1)$  and  $(1, 0)$ . The results of Theorem 3.3 are shown in Figs. 5-8, where  $p^*$  equals to 0.758, 0.746, 0.766, 0.743, respectively. For system (1) with  $\delta = 0.022$ , as is shown in Figs. 9 and 10, there exist order-1 heteroclinic cycles for Holling type I and III functional responses, but no periodic solutions for Holling type II and Ivlev type.

For a general predator-prey interaction system with state impulsive harvesting strategies for both populations and a fixed harvesting rate for the predator, we conclude that the harvesting rate for the prey should be maintained at a reasonable range from above results. Otherwise, both species will be driven to extinction and the sustainable development of the ecological system will be disrupted. Once the perturbation for the system appears, both the parameter perturbation and the predator-prey interaction effect play important roles in keeping the balance of the ecology. Moreover, with the economic factor of harvesting the predator and prey species in mind, optimal harvesting strategy should be put forward. Certainly, other kinds of state feedback control, such as the harvesting strategy which is implemented when the total amount of the two species reach a threshold value, can also be considered. These will be included in our further studies.

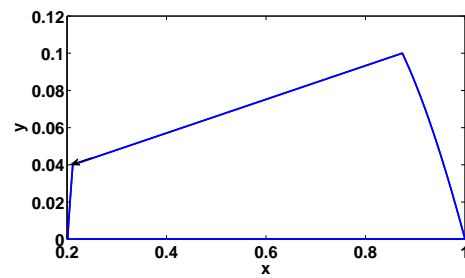
**Acknowledgments.** The authors wish to thank the reviewers and the handling editor for their comments and suggestions, which led to a great improvement to the presentation of this work.

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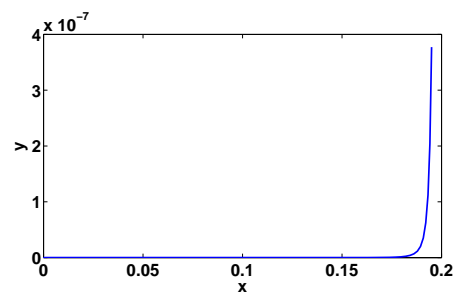
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(a)

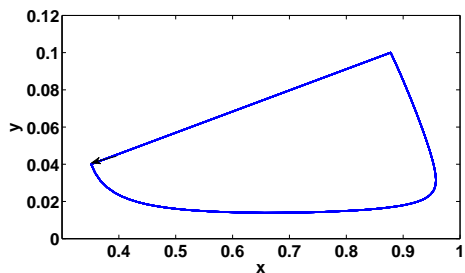


(b)

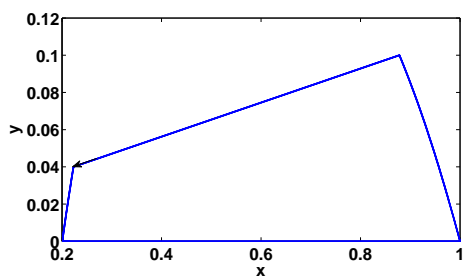


(c)

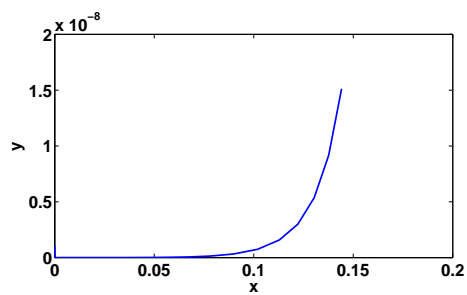
FIGURE 5. The numerical results for system (1) with  $\delta = 0$  and Holling type I functional response: (a)  $p=0.6$ ; (b)  $p=0.758$ ; (c)  $p=0.759$ .



(a)

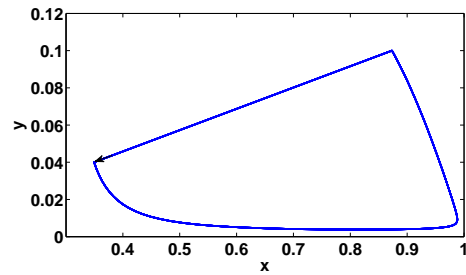


(b)

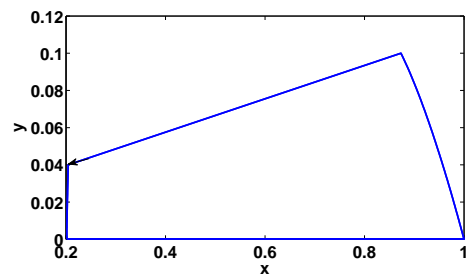


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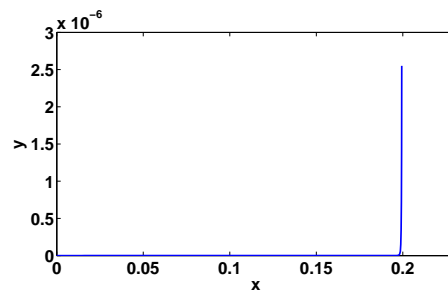
FIGURE 6. The numerical results for system (1) with  $\delta = 0$  and Holling type II functional response: (a)  $p=0.6$ ; (b)  $p=0.746$ ; (c)  $p=0.747$ .



(a)

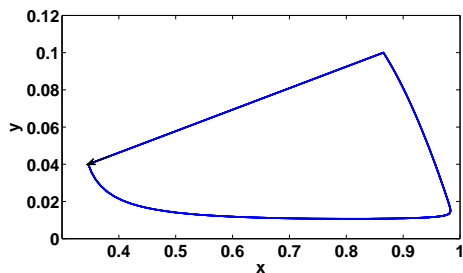


(b)

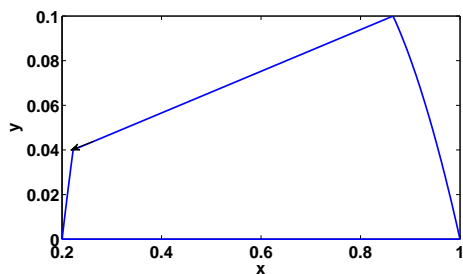


(c)

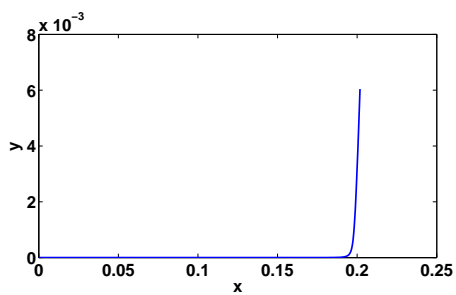
FIGURE 7. The numerical results for system (1) with  $\delta = 0$  and Holling type III functional response: (a)  $p=0.6$ ; (b)  $p=0.766$ ; (c)  $p=0.767$ .



(a)

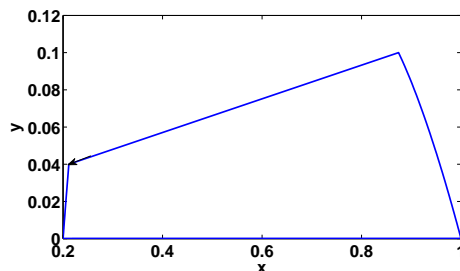


(b)

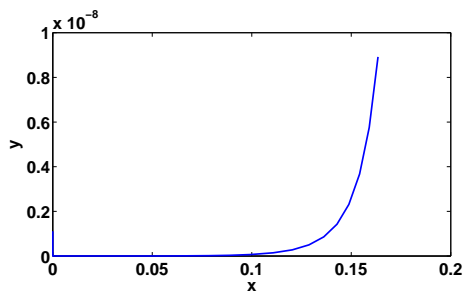


(c)

FIGURE 8. The numerical results for system (1) with  $\delta = 0$  and Ivlev type functional response: (a)  $p=0.6$ ; (b)  $p=0.743$ ; (c)  $p=0.744$ .



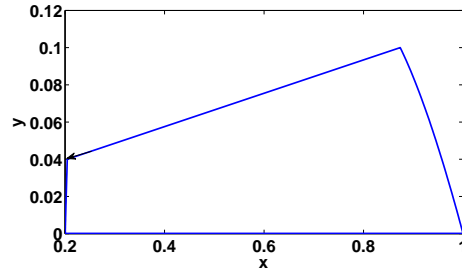
(a)



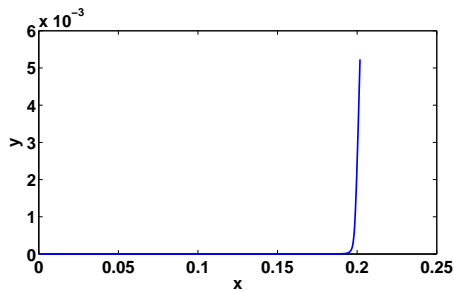
(b)

FIGURE 9. The numerical results for system (1) with  $\delta = 0.022 > 0$ : (a) Holling type I; (b) Holling type II.

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(a)



(b)

FIGURE 10. The numerical results for system (1) with  $\delta = 0.022 > 0$ : (a) Holling type III; (b) Ivlev type.

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*E-mail address:* [xqz19870705@163.com](mailto:xqz19870705@163.com)

*E-mail address:* [bx dai@csu.edu.cn](mailto:bx dai@csu.edu.cn)