RIEMANN PROBLEMS WITH NON–LOCAL POINT CONSTRAINTS AND CAPACITY DROP

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ABSTRACT. In the present note we discuss in details the Riemann problem for a one-dimensional hyperbolic conservation law subject to a point constraint. We investigate how the regularity of the constraint operator impacts the well–posedness of the problem, namely in the case, relevant for numerical applications, of a discretized exit capacity. We devote particular attention to the case in which the constraint is given by a non–local operator depending on the solution itself. We provide several explicit examples.

We also give the detailed proof of some results announced in the paper [Andreianov, Donadello, Rosini, *Crowd dynamics and conservation laws with non-local constraints and capacity drop*], which is devoted to existence and stability for a more general class of Cauchy problems subject to Lipschitz continuous non–local point constraints.

1. **Introduction.**

1.1. **Point constraints in traffic modeling.** Traffic modeling is an exciting and fast–developing field of research with plentiful applications to real life. While this subject was initially limited to the description and the management of vehicular traffic, we see a growing interest nowadays on different applications as crowd dynamics and bio–mathematics. This note is related to an extensive on–going research project concerning the theoretical and the numerical study of macroscopic models for which the definition of solution involves an artificial limitation of the flux in a finite number of points. From the modeling point of view, this may correspond to

a narrow exit in crowd modeling, a toll gate in vehicular traffic, a cell membrane in bio-medical modeling.

In the pioneering paper [7], R. Colombo and P. Goatin introduced point constraints in the classical one-dimensional road traffic model proposed (independently) by M. Lighthill, G. Whitham [20] and P. Richards [22], with the goal to model the presence of obstacles on the road as toll gates and road lights. This model reads as

$$\partial_t \rho + \partial_x f(\rho) = 0 \qquad\qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \qquad\qquad \text{(1a)}$$

$$f\left(\rho(t,0\pm)\right) \le q(t) \qquad\qquad t \in \mathbb{R}_+ \qquad\qquad \text{(1b)}$$

$$\rho(0,x) = \rho_0(x) \qquad\qquad x \in \mathbb{R} \,, \qquad\qquad \text{(1c)}$$

where $\rho = \rho(t,x) \in [0,R]$ is the unknown (mean) density at time $t \in \mathbb{R}_+$ of vehicles moving along the road parameterized by the coordinate $x \in \mathbb{R}$. Then, $R \in \mathbb{R}_+$ is the maximal road density, $f : [0,R] \to \mathbb{R}$ is the nonlinearity relating the flux in the direction of increasing x to the density, $q : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function prescribing the maximal flow allowed through the point $x = 0$, and $\rho_0 : \mathbb{R} \to [0,R]$ is the initial (mean) density. Finally, $\rho(t,0-)$ denotes the left measure theoretic trace along the constraint implicitly defined by

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{+\infty} \int_{-\varepsilon}^0 |\rho(t,x) - \rho(t,0-)| \ \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t = 0$$

for all $\phi \in \mathbf{C_c^\infty}(\mathbb{R}^2; \mathbb{R})$. The right measure theoretic trace, $\rho(t,0+)$, is defined analogously.

In the above setting, the authors of [7] were able to prove existence and well–posedness of solutions. Further theoretical considerations and numerics associated with this model have been developed in [3].

In view of the applications, however, we also need to consider the case in which the evolution of the constraint function q is not given beforehand, but depends on the solution ρ itself in a neighborhood of $x = 0$. In such situation we say that the point constraint is non–local. In this way we obtain crowd and cell membrane dynamics models described by non–local PDEs for which the existence and well–posedness of solutions are not trivial matter. Nevertheless, this case has a practical interest. In crowd dynamics, as an example, the experimental observations by E. Cepolina in [6] prove that the irrational behavior of pedestrians at bottlenecks ends up by reducing the maximal possible outflow. This phenomenon, called *capacity drop*, is also related to other effects observed in crowd dynamics, such *Faster Is Slower* and the Braess' paradox.

In full generality, we may consider the constraint function q as follows

$$q(t) = \mathcal{Q}[\rho](t) \,,$$

where $\mathcal{Q} : \mathbf{C^0}\left([0,T]; \mathbf{L^1}\left(\mathbb{R}; [0,R]\right)\right) \to \mathbf{L^1}\left([0,T]; [0, \max_{[0,R]} f]\right)$. The minimal regularity properties to impose on $\mathcal{Q}$ in order to achieve well–posedness of solutions are not known at the moment, and they are the object of one of our current research projects.

1.2. **An example of non–local point constraint.** In the paper [4], B. Andreianov, C. Donadello and M. D. Rosini proposed a model which generalizes the one in [7] and consists of a Cauchy problem for a one-dimensional hyperbolic conservation law as 1 subject to a non–local point constraint of the form

$$q(t) = p\left(\int_{\mathbb{R}_-} w(x)\,\rho(t,x)\,\mathrm{d}x\right) \quad t \in \mathbb{R}_+\ . \tag{2}$$

Here $p : \mathbb{R}_+ \to \mathbb{R}_+$ prescribes the maximal flow allowed through an exit placed in $x = 0$ as a function of the weighted average density of pedestrians, ρ, in a left neighborhood of the exit and $w : \mathbb{R}_- \to \mathbb{R}_+$ is the weight function used to average the density. The authors of [4] proved well-posedness of the Cauchy problem in $\mathbf{L}^\infty(\mathbb{R}; [0, R])$ for the model under the following assumptions on the regularity of f, w and p, see Figure 1,

(F) $f \in \mathbf{Lip}([0, R]; \mathbb{R}_+)$, $f(0) = 0 = f(R)$ and there exists $\bar{\rho} \in\ ]0, R[$ such that $f'(\rho)\,(\bar{\rho} - \rho) > 0$ for a.e. $\rho \in [0, R] \setminus \{\bar{\rho}\}$.

(W) $w \in \mathbf{L}^\infty(\mathbb{R}_-; \mathbb{R}_+)$ is an increasing map, $\|w\|_{\mathbf{L}^1(\mathbb{R}_-; \mathbb{R}_+)} = 1$ and there exists $\mathrm{i}_w > 0$ such that $w(x) = 0$ for any $x \leq -\mathrm{i}_w$.

(P1) p belongs to $\mathbf{Lip}([0, R]\,;\,]0, f(\bar{\rho})])$ and it is a non-increasing map.

The above assumptions have the following "physical" motivations. By **(F)** the flow $\rho \mapsto f(\rho)$ is bell-shaped and null if the density or the velocity of the pedestrians are null. This is a typical assumption in first order models. Assumption **(W)** means that the possible loss of efficiency of the exit is related to the magnitude of the density of pedestrians upstream the exit. Moreover, assumption **(W)** says that when high values of the density are achieved sufficiently far from the exit, they do not influence directly the evacuation rate. This is reasonable because when a region of high density is followed by a region of low density a rarefaction wave appears between the two. Then, the values of the density at the exit are lower in general than the maximal ones in the high density region. Finally, assumption **(P1)** says that the maximal flow through the exit cannot exceed the capacity of the corridor, $f(\bar{\rho})$, nor it can reach zero even if it reduces (because of the clogging) when people accumulate upstream the exit. The hypothesis that p is Lipschitz continuous is needed in the proof of the well posedness result, but it does not have a specific meaning from the point of view of the modelization.

We recall that, in the previous literature, the only macroscopic model able to reproduce the capacity drop at bottlenecks is the CR model introduced by R. Colombo and M. D. Rosini in [9]. The Riemann solver for the model described in [9] is fairly intricate; in this note, our main goal is to describe in an exhaustive way the Riemann solver (or, rather, solvers) for the model introduced in [4]. Notice that, in contrast to [9], a specific nonclassical Riemann solver has to be used only at the exit point $x = 0$ while the simple classical Riemann solver is used elsewhere.

The notion of solution we adopt is a natural extension of the one introduced in [7], for a constrained Cauchy problem of the form 1.

Definition 1.1. Assume conditions **(F)**, **(W)** and that p is a non–increasing, possibly multivalued, map with values in $]0, f(\bar{\rho})]$. A map $\rho \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}; [0, R]) \cap \mathbf{C}^0(\mathbb{R}_+; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; [0, R]))$ is an entropy weak solution to 1, 2 if the following conditions hold:

1. There exists $q \in \mathbf{L}^\infty(\mathbb{R}_+; [0, f(\bar{\rho})])$ such that for every test function $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}^2; \mathbb{R}_+)$ and for every $k \in [0, R]$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} [|\rho - k|\partial_t\phi + \mathrm{sign}(\rho - k)\,(f(\rho) - f(k))\,\partial_x\phi]\ \mathrm{d}x\,\mathrm{d}t \tag{3a}$$

$$+ 2 \int_{\mathbb{R}_+} \left[1 - \frac{q\,(t)}{f(\bar{\rho})} \right] f(k)\ \phi(t,0)\ \mathrm{d}t \tag{3b}$$

$$+ \int_{\mathbb{R}} |\rho_0(x) - k|\ \phi(0,x)\ \mathrm{d}x \geq 0\ , \tag{3c}$$

and

$$f\left(\rho(t,0\pm) \right) \leq q\,(t)\ \text{ for a.e. } t \in \mathbb{R}_+\ . \tag{3d}$$

2. In addition q is linked to ρ by the relation 2.

If q is given *a priori*, then 3 is the definition of entropy weak solution to problem 1. We refer to Proposition 2.6 in [3] for a series of equivalent formulations of conditions 3. Theorem 2.2 in [3], which is a reformulation of the results in [21, 24], guarantees the existence of traces at $x = 0$ for any entropy weak solution.

The next proposition, proved in [4], lists the basic properties of an entropy weak solution of 1, 2, for the case where p is not multivalued.

Proposition 1. *Let $[t \mapsto \rho(t)]$ be an entropy weak solution of 1, 2 in the sense of Definition 1.1. Then*

(1) *It is also a weak solution of the Cauchy problem 1a, 1c.*
(2) *Any discontinuity satisfies the Rankine–Hugoniot jump condition, see [12].*
(3) *$[t \mapsto \rho(t)]$ is a classical Kružkov solution in $\mathbb{R}_\pm \times \mathbb{R}$, see [17].*
(4) *Nonclassical discontinuities, see [18], may occur only at the constraint location $x = 0$, and in this case the flow at $x = 0$ is the maximal flow allowed by the constraint. Namely, if the solution contains a nonclassical discontinuity for all times $t \in I$, I open in $\mathbb{R}_+$, then for a.e. t in I*

$$f\left(\rho(t,0-) \right) = f\left(\rho(t,0+) \right) = p\left(\int_{\mathbb{R}_-} w(x)\ \rho(t,x)\ \mathrm{d}x \right)\ . \tag{4}$$

If the constraint function p is multivalued the equalities in 2 and 4 should be interpreted as inclusions, and the result of the proposition remains true.

The existence result in [4] is achieved by a procedure which couples the operator splitting method [13], with the wave–front tracking algorithm, [11]. This approach was first adopted by D. Amadori and W. Shen in [1]. It allows us to approximate our problem by a problem with "frozen" constraint, as 1, at each time step.

The regularity of p plays a central role in the well-posedness result. While existence still holds for the Cauchy problem when p is merely continuous, it is difficult to justify uniqueness in this case. Further, in [4], the authors give some basic examples illustrating that solutions of a Riemann problem for the case of a non-decreasing piecewise constant p, see **(P2)** below, may fail to be unique, $\mathbf{L^1_{loc}}$–continuous and consistent.

The case in which p is piecewise constant is extremely important both for the theoretical study of the problem and its numerical applications. First, it is related to the construction of the Riemann solver, which is the basic building block for the wave–front tracking algorithm, a precious tool in the study of existence and stability of the solutions for the general Cauchy problem. Moreover, the piecewise constant case is essentially the only case in which solutions can be computed explicitly, which is an undeniable *atout* when looking for examples and applications. To this aim, it is relevant to provide a detailed study of the different pathological behaviors one may encounter. A piecewise constraint function may also emerge as the discretization of a Lipschitz continuous function $\mathfrak{p}$. Let $(p^h)_h$ be a sequence

of piecewise constant functions such that $\left\| p^h - \mathfrak{p} \right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R}_+)} \leq h$. While for every fixed $h > 0$ the solutions corresponding to p^h may fail to be unique, we still observe convergence of any of the solutions corresponding to p^h to the unique solution of the problem with constraint function $\mathfrak{p}$, as h (the discretization step along the ordinate axis) goes to zero. We refer to [16] for an analogous situation, and to Remark 3 for a more detailed discussion.

The paper is organized as follows. In the following section we develop a detailed proof of the fact that, when p is piecewise constant, the Riemann solver for 1 with a constraint of the form 2, is not unique and does not satisfy the minimal requirements needed to develop the classical wave–front tracking approach. Additionally, we compare the two extreme Riemann solvers: the one that minimizes the capacity drop, and the one that maximizes it. In particular, for any time $T > 0$ we estimate the distance between the profiles of the solutions produced by the two Riemann solvers starting from the same initial condition. In Section 3, we describe a numerical scheme for 1 based on finite volume methods as in [3]. We then illustrate the non-uniqueness of the Riemann solver by providing some numerical examples showing the instability of the scheme. For a fixed given initial datum, we obtain different solutions by modifying the size of the discretization steps. Finally Section 4 is devoted to some technical proofs.

2. **The constrained Riemann problem.** In this section we study constrained Riemann problems of the form

$$\partial_t \rho + \partial_x f(\rho) = 0 \qquad\qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \qquad (5\text{a})$$

$$f\left(\rho(t,0\pm)\right) \leq q(t) = p\left(\int_{\mathbb{R}_-} w(x)\, \rho(t,x)\, \mathrm{d}x\right) \qquad t \in \mathbb{R}_+ \qquad (5\text{b})$$

$$\rho(0,x) = \begin{cases} \rho_L & \text{if } x < 0 \\ \rho_R & \text{if } x \geq 0 \end{cases} \qquad\qquad x \in \mathbb{R}\,, \qquad (5\text{c})$$

with $\rho_L, \rho_R \in [0, R]$. The flux f and the weight function w satisfy **(F)** and **(W)**, moreover, we adopt the following assumption on p (instead of **(P1)**) to allow an explicit construction of solutions to 5

(P2) $p : [0, R] \to\,]0, f(\bar{\rho})]$ is piecewise constant non–increasing map with a finite number of jumps.
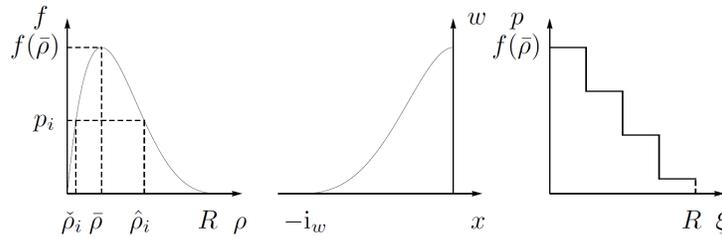
FIGURE 1. Examples of functions satisfying conditions **(F)**, **(W)**, and **(P2)**.

Unfortunately, the regularity of p required by **(P2)** is not enough to apply the well–posedness results in [4]. In particular, Example 2 in [4] illustrates the loss of uniqueness and stability of entropy weak solutions. In this section we present

a systematic study of the possible pathological behaviors. We denote by $\mathcal{R}$ the classical Riemann solver. This means that the map $[(t, x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ is the unique entropy weak solution for the unconstrained problem 5a, 5c, see for example [5] for its construction. Whenever the classical weak solution given by $\mathcal{R}$ does not satisfy the constraint 5b, we replace it by a nonclassical weak solution, see [18] as a general reference,

$$\rho(t, x) = \begin{cases} \mathcal{R}[\rho_L, \hat{\rho}(q)](x/t) & \text{if } x < 0 \\ \mathcal{R}[\check{\rho}(q), \rho_R](x/t) & \text{if } x > 0 \, , \end{cases} \qquad (6)$$

where the maps $\check{\rho}, \hat{\rho} : [0, f(\bar{\rho})] \to [0, R]$ are implicitly defined by

$$f(\check{\rho}(q)) = q = f(\hat{\rho}(q)) \quad \text{and} \quad \check{\rho}(q) \leq \bar{\rho} \leq \hat{\rho}(q) \, .$$
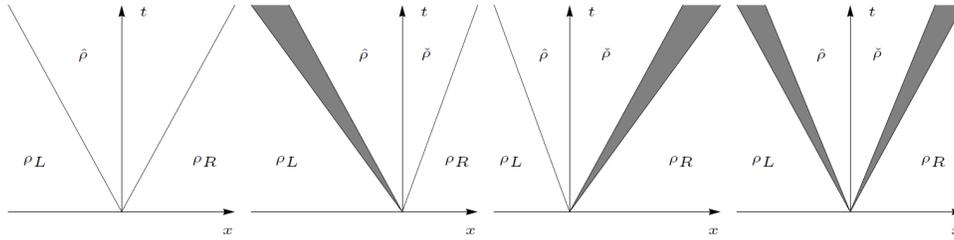
FIGURE 2. The four possible configurations of nonclassical entropy weak solutions of the form 12.

We stress that by Proposition 1 any nonclassical entropy weak solution is a classical entropy weak solution in the Kružkov sense, see [17], [12], in the half–planes $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_+ \times \mathbb{R}_+$. However, the jump at $x = 0$ is a nonclassical shock, in the sense that it does not satisfy the Lax entropy inequalities.

First, we should notice that as soon as the constraint function $t \to q(t)$ is not constant, but not necessarily non-local, the solutions of the Riemann problem may not be self–similar.

Example 2.1. We use a Cauchy problem of the form 1 to model vehicular traffic in presence of a traffic light. Assume $f(\rho) = \rho(1-\rho)$ and $q(t) = 0.25 \sum_{k \in \mathbb{N}} \chi_{[2k, 2k+1[}(t)$. When the traffic light is green, *i.e.* for $t \in [2k, 2k+1[$, the flow at $x = 0$ is free from any constraint. Conversely, when the traffic light is red, *i.e.* for $t \in [2k+1, 2(k+1)[$, the admissible flow at $x = 0$ become zero. This means that the Riemann problem with initial condition at $t = 0$ given by $\rho_L = \rho_R \neq 0$ will not be self–similar, because the constant solution will not satisfy the constraint starting from $t = 1$.

The above example also shows that as soon as we use a nonclassical Riemann solver the corresponding solution does not satisfy the maximum principle and its total variation may eventually increase in time.

Aiming for a general construction of the solutions to 5, we allow p to be a multi–valued piecewise constant function, namely, see Fig. 1, right:

- there exist $\xi_1, \ldots, \xi_n \in \,]0, R[$ and $p_0, \ldots, p_n \in \,]0, f(\bar{\rho})]$, with $\xi_i < \xi_{i+1}$ and $p_i > p_{i+1}$, such that $p(0) = p_0$, $p(R) = p_n$, $p \, \chi_{]\xi_i, \, \xi_{i+1}[} = p_i$ for $i = 0, \ldots, n-1$, $p(\xi_i) = [p_i, p_{i-1}]$ for $i = 1, \ldots, n$, being $\xi_0 = 0$ and $\xi_{n+1} = R$.

In the following we will use the notations $\check{\rho}_i = \check{\rho}(p_i)$ and $\hat{\rho}_i = \hat{\rho}(p_i)$.

As it will become clear in Proposition 2, the possible loss of uniqueness and stability can be easily forecast once the piecewise constant constraint p and the flux f are given. In particular, for some respective configurations of p and f the solution of the Riemann problem exists and is unique, locally in time, for any initial data (ρ_L, ρ_R) in $[0, R]^2$.

Definition 2.2. Introduce the subset of $[0, R]^2$

$$\mathcal{C} = \left\{ (\rho_L, \rho_R) \in [0, R]^2 \ : \ (\rho_L, \rho_R) \text{ satisfies condition } \mathbf{(C)} \right\},$$

where we say that (ρ_L, ρ_R) satisfies condition $\mathbf{(C)}$ if it satisfies one of the following conditions:

(C1): $\rho_L < \rho_R$, $f(\rho_R) < f(\rho_L)$ and $f(\rho_R) \leq p(\rho_L+)$;
(C2): $\rho_L < \rho_R$, $f(\rho_L) \leq f(\rho_R)$ and $f(\rho_L) \leq p(\rho_L+)$;
(C3): $\rho_R \leq \rho_L \leq \bar{\rho}$ and $f(\rho_L) \leq p(\rho_L+)$;
(C4): $\rho_R \leq \bar{\rho} < \rho_L$ and $f(\bar{\rho}) = p(\rho_L+)$;
(C5): $\bar{\rho} < \rho_R \leq \rho_L$, $f(\rho_R) \leq p(\rho_L-)$ and $f(\rho_L) < p(\rho_L+)$.

Analogously, introduce the subset of $[0, R]^2$

$$\mathcal{N} = \left\{ (\rho_L, \rho_R) \in [0, R]^2 \ : \ (\rho_L, \rho_R) \text{ satisfies condition } \mathbf{(N)} \right\},$$

where we say that (ρ_L, ρ_R) satisfies condition $\mathbf{(N)}$ if it satisfies one of the following conditions:

(N1): $\rho_L < \rho_R$ and $f(\rho_L) > f(\rho_R) > p(\rho_L+)$;
(N2): $\rho_L < \rho_R$, $f(\rho_L) \leq f(\rho_R)$ and $f(\rho_L) > p(\rho_L-)$;
(N3): $\rho_R \leq \rho_L \leq \bar{\rho}$ and $f(\rho_L) > p(\rho_L-)$;
(N4a): $\rho_R \leq \bar{\rho} < \rho_L$, $f(\bar{\rho}) \neq p(\rho_L-)$ and $f(\rho_L) < p(\rho_L+)$;
(N4b): $\rho_R \leq \bar{\rho} < \rho_L$, $f(\bar{\rho}) \neq p(\rho_L-)$ and $f(\rho_L) > p(\rho_L-)$;
(N5a): $\bar{\rho} < \rho_R \leq \rho_L$, $f(\rho_R) > p(\rho_L-)$ and $f(\rho_L) < p(\rho_L+)$;
(N5b): $\bar{\rho} < \rho_R \leq \rho_L$ and $f(\rho_L) > p(\rho_L-)$.

Here, $\mathcal{C}$ stands for *classical* and $\mathcal{N}$ for *nonclassical*, in relation with the nature of the shock appearing in the solution of 5 at $x = 0$. Above we let $p(\rho_L-) = \lim_{\rho \uparrow \rho_L} p(\rho)$ and $p(\rho_L+) = \lim_{\rho \downarrow \rho_L} p(\rho)$. Observe that if the constraint function p is constant in a neighborhood of the state ρ_L, then $p(\rho_L-) = p(\rho_L+)$ and this simplifies the above conditions. Also a right or left continuity assumption on p would simplify the above definitions.

The next proposition says that uniqueness holds at least for small times if and only if the initial data are in $\mathcal{C} \cup \mathcal{N}$. It is fundamental to remark that since non–uniqueness is possible only when $p(\rho_L-) \neq p(\rho_L+)$ and $p(\rho_L-) \geq f(\rho_L) \geq p(\rho_L+)$, non–uniqueness concerns at most a finite number of left states and therefore the region $[0, R]^2 \setminus (\mathcal{C} \cup \mathcal{N})$ is the union of a finite number of line segments.

Proposition 2. *Consider the constrained Riemann problem 5.*
• *If $(\rho_L, \rho_R) \in \mathcal{C}$, then the map $[(t, x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ is the unique entropy weak solution at least for $t > 0$ sufficiently small.*
• *If $(\rho_L, \rho_R) \in \mathcal{N}$, then there exists a unique $\bar{p} \in [p(\rho_L+), p(\rho_L-)]$ such that the map*

$$\left[t \mapsto \left\{ \begin{array}{ll} \mathcal{R}[\rho_L, \hat{\rho}(\bar{p})](x/t) & \text{if } x < 0 \\ \mathcal{R}[\check{\rho}(\bar{p}), \rho_R](x/t) & \text{if } x > 0 \end{array} \right. \right]$$

is the unique entropy weak solution at least for $t > 0$ sufficiently small.
• *If $(\rho_L, \rho_R) \in [0, R]^2 \setminus (\mathcal{C} \cup \mathcal{N})$, then the corresponding constrained Riemann problem 5 admits more than one entropy weak solution.*

The proof is deferred to Section 4.1

Remark 1. In some situations, when $(\rho_L, \rho_R) \in [0, R]^2 \setminus (\mathcal{C} \cup \mathcal{N})$, a whole one–parameter family of solutions exists. We refer to Example 2 in [4] for a detailed discussion of this case.

2.1. **Riemann solvers.** Proposition 2 implies that any entropy weak solution of 5 is self–similar for sufficiently small times. Therefore, it makes sense to introduce nonclassical *local* Riemann solvers, see Definition 2.3. Then, the availability of a local Riemann solver allows us to construct a *global* solution to the Riemann problem 5 by a wave–front tracking algorithm in which the jumps in the map $[t \mapsto q(t)]$ are interpreted as non–local interactions.

As the local solutions of the Riemann problem 5 are not unique in general, we are naturally led to question the existence of suitable selection criteria. All the solutions we introduce are solutions in the Kružkov sense in the open half–planes $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{R}_-$, so they satisfy the basic requirement of entropy dissipation. However, coming back to the real situations which our model aims to describe, we argue that the most interesting behaviors to track correspond to the extreme cases in which the flux at the exit is either the highest or the lowest possible from a given initial condition.

If, as an example, our model describes the evacuation of a narrow corridor, it is clear that we can select the unique optimal solution corresponding to the highest admissible values of the flux at the exit, taking into account 4. By opposition to the next case, we describe this situation as *quiet* behaviour. In analogy to the discussion in [14] we interpret all other possible solutions as consequences of an irrational behavior, which in literature is often described as *panic*. In particular, we can use the solution corresponding to the lowest admissible values of the flux at the exit to find an upper bound for the evacuation time.

From now on we restrict ourselves to the case in which $p(\xi_i)$ can only take the values p_i and p_{i+1} and not the intermediate values.

Definition 2.3. Two Riemann solvers $\mathcal{R}^q$ and $\mathcal{R}^p$ for 5 are defined as follows for $t > 0$ sufficiently small and $x \in \mathbb{R}$:

(C): If $(\rho_L, \rho_R) \in \mathcal{C}$ then

$$\mathcal{R}^q[\rho_L, \rho_R](t, x) = \mathcal{R}^p[\rho_L, \rho_R](t, x) = \mathcal{R}[\rho_L, \rho_R](x/t) \ .$$

(N): If $(\rho_L, \rho_R) \in \mathcal{N}$ then

$$\mathcal{R}^q[\rho_L, \rho_R](t, x) = \mathcal{R}^p[\rho_L, \rho_R](t, x) = \begin{cases} \mathcal{R}[\rho_L, \hat{\rho}(\bar{p})](x/t) & \text{if } x < 0 \\ \mathcal{R}[\check{\rho}(\bar{p}), \rho_R](x/t) & \text{if } x > 0 \ , \end{cases}$$

where $\bar{p} = p(\rho_L -)$ if (ρ_L, ρ_R) satisfies **(N4a)** or **(N5a)**, otherwise $\bar{p} = p(\rho_L +)$.

(CN2), (CN3), (CNN5): If (ρ_L, ρ_R) satisfies one of these sets of conditions then

$$\mathcal{R}^q[\rho_L, \rho_R](t, x) = \mathcal{R}[\rho_L, \rho_R](x/t) \ ,$$

$$\mathcal{R}^p[\rho_L, \rho_R](t, x) = \begin{cases} \mathcal{R}[\rho_L, \hat{\rho}(p(\rho_L +))](x/t) & \text{if } x < 0 \\ \mathcal{R}[\check{\rho}(p(\rho_L +)), \rho_R](x/t) & \text{if } x > 0 \ . \end{cases}$$

(NNN4), (NNN5): If (ρ_L, ρ_R) satisfies one of these sets of conditions then $\mathcal{R}^q[\rho_L, \rho_R](t, x)$ takes the form 12 with $\bar{p} = p(\rho_L-)$ and $\mathcal{R}^p[\rho_L, \rho_R](t, x)$ takes the form 12 with $\bar{p} = p(\rho_L+)$.

In the next proposition we collect the main properties of the Riemann solvers $\mathcal{R}^q$ and $\mathcal{R}^p$. In particular **(R6)** means that the Riemann solver $\mathcal{R}^q$ is the one which allows for the fastest evacuation, while $\mathcal{R}^p$ is associated with the slowest one.

Proposition 3. *Let $(\rho_L, \rho_R) \in [0, R]^2$. Then, for $\star = q, p$:*
(R1) *$[(t, x) \mapsto \mathcal{R}^\star[\rho_L, \rho_R](t, x)]$ is a weak solution to 5a, 5c.*
(R2) *$\mathcal{R}^\star[\rho_L, \rho_R]$ satisfies the constraint 5b in the sense that*

$$f\left(\mathcal{R}^\star[\rho_L, \rho_R](t, 0\pm)\right) \le p\left(\int_{\mathbb{R}_-} w(x)\ \mathcal{R}^\star[\rho_L, \rho_R](t, x)\ \mathrm{d}x\right).$$

(R3) *$\mathcal{R}^\star[\rho_L, \rho_R](t) \in \mathbf{BV}(\mathbb{R}; [0, R])$.*
(R4) *The map $\mathcal{R}^\star : [0, R]^2 \to \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}_+ \times \mathbb{R}; [0, R])$ is continuous in $\mathcal{C} \cup \mathcal{N}$ but not in all $[0, R]^2$.*
(R5) *$\mathcal{R}^\star$ is consistent, see [7], [8] and the comment below.*
(R6) *$\mathcal{R}^q[\rho_L, \rho_R]$ maximizes the flux at the exit, in the sense that if $\mathcal{E}$ is the set of all entropy weak solutions of the Riemann problem 5, we have*

$$\max_{\rho \in \mathcal{E}} \{f(\rho(t, 0\pm))\} = f\left(\mathcal{R}^q[\rho_L, \rho_R](0\pm)\right).$$

Analogously, $\mathcal{R}^p[\rho_L, \rho_R]$ minimizes the flux at the exit, in the sense that

$$\min_{\rho \in \mathcal{E}} \{f(\rho(t, 0\pm))\} = f\left(\mathcal{R}^p[\rho_L, \rho_R](0\pm)\right).$$

We recall that a Riemann solver is said to be consistent when the juxtaposition of the solutions of two Riemann problems with respective initial conditions (ρ_L, ρ_M) and (ρ_M, ρ_R) is the solution of the Riemann problem with datum (ρ_L, ρ_R). Moreover the vice versa also holds true, in the sense that whenever the state ρ_M is an intermediate state in the solution of the Riemann problem with initial condition (ρ_L, ρ_R), then the solution consist of exactly the same states and waves which we would obtain by solving side by side the two Riemann problems with data (ρ_L, ρ_M) and (ρ_M, ρ_R).

The proof of Proposition 3 is deferred to Section 4.2.

Remark 2. It is important to observe that even if $p(\xi_i)$ can only take the two values p_i and p_{i+1}, this is not enough to rule out the existence of infinitely many different solutions as the ones described in Example 2 of [4], in the case $p_i > f(\xi_i) = p_{i+1}$. However, each of the extremes Riemann solvers $\mathcal{R}^\star$, $\star = p, q$, selects one of them because it sticks to the constant level of constraint prescribed by Definition 2.3, the level p_i for $\mathcal{R}^q$ and the level p_{i+1} for $\mathcal{R}^p$, until a non–local interaction takes place.

2.2. **On the use of Riemann solvers $\mathcal{R}^\star$, $\star = q, p$.** Although the Riemann solvers $\mathcal{R}^\star$ are not $\mathbf{L}^1_{\mathbf{loc}}$–continuous, an existence result for the Cauchy problem 1 can be obtained from a wave–front tracking algorithm based on $\mathcal{R}^\star$, see for instance [10], [23]. Such approach using $\mathcal{R}^\star$ does not require the operator splitting method. However, the non–local nature of the approximating problems prevents us from a direct application of the Riemann solvers $\mathcal{R}^\star$. In fact, even in a arbitrary small neighborhood of $x = 0$, to prolong the approximating solution ρ^n beyond a time $t = \bar{t} > 0$ it is not sufficient to know the traces $\rho^n(\bar{t}, 0-)$, $\rho^n(\bar{t}, 0+)$, but

also the value $\int_{-\mathrm{i}_w}^{0} w(x)\,\rho^n(\bar{t}, x)\,\mathrm{d}x$ is needed. Roughly speaking, because of the non–local character of the constraint one cannot merely juxtapose the solution to the Riemann problem associated with the values of the traces at $x = 0$ with the solution to the Riemann problems away from the constraint. Finally, also jumps in $[t \mapsto p\,(\xi(t))]$ have to be considered as (non–local) interactions. Therefore, the approach using $\mathcal{R}^\star$ is considerably heavier and more technical than the one presented in [4].

2.3. **On the comparison between the Riemann solvers $\mathcal{R}^p$ and $\mathcal{R}^q$.** In this section we aim at comparing the solutions obtained by the two Riemann solvers introduced above, starting from the same initial condition (ρ_L, ρ_R). We are particularly interested in the situation where a given function $\mathfrak{p}$ satisfying the assumption **(P1)** of [4] is discretized with step h along the ordinate axis. In view of the uniqueness result for the Riemann problem with Lipschitz continuous constraint function $\mathfrak{p}$ and of the non-uniqueness of Riemann solver with discretized (piecewise constant) constraint functions p^h, one expects that the discrepancy between solutions obtained with two different Riemann solvers vanishes, as the maximal jump size h vanishes. More generally, h can take the sense of the maximal amplitude of a jump in the constraint function. Beyond the simple fact of convergence, as $h \to 0$, another aspect of the problem is to upper bound the rate of growth of the discrepancy w.r.t h and to the time t. In what follows, we prove estimates of the form

$$\|\mathcal{R}^p[\rho_L, \rho_R](t, \cdot) - \mathcal{R}^q[\rho_L, \rho_R](t, \cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \le \omega(h) G(t) \tag{7}$$

where ω and G are increasing continuous functions vanishing at zero; in most cases, the optimal estimate corresponds to the case of linear functions ω and G.

We start with the following easy lemma describing a particular discretization of the kind used in of [4, Section 4.1].

Lemma 2.4. *Let $\mathfrak{p}$ be a non-increasing Lipschitz continuous function on $[0, R]$. Then for all $h > 0$ there exists a piecewise constant function p^h on $[0, R]$ such that $\left\|p^h - \mathfrak{p}\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le h$ and*

$$\left|p^h(\rho_1) - p^h(\rho_2)\right| \le h + \mathrm{Lip}(\mathfrak{p})\,|\rho_1 - \rho_2|\,, \tag{8}$$

where $\mathrm{Lip}(\mathfrak{p})$ is the Lipschitz constant of $\mathfrak{p}$.

Proof. Let $\mathrm{Ent}(r)$ denote the integer part of a real number r. Observe that for all r, s, there holds $|\mathrm{Ent}(r) - \mathrm{Ent}(s)| \le 1 + |r - s|$. We set $p^h(\rho) := h\left(\mathrm{Ent}\left(\frac{\mathfrak{p}(\rho)}{h}\right) + 1\right)$. This readily yields (8), due to the bound $|\mathfrak{p}(\rho_1) - \mathfrak{p}(\rho_2)| \le \mathrm{Lip}(\mathfrak{p})|\rho_1 - \rho_2|$. Moreover, by construction, for all $\rho \in [0, R]$ one has $\left|p^h(\rho) - \mathfrak{p}(\rho)\right| \le h$. □

Now, assume we have a family of functions $(p^h)_h$ such that p^h is h-close, uniformly on $[0, R]$, to a Lipschitz constraint function $\mathfrak{p}$ (as in Lemma 2.4). Then the error in $\|\cdot\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}$ between any two solutions that we denote by ρ_1^h and ρ_2^h (for instance, the difference between solutions obtained with the solvers $\mathcal{R}^p$ and $\mathcal{R}^q$) at time t can be bounded like in 7. Indeed, we can use the continuous dependence argument of [4, Theorem 3.1] for a direct comparison of ρ_1^h and ρ_2^h, where the bound

$$|\mathfrak{p}(\rho_1) - \mathfrak{p}(\rho_2)| \le \mathrm{Lip}(\mathfrak{p})\,|\rho_1 - \rho_2| \tag{9}$$

is replaced by the bound 8 corresponding to the notation of Lemma 2.4. Then we see that, starting from the same Riemann data (or, more generally, from the

same Cauchy data) $\rho_1(0,\cdot) = \rho_2(0,\cdot)$, from the fundamental stability estimate of Proposition 2.10 of [3], using 8 and the Gronwall inequality one easily gets the rough bound

$$\|\rho_1(t) - \rho_2(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \leq \frac{h}{2\mathrm{Lip}(\mathfrak{p})} \left[\exp(2\mathrm{Lip}(\mathfrak{p})t) - 1\right]. \tag{10}$$

Further, an indirect comparison brings a better bound. Indeed, with the same arguments of [4, Theorem 3.1, Proposition 2.10] we see that both solutions ρ_i^h, $i = 1, 2$ corresponding to a discretized constraint p^h verify

$$\left\|\rho_i^h(t) - \rho(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \leq 2\mathrm{Lip}(\mathfrak{p})th,$$

where ρ is the *unique* solution corresponding to the Lipschitz continuous constraint $\mathfrak{p}$. It follows by the triangular inequality that the right-hand side of (10) can be replaced by $4\mathrm{Lip}(\mathfrak{p})th$.

Now, let us address the case where the constraint function p is a given non-increasing piecewise constant function, where h will now denote the maximal jump size in p. Let us show that in the case where different Riemann solvers co-exist, the $\mathbf{L^1}$ distance between the associated solutions grows linearly in time $t \in [0, T]$ and in most cases, it grows at most linearly in h (see also the numerical experiment on Figure 6).

It is clear from Definition 2.3 that as soon as (ρ_L, ρ_R) belongs to $\mathcal{C} \cup \mathcal{N}$ the solutions obtained by the two Riemann solvers coincide. In order to keep our presentation as light as possible, we focus on only one of the possible cases in which $\mathcal{R}^p$ and $\mathcal{R}^q$ differ. All other cases can be handed in a similar way. Assume
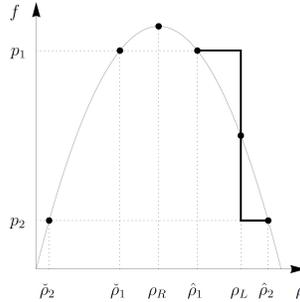
FIGURE 3. The flux f, the constraint function $p|_{]\hat{\rho}_1,\hat{\rho}_2[}$ and the values of the density ρ considered in Section 2.3.

that $p(\xi) = p_1 \, \chi_{[0,\bar{\xi}]}(\xi) + p_2 \, \chi_{]\bar{\xi},R]}(\xi)$, where $\bar{f} > p_1 > f(\bar{\xi}) > p_2 > 0$, and $\rho_L = \bar{\xi}$, $\rho_R = \bar{\rho}$, see Figure 3. We only consider solutions in a small interval of time $[0, T]$ in which they are self–similar. Then we get

$$\mathcal{R}^p[\rho_L, \rho_R](t, x) = \begin{cases} \mathcal{R}[\rho_L, \hat{\rho}_2](x/t) & \text{if } x < 0 \\ \mathcal{R}[\check{\rho}_2, \bar{\rho}](x/t) & \text{if } x > 0 \end{cases},$$

$$\mathcal{R}^q[\rho_L, \rho_R](t, x) = \begin{cases} \mathcal{R}[\rho_L, \hat{\rho}_1](x/t) & \text{if } x < 0 \\ \mathcal{R}[\check{\rho}_1, \bar{\rho}](x/t) & \text{if } x > 0 \end{cases},$$

where the values $\hat{\rho}_i$ and $\check{\rho}_i$, for $i = 1, 2$, are implicitly defined by the relations $f(\hat{\rho}_i) = f(\check{\rho}_i) = p_i$ and $\check{\rho}_i < \bar{\rho} < \hat{\rho}_i$. More explicitly, we can say that the solution corresponding to $\mathcal{R}^p$ consists of a shock of negative speed λ_p between ρ_L and $\hat{\rho}_2$, a stationary nonclassical shock between $\hat{\rho}_2$ and $\check{\rho}_2$ and a shock of positive speed μ_p

between $\check{\rho}_2$ and $\bar{\rho}$. The solution corresponding to $\mathcal{R}^q$ consists of a rarefaction wave between ρ_L and $\hat{\rho}_1$, a stationary nonclassical shock between $\hat{\rho}_1$ and $\check{\rho}_1$ and a shock of positive speed μ_q between $\check{\rho}_1$ and $\bar{\rho}$, see Figure 4.

As the characteristics of this problem propagate with finite speed, we expect the solutions associated with the two solvers coincide outside a bounded interval. The geometry of the problem, see Figure 3, implies that $\mu_p > \mu_q$ and that λ_p is smaller than all the propagation speeds in the rarefaction wave between ρ_L and $\hat{\rho}_1$. Therefore, at time $t \in [0, T]$ fixed, the two solutions coincide outside the interval $[\lambda_p t, \mu_p t]$.

Thanks to self-similarity, the value of the distance

$$\|\mathcal{R}^p[\rho_L, \rho_R](t) - \mathcal{R}^q[\rho_L, \rho_R](t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}$$

is proportional to t and to the value of the area between the profiles of solutions at time $t = 1$, see Figure 4.
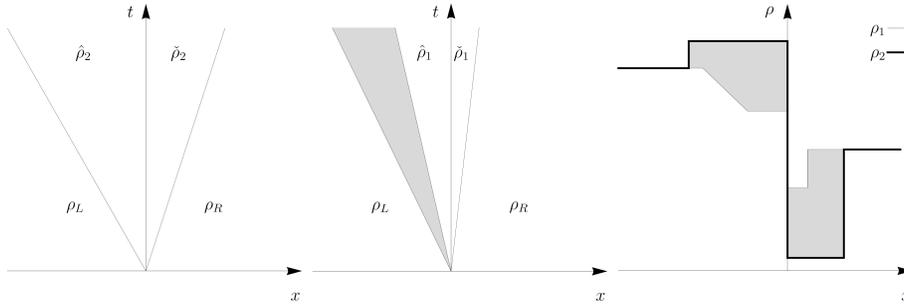
FIGURE 4. The solutions corresponding to $\mathcal{R}^p$, left, and to $\mathcal{R}^q$, center, and the comparison between their profiles at fixed time, right, as described in Section 2.3.

Following the same technique as in [5, Chapter 7], we can estimate the distance between the solutions profiles at a fixed time $t \in [0, T]$. For the reader convenience, we recall that the propagation speed of the shock discontinuity between the states ρ_a and ρ_b is given by the Rankine-Hugoniot condition

$$\sigma(\rho_a, \rho_b) = \frac{f(\rho_a) - f(\rho_b)}{\rho_a - \rho_b} \ , \tag{11}$$

and that the propagation speed of the characteristics in a rarefaction wave joining the states ρ_a and ρ_b varies between the values $f'(\rho_a)$ and $f'(\rho_b)$. Also, by definition $f(\hat{\rho}_i) = f(\check{\rho}_i) = p_i$, for $i = 1, 2$. A direct calculation gives us

$$\|\mathcal{R}^p[\rho_L, \rho_R](t, \cdot) - \mathcal{R}^q[\rho_L, \rho_R](t, \cdot)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}$$
$$= \|\mathcal{R}[\rho_L, \hat{\rho}_2](\cdot/t) - \mathcal{R}[\rho_L, \hat{\rho}_1](\cdot/t)\|_{\mathbf{L^1}(\mathbb{R}_-;\mathbb{R})}$$
$$+ \|\mathcal{R}[\check{\rho}_2, \rho_R](\cdot/t) - \mathcal{R}[\check{\rho}_1, \rho_R](\cdot/t)\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})}$$
$$\leq \{-\lambda_p (\hat{\rho}_2 - \hat{\rho}_1) + \mu_q (\check{\rho}_1 - \check{\rho}_2) + (\mu_p - \mu_q)(\bar{\rho} - \check{\rho}_2)\} t$$
$$= \{-\lambda_p (\hat{\rho}_2 - \hat{\rho}_1) + p_1 - p_2\} t$$
$$\leq t \, \omega(|p_1 - p_2|) \,,$$

indeed, since $\hat{\rho}_2 - \hat{\rho}_1$ vanishes as $h := p_1 - p_2$ vanishes, the latter quantity is upper bounded by an estimate of the kind 7 with linear function $G(t)$. In particular,

whenever f is smooth with $f'(\rho) \neq 0$ for all $\rho \neq \bar{\rho}$ and $\delta := f(\bar{\rho}) - \max\{p_1, p_2\}$ is strictly positive, we can take $\omega(r) = C\,r$ for a large enough constant C depending on δ and find a bound, linear both in t and h, on the discrepancy between the solutions obtained by the Riemann solvers $\mathcal{R}^p$, $\mathcal{R}^q$.

In conclusion, we can say that for small enough time, the discrepancy at time t between different solutions to the Riemann problem with discontinuous constraint function p with maximal jump size h grows at most as th at least when $\max_{\rho \in [0,R]} p(\rho) < \max_{\rho \in [0,R]} f(\rho)$ (while the growth is at most like $t\omega(h)$ in general, where $\omega(h) \to 0$ as $h \to 0$). Moreover, in the situation described by Lemma 2.4 where p^h approximates a Lipschitz function $\mathfrak{p}$, the discrepancy cannot exceed a constant times $\mathrm{Lip}(\mathfrak{p})th$.

Remark 3. Observe that similar phenomenon of convergence of ill-posed, but practically convenient discretized problems to a well-posed limit problem occurs, e.g., in the paper [16] by Isaacson and Temple. Indeed, in the context of scalar conservation laws with heterogeneous resonant damping term

$$\partial_t u + \partial_x f(u) = \partial_x a g(u),$$

the well-posedness is a classical result for a Lipschitz continuous a, while jump discontinuous a may lead to non-uniqueness of the appropriately reformulated Riemann problem. The discretization put forward in [16] involves approximation of Lipschitz $a(\cdot)$ by piecewise constant $a^h(\cdot)$, which is quite analogous to the approach pursued in our paper. However, contrarily to the conclusions of [16], we do not select a preferred Riemann solver: both the "pessimistic" ($\mathcal{R}^p$) and the "optimistic" ($\mathcal{R}^q$) scenarios in our case appear as useful in modeling and optimization of pedestrian traffic.

3. **Numerical results.** We present here some numerical experiments in order to illustrate the results of the above section. Let us first describe the scheme which is based on a finite volume method following the ideas of [3]. We define the points $(x_{j+1/2})_{j \in \mathbb{Z}}$, the cells $K_j = \left[x_{j-1/2}, x_{j+1/2}\right[$, the center of the cells $x_j = \frac{1}{2}(x_{j-1/2} + x_{j+1/2})$, the constant space step $\Delta x = x_{j+1/2} - x_{j-1/2}$. We introduce the index j_c such that $x_{j_c+1/2} = 0$ the position of the constraint. We define the time discretization $t^n = n\Delta t$, where Δt is the constant time step and $n \in \mathbb{N}$. We denote by ρ_j^n the unknow which is an approximation of the average of $\rho(t^n, .)$ on the cell K_j and we set

$$\rho_j^0 = \frac{1}{\Delta x} \int_{K_j} \rho_0(x)\,dx.$$

Let q^n be the approximation of $q(t^n)$ defined by

$$q^n = p\left(\Delta x \sum_{j \leq j_c} w(x_j)\rho_j^n\right).$$

Then the scheme can be written into the form

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_{j+1/2}^n - \mathcal{G}_{j-1/2}^n\right),$$

where

$$\mathcal{G}_{j+1/2}^n = \begin{cases} \mathcal{F}_{j+1/2}^n & \text{if } j \neq j_c \\ \min(\mathcal{F}_{j+1/2}^n, q^n) & \text{if } j = j_c, \end{cases}$$

and where $\mathcal{F}^n_{j+1/2} = F(\rho^n_j, \rho^n_{j+1})$ is a monotone, consistent numerical flux. We will justify in the future work [2] convergence of this scheme to an entropy solution of the nonlocally constrained problem in the sense of Definition 1.1, where the constraint function p must be taken multi-valued.

While it is delicate or even impossible to identify a unique Riemann solver to which the scheme would converge, we can use the simulations on Figure 5 to illustrate the fact that non-uniqueness for the Riemann problem leads to the instability of the scheme for a specific given initial datum.

For the examples, we consider the flux $f(\rho) = \rho\,(1 - \rho)$. The domain of computation is $[-5, 5]$, the constraint function is $p(\xi) = p_1\,\chi_{[0,0.8]}(\xi) + p_2\,\chi_{]0.8,1]}(\xi)$, where $p_1 = 0.1875$, $p_2 = 0.05$, the weight function is $w(x) = 2(x + 1)\,\chi_{]-1,0]}(x)$. The final time of computation is $T = 1$. We consider the following initial state

$$\rho_0(x) = \begin{cases} 0.8 & \text{if } x < 0 \\ 0.5 & \text{if } x > 0\,. \end{cases}$$

The numerical flux F is chosen to be the standard Godunov one (see e.g. [15, 19]):

$$F(a, b) = \begin{cases} \min_{[a,b]} f & \text{if } a \le b \\ \max_{[b,a]} f & \text{if } a > b. \end{cases}$$

We fix the time step $\Delta t = 6.25 \times 10^{-4}$. In Figure 5 is shown, at the final time, the two different solutions computed by the scheme when varying the space step. The first solution is obtained for $\Delta x = 0.025, 0.0125, 0.01$ while the second one is obtained for $\Delta x = \frac{1}{60}, \frac{1}{90}, \frac{1}{120}$.

Nonetheless, as shown in Figure 6, in practice the instability is limited to a behaviour of kind

$$\|\rho_1 - \rho_2\|_{\mathbf{L^1}([-5,5];\mathbb{R})} \le C\,h\,,$$

where ρ_1 and ρ_2 denote the first and the second solution respectively, h is the maximal size of jump in p and $C > 0$ is a constant.

Indeed we reported in Figure 6 the computation of the $\mathbf{L^1}$-discrete norms of the difference $\rho_1 - \rho_2$ when we take, $p_2 = 0.05, 0.075, 0.1, 0.125$ and 0.15 in the definition of the constraint function. The solution ρ_1 is computed with the space step $\Delta x = 0.025$ and ρ_2 is computed with $\Delta x = \frac{1}{120}$. Using logarithmic scales, we deduce that the distance between the two solutions is approximatively proportional to $|p_1 - p_2|$.

4. Proofs.

4.1. Proof of Proposition 2. First, we introduce the notation

$$\xi(t) = \int_{\mathbb{R}_-} w(x)\,\rho(t, x)\,\mathrm{d}x\,.$$

Therefore $\xi(0) = \rho_L$ and the map $[t \mapsto \xi(t)]$ is continuous.

We stress that any nonclassical entropy weak solution in the sense of Definition 1.1 is also a classical entropy weak solution in the Kružkov sense in the half–planes $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_+ \times \mathbb{R}_+$. Therefore, at least for $t > 0$ sufficiently small, by Proposition 1, assumption (**P2**) and the continuity of the map $[t \mapsto \xi(t)]$, any nonclassical entropy weak solution of 5 must have the form, see Fig. 2,

$$\rho(t, x) = \begin{cases} \mathcal{R}[\rho_L, \hat{\rho}(\bar{p})](x/t) & \text{if } x < 0 \\ \mathcal{R}[\check{\rho}(\bar{p}), \rho_R](x/t) & \text{if } x > 0\,. \end{cases} \tag{12a}$$
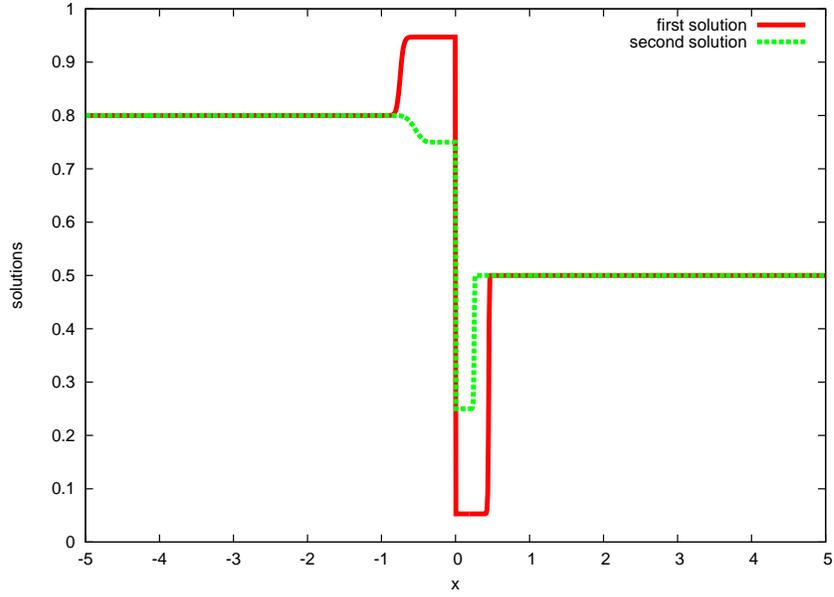
FIGURE 5. The obtained solutions at $T = 1$ when varying Δx
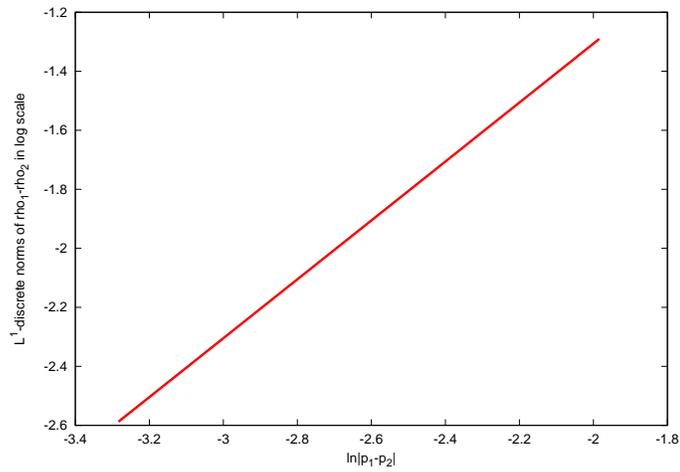
FIGURE 6. The norm $\|\rho_1 - \rho_2\|_{\mathbf{L}^1}$ with respect to $|p_1 - p_2|$ in log/log scale.

Observe that 12a is uniquely identified once we know $\bar{p}$ which, by 4, satisfies

$$\bar{p} = f\left(\check{\rho}(\bar{p})\right) = f\left(\hat{\rho}(\bar{p})\right) \ . \tag{12b}$$

We recall that 12b means in particular that the Rankine–Hugoniot jump condition is satisfied at $x = 0$ even when the solution to the Riemann problem is nonclassical. As a consequence of 12b, of assumption **(P2)** and of the continuity of $[t \mapsto \xi(t)]$, we have that

$$\bar{p} \in [p(\rho_L+), p(\rho_L-)] \, . \tag{12c}$$

This implies that if $p(\rho_L+) = p(\rho_L-)$, then $p(\xi)$ is constant in a neighborhood of ρ_L and, since the solution is in $\mathbf{C^0}\left(\mathbb{R}_+; \mathbf{L^1_{loc}}(\mathbb{R}; [0, R])\right)$, uniqueness is ensured by the results in [7]. However, the continuity of p at ρ_L is not a necessary condition for uniqueness as we show in the following section.

4.1.1. *Cases in which uniqueness holds.* In this section we prove that:

If $(\rho_L, \rho_R) \in \mathcal{C}$**:** the corresponding classical solution satisfies 5 for all $t > 0$ sufficiently small and it is not possible to construct a different solution.

If $(\rho_L, \rho_R) \in \mathcal{N}$**:** the corresponding classical solution does not satisfy 5b, and there exists a unique nonclassical solution that satisfies 5.

We list here two basic properties which will be of great help in the following case by case analysis.

By assumption **(P2)** and the continuity of the map $[t \mapsto \xi(t)]$ we have that for any $t > 0$ sufficiently small

bp1: if $\xi(t) < \rho_L$, then $p(\xi(t)) \equiv p(\rho_L-)$;
bp2: if $\rho_L < \xi(t)$, then $p(\xi(t)) \equiv p(\rho_L+)$.

The case $\xi(t) \equiv \rho_L$ is somehow special and has to be studied separately for each specific case.

Second, when the solution is nonclassical, due to the finite speed of propagation of the waves, the assumption **(P2)** and properties **bp1** and **bp2**, we have

np1: if $\mathcal{R}\left[\rho_L, \hat{\rho}(\bar{p})\right](x) \equiv \rho_L$ for $x < 0$, then $\bar{p} = f(\rho_L) \in [p(\rho_L+), p(\rho_L-)]$;
np2: if $\bar{p} \neq f(\rho_L)$ and $\rho_L < \hat{\rho}(\bar{p})$, then $\bar{p} = p(\rho_L+)$;
np3: if $\bar{p} \neq f(\rho_L)$ and $\hat{\rho}(\bar{p}) < \rho_L$, then $\bar{p} = p(\rho_L-)$;
np4: if p is continuous in ρ_L, namely $p(\rho_L-) = p(\rho_L+)$, then $\bar{p} = p(\rho_L)$.

Now we start the description of the possible cases and we proceed as follows. First, we show that for any initial datum satisfying **(Ci)**, $i = 1, \ldots, 5$, the problem actually has a unique solution and that the solution is classical. Second, we take into consideration the corresponding case **(Ni)**, for which we prove that the classical solution is not suitable and that there exists a unique nonclassical solution.

In general the solutions to the constrained Riemann problem 5 are not self–similar. All the cases listed below describe self–similar solutions because we let the solutions evolve only on a small interval of time.

(C1) In this case $[(t, x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ performs a shock with negative speed $\sigma(\rho_L, \rho_R)$ and satisfies 5b because $f(\rho_R) \leq p(\rho_L+)$ and $p(\xi(t)) \equiv p(\rho_L+)$ by **bp2**. Assume that there exists a nonclassical solution of the form 12. Observe that the assumptions $\rho_L < \rho_R$ and $f(\rho_R) < f(\rho_L)$ together imply that $\bar{\rho} < \rho_R$. Then $\check{\rho}(\bar{p}) \leq \bar{\rho} < \rho_R$ and $\mathcal{R}[\check{\rho}(\bar{p}), \rho_R]$ is given by a shock with non negative speed if and only if $\bar{p} \leq f(\rho_R)$, or equivalently, $\rho_R \leq \hat{\rho}(\bar{p})$. As a consequence, $\bar{p} \leq f(\rho_R) < f(\rho_L)$, $\rho_L < \rho_R \leq \hat{\rho}(\bar{p})$ and by **np2** $\bar{p}$ coincides with

$p(\rho_L+)$. In conclusion we have $\bar{p} \leq f(\rho_R) \leq p(\rho_L+) = \bar{p}$, namely $f(\rho_R) = \bar{p}$ and the nonclassical solution coincides with the classical one.

(N1) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ does not satisfy 5b because $f(\rho_R) > p(\rho_L+)$, see case **(C1)**. Therefore, there does not exist any classical solution and we can consider only nonclassical solutions of the form 12. If p is continuous in ρ_L, then by **np4** we have that $\bar{p} = p(\rho_L)$. If p experiences a jump at ρ_L then, one may wonder which value in $[p(\rho_L+), p(\rho_L-)]$ has to be chosen as $\bar{p}$. As in the case **(C1)**, the assumptions imply that $\bar{\rho} < \rho_R$ and then that $\check{\rho}(\bar{p}) < \rho_R$ and $\bar{p} \leq f(\rho_R)$. Then $\bar{p}$ is strictly smaller than $f(\rho_L)$ and $\hat{\rho}(\bar{p}) > \rho_L$. As a consequence, property **np2** forces us to choose the unique possible value of $\bar{p}$, which is $p(\rho_L+)$.

(C2) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ performs a shock with non negative speed $\sigma(\rho_L, \rho_R)$ and it satisfies 5b because $f(\rho_L) \leq p(\rho_L+)$. Assume that there exists a nonclassical solution of the form 12. Observe that the assumptions $\rho_L < \rho_R$ and $f(\rho_R) \geq f(\rho_L)$ together imply that $\bar{\rho} > \rho_L$. Then $\hat{\rho}(\bar{p}) \geq \bar{\rho} > \rho_L$ and $\mathcal{R}[\rho_L, \hat{\rho}(\bar{p})]$ is given by a shock with non positive speed if and only if $\bar{p} \leq f(\rho_L)$. Thus $\bar{p} \leq f(\rho_L) \leq p(\rho_L+)$ and this implies by 12c that $\bar{p} = f(\rho_L) = p(\rho_L+)$ and that the nonclassical solution coincides with the classical one.

(N2) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ does not satisfy 5b because $f(\rho_L) > p(\rho_L-)$, see case **(C2)**. Therefore, there does not exist any classical solution and we can consider only nonclassical solutions of the form 12. As in the case **(C2)**, the assumptions imply $\hat{\rho}(\bar{p}) \geq \bar{\rho} > \rho_L$. Furthermore, by 12c we have $\bar{p} \leq p(\rho_L-) < f(\rho_L)$, and as a consequence, property **np2** forces us to choose $\bar{p} = p(\rho_L+)$.

(C3) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ performs a possible null rarefaction on the right of the constraint and it satisfies 5b because $f(\rho_L) \leq p(\rho_L+)$. Assume that there exists a nonclassical solution of the form 12. Since $\rho_L \leq \bar{\rho} \leq \hat{\rho}(\bar{p})$, $\mathcal{R}[\rho_L, \hat{\rho}(\bar{p})]$ is given by a shock that has non positive speed if and only if $\bar{p} \leq f(\rho_L)$. Therefore $\bar{p} \leq f(\rho_L) \leq p(\rho_L+)$ and this by 12c implies that $\bar{p} = f(\rho_L) = p(\rho_L+)$ and that the nonclassical solution coincides with the classical one.

(N3) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ does not satisfy 5b because $f(\rho_L) > p(\rho_L-)$, see case **(C3)**. Therefore, there does not exist any classical solution and we can consider only nonclassical solutions of the form 12. By hypothesis and 12c we have $f(\rho_L) > p(\rho_L-) \geq \bar{p}$. Therefore $\rho_L \leq \bar{\rho} < \hat{\rho}(\bar{p})$ and by **np2** we have $\bar{p} = p(\rho_L+)$.

(C4) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ performs a rarefaction with speeds between $\lambda(\rho_L) < 0$ and $\lambda(\rho_R) \geq 0$ and it satisfies 5b because $f(\bar{\rho}) = p(\rho_L+)$ implies that $p(\rho) = f(\bar{\rho})$ for all $\rho \leq \rho_L$. Moreover, it implies also that p is continuous in ρ_L and therefore, by **np4**, any nonclassical solution of the form 12 must have $\bar{p} = p(\rho_L) = f(\bar{\rho})$, but in this case the nonclassical solution coincides with the classical one.

(N4) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ does not satisfy 5b because $f(\bar{\rho}) > p(\rho_L-)$, see case **(C4)**. Therefore, there does not exist any classical solution and we can consider only nonclassical solutions of the form 12.

    (N4a) By assumption and 12c $f(\rho_L) < p(\rho_L+) \leq \bar{p}$ and therefore $\hat{\rho}(\bar{p}) < \rho_L$ and by **np3** we have $\bar{p} = p(\rho_L-)$.

(N4b) By assumption and 12c $f(\rho_L) > p(\rho_L-) \geq \bar{p}$ and therefore $\hat{\rho}(\bar{p}) > \rho_L$ and by **np2** we have $\bar{p} = p(\rho_L+)$.

(C5) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ performs a possible null rarefaction on the left of the constraint and it satisfies 5b because $f(\rho_R) \leq p(\rho_L-)$ and $p(\xi(t)) \equiv p(\rho_L-)$ by **bp1**. Assume that there exists a nonclassical solution of the form 12. Since by assumption and 12c $\bar{p} \geq p(\rho_L+) > f(\rho_L)$, we have $\hat{\rho}(\bar{p}) < \rho_L$ and by **np3** $\bar{p} = p(\rho_L-)$, but in this case the nonclassical solution coincides with the classical one.

(N5a) In this case $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$ does not satisfy 5b because $f(\rho_R) > p(\rho_L-)$, see case **(C5)**. Therefore, there does not exist any classical solution and we can consider only nonclassical solutions of the form 12.

  (N5b) By assumption and 12c, $f(\rho_L) < p(\rho_L+) \leq \bar{p}$ and therefore $\hat{\rho}(\bar{p}) < \rho_L$ and by **np3** we have $\bar{p} = p(\rho_L-)$.

  (N5b) By assumption and 12c, $f(\rho_L) > p(\rho_L-) \geq \bar{p}$ and therefore $\hat{\rho}(\bar{p}) < \rho_L$ and by **np2** we have $\bar{p} = p(\rho_L+)$.

4.1.2. *Cases in which uniqueness is violated.* Now we list the "pathological" cases, where we have more than one admissible solution. We stress once again that a necessary condition for non–uniqueness is $p(\rho_L-) \neq p(\rho_L+)$ and $p(\rho_L-) \geq f(\rho_L) \geq p(\rho_L+)$.

  (CN2): If $\rho_L < \rho_R$, $f(\rho_L) \leq f(\rho_R)$ and $p(\rho_L+) < f(\rho_L) \leq p(\rho_L-)$, then the classical solution $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$, which consists of a shock with non negative speed, as well as the nonclassical solution 12, with $\bar{p} = p(\rho_L+)$, are distinct solutions of 5.

  (CN3): If $\rho_R \leq \rho_L \leq \bar{\rho}$ and $p(\rho_L+) < f(\rho_L) \leq p(\rho_L-)$, then the classical solution $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$, which consists of a possible null rarefaction on the right of the constraint, as well as the nonclassical solution 12, with $\bar{p} = p(\rho_L+)$, are distinct solutions of 5.

  (NNN4): If $\rho_R \leq \bar{\rho} < \rho_L$, $p(\rho_L-) \neq p(\rho_L+)$ and $p(\rho_L+) \leq f(\rho_L) \leq p(\rho_L-)$, then the nonclassical solutions of the form 12 which corresponds to $\bar{p}$ in the set $\{p(\rho_L+), f(\rho_L), p(\rho_L-)\}$ satisfy 5. This is the situation considered in the Example 2 in [4]. Observe that such solutions are distinct as far as they correspond to distinct constraint levels $\bar{p}$, and that in any case there exist at least two distinct nonclassical solutions.

  (CNN5): If $\bar{\rho} < \rho_R \leq \rho_L$, $f(\rho_R) \leq p(\rho_L-)$, $p(\rho_L-) \neq p(\rho_L+)$ and $p(\rho_L+) \leq f(\rho_L)$, then the classical solution $[(t,x) \mapsto \mathcal{R}[\rho_L, \rho_R](x/t)]$, which consists of a possible null rarefaction on the left of the constraint, as well as the nonclassical solutions of the form 12 corresponding to $\bar{p} \in \{p(\rho_L+), f(\rho_L)\}$ satisfy 5. Observe that the two nonclassical solutions are distinct as far as they correspond to distinct constraint levels $\bar{p}$, and that in any case there exist at least two distinct solutions, one classical and one nonclassical.

  (NNN5): If $\bar{\rho} < \rho_R < \rho_L$, $f(\rho_R) > p(\rho_L-) \geq f(\rho_L) \geq p(\rho_L+)$ and $p(\rho_L-) \neq p(\rho_L+)$, then the nonclassical solutions of the form 12 corresponding to $\bar{p} \in \{p(\rho_L+), f(\rho_L), p(\rho_L-)\}$ satisfy 5. Observe that such solutions are distinct as far as they correspond to distinct constraint levels $\bar{p}$, and that in any case there exist at least two distinct nonclassical solutions.

4.2. **Proof of Proposition 3.**

(R1) Any solution given by $\mathcal{R}^\star$ coincides on each side of the constraint with a solution given by the classical Riemann solver $\mathcal{R}$. Therefore it satisfies the

Rankine–Hugoniot jump condition along any of its discontinuities away from the constraint. Finally, by definition of $\hat{\rho}$ and $\check{\rho}$, it satisfies the Rankine–Hugoniot jump condition also along the constraint.

(R2) It is clear by the proof of Proposition 2.

(R3) It is proved as in **(R1)** since any classical solution is in **BV**.

(R4) As was proved in [7], $\mathcal{R}^\star$ is continuous on $\mathcal{C} \cup \mathcal{N}$. If (ρ_L, ρ_R) is not in $\mathcal{C} \cup \mathcal{N}$ then p experiences a jump at $\xi = \rho_L$. Therefore, the local in time solutions of the Riemann problem for the initial conditions $(\rho_L + \varepsilon, \rho_R)$ and $(\rho_L - \varepsilon, \rho_R)$ are different and only one of the two converges to $\mathcal{R}^\star[\rho_L, \rho_R]$ as $\varepsilon > 0$ goes to zero.

(R5) We first stress once again that we can discuss the consistency property of our Riemann solvers only locally in time because, in general, the solutions may be not even self–similar globally in time. However, locally in time, the efficiency of the exit can be assumed to be constant and it is thus sufficient to proceed as in [7].

(R6) It is clear by the proof of Proposition 2.

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