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BASIC STAGE STRUCTURE MEASURE VALUED EVOLUTIONARY GAME MODEL

John Cleveland

University of Wisconsin-Richland
1200 Hwy 14 West
Richland Center, WI 53581-1399, USA

Abstract. The ideas and techniques developed in [12, 3] are extended to a basic stage structured model. Each strategy consists of two stages: a Juvenile (L for larvae), and Adult (A). A general model of this basic stage structure is formulated as a dynamical system on the state space of finite signed measures. Nonnegativity, well-posedness and uniform eventual boundedness are established under biologically natural conditions on the rates. Similar to [12] we also have the unifying of discrete and continuous systems and the containment of the classic nonlinearities.

1. **Introduction.**

In this paper an Evolutionary Game (EG) is defined as a game in which the strategy profiles evolve over time under evolutionary forces (EF) i.e. birth, mortality, mutation, selection (replication), recombination, drift etc... More specifically, in this paper each strategy class is divided into two stages and we concentrate on the five evolutionary forces: Birth, Death, Transition, Selection and Mutation.

It is well known that the solutions of many such EG models constructed on the state space of continuous or integrable functions converge to a Dirac measure concentrated at the fittest strategy or trait [1, 2, 8, 9, 13, 21, 22, 23, 26]. This is particularly the case where competitive exclusion is the evolutionary outcome. Consider the following EG model of generalized logistic growth with pure selection (i.e. strategies replicate themselves exactly and no mutation occurs) which was developed and analyzed in [2]:

$$\frac{d}{dt}x(t,q) = x(t,q)(q_1 - q_2 X(t)), \tag{1}$$

where $X(t) = \int_Q x(t,q)dq$ is the total population, $Q \subset \text{int}(\mathbb{R}^2_+)$ is compact and the state space is the set of continuous real valued functions $C(Q)$. Each $q = (q_1, q_2) \in Q$ is a two tuple where q_1 is an intrinsic replication rate and q_2 is an intrinsic mortality rate. The solution to this model converges to a Dirac mass centered at the fittest q-class. This is the class with the highest birth to death ratio $\frac{q_1}{q_2}$, and this convergence is in a topology called *weak** (point wise convergence of functions) [2].

In [21, ch.2], these measure-valued limits are illustrated in a biological and adaptive dynamics environment. This convergence is in the weak* topology [2]. Thus, the asymptotic limit of the solution is not in such state spaces (function spaces); it is

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a measure. Among other things this precludes stability analysis which traditionally requires the equilibrium point to be in the state space. Some models (e.g. [1], [26]) have addressed this problem. In [1], the authors formulated a *pure* selection model on the space of finite signed measures with density dependent birth and mortality functions and a 2-dimensional strategy space. They discussed existence-uniqueness of solutions and studied the long term behavior of the model. Another very important drawback with the previous literature (function space approach) that this abstraction allows us to resolve is that we can handle a nonlinear mutation term. Here our mutation term is nonlinear because it comprises a product of a mutation kernel and a density dependent replication rate. The nonlinearity of this term is due to the density dependence of the replication rate. The previous methods have a mutation term that is linear. This is done in order to use the Perron Frobenius theory when one wishes to study asymptotic behavior of the model once wellposedness has been established [9].

Another way to think of these types of models is as being derived from stochastic individual based models. One then takes limits, in specific orders, on population size, mutation rate, mutation step, while rescaling time accordingly to arrive at a macroscopic model [10, 15, 17]. Here I use the methods developed in [1, 2, 12, 3].

Modeling tumor growth, cancer therapy and viral evolution are immediate applications. For example, tumor heterogeneity is one main cause of tumor robustness. Tumors are robust in the sense that tumors are systems that tend to maintain stable functioning despite various perturbations. While tumor heterogeneity describes the existence of distinct subpopulations of tumor cells with specific characteristics within a single neoplasm. The mutation between the subpopulations is one major factor that makes the tumor robust. To date there is no unifying framework in mathematical modeling of carcinogenesis that would account for **parametric heterogeneity** [16]. To introduce distributed parameters (heterogeneity) and mutation is essential as we know that cancer recurrence, tumor dormancy and other dynamics can appear in heterogeneous settings and not in homogeneous settings. Increasing technological sophistication has led to a resurgence of using oncolytic viruses in cancer therapy. So in formulating a cancer therapy it is useful to know that in principle *a heterogeneous oncolytic virus (a virus with more than one strain) must be used to eradicate a tumor cell.*

Starting from the papers [1, 2, 12, 3] there are many directions to go in extending the basic theory of measure valued evolutionary games. The stage-structured direction is both natural and there is a need for more ideas in developing a general framework for studying predictive evolution. According to [5] there is no general theoretical framework existing for understanding or predicting evolution in stage-structured populations. Evolution occurs when organisms exhibit differences in the vital rates of birth, death, and dispersal that are at least partly heritable. The best-developed body of evolutionary theory that accounts for interindividual variability in vital rates is for age-structured populations [11, 18]. Many factors other than age, such as sex, body size, location, developmental stage, the magnitude of nutritional reserves, and measures of physiological condition, can be better predictors of birth and death.

This paper is organized as follows: In section 2 we go over some background material needed for our study of Evolutionary Games. In section 3, we develop the basic stage structured model. It is a juvenile adult model. In section 4, we prove the well-posedness of the model developed in section 3. In section 5, we show that the model covers both the discrete and continuous case, pure selection and mutation. In section 6 we begin the next phase of this project, asymptotic analysis, proving a very basic result. In section 7, we have a conclusion.

2. **Background material.** In this section, we state assumptions and define notation that we will use throughout the paper. Here $\mathcal{M} = \mathcal{M}(Q)$ are the finite signed Borel measures on $[Q, d]$, a compact metric space. We will consider two norms on $\mathcal{M}$ under which it becomes a NLS or normed linear space. They are $\| \cdot \|_{BL}^*$, the bounded lipschitz dual norm and $\| \cdot \|_V$ which denotes total variation.

Let $C(Q) = C(Q; \mathbb{R})$, the Banach space of continuous real valued functions on Q under the supremum norm $\| \cdot \|_\infty$,

$$\|f\|_\infty = \sup_{q \in Q} |f(q)|.$$

If $\nu \in \mathcal{M}$, then [4, pg. 185] relays that

$$\|\nu\|_V = \sup_{f \in C(Q;\mathbb{R}), \|f\|_\infty \leq 1} |\nu(f)|.$$

Let $BL(Q; \mathbb{R})$ denote the $\mathbb{R}$ -valued bounded Lipschitz functions on Q under the norm

$$\|f\|_{BL} = \|f\|_\infty + \|f\|_{Lip}$$

where

$$\|f\|_{Lip} = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in Q, x \neq y \right\}.$$

Now since $BL(Q; \mathbb{R}) \subset C(Q; \mathbb{R})$, then $\mathcal{M}(Q) = C^*(Q; \mathbb{R}) \subset BL^*(Q; \mathbb{R})$ and we can view $\mathcal{M} \subset [BL^*(Q), \| \cdot \|_{BL}^*]$, where

$$\|\mu\|_{BL}^* = \sup \left\{ \| \int_Q f d\mu \| : \|f\|_{BL} \leq 1 \right\}.$$

This makes $\mathcal{M}$ into a NLS which we denote as $\mathcal{M}_w$ and on the cone, $\mathcal{M}_{+,w}$, this norm generates the *weak** topology [14]. The duality $< C(Q), \mathcal{M} >$ given by $< f, \mu > \mapsto \int_Q f(q) d\mu$ generates the *weak** topology on $\mathcal{M}$, i.e., the locally convex TVS (topological vector space) $(\mathcal{M}, \sigma(\mathcal{M}, C(Q))$. Here if $(\mu_n)_{n \in \mathbb{N}}, \mu \in \mathcal{M}$ convergence is defined by

$$\mu_n \stackrel{w*}{\to} \mu$$

provided

$$|\mu_n(f) - \mu(f)| \to 0 \text{ for every } f \in C(Q; \mathbb{R}).$$

Or if $\rho_f(\mu) = | < \mu, f > | = |\mu(f)|$ is the seminorm defined by the duality, then $\rho_f(\mu_n - \mu) \to 0$ for each $f \in C(Q; \mathbb{R})$ as $n \to \infty$.

Note that we can use sequences to detect limits on $\mathcal{M}_{+,w}$, because on $\mathcal{M}_{+,w}$ the *weak** topology is completely metrizable e.g. with $\| \cdot \|_{BL}^*$ [14].

Obviously if $\mu \in \mathcal{M}$, then $\|\mu\|_{BL}^* \leq \|\mu\|_V$. We will use this fact liberally in the estimates to follow.

$\overline{B_a[\nu]}$ will denote the closed total variation ball of radius a around $\nu \in \mathcal{M}$.

If $\| \cdot \|$ represents one of the two norms defined above, then we can also put a NLS structure on $[\mathcal{M}, \| \cdot \|] \times [\mathcal{M}, \| \cdot \|] = [\mathcal{M}, \| \cdot \|]^2$ under the norm

$$\|(\mu, \nu)\|_{\cdot^2} = \|\mu\| + \|\nu\|.$$

Note that if we are using total variation, $\| \cdot \|_V$, then both $\mathcal{M}$ and $\mathcal{M} \times \mathcal{M}$ are Banach spaces. If the total variation norm is denoted $\| \cdot \|_V$, then $\mathcal{M}_V$ will denote the Banach space of the finite signed measures with the total variation norm. If

$S \subseteq \mathcal{M}$, S_w denotes the same set under the *weak** topology and S_V the same set under total variation. If no topology is indicated then S is simply a subset of the Riesz space of ordered measures. Also $S_+ = S \cap \mathcal{M}_+$.

If $\mathcal{P}_w$ denotes the probability measures under the *weak** topology, then $C(Q, \mathcal{P}_w(Q))$, the continuous $\mathcal{P}_w$ valued functions on Q with the topology of uniform convergence is a complete metric space. That is if $\gamma \in C(Q, \mathcal{P}_w)$, then

$$||\gamma||_\infty = \sup_{q\in Q} \|\gamma(q)\|^*_{BL}.$$

Suppose f is continuous. If $\gamma \in C(Q, \mathcal{P}_w(Q))$, $g \in C(Q)$, then

$$\Big(\int_Q f(\hat{q})\gamma(\hat{q})\mu(d\hat{q})\Big)[g] = \int_Q f(\hat{q})\gamma(\hat{q})[g]\mu(d\hat{q}).$$

Also the measure $E \mapsto \int_E f(q)\mu(dq)$, denoted $< \int f(q)\mu(dq), \cdot >$, as a functional has the action: $g \mapsto \int_Q g(q)f(q)\mu(dq)$ for $g \in C(Q)$.

If $\mathcal{X}$ is any set, and $\|\cdot\|$ is one of the two previously defined norms on $\mathcal{M}$ and $f = (f_L, f_A)$ is a **bounded map** from $\mathcal{X}$ into $\mathcal{M}^2$ then under the sup norm, i.e.,

$$\|f\|_S = \sup_{x\in\mathcal{X}} \|f_L(x)\| + \sup_{x\in\mathcal{X}} \|f_A(x)\| \tag{2}$$

we obtain another Banach space denoted $\mathcal{BM}(\mathcal{X}) := (\mathcal{B}M(\mathcal{X}), \|\cdot\|_S)$. $\mathcal{BM}(\mathcal{X})$ is the space in which we are *always* working and should be kept in mind when we begin the fixed point argument as there are several topologies being used. For our dynamical system purposes, if $a, b > 0$, $m_0 = (m_{0L}, m_{0A})$ and $(0, m_0) \in \mathbb{R}_+ \times \mathcal{M} \times \mathcal{M}$ are given, we are interested in the set $\mathcal{X} = [-b, b] \times (\overline{B_a[m_{0L}]})_{+,w} \times (\overline{B_a[m_{0A}]})_{+,w} \times C(Q, \mathcal{P}_w)$. Let's denote by $\mathcal{C}\Big([-b, b] \times (\overline{B_a[m_{0L}]})_{+,w} \times (\overline{B_a[m_{0A}]})_{+,w} \times C(Q, \mathcal{P}_w); (\overline{B_{2a}[m_{0L}]})_w \times (\overline{B_{2a}[m_{0A}]})_w\Big)$ the closed subcollection of continuous maps into $(\overline{B_{2a}[m_{0L}]})_w \times (\overline{B_{2a}[m_{0A}]})_w$. Then it is an exercise to show that $(M(a, b, m_0), \|\cdot\|_S)$ where

$$\begin{aligned} M(a, b, m_0) = \{\alpha \in \mathcal{BM}(\mathcal{X})|\ \alpha \in \mathcal{C}\Big([-b, b] \times (\overline{B_a[m_{0L}]})_{+,w} \times (\overline{B_a[m_{0A}]})_{+,w} \times \\ C(Q, \mathcal{P}_w); (\overline{B_{2a}[m_{0L}]})_w \times (\overline{B_{2a}[m_{0A}]})_w\Big), \alpha \geq 0, \alpha(0; u, \gamma) = u\}. \end{aligned} \tag{3}$$

is a nonempty closed metric subspace of the complete metric space $\mathcal{BM}\Big([-b, b] \times \overline{B_a[m_{0L}]}_+ \times \overline{B_a[m_{0A}]}_+ \times C(Q, \mathcal{P}_w)\Big)$.

Also, for any time dependent mapping, $f(t)$, we let $f'(t) = \frac{df}{dt}(t)$

3. **Juvenile adult model.** We have $([Q, d], B_Q, P)$ where $[Q, d]$ is a compact metric space, B_Q are the Borel sets on $[Q, d]$ and P is a probability measure on the Measurable Space $([Q, d], B_Q)$ representing an initial weighting on the strategies. One can think of Q as a compact subset of $\mathbb{R}^n$ and P as a probability measure (initial weighting) on this set.

As mentioned in the first paragraph, the evolutionary forces that act on our population are: $\alpha(q, L, A), \mu(q, L, A)$ the per capita mortality rates of adult and juvenile populations of strategy q respectively. Likewise $\beta(q, L, A)$ is a per capita birth rate of the q -strategy, while $f(q, L, A)$ is the density dependent transition rate from q strategy juveniles to q-strategy adults. We assume the following regularity.

Assumption 11.1. $f, \alpha, \beta, \mu : Q \times [0, \infty)^2 \to (0, \infty)$ *are continuous functions with the following additional properties:*

(1) $f(q,\cdot), \beta(q,\cdot)$ *are nonincreasing and Lipschitz continuous uniformly for* $q \in Q$.
(2) $\alpha(q,\cdot), \mu(q,\cdot)$ *are nondecreasing and Lipschitz continuous uniformly for* $q \in Q$.
(3) $\gamma \in C(Q, \mathcal{P}_w(Q))$ *and* $\gamma(\hat{q})(E)$ *is the proportion of* $\hat{q}$ *-strategy offspring that adopt strategies that are in the Borel set* E. *Since* $\gamma(q)$ *is a probability measure for* $q \in Q$, *it can also be viewed as a continuous linear functional and the notation* $\gamma(q)[g]$ *will denote the value of this measure on* $g \in C(Q)$, *if* $C(Q)$ *represents the real valued functions on* Q.

Similar to [12] we start with a density version of the juvenile adult model taken from [25, pg. 154] and then integrate and use Fubini to obtain a measure theoretic model. In (4) below, let $L(t,q)$ denote the q -strategy juvenile population at time t. Likewise $A(t,q)$ denotes the q - strategy adult population. Also $m(t) = (m_L(t), m_A(t)) \in \mathcal{M} \times \mathcal{M}$ where sometimes we denote

$$m(t)(Q) = (m_L(t)(Q), m_A(t)(Q)).$$

Then

$$\begin{aligned} L'(t,q) &= \int_Q \gamma(\hat{q})(\{q\})\beta(\hat{q}, m(t)(Q))A(t,\hat{q})P(d\hat{q}) - [\mu(q, m(t)(Q)) \\ &\quad + f(q, m(t)(Q))]L(t,q) \\ A'(t,q) &= f(q, m(t)(Q))L(t,q) - \alpha(q, m(t)(Q))A(t,q) \end{aligned} \tag{4}$$

$$\begin{aligned} m_L'(t)(E) &= \int_Q \gamma(q)(E)\beta(q, m(t)(Q))m_A(t)(dq) - \int_E [\mu(q, m(t)(Q)) \\ &\quad + f(q, m(t)(Q)]m_L(t)(dq) \\ m_A'(t)(E) &= \int_E f(q, m(t)(Q))m_L(t)(dq) - \int_E \alpha(q, m(t)(Q))m_A(t)(dq) \end{aligned} \tag{5}$$

This (5) is the full measure theoretic model and $\gamma(q)(E)$ is the proportion of the q -strategy population offspring adopting strategies that are in E, a Borel subset of Q.

If $m = (m_L, m_A) \in \mathcal{M} \times \mathcal{M}$ is as above then let

$$F : \mathcal{M} \times \mathcal{M} \times C(Q, \mathcal{P}_w) \to \mathcal{M} \times \mathcal{M}$$

be given by $F(m,\gamma) = F(m_L, m_A, \gamma) = (F_1(m_L, m_A, \gamma), F_2(m_L, m_A, \gamma))$ where

$$\begin{aligned} &F_1(m,\gamma)(E) \\ =&\int_Q \gamma(q)(E)\beta(q, m(Q))m_A(dq) - \int_E [\mu(q, m(Q)) + f(q, m(Q))]m_L(dq) \\ =&F_{11}(m,\gamma) - F_{12}(m,\gamma) \\ &F_2(m,\gamma)(E) \\ =&\int_E f(q, m(Q))m_L(dq) - \int_E \alpha(q, m(Q))m_A(dq) \\ =&F_{21}(m,\gamma) - F_{22}(m,\gamma). \end{aligned} \tag{6}$$

Then we are interested in the solution to the following IVP (initial value problem).

$$\begin{cases} m'(t; u, \gamma) = F(m, \gamma) \\ m(0; u, \gamma) = u. \end{cases} \tag{7}$$

4. **Wellposedness of measure theoretic dynamics.**

The main result of this paper is as follows:

Theorem 4.1. *Assume Assumption 11.1. There exists a continuous dynamical system* $(\mathcal{M}_{+,w} \times \mathcal{M}_{+,w}, C(Q, \mathcal{P}_w), \varphi)$ *where* $\varphi : \mathbb{R}_+ \times \mathcal{M}_{+,w} \times \mathcal{M}_{+,w} \times C(Q, \mathcal{P}_w) \to \mathcal{M}_{+,w} \times \mathcal{M}_{+,w}$ *satisfies the following:*

1. *The mapping* $(t; u, \gamma) \mapsto \varphi(t; u, \gamma)$ *is continuous.*
2. *For fixed* u, γ, *the mapping* $t \mapsto \varphi(t; u, \gamma)$ *is continuously differentiable in total variation, i.e.,* $\varphi(\cdot; u, \gamma) : \mathbb{R}_+ \to \mathcal{M}_{V,+} \times \mathcal{M}_{V,+}$ *is continuously differentiable.*
3. *For fixed* u, γ, *the mapping* $t \mapsto \varphi(t; u, \gamma)$ *is the unique* solution *to*

$$\begin{cases} m'(t) = F(m, \gamma) \\ m(0) = u. \end{cases} \tag{8}$$

4.1. **Local existence.**

First let $\mathbf{0}$ denote the zero measure and let $F : \mathcal{M}_V \times \mathcal{M}_V \times C(Q; \mathcal{P}_w) \to \mathcal{M}_V \times \mathcal{M}_V$ be as in (6) and (7).

For each $N \in \mathbb{N}$, define F_N as follows. If j is one of the functions α, β, μ or f then we extend j to $Q \times \mathbb{R} \times \mathbb{R}$ by setting $j_N(q, x, y) = j(q, 0, 0)$ for $x, y \le 0$ and make the modification $j_N(q, x, y) = j(q, N, N)$ for $x, y \ge N$. Then $j_N(q, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to (0, \infty)$ is bounded and Lipschitz continuous uniformly for $q \in Q$. Let $F_N(m, \gamma)(E)$ be the redefined vector field obtained by replacing j with j_N.

Below $\|\beta_N\|_{BL} = \|\beta_N(q, \cdot, \cdot)\|_{BL}$, for any $q \in Q$, $\|\beta_N\|_\infty = \|\beta_N(\cdot, 0, 0)\|_\infty$, likewise for f. Also $\|\alpha_N\|_{BL} = \|\alpha_N(q, \cdot, \cdot)\|_{BL}$, for any $q \in Q$, and $\|\alpha_N\|_\infty = \|\alpha_N(\cdot, N, N)\|_\infty$. Likewise for μ_N. However,

$$\|\mu_N + f_N\|_\infty = \|\mu_N(\cdot, N, N) + f_N(\cdot, 0, 0)\|_\infty.$$

We will resolve the following IVP first.

$$\begin{cases} m'(t; u, \gamma) = F_N(m, \gamma) \\ m(0; u, \gamma) = u. \end{cases} \tag{9}$$

Lemma 4.2. *(Lipschitz F_N) Let*

$$F_N : \mathcal{M}_V \times \mathcal{M}_V \times C(Q, \mathcal{P}_w) \to \mathcal{M}_V \times \mathcal{M}_V$$

be as above and let $W \subseteq \mathcal{M}_V$ *be bounded in total variation. Then on* $W_V \times W_V \times C(Q; \mathcal{P}_w)$, $F_N(\cdot, \cdot, \gamma)$ *is bounded and uniformly Lipschitz for* $\gamma \in C(Q, \mathcal{P}_w)$.

Proof. We must show that F_N is bounded and Lipschitz uniformly for $\gamma \in C(Q, \mathcal{P}_w)$. So if $(\rho, \gamma) = (\rho_L, \rho_A, \gamma)$, $(\rho^\dagger, \gamma) = (\rho_L^\dagger, \rho_A^\dagger, \gamma)$, we will find B_L, B_A such that

$$\|F_N(\rho, \gamma) - F_N(\rho^\dagger, \gamma)\|_{V^2} \le B_L \|\rho_L - \rho_L^\dagger\|_V + B_A \|\rho_A - \rho_A^\dagger\|_V.$$

Below if j is α, β, μ or f then

$$j_N = j_N(q, \rho_L(Q), \rho_A(Q)), \;\; j_N^\dagger = j_N(q, \rho_L^\dagger(Q), \rho_A^\dagger(Q)). \tag{10}$$

Now

$$\begin{aligned} \|F_N(\rho, \gamma) - F_N(\rho^\dagger, \gamma)\|_{V^2} =& \big\| [F_{N1}(\rho, \gamma,), F_{N2}(\rho, \gamma] - [F_{N1}(\rho^\dagger, \gamma), F_{N2}(\rho^\dagger, \gamma)] \big\|_{V^2} \\ =& \big\| [F_{N1}(\rho, \gamma) - F_{N1}(\rho^\dagger, \gamma), F_{N2}(\rho, \gamma) - F_{N2}(\rho^\dagger, \gamma)] \big\|_{V^2} \\ =& \big\| \underbrace{F_{N1}(\rho_L, \rho_A, \gamma) - F_{N1}(\rho_L^\dagger, \rho_A^\dagger, \gamma)}_{\text{D1}} \big\|_V \\ &+ \big\| \underbrace{F_{N2}(\rho, \gamma) - F_{N2}(\rho^\dagger, \gamma)}_{\text{D2}} \big\|_V . \end{aligned} \tag{11}$$

Consider the following assignments:

$$D_{11} = F_{N,11}(\rho,\gamma) - F_{N,11}(\rho^{\dagger},\gamma) \qquad D_{12} = F_{N,12}(\rho,\gamma) - F_{N,12}(\rho^{\dagger},\gamma) \tag{12}$$

$$D_{21} = F_{N,21}(\rho,\gamma) - F_{N,21}(\rho^{\dagger},\gamma) \qquad D_{22} = F_{N,22}(\rho,\gamma) - F_{N,22}(\rho^{\dagger},\gamma). \tag{13}$$

Then

$$D1 = D_{11} + D_{12}, \qquad\qquad D2 = D_{21} + D_{22}. \tag{14}$$

We show that F_{N1} (or D1) and F_{N2} (or D2) are bounded and Lipschitz and the result for F_N follows immediately.

D1 : If $g \in C(Q)$, $\|g\|_\infty \le 1$, then $|D1(g)| \le |D11(g)| + |D12(g)|$ where

$$\begin{aligned} D11[\cdot] &= \int_Q \gamma(q)[\cdot]\beta_N(\rho_A - \rho_A^{\dagger})(dq) + \int_Q \gamma(q)[\cdot](\beta_N - \beta_N^{\dagger})\rho_A^{\dagger}(dq) \\ D12[\cdot] &= < \int [\mu_N + f_N](\rho_L - \rho_L^{\dagger})(dq), \cdot > \\ &\quad + < \int [(\mu_N + f_N) - (\mu_N^{\dagger} + f_N^{\dagger})](\rho_L^{\dagger})(dq), \cdot > . \end{aligned} \tag{15}$$

$|D11(g)| \le |\int_Q \gamma(q)[g]\beta_N(\rho_A - \rho_A^{\dagger})(dq)| + |\int_Q \gamma(q)[g](\beta_N - \beta_N^{\dagger})\rho_A^{\dagger}(dq)|$

$= \|\gamma(\cdot)\beta_N(\cdot,0,0)\|_\infty$[1] $|\int_Q \frac{\gamma(q)[g]\beta_N}{\|\gamma(\cdot)\beta_N(\cdot,0,0)\|_\infty}(\rho_A - \rho_A^{\dagger})(dq)| + |\int_Q \big(\gamma(q)[g](\beta_N - \beta_N^{\dagger})\big)$ $\rho_A^{\dagger}(dq)| \le \|\gamma(\cdot)\beta_N(\cdot,0,0)\|_\infty \|\rho_A - \rho_A^{\dagger}\|_V + \int_Q |\beta_N - \beta_N^{\dagger}||\rho_A^{\dagger}|(dq)$ [2]

$\le \|\gamma(\cdot)\beta_N(\cdot,0,0)\|_\infty \|\rho_A - \rho_A^{\dagger}\|_V + \Big(\|\beta_N\|_{BL}\|\rho_L - \rho_L^{\dagger}\|_{BL}^* + \|\beta_N\|_{BL}\|\rho_A - \rho_A^{\dagger}\|_{BL}^*\Big)$ $\|\rho_A^{\dagger}\|_V$

$\le \|\rho_A^{\dagger}\|_V\|\beta_N\|_{BL}\|\rho_L - \rho_L^{\dagger}\|_{BL}^* + \Big(\|\gamma(\cdot)\beta_N(\cdot,0,0)\|_\infty + \|\rho_A^{\dagger}\|_V\|\beta_N\|_{BL}\Big)\|\rho_A - \rho_A^{\dagger}\|_V.$

$$\begin{aligned} D12[\cdot] &= < \int [\mu_N + f_N](\rho_L - \rho_L^{\dagger})(dq), \cdot > \\ &\quad + < \int [(\mu_N + f_N) - (\mu_N^{\dagger} + f_N^{\dagger})](\rho_L^{\dagger})(dq), \cdot > . \end{aligned} \tag{16}$$

$|D12(g)| \le |\int_Q [\mu_N + f_N] g(q)(\rho_L - \rho_L^{\dagger})(dq)| + |\int_Q [(\mu_N + f_N) - (\mu_N^{\dagger} + f_N^{\dagger})] g(q)\rho_L^{\dagger}(dq)|$ $\le \|\mu_N + f_N\|_\infty |\int_Q \frac{[\mu_N + f_N]}{\|\mu_N + f_N\|_\infty} g(q)(\rho_L - \rho_L^{\dagger})(dq)| + \Big([\|\mu_N\|_{BL} + \|f_N\|_{BL}]\|\rho_L - \rho_L^{\dagger}\|_{BL}^* + [\|\mu_N\|_{BL} + \|f_N\|_{BL}]\|\rho_A - \rho_A^{\dagger}\|_{BL}^*\Big)\int_Q |g(q)||\rho_L^{\dagger}|(dq)$

$\le \Big(\|\mu_N + f_N\|_\infty + \|\rho_L^{\dagger}\|_V(\|\mu_N\|_{BL} + \|f_N\|_{BL})\Big)\big\|\rho_L - \rho_L^{\dagger}\big\|_V + \|\rho_L^{\dagger}\|_V(\|\mu_N\|_{BL} + \|f_N\|_{BL})\|\rho_A - \rho_A^{\dagger}\|_{BL}^*.$

Hence $\|D1\|_V \le \Big(\|\rho_A^{\dagger}\|_V\|\beta_N\|_{BL} + \|\mu_N + f_N\|_\infty + \|\rho_L^{\dagger}\|_V(\|\mu_N\|_{BL} + \|f_N\|_{BL})\Big)\|\rho_L - \rho_L^{\dagger}\|_V + \Big(\|\gamma(\cdot)\beta_N(\cdot,0,0)\|_\infty + \|\rho_A^{\dagger}\|_V\|\beta_N\|_{BL} + \|\rho_L^{\dagger}\|_V(\|\mu_N\|_{BL} + \|f_N\|_{BL})\Big)\|\rho_A - \rho_A^{\dagger}\|_V.$

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[1] Here $\|\gamma(\cdot)\beta_N(\cdot,0,0)\|_\infty = \sup_{q\in Q}\|\beta_N(q,0,0)\gamma(q)\|_V = \sup_{g\in C(Q),\|g\|_\infty\le 1, q\in Q}|\gamma(q)[g]\beta_N(q,0,0)|.$

[2] Here $|\rho_A^{\dagger}|$ means the absolute value of the measure $\rho_A^{\dagger}$.

D2: $|D2(g)| \;\le |D21(g)| + |D22(g)|$ where

$$\begin{aligned} D21 &= \int_Q f_N g(q)\rho_L(dq) - \int_Q f_N^\dagger g(q)\rho_L^\dagger(dq) \\ D22 &= \int_Q \alpha_N g(q)\rho_A(dq) - \int_Q \alpha_N^\dagger g(q)\rho_A^\dagger(dq). \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} &|D21(g)| + |D22(g)| \\ \le &\|f_N\|_\infty \Big| \int_Q \frac{f_N}{\|f_N\|_\infty} g(q)(\rho_L - \rho_L^\dagger)(dq)\Big| + \Big| \int_Q [f_N - f_N^\dagger] g(q)\rho_L^\dagger(dq)\Big| \\ &+ \|\alpha_N\|_\infty \Big| \int_Q \frac{\alpha_N}{\|\alpha_N\|_\infty} g(q)(\rho_A - \rho_A^\dagger)(dq)\Big| + \Big| \int_Q [\alpha_N - \alpha_N^\dagger] g(q)\rho_A^\dagger(dq)\Big| \\ \le &\Big(\|f_N\|_\infty + \|\rho_L^\dagger\|_V \|f_N\|_{BL} + \|\rho_A^\dagger\|_V\|\alpha_N\|_{BL}\Big)\|\rho_L - \rho_L^\dagger\|_V \\ &+ \Big(\|\alpha_N\|_\infty + \|\rho_L^\dagger\|_V \|f_N\|_{BL} + \|\rho_A^\dagger\|_V\|\alpha_N\|_{BL}\Big)\|\rho_A - \rho_A^\dagger\|_V. \end{aligned}$$

So if

$$\begin{aligned} B_L(\rho^\dagger) = &\Big(\|\rho_A^\dagger\|_V\|\beta_N\|_{BL} + \|\mu_N + f_N\|_\infty + \|\rho_L^\dagger\|_V(\|\mu_N\|_{BL} + \|f_N\|_{BL})\Big) \\ &+ \Big(\|f_N\|_\infty + \|\rho_L^\dagger\|_V\|f_N\|_{BL} + \|\rho_A^\dagger\|_V\|\alpha_N\|_{BL}\Big), \end{aligned}$$

$$\begin{aligned} B_A(\rho^\dagger) = &\Big(\|\gamma\beta_N(\cdot,0,0)\|_V + \|\rho_A^\dagger\|_V\|\beta_N\|_{BL} + \|\rho_L^\dagger\|_V(\|\mu_N\|_{BL} + \|f_N\|_{BL})\Big) \\ &+ \Big(\|\alpha_N\|_\infty + \|\rho_L^\dagger\|_V\|f_N\|_{BL} + \|\rho_A^\dagger\|_V\|\alpha_N\|_{BL}\Big), \end{aligned}$$

then

$$\|F_N(\rho,\gamma) - F_N(\rho^\dagger,\gamma)\|_{V^2} \le B_L(\rho^\dagger)\|\rho_L - \rho_L^\dagger\|_V + B_A(\rho^\dagger)\|\rho_A - \rho_A^\dagger\|_V. \tag{18}$$

So if C_W is such that $\|\mu\|_V \le C_W$ for $\mu \in W_V$ our result is immediate. □

If $a, b > 0$, $\alpha \in M(a, b, m_0)$, $g \in C(Q)$, then we define

$$\overline{\Delta}_{N,s,t,\alpha(\cdot;u,\gamma)}(\hat{q})[g] = \int_Q e^{-\int_s^t [\mu_N + f_N](q,\alpha(\tau)(Q))d\tau} g(q)\gamma(\hat{q})(dq).$$

Lemma 4.3. *(Estimates) If $\zeta, \xi \in M(\frac{N}{3}, b, \mathbf{0})$, $t_1, t_2, t, s \in [-b, b]$, let $C_1 = \frac{2N}{3}$. If $g \in C(Q)$, $\|g\|_\infty \le 1$ then we have the following estimates:*

1. (a) $\left|\int_{[t_1,t_2]\times Q} \beta_N(\hat{q}, \zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\zeta_A(s)(d\hat{q})ds\right| \le 2bC_1\|\beta_N(\cdot, 0, 0)\|_\infty$
   (b) $\left|\int_{t_1}^{t_2}\int_Q e^{-\int_{t_1}^{t_2}\alpha_N(q,\zeta(\tau)(Q))d\tau} f_N(q, \zeta(s)(Q))g(q)\zeta_L(s)(dq)ds\right| \le 2bC_1\|f_N(\cdot, 0, 0)\|_\infty$
2. (a) $\left|e^{-\int_s^t[\mu_N+f_N](q,\zeta(\tau)(Q))d\tau} - e^{-\int_s^t[\mu_N+f_N](q,\xi(\tau)(Q))d\tau}\right| \le (\|\mu_N\|_{BL} + \|f_N\|_{BL}) \int_s^t \|\zeta(\tau) - \xi(\tau)\|_{BL^2}^* d\tau$
   (b) $\left|e^{-\int_s^t \alpha_N(q,\zeta(\tau)(Q))d\tau} - e^{-\int_s^t \alpha_N(q,\xi(\tau)(Q))d\tau}\right| \le (\|\alpha_N\|_{BL}) \int_s^t \|\zeta(\tau) - \xi(\tau)\|_{BL^2}^* d\tau$

(c) $\left|e^{-\int_s^{t_1}[\mu_N+f_N](q,\zeta(\tau;u_1,\gamma_1)(Q))d\tau} - e^{-\int_s^{t}[\mu_N+f_N](q,\zeta(\tau;u,\gamma)(Q))d\tau}\right| \le (\|\mu_N\|_{BL} + \|f_N\|_{BL}) \int_s^t \|\zeta(\tau;u_1,\gamma_1) - \zeta(\tau;u,\gamma)\|^*_{BL^2} d\tau + (\|\mu_N\|_{BL} + \|f_N\|_{BL})|t_1 - t|$

3. (a) $\left|\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] - \overline{\Delta}_{N,s,t,\xi(\cdot;u,\gamma)}(\hat{q})[g]\right| \le (\|\mu_N\|_{BL} + \|f_N\|_{BL}) b \|\zeta - \xi\|_S$ [3]

(b) $\left|\overline{\Delta}_{N,s,t_1,\zeta(\cdot;u_1,\gamma_1)}(\hat{q})[g] - \overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\right| \le (\|\mu_N\|_{BL} + \|f_N\|_{BL}) \int_s^t \|\zeta(\tau;u_1,\gamma_1) - \zeta(\tau;u,\gamma)\|^*_{BL^2} d\tau + (\|\mu_N\|_{BL} + \|f_N\|_{BL})|t_1 - t| + |\int_Q e^{-\int_s^t[\mu_N+f_N](q,\zeta(\tau;u,\gamma)(Q))d\tau} g(q)[\gamma_1(\hat{q}) - \gamma(\hat{q})](dq)|$

4. (a) $\left|\beta_N(\hat{q},\zeta(s;u,\gamma)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] - \beta_N(\hat{q},\xi(s;u,\gamma)(Q))\overline{\Delta}_{N,s,t,\xi(\cdot;u,\gamma)}(\hat{q})[g]\right| \le \|\beta_N\|_{BL}\|\zeta(s;u,\gamma) - \xi(s;u,\gamma)\|^*_{BL^2} + b\|\beta_N\|_\infty(\|\mu_N\|_{BL} + \|f_N\|_{BL})\|\zeta - \xi\|_S$

(b) $\left|f_N(\hat{q},\zeta(s)(Q))e^{-\int_s^t \alpha_N(q,\zeta(\tau;u,\gamma)(Q))d\tau} - f_N(\hat{q},\xi(s)(Q))e^{-\int_s^t \alpha_N(q,\xi(\tau;u,\gamma)(Q))d\tau}\right| \le \|f_N\|_{BL}\|\zeta(s;u,\gamma) - \xi(s;u,\gamma)\|^*_{BL^2} + b\|f_N\|_\infty(\|\alpha_N\|_{BL})\|\zeta - \xi\|_S$

Proof. 1. Since

$$\left|\beta_N(\hat{q},\zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\right| = |\beta_N(\hat{q},\zeta(s)(Q)) \int_Q g(q) e^{-\int_s^t[\mu_N+f_N](q,\zeta(\tau)(Q))d\tau} \gamma(\hat{q})(dq)| \le \|\beta_N(\cdot,0,0)\|_\infty,$$

$$\left|\int_{[t_1,t_2]\times Q} \beta_N(\hat{q},\zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\zeta_A(s)(d\hat{q})ds\right| \le \int_{[t_1,t_2]\times Q} \left|\beta_N(\hat{q},\zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\right| |\zeta_A(s)|(d\hat{q})ds \le 2bC_1\|\beta_N(\cdot,0,0)\|_\infty.$$

A similar argument holds for the second inequality.

2. (a) Using the mean value theorem on the $C^\infty(\mathbb{R})$ function, e^x, there exists $\theta = \theta(s,t) > 0$, such that

$$\begin{aligned}
&\left|e^{-\int_s^t[\mu_N+f_N](q,\zeta(\tau)(Q))d\tau} - e^{-\int_s^t[\mu_N+f_N](q,\xi(\tau)(Q))d\tau}\right| \\
= \; & e^{-\theta}\left|\int_s^t\Big[[\mu_N + f_N](q,\zeta(\tau)(Q)) - [\mu_N + f_N](q,\xi(\tau)(Q))\Big]d\tau\right| \\
\le \; & (\|\mu_N\|_{BL} + \|f_N\|_{BL})\Big[\int_s^t \|\zeta_L(\tau) - \xi_L(\tau)\|^*_{BL}d\tau + \int_s^t \|\zeta_A(\tau) - \xi_A(\tau)\|^*_{BL}d\tau\Big] \\
= \; & (\|\mu_N\|_{BL} + \|f_N\|_{BL}) \int_s^t \|\zeta(\tau) - \xi(\tau)\|^*_{BL^2}d\tau.
\end{aligned}$$

(b) This argument is similar to (a) above.

(c) Using the mean value theorem on the $C^\infty(\mathbb{R})$ function, e^x, there exists $\theta = \theta(s,t_1,t) > 0$, such that

$$\begin{aligned}
&\left|e^{-\int_s^{t_1}[\mu_N+f_N](q,\zeta(\tau;u_1,\gamma_1)(Q))d\tau} - e^{-\int_s^t[\mu_N+f_N](q,\zeta(\tau;u,\gamma)(Q))d\tau}\right| \\
\le \; & e^{-\theta}\left|\int_s^t\Big[[\mu_N + f_N](q,\zeta(\tau;u_1,\gamma_1)(Q)) - [\mu_N + f_N](q,\zeta(\tau;u,\gamma)(Q))\Big]d\tau\right| \\
& + e^{-\theta}\left|\int_t^{t_1}[\mu_N + f_N](q,\zeta(\tau;u_1,\gamma_1)(Q))d\tau\right| \\
\le \; & \int_s^t(\|\mu_N\|_{BL} + \|f_N\|_{BL})|\zeta_L(\tau;u_1,\gamma_1)(Q) - \zeta_L(\tau;u,\gamma)(Q)|d\tau \\
& + \int_s^t(\|\mu_N\|_{BL} + \|f_N\|_{BL})|\zeta_A(\tau;u_1,\gamma_1)(Q) - \zeta_A(\tau;u,\gamma)(Q)|d\tau \\
& + (\|\mu_N\|_{BL} + \|f_N\|_{BL})|t_1 - t|.
\end{aligned}$$

[3] Here $\|\cdot\|_S$ is taken with respect to total variation norm, see (2),(20).

3. (a)

$$\left|\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] - \overline{\Delta}_{N,s,t,\xi(\cdot;u,\gamma)}(\hat{q})[g]\right| = \Big|\int_Q \big[e^{-\int_s^t [\mu_N+f_N](q,\zeta(\tau)(Q))d\tau} - e^{-\int_s^t [\mu_N+f_N](q,\xi(\tau)(Q))d\tau}\big] g(q)\gamma(\hat{q})(dq)\Big|$$

Since $\|g\|_\infty \leq 1$ and $\gamma(\hat{q})$ is a probability measure our results are immediate from $2(a)$.

(b) $\left|\overline{\Delta}_{N,s,t_1,\zeta(\cdot;u_1,\gamma_1)}(\hat{q})[g] - \overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\right| =$
$\Big| \int_Q e^{-\int_s^{t_1}[\mu_N+f_N](q,\zeta(\tau;u_1,\gamma_1)(Q))d\tau} g(q)\gamma_1(\hat{q})(dq)$
$- \int_Q e^{-\int_s^{t}[\mu_N+f_N](q,\zeta(\tau;u,\gamma)(Q))d\tau} g(q)\gamma(\hat{q})(dq)\Big| \leq$
$\int_Q \Big| e^{-\int_s^{t_1}[\mu_N+f_N](q,\zeta(\tau;u_1,\gamma_1)(Q))d\tau} - e^{-\int_s^{t}[\mu_N+f_N](q,\zeta(\tau;u,\gamma)(Q))d\tau}$
$\Big| |g(q)|\gamma_1(\hat{q})(dq)$
$+ \Big| \int_Q e^{-\int_s^{t}[\mu_N+f_N](q,\zeta(\tau;u,\gamma)(Q))d\tau} g(q)(\gamma_1(\hat{q}) - \gamma(\hat{q}))(dq)\Big|$

Since $\|g\|_\infty \leq 1$ and $\gamma_1(\hat{q})$ is a probability measure using $2(c)$ our results are immediate.

4. (a)

$$\begin{aligned}
&\left|\beta_N(\hat{q},\zeta(s;u,\gamma)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] - \beta_N(\hat{q},\xi(s;u,\gamma)(Q))\overline{\Delta}_{N,s,t,\xi(\cdot;u,\gamma)}(\hat{q})[g]\right| \\
&\quad \leq |\beta_N(\hat{q},\zeta(s;u,\gamma)(Q)) - \beta_N(\hat{q},\xi(s;u,\gamma)(Q))|\left|\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\right| \\
&\quad + \beta_N(\hat{q},\xi(s;u,\gamma)(Q))\left|\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] - \overline{\Delta}_{N,s,t,\xi(\cdot;u,\gamma)}(\hat{q})[g]\right| \\
&\leq \|\beta_N\|_{BL}\|\zeta_L(s) - \xi_L(s)\|^*_{BL} + \|\beta_N\|_{BL}\|\zeta_A(s) - \xi_A(s))\|^*_{BL} + \|\beta_N\|_\infty(\|\mu_N\|_{BL} + \\
&\quad \|f_N\|_{BL})\int_s^t \|\zeta(\tau) - \xi(\tau)\|_{BL^2} d\tau.
\end{aligned}$$

using $2(a)$ and $3(a)$ in the last estimate.

(b) $\left| f_N(\hat{q},\zeta(s)(Q))e^{-\int_s^t \alpha_N(q,\zeta(\tau;u,\gamma)(Q))d\tau} - f_N(\hat{q},\xi(s)(Q))e^{-\int_s^t \alpha_N(q,\xi(\tau;u,\gamma)(Q))d\tau}\right|$

$$\begin{aligned}
&\leq |f_N(\hat{q},\zeta(s)(Q)) - f_N(\hat{q},\xi(s)(Q))|e^{-\int_s^t \alpha_N(q,\zeta(\tau;u,\gamma)(Q))d\tau} + \\
&\quad |e^{-\int_s^t \alpha_N(q,\zeta(\tau;u,\gamma)(Q))d\tau} - e^{-\int_s^t \alpha_N(q,\xi(\tau;u,\gamma)(Q))d\tau}|f_N(\hat{q},\xi(s)(Q))
\end{aligned}$$

$\leq \|f_N\|_{BL}\|\zeta(s) - \xi(s)\|^*_{BL^2} + \|f_N\|_\infty(\|\alpha_N\|_{BL})\int_s^t \|\zeta(\tau) - \xi(\tau)\|^*_{BL^2} d\tau.$
Using $2(b)$ for the last estimate.

□

For the convenience of the reader we mention again a few important notions from section 2.

If $\zeta \in M(a,b,m_0)$, then for $g \in C(Q)$,

$$\overline{\Delta}_{s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] = \int_Q e^{-\int_s^t [\mu+f](q,\zeta(\tau)(Q))d\tau} g(q)\gamma(\hat{q})(dq).$$

and

$$\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] = \int_Q e^{-\int_s^t [\mu_N+f_N](q,\zeta(\tau)(Q))d\tau} g(q)\gamma(\hat{q})(dq).$$

$(M(a,b,m_0), \|\cdot\|_S)$ is defined as

$$M(a,b,m_0) = \Big\{\alpha \in \mathcal{BM}(\mathcal{X}) |\ \alpha \in \mathcal{C}\Big([-b,b] \times (\overline{B_a[m_{0L}]})_{+,w} \times (\overline{B_a[m_{0A}]})_{+,w} \times C(Q,\mathcal{P}_w); (\overline{B_{2a}[m_{0L}]})_w \times (\overline{B_{2a}[m_{0A}]})_w\Big), \alpha \geq 0, \alpha(0;u,\gamma) = u\Big\} \tag{19}$$

and

$$||\zeta||_S = \sup_{(t,u,\gamma)} \|\zeta(t;u,\gamma)\|_{V^2}. \tag{20}$$

Lemma 4.4. *(Fixed Point) Let C_1 be as in Lemma 4.3, $\boldsymbol{0} \times \boldsymbol{0} = \overrightarrow{\boldsymbol{0}}$. Pick b such that :*

- $2b\|\beta_N(\cdot,0,0\|_\infty C_1 < \frac{N}{3}$, $2b\|f_N(\cdot,0,0)\|_\infty C_1 < \frac{N}{3}$.
- $K_L = b\big((\|\mu_N\|_{BL}+\|f_N\|_{BL})C_1+\|\beta_N\|_\infty+C_1\|\beta_N\|_{BL}+C_1 b\|\beta_N\|_\infty(\|\mu_N\|_{BL}+\|f_N\|_{BL})\big) < \frac{1}{2}$ *and* $K_A = b\big(\|\alpha_N\|_{BL}C_1 + \|f_N\|_\infty + C_1(\|f_N\|_{BL} + \|f_N\|_\infty\|\alpha_N\|_{BL} b)\big) < \frac{1}{2}$.

Then

$$S\colon M\Big(\frac{N}{3}, b, \overrightarrow{\boldsymbol{0}}\Big) \to M\Big(\frac{N}{3}, b, \overrightarrow{\boldsymbol{0}}\Big)$$

given by

$$(S\zeta)(t;u,\gamma) = [(S\zeta)_L(t;u,\gamma), (S\zeta)_A(t;u,\gamma)]$$

where if $g \in C(Q)$,

$$(S\zeta)_L(t;u,\gamma)[g] = \int_Q e^{-\int_0^t [\mu_N+f_N](q,\zeta(\tau)(Q))d\tau} g(q)u_L(dq) + \int_0^t [\int_Q \overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g] \beta_N(\hat{q},\zeta(s)(Q))\zeta_A(s)(d\hat{q})]ds \tag{21}$$

$$(S\zeta)_A(t;u,\gamma)[g] = \int_Q e^{-\int_0^t \alpha_N(q,\zeta(\tau)(Q))d\tau} g(q)u_A(dq) + \int_0^t \int_Q e^{-\int_s^t \alpha_N(q,\zeta(\tau)(Q))d\tau} f_N(q,\zeta(s)(Q))g(q)\zeta_L(s)(dq)ds \tag{22}$$

has a unique fixed point.

Proof. Since $(t,u,\gamma) \mapsto [u_L,u_A] \in M\big(\frac{N}{3}, b, \overrightarrow{\mathbf{0}}\big)$, $M\big(\frac{N}{3}, b, \overrightarrow{\mathbf{0}}\big) \neq \emptyset$. If $\zeta \in M\big(\frac{N}{3}, b, \overrightarrow{\mathbf{0}}\big)$, then from the form of (21), (22) $[S\zeta](0;u,\gamma) = u$ and $[S\zeta]$ is nonnegative. Using Lemma 4.3 $1a, b$ we get our first two estimates in demonstrating that $S\zeta \in M\big(\frac{N}{3}, b, \overrightarrow{\mathbf{0}}\big)$. Indeed, we have

$$\begin{aligned}((S\zeta)_L)(t;u,\gamma) &= <\textstyle\int e^{-\int_0^t [\mu_N+f_N](q,\zeta(\tau)(Q))d\tau}(u_L)(dq), \cdot> \\ &+ \Big(\textstyle\int_0^t \int_Q \beta_N(\hat{q},\zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[\cdot]\zeta_A(s)(d\hat{q}) \times ds\Big)\end{aligned}$$

and

$$\begin{aligned}\|(S\zeta)_L(t;u,\gamma)\|_V &\leq \|u_L\|_V + 2b\|\beta_N(\cdot,0,0)\|_\infty C_1 \\ &\leq \tfrac{N}{3} + 2b\|\beta_N(\cdot,0,0)\|_\infty C_1 < \tfrac{2N}{3}.\end{aligned}$$

Likewise,

$$\begin{aligned}(S\zeta)_A(t;u,\gamma) &= <\textstyle\int e^{-\int_0^t \alpha_N(q,\zeta(\tau)(Q))d\tau}(u_A)(dq), \cdot> \\ &+ \textstyle\int_0^t <\int e^{-\int_0^t \alpha_N(q,\zeta(\tau)(Q))d\tau} f_N(q,\zeta(s)(Q))\zeta_L(s)(dq), \cdot> ds\end{aligned}$$

and

$$\begin{aligned}\|(S\zeta)_A(t;u,\gamma)\|_V &\le \|u_A\|_V + 2b\|f_N(\cdot,0,0)\|_\infty C_1\\ &\le \tfrac{N}{3} + 2b\|f_N(\cdot,0,0)\|_\infty C_1 < \tfrac{2N}{3}.\end{aligned}$$

We now show that $[S\zeta]$ is continuous. Since $[S\zeta] = [(S\zeta)_L, (S\zeta)_A]$, this is equivalent to showing that $(S\zeta)_L$ and $(S\zeta)_A$ are both continuous. We will show that $(S\zeta)_L$ is continuous, the case for $(S\zeta)_A$ is easier and similar. This means that if $v^n = (t^n; u^n, \gamma^n)$, $v = (t; u, \gamma) \in [-b, b] \times (\overline{B_{\frac{N}{3},+}[(\mathbf{0})_L]}_w \times (\overline{B_{\frac{N}{3},+}[(\mathbf{0})_A]}_w \times C(Q, \mathcal{P}_w)$ and $v^n \to v$ then

$$(S\zeta)_L(v^n) \to (S\zeta)_L(v)$$

in $(\overline{B_{\frac{2N}{3},+}[(\mathbf{0})_L]})_w$. Below

$$\zeta^n(\cdot) = [\zeta_L(\cdot; u^n, \gamma^n), \zeta_A(\cdot; u^n, \gamma^n)], \zeta(\cdot) = [\zeta_L(\cdot; u, \gamma), \zeta_A(\cdot; u, \gamma)]. \tag{23}$$

Let

$Ia = < \int e^{-\int_0^t [\mu_N + f_N](q, \zeta(\tau)(Q))d\tau}((u^n)_L - u_L)(dq), \cdot >$,

$Ib = < \int [e^{-\int_0^{t^n} [\mu_N + f_N](q, \zeta^n(\tau)(Q))d\tau} - e^{-\int_0^t [\mu_N + f_N](q, \zeta(\tau)(Q))d\tau}](u^n)_L(dq), \cdot >$,

$IIa = \left(\int_t^{t^n} \int_Q \beta_N(\hat q, \zeta^n(s)(Q)) \overline{\Delta}_{N,s,t^n,\zeta(\cdot;u^n,\gamma^n)}(\hat q)[\cdot]\zeta_A(s; u^n, \gamma^n)(d\hat q)ds\right)$,

$IIb1 = \Big(\int_0^t \int_Q \Big(\beta_N(\hat q, \zeta^n(s)(Q)) - \beta_N(\hat q, \zeta(s)(Q))\Big)\overline{\Delta}_{N,s,t^n,\zeta(\cdot;u^n,\gamma^n)}(\hat q)[\cdot]$
$\zeta_A(s; u^n, \gamma^n)(d\hat q)ds\Big)$,

$IIb2 = \Big(\int_0^t \int_Q \beta_N(\hat q, \zeta(s)(Q))(\overline{\Delta}_{N,s,t^n,\zeta(\cdot;u^n,\gamma^n)}(\hat q) - \overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat q))[\cdot]$
$\zeta_A(s; u^n, \gamma^n)(d\hat q)ds\Big)$, and

$IIb3 = \Big(\int_0^t \int_Q \beta_N(\hat q, \zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat q)[\cdot](\zeta_A(s; u^n, \gamma^n) -$
$\zeta_A(s; u, \gamma))(d\hat q)ds\Big)$.

Then,

$$(S\zeta)_L(v^n) - (S\zeta)_L(v) = Ia + Ib + IIa + IIb1 + IIb2 + IIb3.$$

We remind the reader that the *weak** topology is generated by the family of seminorms $\rho_g(\mu) = |\int_Q g d\mu|$, where $g \in C(Q)$. So if ρ_g is a seminorm, we need to show that $\rho_g\big((S\zeta)_L(v^n) - (S\zeta)_L(v)\big) \to 0$, as $n \to \infty$. To this end, we provide an estimate for each of the terms above.

1. $\rho_g(Ia) = |Ia[g]| \to 0$ since $e^{-\int_0^t [\mu_N + f_N](q, \zeta(\tau))d\tau}$ is continuous in q and by hypothesis $(u^n)_L \to u_L$ in $\mathcal{M}_w$.
2. $\rho_g(Ib) = |Ib[g]| \to 0$ since by Lemma 4.3 2 (c) $|Ib[g]| \le C_1(\|\mu_N\|_{BL} + \|f_N\|_{BL}) \int_s^t \|\zeta(\tau; u^n, \gamma^n) - \zeta(\tau; u, \gamma)\|^*_{BL^2} d\tau + (\|\mu_N\|_{BL} + \|f_N\|_{BL})|t^n - t|$. By hypothesis $h^n(\tau) = \|\zeta(\tau; u^n, \gamma^n) - \zeta(\tau; u, \gamma)\|^*_{BL^2} \to 0$ and $|t^n - t| \to 0$, hence by dominated convergence our result is immediate.
3. $\rho_g(IIa) = |IIa[g]| \to 0$ since by hypothesis $t^n \to t$, and
$$h^n(s) = \int_Q \beta_N(\hat q, \zeta^n(s))\overline{\Delta}_{N,s,t^n,\zeta(\cdot;u^n,\gamma^n)}(\hat q)[g]\zeta_A(s; u^n, \gamma^n)(d\hat q)$$
is uniformly bounded.
4. $\rho_g(IIb1) = |IIb1[g]| \to 0$ since $|IIb1[g]| \le \|\beta_N\|_{BL} C_1 b \|\zeta^n(s) - \zeta(s)\|^*_{BL^2}$.

5. $\rho_g(IIb2) = |IIb2[g]| \to 0$ Indeed, let

$$h^n(s) = \int_Q \beta_N(\hat{q}, \zeta(s))(\overline{\Delta}_{N,s,t^n,\zeta(\cdot;u^n,\gamma^n)}(\hat{q}) - \overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q}))[g]\zeta_A(s;u^n,\gamma^n)(d\hat{q}).$$

Now $IIb2[g] = \int_0^t h^n(s)ds$ and

$$|h^n(s)| \leq \int_Q |\beta(\hat{q}, \zeta(s))||(\overline{\Delta}_{N,s,t^n,\zeta(\cdot;u^n,\gamma^n)}(\hat{q}) - \overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q}))[g]|\zeta_A(s;u^n,\gamma^n)(d\hat{q})$$

$\leq C_1\|\beta_N\|_\infty(\|\mu_N\|_{BL} + \|f_N\|_{BL}) \int_s^t \|\zeta(\tau;u^n,\gamma^n) - \zeta(\tau;u,\gamma)\|^*_{BL^2} d\tau$
$+ C_1\|\beta_N\|_{BL}(\|\mu_N\|_{BL} + \|f_N\|_{BL})|t^n - t|$
$+ C_1\|\beta_N\|_{BL}| \int_Q e^{-\int_s^t [\mu_N + f_N](q,\zeta(\tau;u,\gamma))d\tau} g(q)[\gamma^n(\hat{q}) - \gamma(\hat{q})](dq)|$ using Lemma 4.3 3(b).

Using the fact that ζ is continuous along with Dominated Convergence, and the fact that $e^{-\int_s^t [\mu_N + f_N](q,\zeta(\tau;u,\gamma))d\tau} g(q)$ is a continuous function of q we see that $h^n \to 0$.

6. $\rho_g(IIb3) = |IIb3[g]| \to 0$. Indeed,

$|IIb3[g]| =$

$$|\int_0^t \int_Q \beta_N(\hat{q}, \zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g][\zeta_A(s;u^n,\gamma^n) - \zeta_A(s;u,\gamma)](d\hat{q})ds|$$

For fixed s, $\beta_N(\hat{q}, \zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]$ is continuous in $\hat{q}$. Hence if

$$h^n(s) = \int_Q \beta_N(\hat{q}, \zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q})[g]\big(\zeta_A(s;u^n,\gamma^n) - \zeta_A(s;u,\gamma)\big)(d\hat{q}),$$

then by hypothesis $h^n \to 0$ pointwise. Hence our result follows by Dominated Convergence.

Now for the contraction we have the following. We need to find a constant $0 < K < 1$ such that if $\zeta, \xi \in M(\frac{N}{3}, b, \overrightarrow{\mathbf{0}})$, then $||S\zeta - S\xi||_S \leq K||\zeta - \xi||_S$. Now

$$\begin{aligned} ||S\zeta - S\xi||_S &= ||[(S\zeta)_L - (S\xi)_L, (S\zeta)_A - (S\xi)_A]||_S \\ &= \sup_{(t;u,\gamma)} \|(S\zeta)_L(t;u,\gamma) - (S\xi)_L(t;u,\gamma)\|_V + \\ &\quad \sup_{(t;u,\gamma)} \|(S\zeta)_A(t;u,\gamma) - (S\xi)_A(t;u,\gamma)\|_V \\ &= ||(S\zeta)_L - (S\xi)_L||_S + ||(S\zeta)_A - (S\xi)_A||_S \end{aligned}$$

If

$$\begin{aligned} I &= < \int [e^{-\int_0^t [\mu_N + f_N](q,\zeta(\tau)(Q))d\tau} - e^{-\int_0^t [\mu_N + f_N](q,\xi(\tau)(Q))d\tau}] du_L, \cdot > \\ II &= \int_0^t \int_Q \beta_N(\hat{q}, \xi(s)(Q))\overline{\Delta}_{N,s,t,\xi(\cdot;u,\gamma)}(\hat{q})[\cdot](\zeta_A - \xi_A)(s)(d\hat{q})ds, \\ III &= \int_0^t \int_Q \Big\{\beta_N(\hat{q}, \zeta(s)(Q))\overline{\Delta}_{N,s,t,\zeta(\cdot;u,\gamma)}(\hat{q}) - \beta_N(\hat{q}, \xi(s)(Q)) \\ &\quad \overline{\Delta}_{N,s,t,\xi(\cdot;u,\gamma)}(\hat{q})\Big\}[\cdot]\zeta_A(s)(d\hat{q})ds, \end{aligned}$$

then

$$(S\zeta)_L - (S\xi)_L = I + II + III,$$

and

$$|\big((S\zeta)_L - (S\xi)_L\big)[g]| \leq |I[g]| + |II[g]| + |III[g]|.$$

Now

1. $|I[g]| \leq \int_Q |e^{-\int_0^t [\mu_N+f_N](q,\zeta(\tau;u,\gamma)(Q))d\tau} - e^{-\int_0^t [\mu_N+f_N](q,\xi(\tau;u,\gamma)(Q))d\tau}|du_L \leq (\|\mu_N\|_{BL} + \|f_N\|_{BL})bC_1||\zeta - \xi||_S$ using Lemma 4.3 2(a).
2. $|II[g]| \leq \|\beta_N\|_\infty b||\zeta - \xi||_S$
3. $|III[g]| \leq bC_1(\|\beta_N\|_{BL} + b\|\beta_N\|_\infty(\|\mu_N\|_{BL} + \|f_N\|_{BL}))||\zeta - \xi||_S$ using Lemma 4.3 4(a).

Hence if $K_L = b\big((\|\mu_N\|_{BL}+\|f_N\|_{BL})C_1+\|\beta_N\|_\infty+C_1\|\beta_N\|_{BL}+C_1 b\|\beta_N\|_\infty(\|\mu_N\|_{BL} + \|f_N\|_{BL})\big)$, then

$$||(S\zeta)_L - (S\xi)_L||_S \leq K_L||\zeta - \xi||_S.$$

Likewise, for $(S\zeta)_A - (S\xi)_A$ if

$$\begin{aligned}
I &= <\int [e^{-\int_0^t \alpha_N(q,\zeta(\tau;u,\gamma)(Q))d\tau} - e^{-\int_0^t \alpha_N(q,\xi(\tau;u,\gamma)(Q))d\tau}]du_A, \cdot> \\
II &= \int_0^t <\int f_N(\hat{q},\xi(s)(Q))e^{-\int_s^t \alpha_N(q,\xi(\tau;u,\gamma)(Q))d\tau}(\zeta_L - \xi_L)(s)(d\hat{q}), \cdot> ds, \\
III &= \int_0^t <\int \Big\{ f_N(\hat{q},\zeta(s)(Q))e^{-\int_s^t \alpha_N(q,\zeta(\tau;u,\gamma)(Q))d\tau} - \\
&\qquad f_N(\hat{q},\xi(s)(Q))e^{-\int_s^t \alpha_N(q,\xi(\tau;u,\gamma)(Q))d\tau}\Big\} \\
&\quad \zeta_L(s)(d\hat{q}), \cdot> ds,
\end{aligned}$$

then

$$(S\zeta)_A - (S\xi)_A = I + II + III,$$

and

1. $|I[g]| \leq \int_Q |e^{-\int_0^t \alpha_N(q,\zeta(\tau;u,\gamma)(Q))d\tau} - e^{-\int_0^t \alpha_N(q,\xi(\tau;u,\gamma)(Q))d\tau}|u_L(dq) \leq \|\alpha_N\|_{BL} bC_1||\zeta - \xi||_S$ using Lemma 4.3 2(b).
2. $|II[g]| \leq \|f_N\|_\infty b||\zeta - \xi||_S$.
3. $|III[g]| \leq bC_1\Big(\|f_N\|_{BL} + \|f_N\|_\infty\|\alpha_N\|_{BL} b\Big)||\zeta - \xi||_S$ using Lemma 4.3 4(b).

Hence if $K_A = b\big(\|\alpha_N\|_{BL}C_1 + \|f_N\|_\infty + C_1(\|f_N\|_{BL} + \|f_N\|_\infty\|\alpha_N\|_{BL} b)\big)$ then

$$||(S\zeta)_A - (S\xi)_A||_S \leq K_A||\zeta - \xi||_S.$$

Hence, S is a contraction mapping. Therefore, S has a unique fixed point in $M(\frac{N}{3}, b, \overrightarrow{\mathbf{0}})$. □

We will denote this fixed point by φ_N.

Proposition 1. *(Local Solution) For every $N \in \mathbb{N}$, there is $b = b(N) > 0$ and $\varphi_N : [-b,b] \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times C(Q;\mathcal{P}_w) \to (\overline{B_{\frac{2N}{3}}[\mathbf{0}]})_{+,w} \times (\overline{B_{\frac{2N}{3}}[\mathbf{0}]})_{+,w}$ such that*

1. *φ_N is nonnegative and continuous.*
2. *The function φ_N satisfies*

$$\begin{aligned}
\varphi_{NL}(t;u,\gamma) =& <\int e^{-\int_0^t [\mu_N+f_N](q,\varphi_N(\tau)(Q))d\tau} u_L(dq), \cdot> + \\
&\int_0^t \int_Q \overline{\Delta}_{N,s,t,\varphi_N(\cdot;u,\gamma)}(\hat{q})[\cdot]\beta_N(\hat{q},\varphi_N(s)(Q))\varphi_{NA}(s)(d\hat{q})ds \\
\varphi_{NA}(t;u,\gamma) =& <\int e^{-\int_0^t \alpha_N(q,\varphi_N(\tau)(Q))d\tau} u_A(dq), \cdot> + \\
&\int_0^t <\int e^{-\int_s^t \alpha_N(q,\varphi_N(\tau)(Q))d\tau} f_N(\hat{q},\varphi_N(s)(Q)) \\
&\varphi_{NL}(s)(d\hat{q}), \cdot> ds
\end{aligned} \tag{24}$$

3. It is a local solution to (9).

Proof. 1. This follows directly from Lemma 4.4.

2. This follows directly from Lemma 4.4.
3. We differentiate the integral representation (24) and show that it satisfies (9). By Lemma 4.2, F_N, is bounded and Lipschitz, hence the solution to (9) is unique. So fix u, γ.

   If $\varphi_{NL}(t) = \mu_{1L}(t) + \mu_{2L}(t)$, then $\varphi'_{NL}(t) = \mu'_{1L}(t) + \mu'_{2L}(t)$, where

$$\mu_{1L}(t)[g] = \int_Q e^{-\int_0^t [\mu_N + f_N](q,\varphi_N(\tau;u,\gamma)(Q))d\tau} g(q) u_L(dq) \tag{25}$$

$$\mu_{2L}(t)[g] = \int_0^t [\int_Q \overline{\Delta}_{N,s,t,\varphi_N(\cdot,u,\gamma)}(\hat{q})[g] \beta_N(\hat{q}, \varphi_N(s)(Q)) \varphi_{NA}(s)(d\hat{q})] ds \tag{26}$$

   Clearly

$$\mu'_{1L}(t)[g] = \int -[\mu_N + f_N](q, \varphi_N(t)(Q)) g(q) \mu_{1L}(t)(dq)$$
$$= \mu_{1L}(t)\big[-[\mu_N + f_N](\cdot, \varphi_N(t)(Q)) g(\cdot)\big]$$

   Since

$$\mu_{2L}(t)[g] = \int_0^t \left[\int_Q \beta_N(\hat{q}, \varphi_N(s)(Q)) \int_Q g(q) e^{-\int_s^t [\mu_N + f_N](q,\varphi_N(\tau)(Q))d\tau} \gamma(\hat{q})(dq) \varphi_{NA}(s)(d\hat{q})\right] ds,$$

   then using the Liebnitz rule for differentiation of the integral

$$\begin{aligned}
\mu'_{2L}(t)[g] = &\int_0^t \left[\int_Q \beta_N(\hat{q}, \varphi_N(s)(Q)) \int_Q g(q)(-[\mu_N + f_N](q, \varphi_N(t)(Q))\right.\\
&\left. e^{-\int_s^t [\mu_N + f_N](q,\varphi_N(\tau)(Q))d\tau} \gamma(\hat{q})(dq) \varphi_{NA}(s)(d\hat{q})\right] ds \; +\\
&\int_Q \beta_N(\hat{q}, \varphi_N(t)(Q)) \int_Q g(q) \gamma(\hat{q})(dq) \varphi_{NA}(t)(d\hat{q})\\
=&\mu_{2L}(t)\big[-[\mu_N + f_N](\cdot, \varphi_N(t)(Q)) g(\cdot)\big] + \int_Q \beta_N(\hat{q}, \varphi_N(t)(Q)) \gamma(\hat{q})[g]\\
&\varphi_{NA}(t)(d\hat{q})
\end{aligned} \tag{27}$$

   Hence,

$$\varphi'_{NL}(t)[g] = \int_Q \beta_N(\hat{q}, \varphi_N(t)(Q)) \gamma(\hat{q})[g] \varphi_{NA}(t)(d\hat{q}) -$$
$$\int [\mu_N + f_N](q, \varphi_{NL}(t)(Q)) \varphi_{NL}(t)(dq) = F_{N1}(\varphi_N, \gamma)[g]$$

   Using similar arguments we can show that

$$\varphi'_{NA}(t)[g] = \int_Q f_N(q, \varphi_N(t)(Q)) g(q) \varphi_{NL}(dq) - \int_Q \alpha_N(q, \varphi_N(t)(Q)) g(q) \varphi_{NA}(dq)$$
$$= F_{N2}(\varphi_N, \gamma)[g]$$

□

The next theorem is concerned with

$$x' = f(t,x),\ x(t_0) = x_0, \tag{28}$$

where $f \in C[\mathbb{R}_+ \times E, E]$, E being a Banach space.

Theorem 4.5. [19, pg. 454] *Let J be an open interval (a,b) and let U be open in E. Let $f : J \times U \to E$ be a continuous map which is Lipschitz on U uniformly for every compact subinterval of J. Let α be an integral curve of f, defined on a maximal open subinterval (a_0, b_0) of J. Assume:*

(i) There exists $\epsilon > 0$ such that the closure

$$\overline{\alpha((b_0 - \epsilon, b_0))}$$

is contained in U.

(ii) There exists a number $C > 0$ such that $|f(t,\alpha(t))| \le C$ for all t in $(b_0 - \epsilon, b_0)$.

Then $b_0 = b$.

4.2. **Proof of Theorem 4.1.** Let $N \in \mathbb{N}$, be given and let φ_N be as in Proposition 1.

Claim. *φ_N can be extended to a continuous function*

$$\varphi_N : [0,\infty) \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times C(Q;\mathcal{P}_w) \to \mathcal{M}_{+,w} \times \mathcal{M}_{+,w}.$$

Indeed, let b_0, be maximal such that

$$\varphi_N : [0,b_0) \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times C(Q;\mathcal{P}_w) \to \mathcal{M}_{+,w} \times \mathcal{M}_{+,w}$$

satisfies the 3 items in Proposition 1.

Now if $N(t) = \|\varphi_N(t;u,\gamma)\|_{V^2}$, then $N(t)$ is bounded on any finite interval. Indeed,

$$N(t) = \|\varphi_N(t;u,\gamma)\|_{V^2} = \varphi_{NL}(t;u,\gamma)(Q) + \varphi_{NA}(t;u,\gamma)(Q),$$

and

$$N'(t) = \varphi'_{NL}(t;u,\gamma)(Q) + \varphi'_{NA}(t;u,\gamma)(Q).$$

$$\begin{aligned} N'(t) = \int_Q \beta(q,\varphi_N(t)(Q))\varphi_{NA}(t)(dq) - \int_Q [\mu(q,\varphi_N(t)(Q))]\varphi_{NL}(t)(dq) - \\ \int_Q \alpha(q,\varphi_N(t)(Q))\varphi_{NA}(t)(dq) \le \int_Q \beta(q,\varphi_N(t)(Q))\varphi_{NA}(t)(dq) \\ \le \|\beta_N\|_\infty \varphi_{NA}(t;u,\gamma)(Q) \le \|\beta_N\|_\infty N(t). \end{aligned} \tag{29}$$

Hence $N(t) \le N(0)e^{\|\beta_N\|_\infty t}$ and $N(t)$ is bounded and hence so are φ_{NL} and φ_{NA} on any finite interval.

Let $\varphi_N^\diamond = \sup_{t\in[0,b_0)} N(t)$ and let $M > \varphi_N^\diamond$. Then by Lemma 4.2

$$F_N : \mathbb{R}_+ \times B_{2M}[\mathbf{0}] \times B_{2M}[\mathbf{0}] \times C(Q;\mathcal{P}_w) \to \mathcal{M}$$

is bounded and Lipschitz and there is an $\epsilon > 0$ such that $\overline{\varphi_N(b_0 - \epsilon, b_0)} \in B_{2M}[\mathbf{0}] \times B_{2M}[\mathbf{0}]$ and $\|F_N(\varphi_N(t))\|_V \le C$ for some constant C. Hence by Theorem 4.5, $b_0 = \infty$.

Likewise any nonnegative solution to (7) is bounded on any finite interval. Indeed, if $\phi(t;u,\gamma)$ is a solution to (7) and $M(t) = \|\phi(t;u,\gamma)\|_{V^2} = \phi_L(t;u,\gamma)(Q) + \phi_A(t;u,\gamma)(Q)$, then

$$M'(t) = \phi'_L(t;u,\gamma)(Q) + \phi'_A(t;u,\gamma)(Q).$$

$$\begin{aligned} M'(t) = &\int_Q \beta(q, \phi(t)(Q))\phi_A(t)(dq) - \int_Q [\mu(q, \phi(t)(Q))]\phi_L(t)(dq) - \\ &\int_Q \alpha(q, \phi(t)(Q))\phi_A(t)(dq) \leq \int_Q \beta(q, \phi(t)(Q))\phi_A(t)(dq) \\ &\leq \|\beta(\cdot, 0, 0)\|_\infty \phi_A(t; u, \gamma)(Q) \leq \|\beta(\cdot, 0, 0)\|_\infty M(t). \end{aligned} \tag{30}$$

Now for any $N \in \mathbb{N}$, if $b(N)$ is as in Proposition 1 then on $[-b(N), b(N)]$, φ_N is nonnegative and

$$F_N(\varphi_N(t; u, \gamma), \gamma) = F(\varphi_N(t; u, \gamma), \gamma)$$

and hence φ_N is a solution for (7) for initial measures in $(\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w}$.

So fix such an N and let $\phi(t; u, \gamma)$ be this local solution to (7), and let b_0, be maximal such that

$$\phi : [0, b_0) \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times C(Q; \mathcal{P}_w) \to \mathcal{M}_{+,w} \times \mathcal{M}_{+,w}$$

is continuous and satisfies the 3 items in Proposition 1.

So once again $\phi(t; u, \gamma)$ is bounded on $[0, b_0)$ and if $\phi^\diamond = \sup_{t \in [0, b_0)} M(t)$ and $\widetilde{M} > 2 \max\{N, \phi^\diamond\}$, then on $[0, b_0) \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{+,w} \times C(Q; \mathcal{P}_w)$

$$\begin{aligned} \phi' &= F(\phi, \gamma) = F_{\widetilde{M}}(\phi, \gamma) \\ \varphi'_{\widetilde{M}} &= F_{\widetilde{M}}(\varphi_{\widetilde{M}}, \gamma). \end{aligned} \tag{31}$$

From uniqueness of solution, we have $\phi = \varphi_{\widetilde{M}}$. However, $\varphi_{\widetilde{M}}$ is defined on all of $[0, \infty)$ and $F_{\widetilde{M}}(\varphi_{\widetilde{M}}(b_0; u, \gamma), \gamma) = F(\varphi_{\widetilde{M}}(b_0; u, \gamma), \gamma)$ since $\|\varphi_{\widetilde{M}}(b_0; u, \gamma)\|_{V^2} = \lim_{b \to b_0} \|\phi(b; u, \gamma)\|_{V^2} \leq \frac{\widetilde{M}}{2}$. If we replace 0 with b_0 in Lemma 1, then we see that we can extend ϕ unless $b_0 = \infty$.

Since $(0, \mathbf{0} \times \mathbf{0}) \in \mathbb{R}_+ \times \mathcal{M}_+ \times \mathcal{M}_+$, and $\mathbb{R}_+ \times \mathcal{M}_{+,w} \times \mathcal{M}_{+w} \times C(Q, \mathcal{P}_w) = \bigcup_{N \in \mathbb{Z}_+} \mathbb{R}_+ \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{w,+} \times (\overline{B_{\frac{N}{3}}[\mathbf{0}]})_{w,+} \times C(Q, \mathcal{P}_w)$, we let $\varphi = \bigcup_{N \in \mathbb{Z}_+} \varphi_N$ and Theorem 4.1 is immediate.

5. **Discrete and continuous models.** Similar to the models in [1, 12, 3] this model encompasses both discrete and continuous models. By choosing the right combination of initial condition and mutation kernel we can reduce the abstract model to a pure selection model, an absolutely continuous model or a system of ode's. For example, the pure selection model below :

$$\begin{aligned} m'_L(t)(E) = &\int_E \beta(q, m(t)(Q)) m_A(t)(dq) - \int_E [\mu(q, m(t)(Q)) + f(q, m(t)(Q)] \\ &m_L(t)(dq) \\ m'_A(t)(E) = &\int_E f(q, m(t)(Q)) m_L(t)(dq) - \int_E \alpha(q, m(t)(Q)) \\ &m_A(t)(dq) \end{aligned} \tag{32}$$

results from the selection of $\gamma(q) = \delta_q \in C(Q, P_w)$.

For the absolutely continuous and the discrete model formulations we direct the reader to [12].

6. **Uniform eventual boundedness.** A system $x' = F(x)$ is called dissipative and its solution uniformly eventually bounded, if all solutions exist for all forward times and if there exists some $c > 0$ such that

$$\limsup_{t \to \infty} ||x(t)|| < c$$

for all solutions x. We have the following which is essentially the argument in [25, Chpt. 11, pg.155] for densities.

Theorem 6.1. *Suppose* $\inf_{q\in Q}\mu(q,0,0)>0$ *and for any* $q\in Q$,

$$\limsup_{A\to\infty}\frac{\beta(q,0,A)}{\alpha(q,0,A)}<1. \tag{33}$$

Then the solutions to (7) *are uniformly eventually bounded.*

Proof. Below if ϕ is a solution to (7), then $L=\phi_L(Q)$ and $A=\phi_A(Q)$. (33) implies that there exists $\epsilon\in(0,1)$ and some $N^\sharp>0$ such that

$$\frac{\beta(q,L,A)}{\alpha(q,L,A)}\le 1-\epsilon$$

whenever $A\ge N^\sharp, q\in Q$. This implies that $\Big(\beta(q,L,A)-(1-\epsilon)\alpha(q,L,A)\Big)A\le 0$ for $A\ge N^\sharp$, $q\in Q$. So for some $c>0$, $\Big(\beta(q,L,A)-(1-\epsilon)\alpha(q,L,A)\Big)A\le c$ for all $A\ge 0, q\in Q$. Let

$$M(t)=\|\phi(t;u,\gamma)\|_{V^2}=\phi_L(t;u,\gamma)(Q)+\phi_A(t;u,\gamma)(Q)=L(t)+A(t),$$

then

$$M'(t)=\phi'_L(t;u,\gamma)(Q)+\phi'_A(t;u,\gamma)(Q)=L'(t)+A'(t)$$

and

$$\begin{aligned}M'(t)&=\int_Q[\beta(q,\phi(t)(Q))-\alpha(q,\phi(t)(Q))]\phi_A(t)(dq)-\int_Q[\mu(q,\phi(t)(Q))]\phi_L(t)(dq)\\&\le\int_Q[\beta(q,0,0)-\alpha(q,0,0)]\phi_A(t)(dq)-\inf_{q\in Q}\mu(q,0,0)L(t)\\&\le[\|\beta(\cdot,0,0)-\alpha(\cdot,0,0)\|_\infty]A(t)-\inf_{q\in Q}\mu(q,0,0)L(t)\\&\le c-\epsilon\|\alpha(\cdot,0,0)\|_\infty A(t)-\inf_{q\in Q}\mu(q,0,0)L(t)\le c-\delta M(t)\text{ for some }\delta>0.\end{aligned} \tag{34}$$

Hence $M(t)\le\max\{M(0),\frac{c}{\delta}\}$ and $\limsup_{t\to\infty}M(t)\le\frac{c}{\delta}$. □

7. **Conclusion.** In this paper we have shown that one can form a basic two stage juvenile adult model on the space of measures with parametric heterogeneity. It is an extension of the model developed in [25, Chpt. 11, pg.152]. The [25, Chpt. 11, pg.152] model has been extended in that this measure theoretic model allows for parameter heterogeneity and this model is phrased as an abstract game and hence can be applied to any game where these dynamics serve as a model and not just larvae and adult. It also encompasses both discrete and absolutely continuous models, pure selection and mutation. This model also has a nonlinear mutation term and it is continuous in this nonlinear term. The essential piece of the mutation kernel, γ, is modeled as a continuous mapping from the strategy space into the probability measures. It signifies the distribution of the offspring of a single strategy amongst the totality of strategies. This extension of state space (to measures) and the meaning and generalization of strategy (to any game strategy) and the mutation term (to possibly nonlinear) are also an extension of the models found in [8, 9]. In these references, the evolutionary strategy is maturation age and the populations are modeled as densities. Asymptotically, under biologically natural hypotheses on

the vital rates, this paper shows that there is a bounded attractor to which all trajectories tend i.e. uniform eventual boundedness or point dissipativeness.

The main novelty of this work over the previous works [1, 12, 3] is the two stage structuring. This model now takes values in a product of measure spaces. This paper goes through the details of the differences that this causes. Mainly these differences are shown in the topologies formed to deal with the extra structure and the resulting estimates. Also this model is quite different from the abovementioned from the standpoint of asymptotic analysis. For example, this model is not simply evolving under Selection and Mutation, there is also the evolutionary force of Transition that must be taken into account. This will play a more vital role when I take up the asymptotic behavior in future works. For example, the density dependent net reproductive number is now:

$$\mathcal{R}(q,L,A) = \frac{\beta(q,L,A)}{\alpha(q,L,A)} \frac{f(q,L,A)}{[\mu+f](q,L,A)}.$$

The net reproductive number is essential in determining which strategy the model is selecting upon. The next direction is to determine the asymptotic behavior of the model for specific kernels, in particular, for pure selection. Pure selection results from the choice of $\gamma(q) = \delta_q \in C(Q, \mathcal{P}_w)$.

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E-mail address: jcleve72@gmail.com