

STABILITY AND OPTIMIZATION IN STRUCTURED POPULATION MODELS ON GRAPHS

RINALDO M. COLOMBO

INdAM Unit
University of Brescia
Brescia, Italy

MAURO GARAVELLO

Department of Mathematics and Applications
University of Milano Bicocca
Milano, Italy

ABSTRACT. We prove existence and uniqueness of solutions, continuous dependence from the initial datum and stability with respect to the boundary condition in a class of initial–boundary value problems for systems of balance laws. The particular choice of the boundary condition allows to comprehend models with very different structures. In particular, we consider a juvenile–adult model, the problem of the optimal mating ratio and a model for the optimal management of biological resources. The stability result obtained allows to tackle various optimal management/control problems, providing sufficient conditions for the existence of optimal choices/controls.

1. **Introduction.** This paper is devoted to the following initial–boundary value problem for a system of balance laws in one space dimension:

$$\begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) = d_i(t, x) u_i & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ g_i(t, 0) u_i(t, 0+) = \mathcal{B}_i(t, u_1(t), \dots, u_n(t)) & t \in \mathbb{R}^+ \\ u_i(0, x) = u_i^o(x) & x \in \mathbb{R}^+ \end{cases} \quad (1.1)$$

Here, $i = 1, \dots, n$ and $t \in \mathbb{R}^+$ is time. The “space” variable x varies in \mathbb{R}^+ and in the applications of (1.1) will have the meaning of a *biological age*, or *size*. The unknowns u_1, \dots, u_n are the densities of the biological species under consideration. The scalar functions g_1, \dots, g_n are *growth functions*, $-d_1, \dots, -d_n$ are the individual mortality rates and u_1^o, \dots, u_n^o constitute the initial data. A key role is played by our choice of the *per capita* birth function \mathcal{B}_i , for $i = 1, \dots, n$, which we assume of the form

$$\begin{aligned} \mathcal{B}_i(t, u_1, \dots, u_n) &= \alpha_i(t, u_1(\bar{x}_1-), \dots, u_n(\bar{x}_n-)) \\ &+ \beta_i \left(\int_{I_1} w_1(x) u_1(x) dx, \dots, \int_{I_n} w_n(x) u_n(x) dx \right) \end{aligned} \quad (1.2)$$

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for suitable functions α_i, β_i , weights w_i , points $\bar{x}_i > 0$ and measurable $I_i \subseteq \mathbb{R}^+$, for $i = 1, \dots, n$.

The literature on equations similar to (1.1) is vast. We refer for instance to the exhaustive monograph [15] or to the more recent edition of [5] and to the references therein. Specific features of (1.1) are that it is a *system*, boundary conditions may contain both a *local* term, the α_i , and a *nonlocal* term, the β_i .

From the analytic point of view, in the present treatment we emphasize the role of the total variation, setting the main result in **BV**. In particular, this allows to consider a function of the type (1.2) and to prove that the boundary data are attained in the sense of traces, also due to the boundary being non characteristic. In this setting, the stability of solutions with respect to α_i and β_i is also obtained. Moreover, the techniques used in the sequel can easily be extended to more general source terms as well as to situations where also the space distribution needs to be taken into account.

From the modeling point of view, the use of boundary conditions of the type (1.2) unifies the treatment of rather diverse situations. First, it comprises the standard case always covered in the literature on renewal equations, where the independent variable x varies along a segment or a half line, see Figure 1, left. The dependent



FIGURE 1. Biological structures comprised in (1.1)–(1.2). Left, a standard *linear* setting and, right, a *juvenile-adult* situation.

variable u represents the population density that at time t is of size (or age) x .

A more complicate structure was recently considered in [1], see Figure 1, right. There, the size/age biological variable varies along a graph consisting of 2 distinct sets, corresponding to the juvenile and to the adult stages in the development of the considered species. Here, we are able to deal also with this situation, as depicted in Figure 2, right.

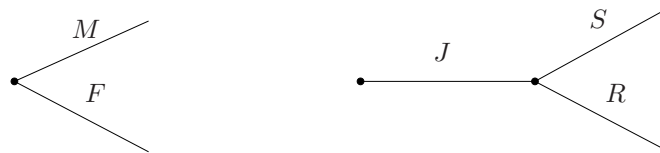


FIGURE 2. Biological structures comprised in (1.1)–(1.2). Left, a framework corresponding, for instance, to sexual reproduction: the two branches correspond to males M and females F . Right, a structure possibly accounting for the exploitation of biological resources: when juveniles reach the adult stage, they are split into a part S which is, say, sold and a part R used for reproduction.

The finite propagation speed intrinsic to models of the type (1.1) clearly allows to combine various instances of the graphs above. Other situations of biological interest can be for instance a three stage linear structure or a tree shaped one,

see Figure 3. These schemes, as well as many others, all fit into the scope of Theorem 2.4 below. In this connection, we recall that similar network structures are widely considered in the framework of vehicular traffic modeling, see [10].



FIGURE 3. General graphs for further biological structures comprised in (1.1)–(1.2).

In the case of nonlinear systems of balance laws with flow independent from the space variable, the initial boundary value problem has been widely investigated, see for instance [8]. For the relations between the problems with boundaries and with junction see [9, Proposition 4.2].

The present treatment is self-contained. Section 2 is devoted to the analytically results. Specific applications are in Section 3, where sample numerical integrations are also provided. All technical details are deferred to Section 4.

2. Analytic results. Throughout, we use the standard notation $\mathbb{R}^+ = [0, +\infty[$ and $\mathring{\mathbb{R}}^+ =]0, +\infty[$. When A and B are suitable subsets of \mathbb{R}^m , $\mathbf{C}^0(A; B)$, respectively $\mathbf{C}^{0,1}(A; B)$, $\mathbf{L}^1(A; B)$ or $\mathbf{L}^\infty(A; B)$, is the set of continuous, respectively Lipschitz continuous, Lebesgue integrable or essentially bounded, maps defined on A and attaining values in B . For the basic theory of **BV** functions we refer to [3].

When referring to a function $u: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$, the first argument is time, the second is the biological age/size variable. If $I \subseteq \mathbb{R}^+$ is an interval, we denote

$$\begin{aligned} \text{TV}(u(t, \cdot)) &= \sup \left\{ \sum_{h=1}^N |u(t, x_h) - u(t, x_{h-1})| : \begin{array}{l} N \in \mathbb{N} \\ x_0 < x_1 < \dots < x_N \end{array} \right\}, \\ \text{TV}(u(t, \cdot); I) &= \sup \left\{ \sum_{h=1}^N |u(t, x_h) - u(t, x_{h-1})| : \begin{array}{l} N \in \mathbb{N} \\ x_0 < x_1 < \dots < x_N \\ x_1, x_2, \dots, x_N \in I \end{array} \right\}, \\ \text{TV}(u(\cdot, x); I) &= \sup \left\{ \sum_{h=1}^N |u(t_h, x) - u(t_{h-1}, x)| : \begin{array}{l} N \in \mathbb{N} \\ t_1 < t_2 < \dots < t_N \\ t_0, t_1, \dots, t_N \in I \end{array} \right\}. \end{aligned}$$

Preliminary, we consider the following initial–boundary value problem for a linear scalar balance law, or *renewal equation* in [15, Chapter 3]:

$$\begin{cases} \partial_t u + \partial_x (g(t, x) u) = d(t, x) u & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0, x) = u_o(x) & x \in \mathbb{R}^+ \\ g(t, 0) u(t, 0+) = b(t) & t \in \mathbb{R}^+ \end{cases} \quad (2.1)$$

under the following assumptions

- (b): $b \in \mathbf{BV}_{\text{loc}}(\mathbb{R}^+; \mathbb{R})$;
- (g): $g \in \mathbf{C}^1(\mathbb{R}^+ \times \mathbb{R}^+; [\check{g}, \hat{g}])$ for $\hat{g} > \check{g} > 0$ and $\begin{cases} \sup_{t \in \mathbb{R}^+} \text{TV}(g(t, \cdot)) < +\infty, \\ \sup_{t \in \mathbb{R}^+} \text{TV}(\partial_x g(t, \cdot)) < +\infty; \end{cases}$
- (d): $d \in (\mathbf{C}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R})$, $\sup_{t \in \mathbb{R}^+} \text{TV}(d(t, \cdot)) < +\infty$.

The solutions to (2.1) can be written in terms of the ordinary differential equation $\dot{x} = g(t, x)$. If g satisfies (g), we can introduce the globally defined maps

$$\begin{aligned} t &\rightarrow X(t; t_o, x_o) \text{ that solves } \begin{cases} \dot{x} = g(t, x) \\ x(t_o) = x_o \end{cases} \quad \text{and} \\ x &\rightarrow T(x; t_o, x_o) \text{ that solves } \begin{cases} t' = \frac{1}{g(t, x)} \\ t(x_o) = t_o. \end{cases} \end{aligned} \quad (2.2)$$

Denote $\gamma(t) = X(t; 0, 0)$, its inverse being $t = \Gamma(x)$. Note that

$$\text{if } x \geq \gamma(t) \text{ then } X(0; t, x) \in [0, x] \quad \text{and} \quad \text{if } x < \gamma(t) \text{ then } T(0; t, x) \in [0, t].$$

Recall the following definition of solution to (2.1), see also [4, 6, 12, 15, 19].

Definition 2.1. Assume that (b), (g) and (d) hold. Choose an initial datum $u_o \in \mathbf{L}^1(\mathbb{R}^+; \mathbb{R})$. The function $u \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}))$ is a *solution* to (2.1) if

1. for all $\varphi \in \mathbf{C}_c^1(\mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R})$,

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} [u(t, x) \partial_t \varphi(t, x) + g(t, x) u(t, x) \partial_x \varphi(t, x) + d(t, x) u(t, x) \varphi(t, x)] dt dx = 0;$$

2. $u(0, x) = u_o(x)$ for a.e. $x \in \mathbb{R}^+$;
3. for a.e. $t \in \mathbb{R}^+$, $\lim_{x \rightarrow 0^+} g(t, x) u(t, x) = b(t)$.

The following Lemma summarizes various properties of the solution to (2.1), see also [15]. Here, we stress the role of **BV** estimates. The proof is deferred to Section 4.

Lemma 2.2. *Let (b), (g) and (d) hold. Then, for any $u_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R})$, the map $u: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by*

$$u(t, x) = \begin{cases} \begin{aligned} &u_o(X(0; t, x)) \\ &\times \exp \left[\int_0^t \left(d(\tau, X(\tau; t, x)) - \partial_x g(\tau, X(\tau; t, x)) \right) d\tau \right] \end{aligned} & x > \gamma(t) \\ \begin{aligned} &\frac{b(T(0; t, x))}{g(T(0; t, x), 0)} \\ &\times \exp \left[\int_{T(0; t, x)}^t \left(d(\tau, X(\tau; t, x)) - \partial_x g(\tau, X(\tau; t, x)) \right) d\tau \right] \end{aligned} & x < \gamma(t) \end{cases} \quad (2.3)$$

solves (2.1) in the sense of Definition 2.1. Moreover, there exists a constant C dependent only on g and d , see (4.6), such that the following a priori estimates hold for all $t \in \mathbb{R}^+$:

$$\|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} + \frac{1}{\check{g}} \|b\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})} \right) e^{Ct} \quad (2.4)$$

$$\|u(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} + \frac{1}{\check{g}} \|b\|_{\mathbf{L}^1([0, t]; \mathbb{R})} \right) e^{Ct} \quad (2.5)$$

$$\text{TV}(u(t)) \leq \left[\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} + \text{TV}(u_o) \right] e^{Ct}$$

$$+ \left[\frac{C + \check{g}}{\check{g}^2} \|b\|_{\mathbf{L}^\infty([0,t];\mathbb{R})} + \frac{1}{\check{g}} \text{TV}(b; [0, t]) \right] e^{Ct} \quad (2.6)$$

$$\text{TV}(u(\cdot, x); [0, t]) \leq \begin{cases} \|u_o\|_{\mathbf{L}^\infty([X(0;t,x),x];\mathbb{R})} e^{Ct} \\ \quad + \text{TV}(u_o; [X(0;t,x), x]) e^{Ct} & x > \gamma(t) \\ \left(\|u_o\|_{\mathbf{L}^\infty([0,x];\mathbb{R})} + \text{TV}(u_o; [0, x]) \right) e^{Ct} \\ \quad + \frac{1}{\check{g}} \text{TV}(b; [0, T(0;t,x)]) e^{Ct} \\ \quad + \frac{Ct}{\check{g}^2} \|b\|_{\mathbf{L}^\infty([0,T(0;t,x)];\mathbb{R})} e^{Ct} & x < \gamma(t). \end{cases} \quad (2.7)$$

Moreover, for any interval $I \subseteq \mathbb{R}^+$ and for any $w \in \mathbf{C}^1(I; [-W, W])$ for a $W > 0$,

$$\text{TV} \left(\int_I w(x) u(\cdot, x) dx; [0, t] \right) \leq C W \int_0^t \left[\|u(\tau)\|_{\mathbf{L}^\infty(I;\mathbb{R})} + \text{TV}(u(\tau, \cdot); I) \right] d\tau. \quad (2.8)$$

For every $t \in \mathbb{R}^+$, there exists a positive \mathcal{L} dependent on \check{g}, \hat{g}, C and $\text{TV}(b; [0, t])$, $\|b\|_{\mathbf{L}^\infty([0,t];\mathbb{R})}$, such that, for $t', t'' \in [0, t]$,

$$\|u(t') - u(t'')\|_{\mathbf{L}^1(\mathbb{R}^+;\mathbb{R})} \leq \mathcal{L} |t'' - t'|. \quad (2.9)$$

For $u'_o, u''_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+;\mathbb{R})$ and b', b'' as in (b), the solutions u' and u'' to

$$\begin{cases} \partial_t u + \partial_x (g(t, x) u) = d(t, x) u \\ u(0, x) = u'_o(x) \\ g(t, 0) u(t, 0+) = b'(t) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t u + \partial_x (g(t, x) u) = d(t, x) u \\ u(0, x) = u''_o(x) \\ g(t, 0) u(t, 0+) = b''(t) \end{cases} \quad (2.10)$$

satisfy the stability and monotonicity estimates

$$\|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}^+;\mathbb{R})} \leq \left[\|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}^+;\mathbb{R})} + \frac{1}{\check{g}} \|b' - b''\|_{\mathbf{L}^1([0,t];\mathbb{R})} \right] e^{Ct}, \quad (2.11)$$

$$\left. \begin{array}{l} u'_o(x) \leq u''_o(x) \quad \forall x \in \mathbb{R}^+ \\ b'(t) \leq b''(t) \quad \forall t \in \mathbb{R}^+ \end{array} \right\} \Rightarrow u'(t, x) \leq u''(t, x) \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+. \quad (2.12)$$

Recall that in the present case of a linear conservation law, the definition of weak solution at 2. is equivalent to the definition of Kruřkov solution [12, Definition 1].

It is immediate to verify that for $u_o = 0$ and $b = 0$, problem (2.3) admits the solution $u = 0$. Hence, the monotonicity property (2.12) also ensures that non-negative initial and boundary data in (1.1)–(1.2) lead to non-negative solutions.

In order to pass to system (1.1), we need the following notation for norms and total variations of functions attaining values in \mathbb{R}^n :

$$\|u\|_{\mathbf{L}^1(\mathbb{R}^+;\mathbb{R}^n)} = \sum_{i=1}^n \|u_i\|_{\mathbf{L}^1(\mathbb{R}^+;\mathbb{R})}, \quad \|u\|_{\mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R}^n)} = \sum_{i=1}^n \|u_i\|_{\mathbf{L}^\infty(\mathbb{R}^+;\mathbb{R})}, \\ \text{TV}(u) = \sum_{i=1}^n \text{TV}(u_i).$$

As a reference for the usual definition of weak solution to scalar conservation laws, see [4, 12].

Definition 2.3. Let $T > 0$. Consider (1.1) with g_1, \dots, g_n satisfying assumptions (g), d_1, \dots, d_n satisfying (d) and the maps $\alpha \equiv (\alpha_1, \dots, \alpha_n)$, $\beta \equiv (\beta_1, \dots, \beta_n)$ and (w_1, \dots, w_n) satisfy

$$\begin{array}{ll} \alpha \in \mathbf{C}^{0,1}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^n) & \alpha(t, 0) = 0 \\ \beta \in \mathbf{C}^{0,1}(\mathbb{R}^n; \mathbb{R}^n) & \beta(0) = 0 \\ w_i \in \mathbf{C}^1(I_i; [-W, W]) & i = 1, \dots, n, \end{array} \quad (2.13)$$

where $W > 0$ and I_1, \dots, I_n are real intervals. Fix an initial datum $u_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R}^n)$. A map

$$u \in \mathbf{C}^0\left([0, T]; (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R}^n)\right)$$

is a *solution* to (1.1)–(1.2) if, setting

$$\begin{aligned} b_i(t) &= \alpha_i(t, u_1(t, \bar{x}_1-), \dots, u_n(t, \bar{x}_n-)) \\ &\quad + \beta_i\left(\int_{I_1} w_1(x) u_1(t, x) dx, \dots, \int_{I_n} w_n(x) u_n(t, x) dx\right) \end{aligned}$$

for all $i = 1, \dots, n$, the i -th component u_i is a solution to

$$\begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) = d_i(t, x) u_i & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u_i(0, x) = u_o^i(x) & x \in \mathbb{R}^+ \\ g_i(t, 0) u_i(t, 0+) = b_i(t) & t \in \mathbb{R}^+ \end{cases} \quad (2.14)$$

in the sense of Definition 2.1.

The following result ensures the well posedness of (1.1)–(1.2). Its proof is presented in Section 4.

Theorem 2.4. *Let $n \in \mathbb{N} \setminus \{0\}$, $\bar{x}_1, \dots, \bar{x}_n \in \mathbb{R}^+$, g_1, \dots, g_n satisfy (g) and d_1, \dots, d_n satisfy (d). Assume that the maps $\alpha \equiv (\alpha_1, \dots, \alpha_n)$, $\beta \equiv (\beta_1, \dots, \beta_n)$ and (w_1, \dots, w_n) satisfy (2.13), where $W > 0$ and I_1, \dots, I_n are real intervals. Then, for any $u_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R}^n)$, the problem (1.1) admits a unique solution in the sense of Definition 2.3. Moreover, there exists an increasing function $\mathcal{K} \in \mathbf{C}^0(\mathbb{R}^+; \mathbb{R}^+)$ dependent only on $\mathbf{Lip}(\alpha)$, $\mathbf{Lip}(\beta)$, W and on C in (4.6) such that for any initial data $u'_o, u''_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R}^n)$, the corresponding solutions u' and u'' satisfy*

$$\begin{aligned} &\|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} \\ &\leq \mathcal{K}(t) \left(\|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^m)} + t \|u'_o - u''_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^m)} \right) \quad \text{and} \end{aligned} \quad (2.15)$$

$$\begin{aligned} &\|u'(t) - u''(t)\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} \\ &\leq \mathcal{K}(t) \left(\|u'_o - u''_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^m)} + \|u'_o - u''_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^m)} \right). \end{aligned} \quad (2.16)$$

Moreover, if $u_o = 0$, then the solution is $u(t, x) = 0$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$.

We now state separately the stability of solutions to (1.1)–(1.2) with respect to the birth function \mathcal{B} . This result plays a key role in the optimization problems considered below.

Theorem 2.5. *Let both systems*

$$\begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) = d_i(t, x) u_i \\ g_i(t, 0) u_i(t, 0+) = \mathcal{B}'_i(t, u_1(t), \dots, u_n(t)) \\ u_i(0, x) = u_i^o(x) \end{cases} \quad \text{and} \quad (2.17)$$

$$\begin{cases} \partial_t u_i + \partial_x (g_i(t, x) u_i) = d_i(t, x) u_i \\ g_i(t, 0) u_i(t, 0+) = \mathcal{B}''_i(t, u_1(t), \dots, u_n(t)) \\ u_i(0, x) = u_i^o(x) \end{cases}$$

with

$$\mathcal{B}'_i(t, u_1, \dots, u_n) = \alpha'_i(t, u_1(\bar{x}_1-), \dots, u_n(\bar{x}_n-))$$

$$\begin{aligned} & +\beta'_i \left(\int_{I_1} w'_1(x) u_1(x) dx, \dots, \int_{I_n} w'_n(x) u_n(x) dx \right), \\ \mathcal{B}''_i(t, u_1, \dots, u_n) & = \alpha''_i(t, u_1(\bar{x}_1-), \dots, u_n(\bar{x}_n-)) \\ & +\beta''_i \left(\int_{I_1} w''_1(x) u_1(x) dx, \dots, \int_{I_n} w''_n(x) u_n(x) dx \right) \end{aligned}$$

satisfy the assumptions of Theorem 2.4. Then, the corresponding solutions u' and u'' are such that

$$\begin{aligned} \|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} & \leq \mathcal{H}(t) \|\alpha' - \alpha''\|_{\mathbf{C}^0(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^n)} \\ & + \mathcal{H}(t) \|\beta' - \beta''\|_{\mathbf{C}^0(\mathbb{R}^n; \mathbb{R}^n)} \\ & + \mathcal{H}(t) \sum_{j=1}^n \|w'_j - w''_j\|_{\mathbf{C}^0(I_j; \mathbb{R})} \end{aligned} \tag{2.18}$$

where $\mathcal{H} \in \mathbf{C}^0(\mathbb{R}^+; \mathbb{R}^+)$ is such that $\mathcal{H}(0) = 0$.

The proof is deferred to Section 4.

In applications of Theorem 2.4 to systems motivated by, for instance, structured population biology, further assumptions are natural and lead to further reasonable properties.

Proposition 2.6. *Under the assumptions of Theorem 2.4, if the boundary functions and the initial data are such that*

$$\begin{aligned} \partial_{u_j} \alpha_i & \geq 0 & \text{for all } i, j & = 1, \dots, n, \\ \partial_{w_j} \beta_i & \geq 0 & \text{for all } i, j & = 1, \dots, n, \\ w_i & \geq 0 & \text{for all } i & = 1, \dots, n, \\ (u'_o)_i & \geq (u''_o)_i & \text{for all } i & = 1, \dots, n, \end{aligned}$$

then, the corresponding solutions satisfy $u'_i(t, x) \geq u''_i(t, x)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $i = 1, \dots, n$. In particular, if $(u'_o)_i \geq 0$ for $i = 1, \dots, n$, then $u'_i(t, x) \geq 0$ for $i = 1, \dots, n$.

The proof follows immediately from Theorem 2.4 and from (2.12) and it is omitted.

3. Applications. This section is devoted to sample applications of Theorem 2.4 and Theorem 2.5 to models inspired by structured population biology. We selected three cases corresponding to three different graphs, namely those in Figure 1, right, and in Figure 2.

First, the well posedness ensured by Theorem 2.4 provides a ground for the reliability of each model. Then, the stability result in Theorem 2.5 allows to consider further problems. On the one hand, it ensures the existence of a choice of parameters in the equations that lead to solution that best approximate a given set of data. On the other hand, it allows to tackle the problem of optimal mating ratio in a population with sexual reproduction. Finally, we consider the problem of the optimal management of a biological resource. In the former case, the presentation is based on [1, 2] where a sensitivity analysis for a model belonging to the class (1.1)–(1.2) is proved. In the latter cases, we provide numerical integrations showing further qualitative properties of the models considered.

3.1. A nonautonomous juvenile–adult model. In (1.1)–(1.2), setting $n = 2$ and with reference to the structure in Figure 1, right,

$$\begin{aligned} u_1(t, x) &= J(t, x) & g_1(t, x) &= 1 & \alpha_1(t, u_1, u_2) &= 0 \\ u_2(t, x) &= A(t, x + x_{\min}) & g_2(t, x) &= g(t, x + x_{\min}) & \alpha_2(t, u_1, u_2) &= u_1 \\ \bar{x}_1 &= a_{\max} & d_1(t, x) &= -\nu(t, x) & \beta_1(w_1, w_2) &= w_2 \\ \bar{x}_2 &= 0 & d_2(t, x) &= -\mu(t, x + x_{\min}) & \beta_2(w_1, w_2) &= 0 \end{aligned} \quad (3.1)$$

with moreover $I_2 = [0, x_{\max} - x_{\min}]$, we recover [1, Formula (2.1)] in the case $\beta = 1$, which we state here for completeness:

$$\left\{ \begin{array}{ll} \partial_t J + \partial_a J = -\nu(t, a) J & (t, a) \in \mathbb{R}^+ \times [0, a_{\max}] \\ \partial_t A + \partial_x (g(t, x) A) = -\mu(t, x) A & (t, x) \in \mathbb{R}^+ \times [x_{\min}, x_{\max}] \\ J(t, 0) = \int_{x_{\min}}^{x_{\max}} A(t, x) dx & t \in \mathbb{R}^+ \\ g(t, x_{\min}) A(t, x_{\min}) = J(t, a_{\max}) & t \in \mathbb{R}^+ \\ J(0, a) = J_o(a) & a \in [0, a_{\max}] \\ A(0, x) = A_o(x) & x \in [x_{\min}, x_{\max}]. \end{array} \right. \quad (3.2)$$

Theorem 2.4 then applies and ensures the well posedness of (3.2) under assumptions slightly different from those in [1].

Corollary 3.1. *In (3.2), assume that*

$$\begin{aligned} \nu &\in (\mathbf{C}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^+ \times [0, a_{\max}]; \mathbb{R}) & \text{and} & \sup_{t \in \mathbb{R}^+} \text{TV}(\nu(t, \cdot)) < +\infty \\ \mu &\in (\mathbf{C}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^+ \times [x_{\min}, x_{\max}]; \mathbb{R}) & \text{and} & \sup_{t \in \mathbb{R}^+} \text{TV}(\mu(t, \cdot)) < +\infty \\ g &\in \mathbf{C}^1(\mathbb{R}^+ \times [x_{\min}, x_{\max}]; [\check{g}, \hat{g}]) & \text{and} & \begin{cases} \sup_{t \in \mathbb{R}^+} \text{TV}(g(t, \cdot)) < +\infty \\ \sup_{t \in \mathbb{R}^+} \text{TV}(\partial_x g(t, \cdot)) < +\infty \end{cases} \\ J_o &\in \mathbf{BV}([0, a_{\max}]; \mathbb{R}^+) \\ A_o &\in \mathbf{BV}([x_{\min}, x_{\max}]; \mathbb{R}^+). \end{aligned}$$

Then, problem (3.2) admits a unique solution in the sense of Definition 2.3, the continuous dependence estimates (2.15)–(2.16) and the stability estimate (2.18) apply.

For completeness, we remark that the model in [1] contains the following slightly more general boundary inflow:

$$J(t, 0) = \int_{x_{\min}}^{x_{\max}} \beta(t, x) A(t, x) dx .$$

As soon as $\beta \in \mathbf{C}^1(\mathbb{R}^+ \times [x_{\min}, x_{\max}]; [\check{\beta}, +\infty])$ for a suitable $\check{\beta} > 0$, the change of variables

$$\mathcal{A}(t, x) = \beta(t, x) A(t, x) \quad (3.3)$$

still allows to apply Theorem 2.4. Indeed, with this variable, the second equation in (3.2) becomes

$$\partial_t \mathcal{A} + \partial_x (g(t, x) \mathcal{A}) = (\partial_t \beta(t, x) + g(t, x) \partial_x \beta(t, x) - \mu(t, x)) \mathcal{A},$$

which is again of the type (2.1) and hence Theorem 2.4 can still be applied.

The stability proved above allows to tackle the problem of parameter identification. Indeed, through a Weierstraß argument based on Theorem 2.5, one can prove the existence of a set of parameters in (3.2) that minimizes a continuous functional representing the distance between the computed solution and a set of experimental data. For a detailed sensitivity analysis for a juvenile–adult model we refer to [2]. A different approach to a juvenile–adult model is in [7].

3.2. Optimal mating ratio. Consider a species consisting of males and females, whose densities at time t and age a are described through the functions $M = M(t, a)$ and $F = F(t, a)$ on a structure as that in Figure 2, left. A natural model is then

$$\begin{cases} \partial_t M + \partial_a M = -\kappa \mu M \\ \partial_t F + \partial_a F = -(1 - \kappa) \mu F \\ M(t, 0) + F(t, 0) = \nu \min \left\{ \vartheta \int_{m_1}^{m_2} M(t, a) da, (1 - \vartheta) \int_{f_1}^{f_2} F(t, a) da \right\} \\ \eta M(t, 0) = (1 - \eta) F(t, 0) \\ M(0, a) = M_o(a) \\ F(0, a) = F_o(a). \end{cases} \quad (3.4)$$

Here, $\kappa \mu$, respectively $(1 - \kappa) \mu$, is the mortality rate of males, respectively females, with $\mu > 0$ and $\kappa \in [0, 1]$. The positive parameter $\eta \in [0, 1]$ defines the ratio of male to female newborns, in the sense that every ηM males, $(1 - \eta) F$ females are born. The constant ν is the fertility rate. We describe the mating ratio at age a through the parameter ϑ , with $\vartheta \in [0, 1]$ as follows. The fertile ages are those in the intervals $[m_1, m_2]$ for males and $[f_1, f_2]$ for females, where m_1, m_2, f_1, f_2 are positive constants. According to (3.4), all individuals in their fertile age might contribute to reproduction provided the condition imposed by the presence of the mating ratio ϑ is met. If $\vartheta \int_{m_1}^{m_2} M(t, a) da$ exceeds $(1 - \vartheta) \int_{f_1}^{f_2} F(t, a) da$, then only $\frac{1-\vartheta}{\vartheta} \int_{f_1}^{f_2} F(t, a) da$ males contribute to the overall population's fertility.

Problem (3.4) fits into (1.1)–(1.2) setting

$$\begin{array}{lll} u_1 = M & g_1 = 1 & d_1(t, x) = -\kappa \\ u_2 = F & g_2 = 1 & d_2(t, x) = -(1 - \kappa) \mu \\ I_1 = [m_1, m_2] & \alpha_1 = 0 & \beta_1(w_1, w_2) = (1 - \eta) \nu \min\{\vartheta w_1, (1 - \vartheta)w_2\} \\ I_2 = [f_1, f_2] & \alpha_2 = 0 & \beta_2(w_1, w_2) = \eta \nu \min\{\vartheta w_1, (1 - \vartheta)w_2\}. \end{array}$$

Corollary 3.2. Let $\mu, \nu \in \mathbb{R}^+$; $\eta, \vartheta, \kappa \in [0, 1]$; $m_1, m_2, f_1, f_2 \in \mathbb{R}^+$ with $m_1 < m_2$ and $f_1 < f_2$. For $M_o, F_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R})$, problem (3.4) has a unique solution in the sense of Definition 2.3, the continuous dependence estimates (2.15)–(2.16) and the stability estimate (2.18) apply.

The proof is immediate and, hence, omitted. Here, we note that the presence of \mathbf{C}^1 positive weights in the integrands defining the boundary data can be recovered through a change of variables entirely similar to that in (3.3).

A first immediate property of the solutions to (3.4) is that a zero initial density in either of the two sexes leads to the extinction of the other at exponential speed.

Several different optimization problems can be tackled in the framework of (3.4). It is possible to investigate the relations between the parameters κ (identifying relative mortality), η (the relative natality) and ϑ (the mating ratio). Below, we look for the optimal mating ratio for given relative natality and mortality coefficients.

To this aim, consider the instantaneous average fertility rate over the fertile population

$$R = \frac{\nu \min \left\{ \vartheta \int_{m_1}^{m_2} M(t, a) da, (1 - \vartheta) \int_{f_1}^{f_2} F(t, a) da \right\}}{\int_{m_1}^{m_2} M(t, a) da + \int_{f_1}^{f_2} F(t, a) da}. \quad (3.5)$$

Remark that the functions M and F in (3.5) are solutions to (3.4), hence they depend on the mating ratio ϑ that enters the boundary condition throughout the

time interval $[0, t]$. It is natural to assume that a key role is played by the maximal value of R , which is obtained by the choice

$$\vartheta = \frac{\int_{f_1}^{f_2} F(t, a) \, da}{\int_{m_1}^{m_2} M(t, a) \, da + \int_{f_1}^{f_2} F(t, a) \, da}. \quad (3.6)$$

Remarkably, this leads to the maximal fertility rate

$$R = \nu \frac{\left(\int_{m_1}^{m_2} M(t, a) \, da \right) \left(\int_{f_1}^{f_2} F(t, a) \, da \right)}{\left(\int_{m_1}^{m_2} M(t, a) \, da + \int_{f_1}^{f_2} F(t, a) \, da \right)^2}$$

coherently with the classical *harmonic mean* law, see [11, 14, 16, 17, 18, 20].

On the other hand, the right hand sides in (3.5) and (3.6) are time dependent and it can be hardly accepted that ϑ is instantaneously adjusted to the value that maximizes R . More reasonably, one may imagine that ϑ is optimal¹ over a suitably long time interval. We are thus lead to introduce the utility function

$$\mathcal{R}(\vartheta; T, M_o, F_o) = \frac{1}{T} \int_0^T \nu \min \left\{ \vartheta \int_{m_1}^{m_2} M(t, a) \, da, (1 - \vartheta) \int_{f_1}^{f_2} F(t, a) \, da \right\} \frac{dt}{\int_{m_1}^{m_2} M(t, a) \, da + \int_{f_1}^{f_2} F(t, a) \, da}. \quad (3.7)$$

We thus consider the problem

$$\text{find } \vartheta \text{ that maximizes } \mathcal{R}(\vartheta; T, M_o, F_o).$$

A straightforward corollary of Theorem 2.5 ensures the existence of one such ϑ .

Corollary 3.3. *Under the assumptions of Corollary 3.2, for any $T \in \mathring{\mathbb{R}}^+$ and any initial datum $(M_o, F_o) \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R})$ there exists a $\vartheta_* \in]0, 1[$ such that*

$$\mathcal{R}(\vartheta_*; T, M_o, F_o) = \max_{\vartheta \in [0, 1]} \mathcal{R}(\vartheta; T, M_o, F_o).$$

The proof is immediate: thanks to Theorem 2.5, the function $\vartheta \rightarrow \mathcal{R}(\vartheta; T, M_o, F_o)$ is continuous for any choice of $T \in \mathbb{R}^+$ and $(M_o, F_o) \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R})$. By the compactness of $[0, 1]$, Weierstraß Theorem ensures the existence of ϑ_* . Moreover, since $\mathcal{R}(0; T, M_o, F_o) = \mathcal{R}(1; T, M_o, F_o) = 0$, we also have $\vartheta_* \in]0, 1[$.

It can be of interest to note that M and F may well increase exponentially with time, but $\mathcal{R}(\vartheta; T, M_o, F_o) \in [0, \nu]$ for all $T \in \mathring{\mathbb{R}}^+$ and $(M_o, F_o) \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R})$.

As a specific example, we consider the situation identified by the following choices of functions and parameters in (3.4)–(3.7):

$$\begin{array}{llll} \kappa & = & 0.600 & \mu & = & 0.020 & m_1 & = & 18 & f_1 & = & 16 \\ \eta & = & 0.485 & \nu & = & 5 & m_2 & = & 60 & f_2 & = & 55 \end{array} \quad (3.8)$$

and we consider ϑ as a control parameter in $[0, 1]$. As initial datum we choose

$$M_o(a) = 10 \quad \text{and} \quad F_o(a) = 10 \quad \text{for all } a. \quad (3.9)$$

The graph of the average fertility rate $\mathcal{R}(\vartheta; T, M_o, F_o)$ as a function of ϑ for $T = 500$ is in Figure 4. The outcome shows a reasonable qualitative behavior. As $\vartheta \rightarrow 0$ or $\vartheta \rightarrow 1$, the number of newborns goes to 0; hence the population extinguish. Near to $\vartheta = 0.77$ there is an optimal choice for the parameter ϑ with respect to the average fertility rate (3.7), which yields a maximal value of 0.83, see Figure 5

¹Here and in the sequel, *optimal* is understood in the sense that it is the value that maximizes R . However, an excessive natality rate might turn out to be not *optimal* from the biological point of view.

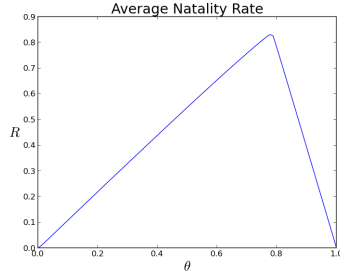


FIGURE 4. Average fertility rate (3.7) along the solutions to (3.4)–(3.8) with initial datum (3.9) plotted as a function of ϑ . For $\vartheta = 0$ or $\vartheta = 1$, there is no reproduction and the population extinguishes. For $\vartheta \approx 0.77$, the average natality rate reaches its maximum value, which is approximately equal to 0.83.

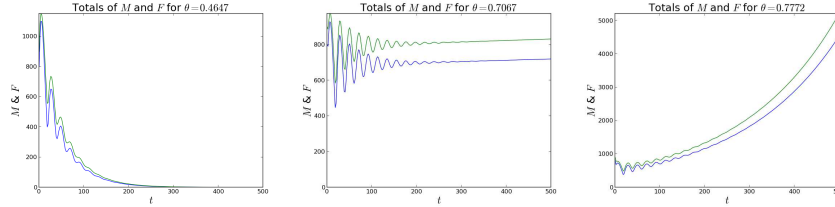


FIGURE 5. Graphs of the solutions to (3.4)–(3.8)–(3.9). The integrals $\int_0^{80} M(t, a) da$ (blue) and $\int_0^{90} F(t, a) da$ (green) as a function of time. Left, for $\vartheta = 0.4647$, the population vanishes. Middle, with $\vartheta = 0.7067$, there is an equilibrium and right, for $\vartheta = 0.7772$, the population grows exponentially. The latter value gives the maximal average natality rate, see Figure 4.

For completeness, we precise that the numerical integration above was obtained using a Lax–Friedrichs algorithm, see [13, § 12.5], with space mesh $\Delta a = 0.04167$.

3.3. Management of a biological resource. In biological resource management, one typically rears/breeds a species up to a suitable stage, then part of the population is sold and part is used for reproduction. The equations (1.1)–(1.2) comprehend this situation. Indeed, call $J = J(t, a)$ the density of the juveniles at time t of age or size a . Juveniles reaching the age/size \bar{a} are then selected. The density $S = S(t, a)$ refers to those individuals that are going to be sold, while $R = R(t, a)$ stands for the density of those reserved for reproduction purposes. One is thus lead to the following model, defined on the structure in Figure 2, right:

$$\left\{ \begin{array}{ll}
 \partial_t J + \partial_a (g_J(t, a) J) = d_J(t, a) J & (t, a) \in \mathbb{R}^+ \times [0, \bar{a}] \\
 \partial_t S + \partial_a (g_S(t, a) S) = d_S(t, a) S & (t, a) \in \mathbb{R}^+ \times [\bar{a}, +\infty[\\
 \partial_t R + \partial_a (g_R(t, a) R) = d_R(t, a) R & (t, a) \in \mathbb{R}^+ \times [\bar{a}, +\infty[\\
 g_J(t, 0) J(t, 0) = \beta \left(\int_{\bar{a}}^{a_{\max}} R(t, x) dx \right) & t \in \mathbb{R}^+ \\
 g_S(t, \bar{a}) S(t, \bar{a}) = \eta g_J(t, \bar{a}) J(t, \bar{a}) & t \in \mathbb{R}^+ \\
 g_R(t, \bar{a}) R(t, \bar{a}) = (1 - \eta) g_J(t, \bar{a}) J(t, \bar{a}) & t \in \mathbb{R}^+ \\
 J(0, a) = J_o(a) & a \in [0, \bar{a}] \\
 S(0, a) = S_o(a) & a \in [\bar{a}, +\infty[\\
 R(0, a) = R_o(a) & a \in [\bar{a}, +\infty[.
 \end{array} \right. \tag{3.10}$$

Above, we used the obvious notation for growth and mortality functions g_J, g_S, g_R and d_J, d_S, d_R . The birth rate is described through the function β . A key role is played by the parameter $\eta \in [0, 1]$ which quantifies the percentage of juveniles selected for the market.

System (3.10) fits into (1.1)–(1.2) setting $w \equiv (w_1, w_2, w_3)$ and

$$\begin{aligned} u_1(t, s) &= J(t, x) & g_1(t, x) &= g_J(t, x) & \alpha_1(t, w) &= 0 \\ u_2(t, x) &= S(t, x + \bar{a}) & g_2(t, x) &= g_S(t, x + \bar{a}) & \alpha_2(t, w) &= \eta w_1 g_1(t, \bar{x}_1) \\ u_3(t, x) &= R(t, x + \bar{a}) & g_3(t, x) &= g_R(t, x + \bar{a}) & \alpha_3(t, w) &= (1 - \eta) w_1 g_1(t, \bar{x}_1) \\ & & d_1(t, x) &= d_J(t, x) & \beta_1(w) &= \beta(w_3) \\ \bar{x}_1 &= \bar{a} & d_2(t, x) &= d_S(t, x + \bar{a}) & \beta_2(w) &= 0 \\ I_3 &= [\bar{a}, a_{\max}] & d_3(t, x) &= d_R(t, x + \bar{a}) & \beta_3(w) &= 0 \end{aligned}$$

Corollary 3.4. *Let g_J, g_S, g_R satisfy (g) for suitable $\check{g}, \hat{g} \in \mathbb{R}^+$ with $\hat{g} > \check{g} > 0$. Let d_J, d_S, d_R satisfy (d). Let $\beta \in \mathbf{C}^{0,1}(\mathbb{R}^+; \mathbb{R})$ be such that $\beta(0) = 0$. For any $\eta \in [0, 1]$ and any initial data $J_o \in \mathbf{BV}([0, \bar{a}]; \mathbb{R}^+)$ and $S_o, R_o \in (\mathbf{L}^1 \cap \mathbf{BV})([\bar{a}, +\infty[, \mathbb{R}^+)$, system (3.10) admits a unique non negative solution and the stability estimates in Theorem 2.4 apply.*

A natural question based on model (3.10) is: find the optimal percentage η of juveniles that have to be chosen for the market. To this aim, we postulate simple, though reasonable, cost and gain functionals

$$\begin{aligned} \mathcal{C}(\eta; T) &= \int_0^T \int_0^{\bar{a}} C_J(a) J(t, a) da dt \\ &\quad + \int_{\bar{a}}^{a_{\max}} [C_S(a) S(t, a) + C_R(a) R(t, a)] da dt, \quad (3.11) \\ \mathcal{G}(\eta; T) &= \int_0^T \int_{\bar{a}}^{a_{\max}} G(a) S(t, a) da dt. \end{aligned}$$

Here, $C_J(a)$ is the unit cost to grow a juvenile at age a , and similarly C_R, C_S are the costs for the other two groups. The gain obtained selling an adult at age a is $G(a)$. We denoted by $J = J(t, a), S = S(t, a)$ and $R = R(t, a)$ the solution to (3.10) with initial datum J_o and $S_o = 0, R_o = 0$ and with the selection parameter η . A direct consequence of Theorem 2.5 is the following corollary.

Corollary 3.5. *In the same assumptions of Corollary 3.4, for any $T \in \mathbb{R}^+$ and any $J_o, S_o, R_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^+; \mathbb{R})$, there exists an optimal choice η_* such that*

$$\mathcal{G}(\eta_*; T) - \mathcal{C}(\eta_*; T) = \max_{\eta \in [0, 1]} (\mathcal{G}(\eta; T) - \mathcal{C}(\eta; T)).$$

The proof relies on Weierstraß Theorem, exactly as that of Corollary 3.3 and is here omitted.

As a specific example, we consider the situation identified by the following choices of functions and parameters in (3.10)–(3.11):

$$\begin{aligned} g_J(t, a) &= 1 & d_J(t, a) &= 0 & C_J(t, a) &= a & \beta(w) &= 2w \\ g_S(t, a) &= 1 & d_S(t, a) &= -\frac{a-\bar{a}}{2} & C_S(t, a) &= 0 & G(t, a) &= 10 \\ g_R(t, a) &= 1 & d_R(t, a) &= -\frac{a-\bar{a}}{2} & C_R(t, a) &= 0.5 & [\bar{a}, a_{\max}] &= [1, 2] \end{aligned} \quad (3.12)$$

and we consider η as a control parameter in $[0, 1]$. As initial datum we choose

$$J_o(a) = 5, \quad S_o(a) = 0, \quad R_o(a) = 0. \quad (3.13)$$

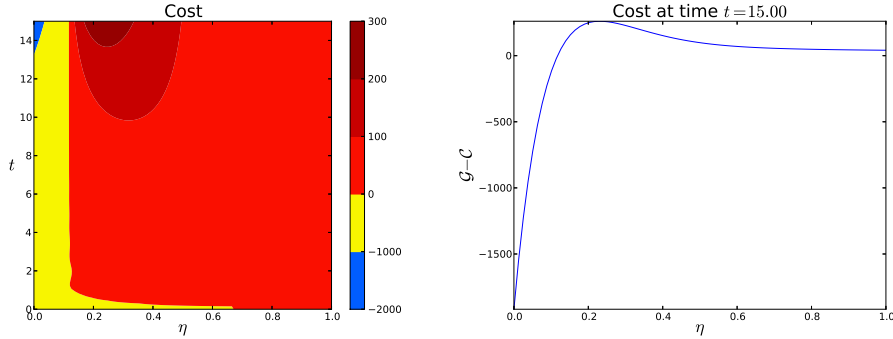


FIGURE 6. Cost (3.11) along the solutions to (3.10)–(3.12) with initial datum (3.13). Left, as a function of η (on the horizontal axis) and t (on the vertical axis). Right, as a function of η at time $t = 15.00$. Recall that for $\eta = 0$, all individual are kept for reproduction and no one is sold. On the contrary, for $\eta = 1$, they are all sold and no one is kept for reproduction.

The graph of the cost $\mathcal{G}(\eta; T) - \mathcal{C}(\eta; T)$ (see (3.11)) for $T = 15$ with respect to λ is in Figure 6. The outcome shows a reasonable qualitative behavior. As $\eta \rightarrow 0$, nothing is sold, all population members are kept for reproduction, the population increases exponentially as also does the functional $\mathcal{G}(\eta; T) - \mathcal{C}(\eta; T)$. On the contrary, for $\eta \rightarrow 1$, all population members are immediately sold giving a positive gain and the population vanishes as also $\mathcal{G} - \mathcal{C}$. Near to $\eta \approx 0.23$ there is an optimal balance, given the chosen unitary costs and gain (3.11)–(3.12).

With the chosen parameters, the optimal choice for η is $\eta_* \approx 0.23$, which yields a gain of about 260.48 at time $t = 15$. As expected, different choices of η have deep influences on the solutions to (3.10)–(3.12)–(3.13), as shown in Figure 7.

For completeness, we precise that the numerical integration above was obtained using a Lax–Friedrichs algorithm, see [13, § 12.5], with space mesh $\Delta a = 0.001$.

4. Technical details. Throughout, when **BV** functions are considered, we refer to a right continuous representative. We now recall the following elementary estimates on **BV** functions.

$$\left. \begin{array}{l} u \in \mathbf{BV}(\mathbb{R}^+; \mathbb{R}) \\ w \in \mathbf{BV}(\mathbb{R}^+; \mathbb{R}) \end{array} \right\} \Rightarrow \text{TV}(uw) \leq \|u\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} \text{TV}(w) + \text{TV}(u) \|w\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} \quad (4.1)$$

$$\left. \begin{array}{l} f \in \mathbf{C}^{0,1}(\mathbb{R}; \mathbb{R}) \\ u \in \mathbf{BV}(\mathbb{R}^+; \mathbb{R}) \end{array} \right\} \Rightarrow \text{TV}(f \circ u) \leq \text{Lip}(f) \text{TV}(u) \quad (4.2)$$

$$\left. \begin{array}{l} u \in \mathbf{BV}(\mathbb{R}^+; \mathbb{R}) \\ f \in \mathbf{BV}(\mathbb{R}^+; [f, +\infty]) \\ \tilde{f} > 0 \end{array} \right\} \Rightarrow \text{TV}\left(\frac{u}{\tilde{f}}\right) \leq \frac{1}{\tilde{f}} \text{TV}(u) + \frac{1}{\tilde{f}^2} \text{TV}(f) \|u\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} \quad (4.3)$$

$$u \in \mathbf{L}^1(\mathbb{R}^+; \mathbf{BV}(\mathbb{R}^+; \mathbb{R})) \Rightarrow \text{TV}\left(\int_0^t u(\tau, \cdot) \, d\tau\right) \leq \int_0^t \text{TV}(u(\tau)) \, d\tau \quad (4.4)$$

$$\left. \begin{array}{l} u \in \mathbf{BV}(\mathbb{R}^+; \mathbb{R}) \\ h \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^+) \end{array} \right\} \Rightarrow \int_{\mathbb{R}^+} |u(x+h(x)) - u(x)| \, dx \leq \text{TV}(u) \|h\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})}. \quad (4.5)$$

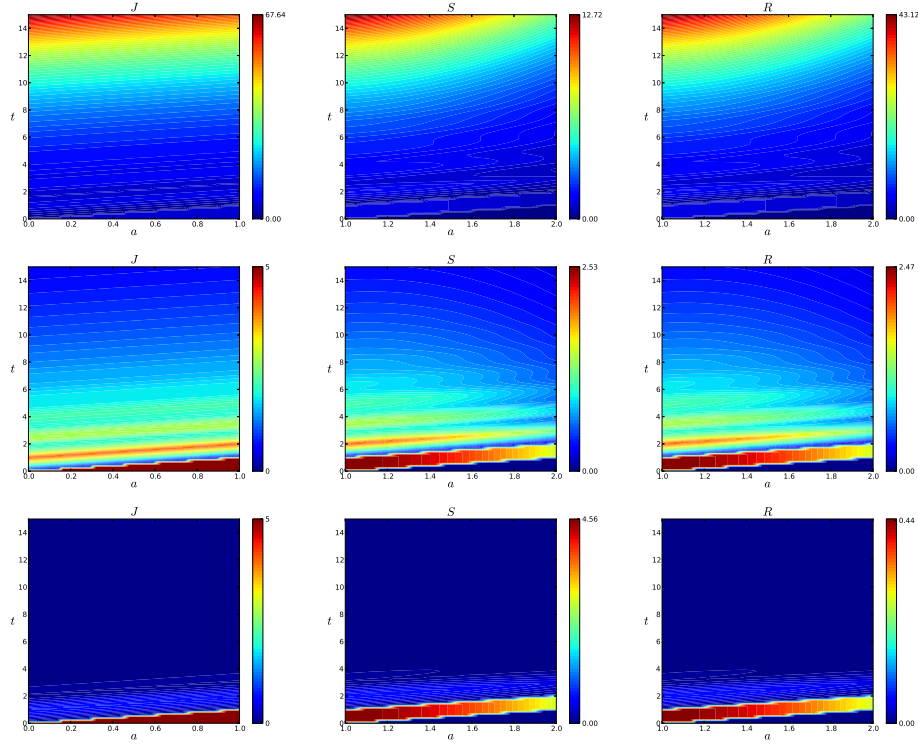


FIGURE 7. Solutions to (3.10)–(3.12)–(3.13). Time varies along the vertical axis and a along the horizontal one. Above, $\eta = 0.23$ is near to the optimal choice. Middle, $\eta = 0.50$ and, below, $\eta = 0.91$.

Inequality (4.1) follows from [3, Formula (3.10)]. The definition of total variation directly implies (4.2), (4.3) and (4.4). For a proof of (4.5) see for instance [6, Lemma 2.3].

Proof of Lemma 2.2. To verify that (2.3) solves (2.1), a standard integration along characteristics is sufficient. The bounds (2.4) and (2.5) are an immediate consequence of (2.3).

Passing to the estimates on the total variation, introduce

$$C = 2 \max \left\{ \begin{array}{ll} \|\partial_x g\|_{\mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R})}, & \|\partial_t g\|_{\mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R})}, \\ \sup_{t \in \mathbb{R}^+} \text{TV}(g(t, \cdot)), & \sup_{t \in \mathbb{R}^+} \text{TV}(\partial_x g(t, \cdot)), \\ \|d\|_{\mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R})}, & \sup_{t \in \mathbb{R}^+} \text{TV}(d(t, \cdot)) \end{array} \right\} \quad (4.6)$$

which is finite by (g) and (d).

Consider the total variation estimates. Using (4.1), (4.2), (4.3), (4.4), compute:

$$\begin{aligned} & \text{TV}(u(t)) \\ &= \text{TV}(u(t, \cdot), [0, \gamma(t)]) + \text{TV}(u(t, \cdot), [\gamma(t), +\infty[) \\ &\leq \text{TV}\left(\frac{b(\cdot)}{g(\cdot, 0)}; [0, t]\right) e^{Ct} + \frac{\|b\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})}}{\tilde{g}} e^{Ct} + \text{TV}(u_o) e^{Ct} + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} e^{Ct} \end{aligned}$$

$$\leq \left[\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} + \text{TV}(u_o) + \frac{1}{\bar{g}} \left(\left(1 + \frac{C}{\bar{g}} \right) \|b\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})} + \text{TV}(b; [0, t]) \right) \right] e^{Ct}$$

proving (2.6). The bound (2.6) follows from (2.3), using (4.1), (4.2) and (4.3). We exploit now (4.4) and [3, Definition 3.4], and in the lines below, for typographical reasons, we denote by J the real interval $[-1, 1]$.

$$\begin{aligned} & \text{TV} \left(\int_I w(x) u(\cdot, x) \, dx ; [0, t] \right) \\ &= \sup \left\{ \int_0^t \int_I w(x) u(\tau, x) \, dx \, \partial_t \varphi(\tau) \, d\tau : \varphi \in \mathbf{C}_c^1([0, t]; J) \right\} \\ &\leq \sup \left\{ \int_0^t \int_I w(x) u(\tau, x) \chi(x) \partial_t \varphi(\tau) \, dx \, d\tau : \begin{array}{l} \varphi \in \mathbf{C}_c^1([0, t]; J) \\ \chi \in \mathbf{C}_c^1(\overset{\circ}{I}; J) \end{array} \right\} \\ &\leq W \sup \left\{ \int_0^t \int_I u(\tau, x) \psi(x) \partial_t \varphi(\tau) \, dx \, d\tau : \begin{array}{l} \varphi \in \mathbf{C}_c^1([0, t]; J) \\ \psi \in \mathbf{C}_c^1(\overset{\circ}{I}; J) \end{array} \right\} \\ &= W \sup \left\{ \int_0^t \int_I \left[g(\tau, x) u(\tau, x) \partial_x \psi(x) \varphi(\tau) \right. \right. \\ &\quad \left. \left. + d(\tau, x) u(\tau, x) \psi(x) \varphi(\tau) \right] \, dx \, d\tau : \begin{array}{l} \varphi \in \mathbf{C}_c^1([0, t]; J) \\ \psi \in \mathbf{C}_c^1(\overset{\circ}{I}; J) \end{array} \right\} \\ &\leq W \sup \left\{ \int_0^t \int_I g(\tau, x) u(\tau, x) \partial_x \psi(x) \varphi(\tau) \, dx \, d\tau : \begin{array}{l} \varphi \in \mathbf{C}_c^1([0, t]; J) \\ \psi \in \mathbf{C}_c^1(\overset{\circ}{I}; J) \end{array} \right\} \\ &\quad + W \sup \left\{ \int_0^t \int_I d(\tau, x) u(\tau, x) \psi(x) \varphi(\tau) \, dx \, d\tau : \begin{array}{l} \varphi \in \mathbf{C}_c^1([0, t]; J) \\ \psi \in \mathbf{C}_c^1(\overset{\circ}{I}; J) \end{array} \right\} \\ &\leq W \sup \left\{ \int_0^t \sup \left\{ \int_I g(\tau, x) u(\tau, x) \partial_x \psi(x) \, dx : \psi \in \mathbf{C}_c^1(\overset{\circ}{I}; J) \right\} \varphi(\tau) \, d\tau : \right. \\ &\quad \left. \varphi \in \mathbf{C}_c^1([0, t]; J) \right\} \\ &\quad + W \sup \left\{ \int_0^t \sup \left\{ \int_I d(\tau, x) u(\tau, x) \psi(x) \, dx : \psi \in \mathbf{C}_c^1(\overset{\circ}{I}; J) \right\} \varphi(\tau) \, d\tau : \right. \\ &\quad \left. \varphi \in \mathbf{C}_c^1([0, t]; J) \right\} \\ &= W \sup \left\{ \int_0^t \text{TV} (g(\tau, \cdot) u(\tau, \cdot)) \, \varphi(\tau) \, d\tau : \varphi \in \mathbf{C}_c^1([0, t]; J) \right\} \\ &\quad + W \sup \left\{ \int_0^t \text{TV} (d(\tau, \cdot) u(\tau, \cdot)) \, \varphi(\tau) \, d\tau : \varphi \in \mathbf{C}_c^1([0, t]; J) \right\} \\ &\leq W \int_0^t \left(\text{TV} (g(\tau, \cdot) u(\tau, \cdot)) + \text{TV} (d(\tau, \cdot) u(\tau, \cdot)) \right) \, d\tau \end{aligned}$$

Apply now (4.1) to obtain:

$$\begin{aligned} & \text{TV} \left(\int_I w(x) u(\cdot, x) \, dx ; [0, t] \right) \\ &\leq W \int_0^t \left(\text{TV} (g(\tau, \cdot)) + \text{TV} (d(\tau, \cdot)) \right) \|u(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} \, d\tau \end{aligned}$$

$$\begin{aligned}
& +W \int_0^t \left(\|g(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} + \|d(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} \right) \text{TV}(u(\tau)) \, d\tau \\
& \leq 2CW \int_0^t \left(\|u(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} + \text{TV}(u(\tau)) \right) \, d\tau,
\end{aligned}$$

completing the proof of (2.8). Concerning the stability bounds, (2.3) implies

$$\begin{aligned}
\int_0^{\gamma(t)} |u(t, x)| \, dx & \leq \frac{1}{\tilde{g}} \int_0^t |b(\tau)| \, d\tau + \int_0^t \int_0^{\gamma(t)} |d(\tau, x) u(\tau, x)| \, dx \, d\tau \\
\int_{\gamma(t)}^{+\infty} |u(t, x)| \, dx & \leq \int_0^{+\infty} |u_o(x)| \, dx + \int_0^t \int_{\gamma(t)}^{+\infty} |d(\tau, x) u(\tau, x)| \, dx \, d\tau \\
\|u(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} & \leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} + \frac{1}{\tilde{g}} \|b\|_{\mathbf{L}^1([0, t]; \mathbb{R})} + C \int_0^t \|u(\tau)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} \, d\tau.
\end{aligned}$$

An application of Gronwall Lemma yields the desired estimate (2.11). Finally, the monotonicity property (2.12) directly follows from (2.3).

To prove (2.9), fix $t', t'' \in \mathbb{R}^+$ with $t' < t''$. Then,

$$\|u(t'') - u(t')\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} = \int_0^{\gamma(t')} |u(t'', x) - u(t', x)| \, dx \quad (4.7)$$

$$+ \int_{\gamma(t')}^{\gamma(t'')} |u(t'', x) - u(t', x)| \, dx \quad (4.8)$$

$$+ \int_{\gamma(t'')}^{+\infty} |u(t'', x) - u(t', x)| \, dx \quad (4.9)$$

and we deal with the three terms separately, using (2.3) as follows. Begin with (4.7):

$$\begin{aligned}
& \int_0^{\gamma(t')} |u(t', x) - u(t'', x)| \, dx \\
& \leq \int_0^{\gamma(t')} \left| \frac{b(T(0; t', x))}{g(T(0; t', x), 0)} \exp \int_{T(0; t', x)}^{t'} \left[d(\tau, X(\tau; t', x)) - \partial_x g(\tau, X(\tau; t', x)) \right] \, d\tau \right. \\
& \quad \left. - \frac{b(T(0; t'', x))}{g(T(0; t'', x), 0)} \exp \int_{T(0; t'', x)}^{t''} \left[d(\tau, X(\tau; t'', x)) - \partial_x g(\tau, X(\tau; t'', x)) \right] \, d\tau \right| \, dx \\
& \leq \int_0^{\gamma(t')} \left| \frac{b(T(0; t', x))}{g(T(0; t', x), 0)} - \frac{b(T(0; t'', x))}{g(T(0; t'', x), 0)} \right| \\
& \quad \times \exp \left(\int_{T(0; t', x)}^{t'} \left(d(\tau, X(\tau; t', x)) - \partial_x g(\tau, X(\tau; t', x)) \right) \, d\tau \right) \, dx \\
& \quad + \int_0^{\gamma(t')} \left| \frac{b(T(0; t'', x))}{g(T(0; t'', x), 0)} \right| \\
& \quad \times \left| \exp \left(\int_{T(0; t', x)}^{t'} \left(d(\tau, X(\tau; t', x)) - \partial_x g(\tau, X(\tau; t', x)) \right) \, d\tau \right) \right.
\end{aligned}$$

$$\begin{aligned}
& - \exp \left(\int_{T(0;t'',x)}^{t''} \left(d(\tau, X(\tau; t'', x)) - \partial_x g(\tau, X(\tau; t'', x)) \right) d\tau \right) \Big| dx \\
\leq & \hat{g} t' \left(\frac{\text{TV}(b; [0, t''])}{\check{g}} + \|b\|_{\mathbf{L}^\infty([0, t'']; \mathbb{R})} \frac{\text{TV}(g(\cdot, 0); [0, t''])}{\check{g}^2} \right) (t'' - t') e^{2C t'} \\
& + 2\hat{g} t' \frac{\|b\|_{\mathbf{L}^\infty([0, t'']; \mathbb{R})}}{\check{g}} e^{2C t''} (t'' - t') \left[\frac{C}{\check{g}} + C \right. \\
& \left. + \hat{g} t'' \left(\|d\|_{\mathbf{L}^\infty([0, t'']; \mathbb{R})} + \text{TV}(d) + \|\partial_x g(\cdot, 0)\|_{\mathbf{L}^\infty([0, t'']; \mathbb{R})} + \text{TV} \partial_x g(\cdot, 0) \right) \right] \\
\leq & \mathcal{L} (t'' - t')
\end{aligned}$$

for a suitable positive \mathcal{L} dependent on $t'', \check{g}, \hat{g}, C$ and $\text{TV}(b; [0, t''])$, $\|b\|_{\mathbf{L}^\infty([0, t'']; \mathbb{R})}$. Passing to (4.8), use (4.6) and (2.5):

$$\begin{aligned}
& \int_{\gamma(t')}^{\gamma(t'')} |u(t'', x) - u(t', x)| dx \\
\leq & 2\hat{g} \max \left\{ \|u(t')\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})}, \|u(t'')\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} \right\} (t'' - t') \\
\leq & 2\hat{g} \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} + \frac{1}{\check{g}} \|b\|_{\mathbf{L}^\infty([0, t'']; \mathbb{R})} \right) e^{C t''} (t'' - t')
\end{aligned}$$

Finally, deal with (4.9) using (4.5):

$$\begin{aligned}
& \int_{\gamma(t'')}^{+\infty} |u(t', x) - u(t'', x)| dx \\
\leq & \int_{\gamma(t'')}^{+\infty} \left| u(t', x) - u(t', X(t'; t'', x)) \right| \\
& \times \exp \left[\int_{t'}^{t''} \left(d(\tau, X(\tau; t', x)) - \partial_x g(\tau, X(\tau; t', x)) \right) d\tau \right] dx \\
\leq & \int_{\gamma(t'')}^{+\infty} |u(t', x) - u(t', X(t'; t'', x))| dx \\
& + \int_{\gamma(t'')}^{+\infty} |u(t', X(t'; t'', x))| \\
& \times \left| \exp \left(\int_{t'}^{t''} \left(d(\tau, X(\tau; t', x)) - \partial_x g(\tau, X(\tau; t', x)) \right) d\tau \right) - 1 \right| dx \\
\leq & \text{TV}(u) \|g(t)\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} (t'' - t') + \|u(t')\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} \left(\exp(2C(t'' - t')) - 1 \right) \\
\leq & \left(\hat{g} \text{TV}(u) + 2C \|u(t')\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} \right) (t'' - t')
\end{aligned}$$

Completing the proof of (2.9). \square

The following elementary lemma is of use below.

Lemma 4.1. *Let $H, K \in \mathbb{R}^+$ and assume that the numbers $B_k \in \mathbb{R}^+$ satisfy $B_{k+1} \leq H + K B_k$ for all $k \in \mathbb{N}$. Then, $B_k \leq K^k B_0 + \frac{1-K^k}{1-K} H$.*

Proof of Theorem 2.4. Fix a time T so that

$$\gamma(T) \in \left] 0, \min_{i=1, \dots, n} \bar{x}_i \right[\quad (4.10)$$

and define $u^0(t, x) = u_o(x)$ for $t \in [0, T]$. Recursively, for $k \geq 1$ let $u^k \equiv (u_1^k, \dots, u_n^k)$ solve

$$\begin{cases} \partial_t u_i^k + \partial_x (g_i(t, x) u_i^k) = d_i(t, x) u_i^k(t) & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ g_i(t, 0) u_i^k(t, 0) = b_i^k(t) & t \in \mathbb{R}^+ \\ u_i(0, x) = u_i^o(x) & x \in \mathbb{R}^+ \end{cases} \quad i = 1, \dots, n \quad (4.11)$$

where

$$\begin{aligned} b_i^k(t) = & \alpha_i \left(t, u_1^{k-1}(t, \bar{x}_1), \dots, u_n^{k-1}(t, \bar{x}_n) \right) \\ & + \beta_i \left(\int_{I_1} w_1(x) u_1^{k-1}(t, x) dx, \dots, \int_{I_n} w_n(x) u_n^{k-1}(t, x) dx \right). \end{aligned} \quad (4.12)$$

Note that **(b)** is satisfied and Lemma 2.2 applies. Indeed, if $k = 1$, then b_i^1 is independent on time. Let $k > 1$, then by (2.13) and (2.8)

$$\begin{aligned} & \text{TV} \left(b_i^k; [0, T] \right) \\ & \leq \text{TV} \left(\alpha_i \left(\cdot, u_j^{k-1}(\cdot, \bar{x})|_{j=1, \dots, n} \right); [0, T] \right) \\ & \quad + \text{TV} \left(\beta_i \left(\int_{I_j} w_j(x) u_j^{k-1}(\cdot, x) dx|_{j=1, \dots, n} \right); [0, T] \right) \\ & \leq \mathbf{Lip}(\alpha) \left(T + \text{TV} \left(u_j^{k-1}(\cdot, \bar{x})|_{j=1, \dots, n}; [0, T] \right) \right) \\ & \quad + \mathbf{Lip}(\beta) \sum_{i=1}^n \text{TV} \left(\int_{I_i} w_i(x) u_i^{k-1}(\cdot, x) dx; [0, T] \right) \\ & \leq \mathbf{Lip}(\alpha) \left(T + \text{TV} \left(u_j^{k-1}(\cdot, \bar{x}_j)|_{j=1, \dots, n}; [0, T] \right) \right) \\ & \quad + CW \mathbf{Lip}(\beta) \sum_{i=1}^n \int_0^t \left[\left\| u_i^{k-1}(\tau) \right\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})} + \text{TV} \left(u_i^{k-1}(\tau) \right) \right] d\tau \\ & \leq \mathbf{Lip}(\alpha) \left(T + \text{TV} \left(u_j^{k-1}(\cdot, \bar{x}_j)|_{j=1, \dots, n}; [0, T] \right) \right) \\ & \quad + CW \mathbf{Lip}(\beta) \int_0^t \left(\left\| u^{k-1}(\tau) \right\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV} \left(u^{k-1}(\tau) \right) \right) d\tau \end{aligned}$$

which is finite by induction. Lemma 2.2 then ensures existence and uniqueness of a solution to (4.11)–(4.12) for any $k > 0$. By construction, (4.10) ensures that

$$u^k(t, x) = u^1(t, x) \quad \text{for all } x > \gamma(t) \quad \text{and } k \geq 1. \quad (4.13)$$

Therefore, also $\alpha_i \left(t, u^k(t, \bar{x}_i) \right) = \alpha_i \left(t, u^1(t, \bar{x}_i) \right)$ for all $t \in [0, T]$, for all $k \geq 1$ and all $i = 1, \dots, n$. Compute now

$$\left\| u_i^{k+1}(t) - u_i^k(t) \right\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})}$$

$$\begin{aligned}
&\leq \frac{1}{\hat{g}} \left\| b_i^{k+1} - b_i^k \right\|_{\mathbf{L}^1([0,t];\mathbb{R})} \\
&\leq \frac{\hat{g}}{\hat{g}} \left\| \alpha_i(\cdot, u^k(\cdot, \bar{x}_i)) - \alpha_i(\cdot, u^{k-1}(\cdot, \bar{x}_i)) \right\|_{\mathbf{L}^1([0,t];\mathbb{R})} \\
&+ \frac{\hat{g}}{\hat{g}} \left\| \beta_i \left(\int_{I_1} w_1(x) u_1^k(\cdot, x) dx, \dots, \int_{I_n} w_n(x) u_n^k(\cdot, x) dx \right) \right. \\
&\quad \left. - \beta_i \left(\int_{I_1} w_1(x) u_1^{k-1}(\cdot, x) dx, \dots, \int_{I_n} w_n(x) u_n^{k-1}(\cdot, x) dx \right) \right\|_{\mathbf{L}^1([0,t];\mathbb{R})} \\
&\leq \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) \sum_{j=1}^n \left\| \int_{I_j} (u_j^k(\cdot, x) - u_j^{k-1}(\cdot, x)) dx \right\|_{\mathbf{L}^1([0,t];\mathbb{R})} \\
&\leq \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) \sum_{j=1}^n \left\| \int_{\mathbb{R}^+} (u_j^k(\cdot, x) - u_j^{k-1}(\cdot, x)) dx \right\|_{\mathbf{L}^1([0,t];\mathbb{R})} \\
&\leq \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) \sum_{j=1}^n \int_0^t \left\| u_j^k(\tau) - u_j^{k-1}(\tau) \right\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R})} d\tau \\
&\leq \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) \int_0^t \left\| u^k(\tau) - u^{k-1}(\tau) \right\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} d\tau \tag{4.14}
\end{aligned}$$

Adding up all the components,

$$\begin{aligned}
\left\| u^{k+1} - u^k \right\|_{\mathbf{C}^0([0,T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))} &\leq n \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) \int_0^T \left\| u^k - u^{k-1} \right\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} d\tau \\
&\leq n \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) T \left\| u^k - u^{k-1} \right\|_{\mathbf{C}^0([0,T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))}
\end{aligned}$$

and, recursively,

$$\begin{aligned}
&\left\| u^{k+1} - u^k \right\|_{\mathbf{C}^0([0,T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))} \\
&\leq \left(n \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) T \right)^k \left\| u^1(\tau) - u^0(\tau) \right\|_{\mathbf{C}^0([0,T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))}.
\end{aligned}$$

Choosing now also $T < 1 / \left(n \frac{\hat{g}}{\hat{g}} W \mathbf{Lip}(\beta) \right)$, the sequence u^k is a Cauchy sequence and we obtain the existence of a map $u^* \in \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))$ which is the limit of the sequence u^k , in the sense that

$$\lim_{k \rightarrow +\infty} \sup_{t \in [0, T]} \left\| u^k(t) - u^*(t) \right\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} = 0. \tag{4.15}$$

To prove that u^* solves (1.1), it is sufficient to check that the boundary condition is attained. Indeed, proving that u^* is a weak solution to the balance law is a standard procedure. Clearly, the initial datum is attained, since $u^k(0) = u_o$ for all k .

Using Lemma 2.2, (2.3), (4.13) and (4.15), for all large $k \in \mathbb{N}$ we have

$$\begin{aligned}
&\left\| b^{k+1} \right\|_{\mathbf{L}^\infty([0,T]; \mathbb{R}^n)} \\
&\leq \mathbf{Lip}(\alpha) \sum_{i=1}^n \left\| u^k(\cdot, \bar{x}_i) \right\|_{\mathbf{L}^\infty([0,T]; \mathbb{R})} + W \mathbf{Lip}(\beta) \left\| u^k \right\|_{\mathbf{C}^0([0,T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))}
\end{aligned}$$

$$\leq \mathbf{Lip}(\alpha) \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{CT} + W \mathbf{Lip}(\beta) (\|u^*\|_{\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))} + 1). \quad (4.16)$$

With the above choice of T , using Lemma 2.2,

$$\begin{aligned} & \text{TV}(b_i^{k+1}; [0, T]) \\ \leq & \mathbf{Lip}(\alpha) T + \mathbf{Lip}(\alpha) \left(\|u_o\|_{\mathbf{L}^\infty([0, \max_j \bar{x}_j; \mathbb{R}^n)} + \text{TV}(u_o; [0, \max_j \bar{x}_j]) \right) e^{CT} \\ & + C \mathbf{Lip}(\beta) \int_0^T \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \frac{1}{\check{g}} \|b^k\|_{\mathbf{L}^\infty([0, \tau]; \mathbb{R}^n)} \right) e^{C\tau} d\tau \\ & + C \mathbf{Lip}(\beta) \int_0^T \left[\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right. \\ & \quad \left. + \frac{1}{\check{g}} \left[\frac{C + \check{g}}{\check{g}} \|b^k\|_{\mathbf{L}^\infty([0, \tau]; \mathbb{R}^n)} + \text{TV}(b^k; [0, \tau]) \right] \right] e^{C\tau} d\tau \\ \leq & \mathbf{Lip}(\alpha) T + \mathbf{Lip}(\alpha) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right) e^{CT} \\ & + 2C \mathbf{Lip}(\beta) \int_0^T \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right) e^{C\tau} d\tau \\ & + \frac{C}{\check{g}} \mathbf{Lip}(\beta) \int_0^T \left(\frac{2\check{g} + C}{\check{g}} \|b^k\|_{\mathbf{L}^\infty([0, \tau]; \mathbb{R}^n)} + \text{TV}(b^k; [0, \tau]) \right) e^{C\tau} d\tau \\ \leq & \mathbf{Lip}(\alpha) T + \mathbf{Lip}(\alpha) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right) e^{CT} \\ & + 2 \mathbf{Lip}(\beta) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right) (e^{CT} - 1) \\ & + \frac{C}{\check{g}} \mathbf{Lip}(\beta) \int_0^T \left(\frac{2\check{g} + C}{\check{g}} \|b^k\|_{\mathbf{L}^\infty([0, \tau]; \mathbb{R}^n)} + \text{TV}(b^k; [0, \tau]) \right) e^{C\tau} d\tau \\ \leq & \mathbf{Lip}(\alpha) T + \mathbf{Lip}(\alpha) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right) e^{CT} \\ & + 2 \mathbf{Lip}(\beta) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right) (e^{CT} - 1) \\ & + \frac{2\check{g} + C}{\check{g}^2} \mathbf{Lip}(\beta) \left(\|b^k\|_{\mathbf{L}^\infty([0, T]; \mathbb{R}^n)} + \text{TV}(b^k; [0, T]) \right) (e^{CT} - 1) \end{aligned}$$

Inserting now the estimate (4.16) in the latter term above, we can apply Lemma 4.1 to the inequality $B_{k+1} \leq H + K B_k$, where

$$\begin{aligned} B_k &= \text{TV}(b^k; [0, T]) \\ H &= n \mathbf{Lip}(\alpha) T \\ &+ n \left(\mathbf{Lip}(\alpha) e^{CT} + 2 \mathbf{Lip}(\beta) (e^{CT} - 1) \right) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \text{TV}(u_o) \right) \\ &+ n \frac{2\check{g} + C}{\check{g}^2} \mathbf{Lip}(\beta) (e^{CT} - 1) \\ &\quad \times \left(\mathbf{Lip}(\alpha) \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{CT} + \mathbf{Lip}(\beta) (\|u^*\|_{\mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))} + 1) \right) \\ &\quad \times (e^{CT} - 1) \end{aligned}$$

$$K = n \frac{2\tilde{g} + C}{\tilde{g}^2} \mathbf{Lip}(\beta) (e^{CT} - 1).$$

Hence, as soon as T is so small that

$$n \frac{2\tilde{g} + C}{\tilde{g}^2} \mathbf{Lip}(\beta) (e^{CT} - 1) < 1,$$

we obtain a bound on $\text{TV}(b^k; [0, T])$ uniform in k . This bound, due to Lemma 2.2, ensures that also $\sup_{t \in [0, T]} \sup_{k \in \mathbb{N}} \text{TV}(u^k(t)) < +\infty$ and, by the lower semicontinuity of the total variation with respect to the \mathbf{L}^1 topology, $\sup_{t \in [0, T]} \text{TV}(u^*(t)) < +\infty$. Therefore, the trace $\lim_{x \rightarrow 0^+} u^*(t, x)$ exists for all $t \in [0, T]$.

The uniform bound on $\text{TV}(b^k; [0, T])$, together with (2.9) and [6, Theorem 2.4], ensure that for a.e. $x \in \mathbb{R}^+$, we have $\lim_{k \rightarrow +\infty} u^k(t, x) = u^*(t, x)$. Choose one such x and observe that:

$$\begin{aligned} & u^*(t, x) \\ = & \lim_{k \rightarrow +\infty} u^k(t, x) \\ = & \lim_{k \rightarrow +\infty} \frac{b^k(T(0; t, x))}{g(T(0; t, x), 0)} \\ & \times \exp\left(\int_{T(0; t, x)}^t (d(\tau, X(\tau; t, x)) - \partial_x g(\tau, X(\tau; t, x))) d\tau\right) \\ = & \frac{\lim_{k \rightarrow +\infty} b^k(T(0; t, x))}{g(T(0; t, x), 0)} \\ & \times \exp\left(\int_{T(0; t, x)}^t (d(\tau, X(\tau; t, x)) - \partial_x g(\tau, X(\tau; t, x))) d\tau\right) \\ = & \frac{\lim_{k \rightarrow +\infty} \mathcal{B}(T(0; t, x), u^{k-1})}{g(T(0; t, x), 0)} \\ & \times \exp\left[\int_{T(0; t, x)}^t (d(\tau, X(\tau; t, x)) - \partial_x g(\tau, X(\tau; t, x))) d\tau\right] \\ = & \frac{\mathcal{B}(T(0; t, x), u)}{g(T(0; t, x), 0)} \exp\left(\int_{T(0; t, x)}^t (d(\tau, X(\tau; t, x)) - \partial_x g(\tau, X(\tau; t, x))) d\tau\right) \end{aligned}$$

where in the last step above we used the convergences:

$$\begin{aligned} \lim_{k \rightarrow +\infty} \alpha_i(t, u_j^k(t, \bar{x}_j)_{|j=1, \dots, n}) &= \alpha_i(t, u^*(t, \bar{x}_j)_{|j=1, \dots, n}) \\ \lim_{k \rightarrow +\infty} \beta_i\left(\int_{I_j} w_j(x) u_j^k(t, x) dx_{|j=1, \dots, n}\right) &= \beta_i\left(\int_{I_j} w_j(x) u^*(t, x) dx_{|j=1, \dots, n}\right) \end{aligned}$$

the former by (4.13) and the latter by (4.15).

The time T chosen above depends only on β , $\min_i \bar{x}_i$, on d and on g . In particular, it is independent from the initial datum. Hence, the above procedure can be iterated, extending u^* to a function defined on all \mathbb{R}^+ , i.e. $u^* \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n))$.

Let now u'_o, u''_o be two initial data. Define $\bar{x} = \min_{i=1, \dots, n} \bar{x}_i$ and $\bar{t} = \Gamma(\bar{x})$. Denote $\bar{I} = [\bar{x}, +\infty[$. To prove the stability estimate, with obvious notation, preliminary compute for $t \in [0, \bar{t}]$:

$$\begin{aligned} \|u''(t) - u'(t)\|_{\mathbf{L}^1(\bar{I}; \mathbb{R}^n)} &\leq \|u''_o - u'_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \\ \|u''(t) - u'(t)\|_{\mathbf{L}^\infty(\bar{I}; \mathbb{R}^n)} &\leq \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \end{aligned}$$

Moreover, by (4.12) and (2.13), for $t \in [0, \bar{t}]$,

$$\begin{aligned} &\|b'' - b'\|_{\mathbf{L}^\infty([0, \bar{t}]; \mathbb{R}^n)} \\ &\leq n \mathbf{Lip}(\alpha) \|u''(t) - u'(t)\|_{\mathbf{L}^\infty(\bar{I}; \mathbb{R}^n)} + nW \mathbf{Lip}(\beta) \|u''(t) - u'(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} \\ &\leq n \mathbf{Lip}(\alpha) \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \\ &\quad + nW \mathbf{Lip}(\beta) \|u''(t) - u'(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)}. \end{aligned} \quad (4.17)$$

$$\begin{aligned} &\|b'' - b'\|_{\mathbf{L}^1([0, \bar{t}]; \mathbb{R}^n)} \\ &\leq n \mathbf{Lip}(\alpha) \frac{e^{Ct} - 1}{C} \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} \\ &\quad + nW \mathbf{Lip}(\beta) \int_0^t \|u''(\tau) - u'(\tau)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} d\tau. \end{aligned} \quad (4.18)$$

Therefore, by (2.9), for $t \in [0, \bar{t}]$,

$$\begin{aligned} \|u''(t) - u'(t)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R}^n)} &\leq \left(\frac{1}{\check{g}} \|b'' - b'\|_{\mathbf{L}^1([0, t]; \mathbb{R})} + \|u''_o - u'_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} \right) e^{Ct} \\ &\leq \frac{n \mathbf{Lip}(\alpha)}{\check{g}} \frac{e^{Ct} - 1}{C} e^{Ct} \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} \\ &\quad + \|u''_o - u'_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \\ &\quad + nW \mathbf{Lip}(\beta) \int_0^t \|u''(\tau) - u'(\tau)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} d\tau e^{Ct}. \end{aligned}$$

An application of Gronwall Lemma yields, for $t \in [0, \bar{t}]$,

$$\begin{aligned} &\|u''(t) - u'(t)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R}^n)} \\ &\leq \left(\frac{n \mathbf{Lip}(\alpha)}{\check{g}} \frac{e^{Ct} - 1}{C} \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} + \|u''_o - u'_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} \right) \\ &\quad \times \exp \left(Ct + nW \mathbf{Lip}(\beta) e^{Ct} \right) \end{aligned}$$

To iterate beyond time \bar{t} , using (2.5), (4.17) and the above bound to estimate

$$\begin{aligned} &\|u''(t) - u'(t)\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^n)} \\ &\leq \left(\frac{1}{\check{g}} \|b'' - b'\|_{\mathbf{L}^\infty([0, t]; \mathbb{R}^n)} + \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} \right) e^{Ct} \\ &\leq \frac{n \mathbf{Lip}(\alpha)}{\check{g}} \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{2Ct} \\ &\quad + \frac{nW \mathbf{Lip}(\beta)}{\check{g}} \|u''(t) - u'(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} + \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \\ &\leq \left(\frac{n \mathbf{Lip}(\alpha)}{\check{g}} e^{Ct} + 1 \right) \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \end{aligned}$$

$$\begin{aligned}
& + \frac{n W \mathbf{Lip}(\beta)}{\check{g}} \|u''(t) - u'(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \\
\leq & \left[\left(\frac{n \mathbf{Lip}(\alpha)}{\check{g}} e^{Ct} + 1 \right) \right. \\
& \left. + \frac{n W \mathbf{Lip}(\beta)}{\check{g}} \frac{n \mathbf{Lip}(\alpha)}{\check{g}} \frac{e^{Ct} - 1}{C} \exp \left(Ct + n W \mathbf{Lip}(\beta) e^{Ct} \right) \right] \\
& \times \|u''_o - u'_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} e^{Ct} \\
& + \frac{n W \mathbf{Lip}(\beta)}{\check{g}} \exp \left(2Ct + n W \mathbf{Lip}(\beta) e^{Ct} \right) \|u''_o - u'_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)}
\end{aligned}$$

completing the proof. \square

Proof of Theorem 2.5. Define, for $i = 1, \dots, n$,

$$\begin{aligned}
b'_i(t) &= \alpha'_i(t, u'_1(t, \bar{x}_1-), \dots, u'_n(t, \bar{x}_n-)) \\
&+ \beta'_i \left(\int_{I_1} w'_1(x) u'_1(t, x) dx, \dots, \int_{I_n} w'_n(x) u'_n(t, x) dx \right), \\
b''_i(t) &= \alpha''_i(t, u''_1(t, \bar{x}_1-), \dots, u''_n(t, \bar{x}_n-)) \\
&+ \beta''_i \left(\int_{I_1} w''_1(x) u''_1(t, x) dx, \dots, \int_{I_n} w''_n(x) u''_n(t, x) dx \right).
\end{aligned}$$

Preliminary, using (2.13), let us estimate the term

$$\begin{aligned}
& \|b' - b''\|_{\mathbf{L}^1([0, t]; \mathbb{R}^n)} \\
\leq & \sum_{i=1}^n \int_0^t \left| \alpha'_i(s, u'_1(s, \bar{x}_1), \dots, u'_n(s, \bar{x}_n)) - \alpha''_i(s, u''_1(s, \bar{x}_1), \dots, u''_n(s, \bar{x}_n)) \right| ds \\
& + \sum_{i=1}^n \int_0^t \left| \beta'_i \left(\int_{I_j} w'_j(x) u'_j(s, x) dx \right) - \beta''_i \left(\int_{I_j} w''_j(x) u''_j(s, x) dx \right) \right| ds \\
\leq & t \|\alpha' - \alpha''\|_{\mathbf{C}^0(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^n)} + \mathbf{Lip}(\alpha'') \sum_{i=1}^n \int_0^t \left| u'_j(s, \bar{x}_j) - u''_j(s, \bar{x}_j) \right| ds \\
& + t \|\beta' - \beta''\|_{\mathbf{C}^0(\mathbb{R}^n; \mathbb{R}^n)} + W \mathbf{Lip}(\beta'') \int_0^t \|u'(s) - u''(s)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} ds \\
& + \mathbf{Lip}(\beta'') \int_0^t \|u'(s)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} ds \sum_{j=1}^n \|w'_j - w''_j\|_{\mathbf{C}^0(I_j; \mathbb{R})}.
\end{aligned}$$

Define $\bar{x} = \min_{i=1, \dots, n} \bar{x}_i$ and $\bar{t} = \Gamma(\bar{x})$. As long as $s \in [0, \bar{t}]$, we have $u'_j(s, \bar{x}_j) = u''_j(s, \bar{x}_j)$. Hence the above estimate leads to

$$\begin{aligned}
& \|b' - b''\|_{\mathbf{L}^1([0, t]; \mathbb{R}^n)} \\
\leq & t \left(\|\alpha' - \alpha''\|_{\mathbf{C}^0(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^n)} + \|\beta' - \beta''\|_{\mathbf{C}^0(\mathbb{R}^n; \mathbb{R}^n)} \right) \\
& + \mathbf{Lip}(\beta'') \int_0^t \|u'(s) - u''(s)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} ds
\end{aligned}$$

$$+ \mathbf{Lip}(\beta'') \int_0^t \mathcal{K}(s) \, ds \left[\|u_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} + t \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} \right] \sum_{j=1}^n \|w'_j - w''_j\|_{\mathbf{C}^0(I_j; \mathbb{R})}$$

for all $t \in [0, \bar{t}]$. In the same time interval,

$$\begin{aligned} & \|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} \\ & \leq \frac{e^{Ct}}{\check{g}} \|b' - b''\|_{\mathbf{L}^1([0, t]; \mathbb{R}^n)} \\ & \leq \frac{t e^{Ct}}{\check{g}} \left(\|\alpha' - \alpha''\|_{\mathbf{C}^0(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^n)} + \|\beta' - \beta''\|_{\mathbf{C}^0(\mathbb{R}^n; \mathbb{R}^n)} \right) \\ & + \frac{\mathbf{Lip}(\beta'') e^{Ct}}{\check{g}} \int_0^t \|u'(s) - u''(s)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} \, ds \\ & + \frac{\mathbf{Lip}(\beta'') \int_0^t \mathcal{K}(s) \, ds e^{Ct}}{\check{g}} \left[\|u_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} + t \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} \right] \sum_{j=1}^n \|w'_j - w''_j\|_{\mathbf{C}^0(I_j; \mathbb{R})} \end{aligned}$$

so that by Gronwall Lemma, for $t \in [0, \bar{t}]$,

$$\begin{aligned} & \|u'(t) - u''(t)\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} \\ & \leq \left[\frac{t e^{Ct}}{\check{g}} \left(\|\alpha' - \alpha''\|_{\mathbf{C}^0(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^n)} + \|\beta' - \beta''\|_{\mathbf{C}^0(\mathbb{R}^n; \mathbb{R}^n)} \right) \right] \exp \left[\frac{\mathbf{Lip}(\beta'') t e^{Ct}}{\check{g}} \right] \\ & + \frac{\mathbf{Lip}(\beta'') \int_0^t \mathcal{K}(s) \, ds e^{Ct}}{\check{g}} \left(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n)} + t \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^n)} \right) \\ & \quad \times \sum_{j=1}^n \|w'_j - w''_j\|_{\mathbf{C}^0(I_j; \mathbb{R})} \exp \left[\frac{\mathbf{Lip}(\beta'') t e^{Ct}}{\check{g}} \right]. \end{aligned}$$

A repeated application of the estimate above on the intervals $[(k-1)\bar{t}, k\bar{t}]$ allows to complete the proof. \square

Proof of Corollary 3.1. Introduce u_1, u_2, \dots as in table (3.1). Then, extend d_1, d_2 and g_2 to $\mathbb{R}^+ \times \mathbb{R}$ maintaining the required regularity and bounds on the total variation. The resulting system fits into (1.1)–(1.2). Hence, (d), (g) and (2.13) hold. Theorem 2.4 applies, ensuring the well posedness of the Cauchy problem. Finally, the solution to (3.2) is obtained restricting the solution to (1.1)–(1.2)–(3.1) to $[0, a_{\max}]$ and to $[x_{\min}, x_{\max}]$. \square

REFERENCES

- [1] A. S. Ackleh and K. Deng, [A nonautonomous juvenile-adult model: Well-posedness and long-time behavior via a comparison principle](#), *SIAM J. Appl. Math.*, **69** (2009), 1644–1661.
- [2] A. S. Ackleh, K. Deng and X. Yang, [Sensitivity analysis for a structured juvenile–adult model](#), *Comput. Math. Appl.*, **64** (2012), 190–200.
- [3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.
- [4] C. Bardos, A. Y. le Roux and J.-C. Nédélec, [First order quasilinear equations with boundary conditions](#), *Comm. Partial Differential Equations*, **4** (1979), 1017–1034.
- [5] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Second edition, Texts in Applied Mathematics, 40, Springer, New York, 2012.

- [6] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem*, Oxford Lecture Series in Mathematics and its Applications, 20, Oxford University Press, Oxford, 2000.
- [7] J. A. Carrillo, S. Cuadrado and B. Perthame, [Adaptive dynamics via Hamilton-Jacobi approach and entropy methods for a juvenile-adult model](#), *Math. Biosci.*, **205** (2007), 137–161.
- [8] R. M. Colombo and G. Guerra, [On general balance laws with boundary](#), *J. Differential Equations*, **248** (2010), 1017–1043.
- [9] R. M. Colombo, G. Guerra, M. Herty and V. Schleper, [Optimal control in networks of pipes and canals](#), *SIAM J. Control Optim.*, **48** (2009), 2032–2050.
- [10] M. Garavello and B. Piccoli, *Traffic Flow on Networks*, AIMS Series on Applied Mathematics, 1, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006.
- [11] N. Keyfitz, The mathematics of sex and marriage, in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 4: Biology and Health*, University of California Press, Berkeley, Calif., 1972, 89–108.
- [12] S. N. Kruzhkov, First order quasilinear equations with several independent variables, *Mat. Sb. (N.S.)*, **81** (1970), 228–255.
- [13] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.
- [14] P. Manfredi, Logistic effects in the two-sex model with “harmonic mean” fertility function, *Genus*, **49** (1993), 43–65.
- [15] B. Perthame, *Transport Equations in Biology*, Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.
- [16] R. Schoen, [The harmonic mean as the basis of a realistic two-sex marriage model](#), *Demography*, **18** (1981), 201–216.
- [17] R. Schoen, [Relationships in a simple harmonic mean two-sex fertility model](#), *Journal of Mathematical Biology*, **18** (1983), 201–211.
- [18] R. Schoen, *Modeling Multigroup Populations*, Springer, 1988.
- [19] D. Serre, *Systems of Conservation Laws. 2. Geometric Structures, Oscillations, and Initial-Boundary Value Problems*, Translated from the 1996 French original by I. N. Sneddon, Cambridge University Press, Cambridge, 2000.
- [20] A. Sundelof and P. Aberg, [Birth functions in stage structured two-sex models](#), *Ecological Modelling*, **193** (2006), 787–795.

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E-mail address: rinaldo.colombo@unibs.it

E-mail address: mauro.garavello@unimib.it