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THRESHOLD DYNAMICS OF A PERIODIC SIR MODEL WITH DELAY IN AN INFECTED COMPARTMENT

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Abstract. Threshold dynamics of epidemic models in periodic environments attract more attention. But there are few papers which are concerned with the case where the infected compartments satisfy a delay differential equation. For this reason, we investigate the dynamical behavior of a periodic SIR model with delay in an infected compartment. We first introduce the basic reproduction number $\mathcal{R}_0$ for the model, and then show that it can act as a threshold parameter that determines the uniform persistence or extinction of the disease. Numerical simulations are performed to confirm the analytical results and illustrate the dependence of $\mathcal{R}_0$ on the seasonality and the latent period.

1. **Introduction.** Since Bacaër and Guernaoui [1] gave the definition of the basic reproduction number $\mathcal{R}_0$ in a periodic environment, the study of the dynamics of time-periodic seasonal models attracts more attention, see [5, 9, 10, 15, 16, 22, 26] and references therein. These studies showed that $\mathcal{R}_0$ can serve as a threshold parameter that assess whether a newly infectious disease can invade a population. That is, the disease eventually disappears if $\mathcal{R}_0 < 1$, while the model has a positive periodic solution and the disease persists if $\mathcal{R}_0 > 1$. One common property of models considered in these studies is that there is no delay in infected compartments, which makes the study of threshold dynamics relatively easy. Due to the difficulty in the analysis, to our knowledge, there is few study of the threshold dynamics of periodic epidemic models where the infected compartments satisfy a delay differential equation, only see [17]. It is necessary to point out that the reference [17] obtained the threshold-type results for periodic delayed models in terms of basic reproduction number R_0. But the method and technique used in this paper are quite different to [17].

It is well-known that for some vector-borne diseases, delay caused by latent period in a vector cannot be ignored in the transmission of infectious diseases. This is because transmission of such disease is through vectors which undergo fast dynamics and a fixed latent period $\tau > 0$, during which the infectious agents develop in the vector. It is only after that time that the infected vector can infect a susceptible human [6, 8]. On the other hand, seasonality is also an important aspect

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of vector-borne diseases. For example, seasonal change in the incidence of malaria epidemics are caused by seasonal variation in mosquito abundance in response to annual variation in temperature and rainfall [7]. Thus, it is not trivial but natural to incorporate the delay and periodic fluctuation into epidemic models.

We consider a disease transmission of SIR type. Let S, I and R be the number of susceptible, infectious and recovered individuals, respectively. The death rates for the classes are μ_1, μ_2 and μ_3, respectively. It is natural biologically to assume that $\mu_1 \leqslant \min\{\mu_2, \mu_3\}$. All recruitment is into the susceptible class at a constant rate Λ. The recovery rate of infectives is γ. To account for seasonal effects, we assume that the contact rate $\beta(t)$ is a continuous, positive and ω-periodic function. Following [6, 8, 11], the vectors can be omitted from the equations by including a delay τ in the force of infection $\beta(t)Sf(I(t-\tau))$, which is a general incidence function.

The preceding assumptions lead to the following delay SIR model:

$$\begin{cases} S' = \Lambda - \mu_1 S - \beta(t)Sf(I(t-\tau)), \\ I' = \beta(t)Sf(I(t-\tau)) - (\mu_2+\gamma)I, \\ R' = \gamma I - \mu_3 R. \end{cases} \tag{1}$$

The function $f(I)$ is assumed to satisfy

(H1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable with $f(0) = 0, f'(0) > 0$ and $f(I) > 0$ for $I \in (0, +\infty)$;
(H2) $f(I) - If'(I) \geqslant 0$.

Under the hypotheses (H1) and (H2), the forms of $\beta(t)Sf(I(t-\tau))$ include various types of incidence rates. If $f(I) = I$, then the incidence rate becomes a bilinear form, which is proposed in [6, 12, 13]. If $f(I) = \frac{1}{1+kI}(k > 0)$, then the incidence rate describes the saturated effects of the prevalence of infectious diseases [11, 23]. Moreover, if $f(I) = \frac{1}{1+\alpha I^2}(\alpha > 0)$ is a nonmonotone function, then the incidence rate describes the psychological or inhibitory effect of the susceptible individuals on the disease when the number of infected individuals becomes large, see [21, 24].

Observe that the variable R does not appear in the first two equations of (1). This allows us to attack (1) by studying the subsystem

$$\begin{cases} S' = \Lambda - \mu_1 S - \beta(t)Sf(I(t-\tau)), \\ I' = \beta(t)Sf(I(t-\tau)) - (\mu_2+\gamma)I. \end{cases} \tag{2}$$

Initial conditions for (2) are chosen at $t = 0$ as

$$S(0) = \phi_1,\ I_0(\theta) = \phi_2(\theta) > 0,\ \theta \in [-\tau, 0], \tag{3}$$

where $\phi = (\phi_1, \phi_2) \in \mathbb{R}_+ \times \mathcal{C}_+$ and $\mathcal{C}_+ := C([-\tau, 0], \mathbb{R}_+)$. For any function $x : [-\tau, b] \to \mathbb{R}^m, b > 0$, we define $x_t \in C([-\tau, 0], \mathbb{R}^m)$ by $x_t(\theta) = x(t+\theta), \forall\, \theta \in [-\tau, 0]$. Let $\hat{}$ denote the constant function, that is, $\hat{x}(\theta) = x$ for all $\theta \in [-\tau, 0]$.

The remainder of this paper is organized as follows. In Section 2, we introduce the basic reproduction number $\mathcal{R}_0$ of (2) by the theory developed in [1]. In Section 3, we show that $\mathcal{R}_0$ can act as a threshold parameter for the uniform persistence and global extinction of the disease. In Section 4, we present numerical simulations which support the analytical results. Moreover, with an approximation method proposed in [4], we demonstrate the difference of $\mathcal{R}_0$ between SIR (ODE) models and SIR (DDE) models with periodic coefficients. The last section is a brief discussion.

2. **Well-posed and the basic reproduction number.** To investigate the dynamics of (2), we first establish the feasible region of the model and show that the model is well-posed.

Lemma 2.1. *Under initial conditions in (3), all solutions of (2) are positive and ultimately uniformly bounded. Further, the compact set*

$$\mathcal{D} := \{(S, I) \in \mathbb{R}_+ \times \mathcal{C}_+ : 0 \leqslant S, I \leqslant \Lambda/\mu\}.$$

is positively invariant for (2), where $\mu = \min\{\mu_1, \mu_2\}$.

Proof. First, we prove that $S(t)$ is positive for all $t \geqslant 0$. Assuming the contrary, and letting $t_1 > 0$ be the first time such that $S(t_1) = 0$, then by the first equation of system (1) we have $S'(t_1) = \Lambda > 0$, and hence $S(t) < 0$ for $t \in (t_1 - \varepsilon, t_1)$ and sufficiently small $\varepsilon > 0$. This contradicts $S(t) > 0$ for $t \in [0, t_1)$. It follows that $S(t) > 0$ for $t > 0$. Solving $I(t)$ in the second equation of (1), we obtain

$$I(t) = \left(I(0) + \int_0^t \beta(\xi) S(\xi) f(I(\xi - \tau)) e^{(\mu_2+\gamma)\xi} d\xi \right) e^{-(\mu_2+\gamma)t}.$$

It follows that $I(t) > 0$ for $t > 0$.

Next we show that positive solutions of (2) are ultimately uniformly bounded for $t \geqslant 0$. Adding all the equations of (2) we get

$$\begin{aligned} (S(t) + I(t))' &= \Lambda - \mu_1 S - \mu_2 I - \gamma I \\ &\leqslant \Lambda - \mu(S + I). \end{aligned}$$

Thus $\limsup_{t\to\infty}(S(t)+I(t)) \leqslant \Lambda/\mu$. This implies that $S(t)$ and $I(t)$ are ultimately uniformly bounded. □

It is easy to see that system (2) always has a disease-free equilibrium $E_0(S^*, 0)$, where $S^* = \frac{\Lambda}{\mu_1}$. We then introduce the basic reproduction number $\mathcal{R}_0$ for system (2) by using the next generation operators approach [1]. The linearized equation for infectious compartment is

$$I'(t) = \beta(t) S^* f'(0) I(t - \tau) - (\mu_2 + \gamma) I(t). \tag{4}$$

Let $i(t) = \beta(t) S^* f'(0) I(t - \tau)$ be the incidence of infectious cases at time t. Then

$$(e^{(\mu_2+\gamma)t} I(t))' = e^{(\mu_2+\gamma)t} i(t). \tag{5}$$

Integrating the equation (5) from 0 to t and making a change of variable, we obtain

$$i(t) = \beta(t) S^* f'(0) I(0) e^{-(\mu_2+\gamma)(t-\tau)} + \beta(t) S^* f'(0) \int_\tau^t i(t - x) e^{-(\mu_2+\gamma)(x-\tau)} dx.$$

This is of the form

$$i(t) = i_0(t) + \int_0^t K(t, x) i(t - x) dx,$$

where

$$\begin{aligned} i_0(t) &= \beta(t) S^* f'(0) I(0) e^{-(\mu_2+\gamma)(t-\tau)}, \\ K(t, x) &= \begin{cases} 0, & \text{if } 0 < x < \tau, \\ \beta(t) S^* f'(0) e^{-(\mu_2+\gamma)(x-\tau)}, & \text{if } x > \tau. \end{cases} \end{aligned} \tag{6}$$

Let $\mathcal{C}_\omega$ be the ordered Banach space of all ω-periodic functions from $\mathbb{R}$ to $\mathbb{R}$, which is equipped with the maximum norm $\|\cdot\|$ and the positive cone $\mathcal{C}_\omega^+ := \{\phi \in \mathcal{C}_\omega : \phi(t) \geqslant 0, \forall\ t \in \mathbb{R}\}$. Then we can define a linear operator $L_0 : \mathcal{C}_\omega \to \mathcal{C}_\omega$ by

$$L_0 : u(t) \mapsto \int_0^\infty K(t,x)u(t-x)dx.$$

Following [1], we define the basic reproduction number as $\mathcal{R}_0 := \rho(L_0)$, the spectral radius of L_0.

Consider a more general operator

$$L_\lambda : u(t) \mapsto \int_0^\infty e^{-\lambda x}K(t,x)u(t-x)dx$$

on the same space $\mathcal{C}_\omega$. Since $K(t,x)$ has no compact support with respect to x, we cannot use directly the periodic renewal theorem in [19] to give the asymptotic behavior of $i(t)$. To this end, we will use Lemmas 1 and 3 in [3]. Using similar arguments in [2], we can show the existence of the Malthusian parameter and the positivity of the total reproductive value. It then follows from [3, Lemmas 1 and 3] that there is a positive ω-periodic function $v(t)$ such that $i(t) \sim e^{\lambda^* t}v(t)$ as $t \to +\infty$, and

$$\int_0^\infty e^{-\lambda^* x}K(t,x)v(t-x)dx = v(t).$$

This number λ^* is called the Malthusian parameter. Then we have the following result.

Lemma 2.2. *(see* [3, Gorollary 1]*)* $\lambda^* > 0$ *(resp.* $\lambda^* = 0, \lambda^* < 0$*) if and only if* $\mathcal{R}_0 > 1$ *(resp.* $\mathcal{R}_0 = 1, \mathcal{R}_0 < 1$*).*

3. **Global dynamics.** In this section, we will show that $\mathcal{R}_0$ as defined above is a threshold quantity that determines whether the disease dies out or uniformly persists.

Theorem 3.1. *If* $\mathcal{R}_0 < 1$*, then the disease-free equilibrium of* (2) *is globally attracting.*

Proof. Let $(S(t), I(t)) = (S(t,\phi), I(t,\phi))$ be the solution of (2) through $\phi \in \mathbb{R}_+ \times \mathcal{C}_+$ at $t = 0$. From the first equation of (2), we obtain

$$S'(t) \leqslant \Lambda - \mu_1 S(t),$$

and thus $\limsup_{t\to\infty} S(t) \leqslant S^*$. Using the comparison theorem, we conclude that for any $\varepsilon > 0$, there exists $\bar{t} > 0$ such that

$$S(t) \leqslant S^* + \varepsilon,\ \forall\ t \geqslant \bar{t}.$$

Then we have

$$I'(t) \leqslant \beta(t)(S^* + \varepsilon)f'(0)I(t-\tau) - (\mu_2 + \gamma)I(t),\ \forall\ t \geqslant \bar{t}.$$

Here, we have used the assumptions on $f(I)$ and the fact that

$$\left(\frac{f(I)}{I}\right)' = \frac{f'(I)I - f(I)}{I^2} \leqslant 0,\ \text{and thus}\ \frac{f(I)}{I} \leqslant \lim_{I\to 0^+}\frac{f(I)}{I} = f'(0).$$

Considering the following comparison equation

$$\bar{I}'(t) = \beta(t)(S^* + \varepsilon)f'(0)\bar{I}(t-\tau) - (\mu_2 + \gamma)\bar{I}(t).$$

Let $\bar{i}(t) = \beta(t)(S^* + \varepsilon)f'(0)\bar{I}(t-\tau)$. Similar to the arguments in Section 2, we have that $\bar{i}(t)$ is of the form

$$\bar{i}(t) = \bar{i}_0(t) + \int_0^t K_\varepsilon(t,x)\bar{i}(t-x)dx,$$

where

$$\bar{i}_0(t) = \beta(t)(S^* + \varepsilon)f'(0)\bar{I}(0)e^{-(\mu_2+\gamma)(t-\tau)},$$
$$K_\varepsilon(t,x) = \begin{cases} 0, & \text{if } 0 < x < \tau, \\ \beta(t)(S^* + \varepsilon)f'(0)e^{-(\mu_2+\gamma)(x-\tau)}, & \text{if } x > \tau. \end{cases}$$

Define $\mathcal{R}_\varepsilon := \rho(L_\varepsilon)$ as the spectral radius of the operator

$$L_\varepsilon : u(t) \mapsto \int_0^\infty K_\varepsilon(t,x)u(t-x)dx.$$

Similar to the analysis of the asymptotic behavior of $i(t)$ in Section 2, it follows from [3, Lemmas 1 and 3] that there is a nonnegative ω-periodic function $v(t,\varepsilon)$ such that $\bar{i}(t) \sim e^{\lambda^*(\varepsilon)t}v(t,\varepsilon)$ as $t \to +\infty$, where $\lambda^*(\varepsilon)$ is a real number such that

$$v(t,\varepsilon) = \int_0^\infty e^{-\lambda^*(\varepsilon)x}K_\varepsilon(t,x)v(t-x,\varepsilon)dx.$$

Since $\lim_{\varepsilon\to 0}\mathcal{R}_\varepsilon = \mathcal{R}_0 < 1$, we can restrict ε small enough such that $R_\varepsilon < 1$. From Lemma 2.2, we know that $\mathcal{R}_\varepsilon = \rho(L_\varepsilon) < 1$ if and only if $\lambda^*(\varepsilon) < 0$. It is easy to see that $\lim_{t\to\infty}\bar{i}(t) = 0$, and thus

$$\lim_{t\to\infty}\bar{I}(t) = 0.$$

Let $\bar{I}(s) = I(s,\phi), s \in [\bar{t}, \bar{t}+\tau]$. By the comparison theorem [18, Theorem 5.1.1], we have $\lim_{t\to\infty} I(t) = 0$. By asymptotically autonomous semiflows [20], it then follows that $S(t) \to S^*$ as $t \to \infty$. The result follows. □

In order to use persistence theory, we define

$$X := \mathbb{R}_+ \times \mathcal{C}_+,\ X_0 := \{\phi = (\phi_1,\phi_2) \in X : \phi_2(0) > 0\},\ \partial X_0 := X \setminus X_0.$$

Let $\Phi(t)(\phi) = (S(t,\phi), I_t(\phi))$ be the periodic semiflow of (2), and $P : X \to X$ be the Poincaré map associated with (2), that is,

$$P(\phi) = \Phi(\omega)(\phi),\ \phi \in X.$$

It is easy to see that

$$P^n(\phi) = \Phi(n\omega)(\phi).$$

Lemma 3.2. *If $\mathcal{R}_0 > 1$, then there is a $\delta > 0$ such that for any $\phi \in X_0$, we have*

$$\limsup_{n\to\infty}\|P^n(\phi) - M_1\| \geqslant \delta,$$

where $M_1 = (S^*, \hat{0})$.

Proof. Note that $f(I)/I$ is a decreasing function on $[0,\infty)$, there exists a positive number σ such that for any $\epsilon \in (0, \min\{f'(0), S^*\})$,

$$f(I)/I \geqslant f'(0) - \epsilon,\ 0 < I < \sigma.$$

Let $0 < \eta < \min\{\sigma,\epsilon\}$. By the fact that $\lim_{\phi\to M_1}(\Phi(t)\phi - \Phi(t)M_1) = 0$ uniformly for $t \in [0,\omega]$, there exists $\delta > 0$ satisfying $\delta < \eta$ such that

$$\|\Phi(t)\phi - \Phi(t)M_1\| < \eta,\ \forall\ t \in [0,\omega], \|\phi - M_1\| < \delta.$$

Assume, by contradiction, that $\limsup_{n\to\infty} \|\Phi(n\omega)\psi - M_1\| < \delta$ for some $\psi \in X_0$. Then there exists an integer $N_1 \geqslant 1$ such that $\|\Phi(n\omega)\psi - M_1\| < \delta$ for all $n \geqslant N_1$. For any $t \geqslant N_1\omega + \tau =: \hat{t}$, we can choose $t' \in [0, \omega]$ such that $t = n\omega + t'$. Then

$$\|\Phi(t)\psi - \Phi(t)M_1\| = \|\Phi(t')\Phi(n\omega)\psi - \Phi(t')M_1\| < \eta,$$

and hence $S(t, \psi) > S^* - \eta$ when $t \geqslant \hat{t}$. Consequently, for $t \geqslant \hat{t}$, it holds that

$$\begin{aligned} I'(t) &= \beta(t)Sf(I(t-\tau)) - (\mu_2 + \gamma)I \\ &\geqslant \beta(t)(f'(0) - \epsilon)(S^* - \epsilon)I(t-\tau) - (\mu_2 + \gamma)I(t). \end{aligned}$$

Now we consider a perturbed system of (4)

$$\hat{I}'(t) = \beta(t)(f'(0) - \epsilon)(S^* - \epsilon)\hat{I}(t-\tau) - (\mu_2 + \gamma)\hat{I}(t).$$

Let $\hat{i}(t) = \beta(t)(f'(0) - \epsilon)(S^* - \epsilon)\hat{I}(t-\tau)$. It is easy to see that $\hat{i}(t)$ satisfies

$$\hat{i}(t) = \hat{i}_0(t) + \int_0^t K_\epsilon(t, x)\hat{i}(t-x)dx$$

with $\hat{i}_0(t)$ and $K_\epsilon(t, x)$ similar to (6) except that $f'(0)$ and S^* are replaced by $f'(0) - \epsilon$ and $S^* - \epsilon$, respectively. Define $\mathcal{R}_\epsilon := \rho(L_\epsilon)$ as the spectral radius of the operator

$$L_\epsilon : u(t) \mapsto \int_0^\infty K_\epsilon(t, x)u(t-x)dx.$$

Since $\lim_{\epsilon\to 0} \mathcal{R}_\epsilon = \mathcal{R}_0 > 1$, we restrict ϵ small enough such that $\mathcal{R}_\epsilon > 1$. By [3, Lemmas 1 and 3], there exists a unique real number $\lambda^*(\epsilon)$ and nonnegative ω-periodic function $v(t, \epsilon)$ such that $\hat{i}(t) \sim e^{\lambda^*(\epsilon)t}v(t, \epsilon)$ as $t \to \infty$, and

$$v(t, \epsilon) = \int_0^\infty e^{-\lambda^*(\epsilon)x}K_\epsilon(t, x)v(t-x, \epsilon)dx.$$

By Lemma 2.2, $\mathcal{R}_\epsilon = \rho(L_\epsilon) > 1$ if and only if $\lambda^*(\epsilon) > 0$. Hence, $\lim_{t\to\infty} \hat{i}(t) = +\infty$, and then

$$\lim_{t\to\infty} \hat{I}(t) = +\infty.$$

Let $\hat{I}(s) = I(s, \psi), s \in [\hat{t}, \hat{t} + \tau]$. By the comparison theorem [18, Theorem 5.1.1], there must be a $t_1 > \hat{t}$ such that

$$\eta < I(t_1, \psi) < \sigma,$$

which is a contradiction. □

Theorem 3.3. *If $\mathcal{R}_0 > 1$, then there is an $\eta > 0$ such that every solution $(S(t, \phi), I(t, \phi))$ of (2) with $\phi \in X_0$ satisfies*

$$\liminf_{t\to\infty} (S(t, \phi), I(t, \phi)) \geqslant (\eta, \eta).$$

Furthermore, system (2) admits at least one strictly positive ω-periodic solution.

Proof. Clearly, both X and X_0 are positively invariant for the semiflow $\Phi(t)$. By Lemma 2.1, P is point dissipative and P^{n_0} is compact whenever $n_0\omega \geqslant 2\tau$. It then follows from [14, Theorem 2.9] that P admits a global attractor in X. Define

$$M_\partial := \{\phi \in \partial X_0 : P^n(\phi) \in \partial X_0, n \geqslant 0\}.$$

We then claim that

$$M_\partial = \{\phi \in \partial X_0 : \phi_2(0) = 0\}. \tag{7}$$

For any $\phi = (\phi_1, \phi_2) \in M_\partial$, since $\{\phi \in \partial X_0 : \phi_2(0) = 0\} \subseteq M_\partial$, it suffices to show that $I(n\omega, \phi) = 0$ for $n \geqslant 0$. If it is not true, there exists an $N_2 \in \mathbb{Z}^+$ such that $I(N_2\omega, \phi) > 0$. Note that $S(t, \phi) > 0$ for $t \geqslant 0$ and the solution $I(t, \phi)$ on the interval $[N_2\omega, N_2\omega + \tau]$ is given by

$$I(t,\phi) = e^{-(\mu_2+\gamma)t}\left(I(N_2\omega,\phi) + \int_{N_2\omega}^{t} e^{(\mu_2+\gamma)s}\beta(s)S(s)f(I(s-\tau))ds\right) > 0.$$

By the techniques of the method of steps for delay differential equations, we obtain that $I(t, \phi) > 0, \forall\ t \geqslant N_2\omega$. Then, $(S(t, \phi), I(t, \phi)) \in X_0,\ \forall\ t \geqslant N_2\omega$. Accordingly, the positive invariance of X_0 implies (7). Clearly, there is exactly one fixed-point M_1 of P in M_∂.

Lemma 3.2 indicates that M_1 is isolated in X and $W^s(M_1) \cap X_0 = \emptyset$, where $W^s(M_1)$ is stable set of M_1. Moreover, there is no cycle in M_∂ from M_1 to M_1. Using [25, Theorem 1.3.1 and Remark 1.3.1], we conclude that $P : X \to X$ is uniformly persistent with respect to X_0. Thus, $\Phi(t) : X \to X$ is also uniformly persistent with respect to X_0 by [25, Theorem 3.1.1]. It then follows from [14, Theorem 4.5] that system (2) admits a ω-periodic solution $(S^\star(t), I^\star(t)) \in X_0$.

It remains to prove the practical uniform persistence. A proof similar to that of [10, Theorem 3.2] shows that there exists $\eta > 0$ such that

$$\liminf_{t\to\infty} \min(S(t,\phi), I(t,\phi)) \geqslant (\eta, \eta),\ \forall\ \phi \in X_0.$$

Thus, $(S^\star(t), I^\star(t))$ is a strictly positive periodic solution. □

4. **Numerical simulation.** In this section, we carry out numerical simulations to illustrate the results of the last section. For simplicity, we consider the following system:

$$\begin{cases} S' = \Lambda - \mu S - \beta(t)SI(t-\tau), \\ I' = \beta(t)SI(t-\tau) - (\mu+\gamma)I, \end{cases} \tag{8}$$

with $\beta(t) = \beta_0(1 + \varepsilon\cos(2\pi t)), |\varepsilon| \leqslant 1$. Let

$$k(x) = \begin{cases} 0, & \text{if } 0 < x < \tau, \\ \beta_0 S^* e^{-(\mu+\gamma)(x-\tau)}, & \text{if } x > \tau. \end{cases}$$

Then

$$k_n = \int_0^\infty k(x)e^{-ni\omega x}dx = \beta_0 S^* \frac{e^{-ni\omega\tau}}{\mu+\gamma+ni\omega}.$$

Using the method of [4], we can compute $\mathcal{R}_0$ as the largest real root of

$$\frac{\mathcal{R}_0}{k_0} - 1 - 2\mathrm{Re}\frac{\varepsilon^2/4}{\frac{\mathcal{R}_0}{k_1} - 1 - \frac{\varepsilon^2/4}{\frac{\mathcal{R}_0}{k_2} - 1 - \frac{\varepsilon^2/4}{\cdots}}} = 0.$$

In the case where the parameters are given, we can get the value of $\mathcal{R}_0$ by using a dichotomy method.

4.1. **Persistence or extinction.** Let $\Lambda = 2,\ \mu = 0.2,\ \beta(t) = \beta_0(1+0.8\cos(2\pi t)), \gamma = 0.6, \tau = 0.5$. The initial conditions are chosen as

$$S(0) = 2,\ I_0(\theta) = 2.4,\ \theta \in [-\tau, 0].$$

Choose $\beta_0 = 0.06$, then $\mathcal{R}_0 \simeq 0.7426$. By Theorem 3.1, the disease will die out. Our numerical simulations confirm this result, see Figure 1(a). Choose $\beta_0 = 0.15$, then $\mathcal{R}_0 \simeq 1.8564$. Hence, from Theorem 3.3, we expect that the disease persists and there is one positive 1-periodic solution. This is illustrated in Figure 1(b).

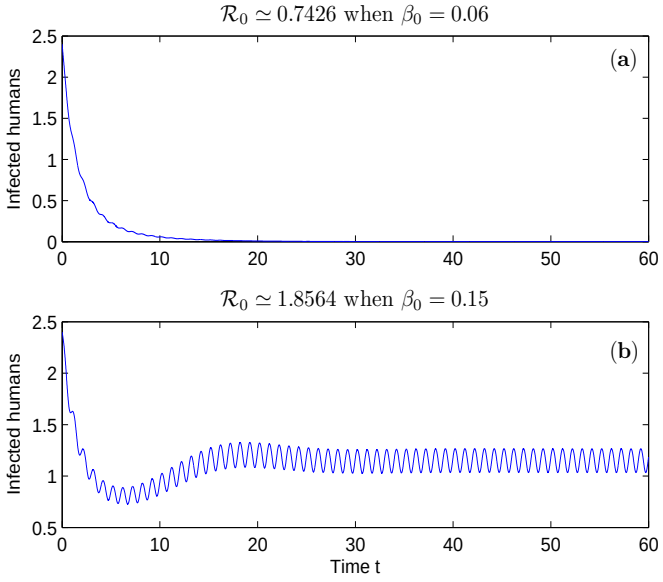

FIGURE 1. Long-term behavior of the infectious host population when $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$.

4.2. **The effect of seasonality and latent period on $\mathcal{R}_0$.** When $\tau = 0$, following the next generation operators approach [1, 22], the basic reproduction number for (8) can be defined as

$$\mathcal{R}_0 = \frac{\bar{\beta}\Lambda}{\mu(\mu+\gamma)}, \qquad \bar{\beta} = \frac{1}{\omega}\int_0^\omega \beta(t)dt.$$

For the case $\beta(t) = \beta_0(1 + \varepsilon\cos(2\pi t))$, it is easy to see that $\mathcal{R}_0$ is independent of ε. However, when $\tau > 0$, $\mathcal{R}_0$ has no closed formula. Now, we numerically analyze the relationship between $\mathcal{R}_0$ and the parameters τ and ε. Fixing $\varepsilon = 0, 0.3, 0.5, 0.8$, respectively, and varying τ in [0, 5], we obtain Figure 2, which demonstrates that:

(1) $\mathcal{R}_0$ is independent of τ when $\varepsilon = 0$. In this case, $\mathcal{R}_0 = \frac{\beta_0\Lambda}{\mu(\mu+\gamma)}$ is indeed the basic reproduction number for autonomous epidemic model (see [12]).

(2) $\mathcal{R}_0$ is a 1-periodic function of τ;

(3) For a fixed τ, $\mathcal{R}_0$ can be an increasing or decreasing function of ε.

This shows that there is a big difference for the epidemic threshold between SIR (ODE) models and SIR (DDE) models with periodic coefficients. Similar observations are also seen in [4, Section 4.3].

5. **Discussion.** In this paper, we consider a time-delayed periodic SIR model with a class of nonlinear incidence. We first introduce the basic reproduction number $\mathcal{R}_0$ for (2), and then show that the disease-free equilibrium is globally asymptotically stable, and the disease dies out if $\mathcal{R}_0 < 1$, while there exists an endemic periodic solution and the disease keeps persistent in the population when $\mathcal{R}_0 > 1$. This threshold result is very important for epidemiologists to control a disease.

We strongly believe that not only for periodic SIR models with delay in one infected compartment but also for epidemic models with delays in multiple infected

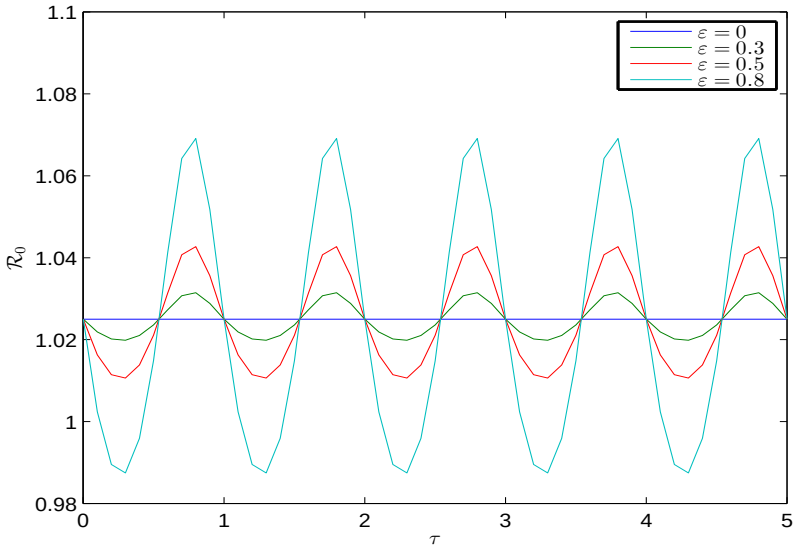

FIGURE 2. The relationship of $\mathcal{R}_0$ and τ as ε changes. The other parameters are fixed as in Figure 1.

compartments or SEIR models, our method might be applied and so $\mathcal{R}_0$ defined by [1, 4] still can act as a threshold parameter which predicts the spread of the disease.

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REFERENCES

[1] N. Bacaër and S. Guernaoui, The epidemic threshold of vector-borne diseases with seasonality, *J. Math. Biol.*, **53** (2006), 421–436.
[2] Z. Bai and Y. Zhou, Threshold dynamics of a delayed SEIRS model with pulse vaccination and general nonlinear incidence, 2014, preprint.
[3] N. Bacaër and E. Ait Dads, Genealogy with seasonality, the basic reproduction number, and the influenza pandemic, *J. Math. Biol.*, **62** (2011), 741–762.
[4] N. Bacaër and R. Ouifki, Growth rate and basic reproduction number for population models with a simple periodic factor, *Math. Biosci.*, **210** (2007), 647–658.
[5] Z. Bai and Y. Zhou, Existence of multiple periodic solutions for an SIR model with seasonality, *Nonlinear Anal.*, **74** (2011), 3548–3555.
[6] K. L. Cooke, Stability analysis for a vector disease model, *Rocky Mountain J. Math.*, **9** (1979), 31–42.
[7] N. C. Grassly and C. Fraser, Seasonal infectious disease epidemiology, *Proc. R. Soc. B.*, **273** (2006), 2541–2550.
[8] G. Huang, Y. Takeuchi, W. Ma and D. Wei, Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate, *Bull. Math. Biol.*, **72** (2010), 1192–1207.
[9] J. Lou, Y. Lou and J. Wu, Threshold virus dynamics with impulsive antiretroviral drug effects, *J. Math. Biol.*, **65** (2012), 623–652.
[10] Y. Lou and X.-Q. Zhao, A climate-based malaria transmission model with structured vector population, *SIAM J. Appl. Math.*, **70** (2010), 2023–2044.
[11] C. C. McCluskey, Global stability for an SIR epidemic model with delay and nonlinear incidence, *Nonlinear Anal. RWA*, **11** (2010), 3106–3109.

[12] C. C. McCluskey, Complete global stability for an SIR epidemic model with delay-distributed or discrete, *Nonlinear Anal. RWA*, **11** (2010), 55–59.
[13] W. Ma, M. Song and Y. Takeuchi, Global stability of an SIR epidemic model with time delay, *Appl. Math. Lett.*, **17** (2004), 1141–1145.
[14] P. Magal and X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM J. Math. Anal.*, **37** (2005), 251–275.
[15] Y. Nakata and T. Kuniya, Global dynamics of a class of SEIRS epidemic models in a periodic environment, *J. Math. Anal. Appl.*, **363** (2010), 230–237.
[16] C. Rebelo, A. Margheri and N. Bacaër, Persistence in seasonally forced epidemiological models, *J. Math. Biol.*, **64** (2012), 933–949.
[17] C. Rebelo, A. Margheri and N. Bacaër, Persistence in some periodic epidemic models with infection age or constant periods of infection, *Discrete Contin. Dyn. Syst. Ser. B*, **19** (2014), 1155–1170.
[18] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Mathematical Surveys and Monographs, 41. American Mathematical Society, Providence, RI, 1995.
[19] H. R. Thieme, Renewal theorems for linear periodic Volterra integral equations, *J. Integral Equations*, **7** (1984), 253–277.
[20] H. R. Thieme, Convergence results and Poincaré-Bendixson trichotomy for asymptotically autonomous differential equations, *J. Math. Biol.*, **30** (1992), 755–763.
[21] Y. Tang, D. Huang, S. Ruan and W. Zhang, Coexistence of limit cycles and homoclinic loops in a SIRS model wiht a nonlinear incidence rate, *SIAM J. Appl. Math.*, **69** (2008), 621–639.
[22] W. Wang and X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, *J. Dyn. Diff. Equat.*, **20** (2008), 699–717.
[23] R. Xu and Z. Ma, Global stability of a SIR epidemic model with nonlinear incidence rate and time delay, *Nonlinear Anal. RWA*, **10** (2009), 3175–3189.
[24] D. Xiao and S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, *Math. Biosci.*, **208** (2007), 419–429.
[25] X.-Q. Zhao, *Dynamical Systems in Population Biology*, Springer-Verlag, New York, 2003.
[26] F. Zhang and X.-Q. Zhao, A periodic epidemic model in a patchy environment, *J. Math. Anal. Appl.*, **325** (2007), 496–516.

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