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DISEASE DYNAMICS FOR THE HOMETOWN OF MIGRANT WORKERS

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Abstract. A recent paper by L. Wang, X. Wang *J. Theoret. Biol.* 300:100–109 (2012) formulated and studied a delay differential equation model for disease dynamics in a region where a portion of the population leaves to work in a different region for an extended fixed period. Upon return, a fraction of the migrant workers have become infected with the disease. The global dynamics were not fully resolved in that paper, but are resolved here. We show that for all parameter values and all delays, the unique equilibrium is globally asymptotically stable, implying that the disease will eventually reach a constant positive level in the population.

1. **Introduction.** Throughout the world, there are millions of migrant workers, who travel to a location away from home to work for an extended period before returning home again. As an example, approximately 4 million migrants from Nepal work in India and an additional 3.2 million work in other countries [5].

In Nepal, the first instance of HIV/AIDS was reported in 1988. By December 2011, over 19,000 cases of HIV infection were officially reported. It is estimated that in 2011 there were approximately 50,200 people living in Nepal with HIV, and that approximately 60% of those were unaware of their infection status [7].

Compared to similar non-migrant groups, male labor migrants and female sex workers who returned from India, especially from Mumbai, showed a higher prevalence of HIV. At one point in the early 2000s, 10% of men returning from Mumbai to the far-west of Nepal were infected with HIV, compared with 3% or less of non-migrant men. Similarly, in 1999 to 2000 the prevalence of HIV among sex workers in Kathmandu was 17%, while 73% of sex workers returning from Mumbai were infected [4].

Clearly, it is important to account for the fact that migrant workers often become infected while abroad, and then bring the infection back to their hometowns.

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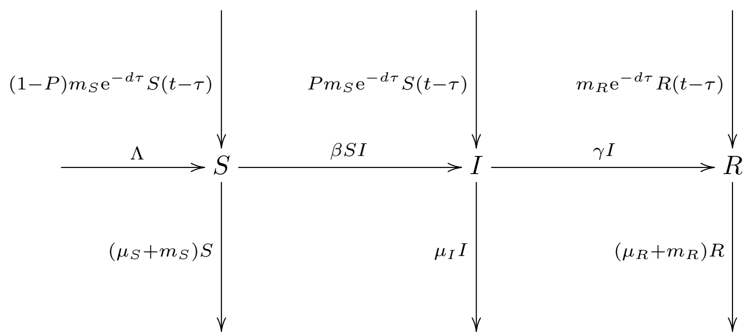
2. **Delay model for migrant workers.** We consider a model that was first studied in [6]. Consider a town of size N from which workers migrate to become temporary workers in another location. The population in the town is divided into three groups; susceptible, infected and recovered with respective sizes $S(t)$, $I(t)$ and $R(t)$. Recruitment of new individuals is into the susceptible class at a constant rate Λ. Mass action incidence is assumed with coefficient β. The per capita death rates of the susceptible, infectious and recovered groups are given by the coefficients μ_S,μ_I and μ_R, respectively. Infected individuals recover with permanent immunity with rate coefficient γ.

Infectious individuals do not migrate. Susceptible and recovered individuals migrate with rate coefficients m_S and m_R, respectively. Migrants spend a duration τ away before returning to their home town. While away, the per capita death rate of the migrants is given by coefficient d. This means that only a fraction $\mathrm{e}^{-d\tau}$ of the migrants return.

It is assumed that a fraction P of the susceptibles that migrate are infected upon return; the remaining fraction $1-P$ are still susceptible upon return. (We ignore the situation whereby susceptibles return to the home town as recovereds. This possibility can be obtained from our results by re-interpreting certain parameters.)

All of the parameters are assumed to positive with $P \in (0, 1)$. The transfer diagram of the model is given below.



The system of delay differential equations for the model is

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - (\mu_S + m_S)S(t) - \beta S(t)I(t) + (1-P)m_S\mathrm{e}^{-d\tau}S(t-\tau)\\ \frac{dI}{dt} &= \beta S(t)I(t) - (\mu_I + \gamma)I(t) + Pm_S\mathrm{e}^{-d\tau}S(t-\tau)\\ \frac{dR}{dt} &= \gamma I(t) - (\mu_R + m_R)R(t) + m_R\mathrm{e}^{-d\tau}R(t-\tau).\end{aligned} \tag{1}$$

The first two equations of (1) do not contain R, so they can be studied in isolation giving

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - (\mu_S + m_S)S(t) - \beta S(t)I(t) + (1-P)m_S\mathrm{e}^{-d\tau}S(t-\tau)\\ \frac{dI}{dt} &= \beta S(t)I(t) - (\mu_I + \gamma)I(t) + Pm_S\mathrm{e}^{-d\tau}S(t-\tau).\end{aligned} \tag{2}$$

The initial conditions for (2) are

$$\begin{aligned}&S(\theta) = \phi(\theta) \geq 0\\ &I(0) > 0,\end{aligned} \tag{3}$$

where $\phi : [-\tau, 0] \to \mathbb{R}_{\geq 0}$ is Lebesgue integrable. The following theorem summarizes some key results from [6].

Theorem 2.1 (Wang and Wang, 2012)**.** *For each initial value (3), a unique solution to (2) exists globally and is positive and bounded. There exists a unique equilibrium* (S^*, I^*)*, which satisfies* $S^*, I^* > 0$ *and is locally asymptotically stable. Furthermore, if*

$$\frac{\beta\Lambda}{[\mu_S + m_S - (1-P)m_S e^{-d\tau}](\mu_I + \gamma)} < 1, \tag{4}$$

then the equilibrium is globally asymptotically stable.

In reading [6], it becomes apparent that condition (4) is an artifact of the proof, rather than a sharp threshold. In the next section, we show that the condition is not necessary and that the unique equilibrium is always globally asymptotically stable.

3. Global stability of the unique equilibrium.

We now prove that the unique equilibrium is always globally asymptotically stable, using a Lyapunov functional similar to that used in [2, 3] and other works.

Theorem 3.1. *The equilibrium* (S^*, I^*) *is globally asymptotically stable.*

Proof. Let

$$g(x) = x - 1 - \ln x \qquad \text{and} \qquad \begin{aligned} V_S(t) &= S^* g\left(\frac{S(t)}{S^*}\right) \\ V_I(t) &= I^* g\left(\frac{I(t)}{I^*}\right) \\ W(t) &= \int_0^\tau g\left(\frac{S(t-s)}{S^*}\right) ds \end{aligned} \tag{5}$$

and

$$U(t) = V_S(t) + V_I(t) + m_S \mathrm{e}^{-d\tau} S^* W(t). \tag{6}$$

We note that $g : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is monotone on each side of 1, with a strict minimum $g(1) = 0$. Thus, U is positive and is strictly minimized at the unique equilibrium.

We now work to show that $\frac{dU}{dt} \leq 0$. In order to calculate $\frac{dU}{dt}$, we will use the equilibrium equations

$$\begin{aligned} 0 &= \Lambda - (\mu_S + m_S)S^* - \beta S^* I^* + (1-P)m_S \mathrm{e}^{-d\tau} S^* \\ 0 &= \beta S^* I^* - (\mu_I + \gamma)I^* + P m_S \mathrm{e}^{-d\tau} S^*. \end{aligned} \tag{7}$$

In what follows, we use the notation $S = S(t)$, $I = I(t)$ and $S_\tau = S(t-\tau)$. We begin by calculating $\frac{dV_S}{dt}$.

$$\begin{aligned} \frac{dV_S}{dt} &= \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} \\ &= \left(1 - \frac{S^*}{S}\right)\left[\Lambda - (\mu_S + m_S)S(t) - \beta S(t)I(t) + (1-P)m_S \mathrm{e}^{-d\tau} S(t-\tau)\right]. \end{aligned}$$

Using (7) to replace Λ gives

$$\begin{aligned}
\frac{dV_S}{dt} &= \left(1-\frac{S^*}{S}\right)\big[(\mu_S+m_S)S^*+\beta S^*I^*-(1-P)m_S\mathrm{e}^{-d\tau}S^* \\
&\qquad\qquad - (\mu_S+m_S)S-\beta SI+(1-P)m_S\mathrm{e}^{-d\tau}S_\tau\big] \\
&= \left(1-\frac{S^*}{S}\right)\left[(\mu_S+m_S)S^*\left(1-\frac{S}{S^*}\right)+\beta S^*I^*\left(1-\frac{SI}{S^*I^*}\right)\right. \\
&\qquad\qquad \left.+(1-P)m_S\mathrm{e}^{-d\tau}S^*\left(\frac{S_\tau}{S^*}-1\right)\right] \\
&= -(\mu_S+m_S)\frac{(S-S^*)^2}{S}+\beta S^*I^*\left(1-\frac{S^*}{S}-\frac{SI}{S^*I^*}+\frac{I}{I^*}\right) \\
&\qquad\qquad +(1-P)m_S\mathrm{e}^{-d\tau}S^*\left(\frac{S_\tau}{S^*}-\frac{S_\tau}{S}-1+\frac{S^*}{S}\right).
\end{aligned}$$

We now calculate $\frac{dV_I}{dt}$, using (7) to replace $(\mu_I+\gamma)$.

$$\begin{aligned}
\frac{dV_I}{dt} &= \left(1-\frac{I^*}{I}\right)\frac{dI}{dt} \\
&= \left(1-\frac{I^*}{I}\right)\big[\beta SI-(\mu_I+\gamma)I+Pm_S\mathrm{e}^{-d\tau}S(t-\tau)\big] \\
&= \left(1-\frac{I^*}{I}\right)\left[\beta SI-\left(\beta S^*+Pm_S\mathrm{e}^{-d\tau}\frac{S^*}{I^*}\right)I+Pm_S\mathrm{e}^{-d\tau}S(t-\tau)\right] \\
&= \left(1-\frac{I^*}{I}\right)\left[\beta S^*I^*\left(\frac{SI}{S^*I^*}-\frac{I}{I^*}\right)+Pm_S\mathrm{e}^{-d\tau}S^*\left(\frac{S_\tau}{S^*}-\frac{I}{I^*}\right)\right] \\
&= \beta S^*I^*\left[\frac{SI}{S^*I^*}-\frac{I}{I^*}-\frac{S}{S^*}+1\right]+Pm_S\mathrm{e}^{-d\tau}S^*\left[\frac{S_\tau}{S^*}-\frac{I}{I^*}-\frac{S_\tau I^*}{S^*I}+1\right].
\end{aligned}$$

Next, let $H(t)=g(X(t))$ and $X(t)=\frac{S(t)}{S^*}$. Then

$$\begin{aligned}
\frac{dW}{dt} &= \frac{d}{dt}\left(\int_0^\tau H(t-s)\,ds\right) \\
&= \int_0^\tau \left[\frac{d}{dt}H(t-s)\right]ds \\
&= \int_0^\tau \left[-\frac{d}{ds}H(t-s)\right]ds \\
&= H(t)-H(t-\tau) \\
&= g(X(t))-g(X(t-\tau)) \\
&= g\left(\frac{S}{S^*}\right)-g\left(\frac{S(t-\tau)}{S^*}\right) \\
&= \frac{S}{S^*}-\frac{S_\tau}{S^*}-\ln\left(\frac{S}{S^*}\right)+\ln\left(\frac{S_\tau}{S^*}\right).
\end{aligned}$$

To make the expressions more concise, we introduce $Q=m_S\mathrm{e}^{-d\tau}S^*$. Then, (6) implies

$$\frac{dU}{dt}=\frac{dV_S}{dt}+\frac{dV_I}{dt}+Q\frac{dW}{dt}$$

$$
\begin{aligned}
&= -(\mu_S + m_S)\frac{(S-S^*)^2}{S} + \beta S^* I^* \left[2 - \frac{S^*}{S} - \frac{S}{S^*}\right] \\
&\quad + (1-P)Q\left[\frac{S_\tau}{S^*} - \frac{S_\tau}{S} - 1 + \frac{S^*}{S}\right] + PQ\left[\frac{S_\tau}{S^*} - \frac{I}{I^*} - \frac{S_\tau I^*}{S^* I} + 1\right] \\
&\quad + Q\left[\frac{S}{S^*} - \frac{S_\tau}{S^*} - \ln\left(\frac{S}{S^*}\right) + \ln\left(\frac{S_\tau}{S^*}\right)\right].
\end{aligned}
$$

Note that $Q = (1-P)Q + PQ$, so that the finally group of terms can be added to the two prior terms. Doing this, and then using properties of logarithms, we find

$$
\begin{aligned}
\frac{dU}{dt} &= -(\mu_S + m_S)\frac{(S-S^*)^2}{S} + \beta S^* I^* \left[2 - \frac{S^*}{S} - \frac{S}{S^*}\right] \\
&\quad + (1-P)Q\left[\frac{S}{S^*} - \frac{S_\tau}{S} - 1 + \frac{S^*}{S} - \ln\left(\frac{S}{S^*}\right) + \ln\left(\frac{S_\tau}{S^*}\right)\right] \\
&\quad + PQ\left[\frac{S}{S^*} - \frac{I}{I^*} - \frac{S_\tau I^*}{S^* I} + 1 - \ln\left(\frac{S}{S^*}\right) + \ln\left(\frac{S_\tau}{S^*}\right)\right] \\
&= -(\mu_S + m_S)\frac{(S-S^*)^2}{S} + \beta S^* I^* \left[2 - \frac{S^*}{S} - \frac{S}{S^*}\right] \\
&\quad + (1-P)Q\left[\frac{S}{S^*} + \frac{S^*}{S} - 2 - g\left(\frac{S_\tau}{S}\right)\right] \\
&\quad + PQ\left[\frac{S}{S^*} + \frac{S^*}{S} - 2 - g\left(\frac{S^*}{S}\right) - g\left(\frac{I}{I^*}\right) - g\left(\frac{S_\tau I^*}{S^* I}\right)\right] \\
&= -(\mu_S + m_S)\frac{(S-S^*)^2}{S} + \beta S^* I^* \left[2 - \frac{S^*}{S} - \frac{S}{S^*}\right] \\
&\quad - (1-P)Qg\left(\frac{S_\tau}{S}\right) - PQ\left[g\left(\frac{S^*}{S}\right) + g\left(\frac{I}{I^*}\right) + g\left(\frac{S_\tau I^*}{S^* I}\right)\right] \\
&\quad + Q\left[\frac{S}{S^*} + \frac{S^*}{S} - 2\right].
\end{aligned}
$$

Recalling that $Q = m_S e^{-d\tau} S^*$, and that $\frac{S}{S^*} + \frac{S^*}{S} - 2 = \frac{(S-S^*)^2}{SS^*}$, it follows that $Q\left[\frac{S}{S^*} + \frac{S^*}{S} - 2\right] \le m_S \frac{(S-S^*)^2}{S}$, and so

$$
\begin{aligned}
\frac{dU}{dt} &\le -\mu_S \frac{(S-S^*)^2}{S} + \beta S^* I^* \left[2 - \frac{S^*}{S} - \frac{S}{S^*}\right] \\
&\quad - (1-P)Qg\left(\frac{S_\tau}{S}\right) - PQ\left[g\left(\frac{S^*}{S}\right) + g\left(\frac{I}{I^*}\right) + g\left(\frac{S_\tau I^*}{S^* I}\right)\right] \\
&\le 0.
\end{aligned}
$$

Furthermore, we only have $\frac{dU}{dt} = 0$ if $S = S^*$ and $I = I^*$. Thus, the largest invariant set on which $\frac{dU}{dt}$ is zero consists of just the equilibrium. Therefore, by LaSalle's Invariance Principle [1], the equilibrium is globally asymptotically stable. □

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