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R_0 AND THE GLOBAL BEHAVIOR OF AN AGE-STRUCTURED SIS EPIDEMIC MODEL WITH PERIODICITY AND VERTICAL TRANSMISSION

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Abstract. In this paper, we study an age-structured SIS epidemic model with periodicity and vertical transmission. We show that the spectral radius of the Fréchet derivative of a nonlinear integral operator plays the role of a threshold value for the global behavior of the model, that is, if the value is less than unity, then the disease-free steady state of the model is globally asymptotically stable, while if the value is greater than unity, then the model has a unique globally asymptotically stable endemic (nontrivial) periodic solution. We also show that the value can coincide with the well-know epidemiological threshold value, the basic reproduction number $\mathcal{R}_0$.

1. **Introduction.** Seasonal fluctuations in the incidence of infectious diseases is an important aspect of epidemics occurrence and an interesting subject in the field of mathematical epidemiology. In this context, systems of nonlinear differential equations with periodic coefficients are a natural mathematical tool for modeling purposes and several authors have adopted this approach to explain the periodic outbreak and the oscillations in the endemic presence of a disease in a population [2, 3, 12, 13, 15, 16, 18, 19, 21].

The basic reproduction number $\mathcal{R}_0$, which is defined as the expected number of secondary cases produced by a typical infected individual during its entire period of infectiousness in a completely susceptible population [8], is known as a good indicator of the future spread pattern of disease. That is, it can be expected that a disease dies out if $\mathcal{R}_0 < 1$, while it remains endemic if $\mathcal{R}_0 > 1$. The mathematical definition of $\mathcal{R}_0$ as the spectral radius of a linear integral operator called the next generation operator was firstly given for autonomous cases [8], and has recently been extended to periodic cases [2, 3, 19] and to more general nonautonomous cases [12, 18]. Since $\mathcal{R}_0$ is defined for linearized systems around the disease-free steady states, it plays the role of a threshold for the local behavior of the original systems.

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However, whether $\mathcal{R}_0$ plays the same role of the threshold and determines the global behavior of these systems is generally an open question, and thus, we have to clarify the relation between $\mathcal{R}_0$ and the global behavior of epidemic systems for each case. In this paper, we investigate such relation for an age-structured SIS epidemic model with periodicity and vertical transmission.

The global behavior of age-structured SIS epidemic models without periodicity was successfully investigated in [4, 5, 6, 9], while the periodic case was investigated in [15], where it has been proved that an endemic (nontrivial) periodic solution is unique and even globally stable if it exists. However, no threshold-like condition for the existence of such a solution was given. The purpose of this paper is to show that the basic reproduction number $\mathcal{R}_0$, obtained for our periodic age-structured SIS epidemic model, plays the role of such a threshold and, applying the previous significant results in [15], we see that $\mathcal{R}_0$ is a threshold for the global behavior of the model. To our knowledge, this is the first study of the relation between $\mathcal{R}_0$ and the global behavior of an age-structured SIS epidemic model with periodicity and vertical transmission (see [11] for the analysis of another age-structured epidemic model with vertical transmission). In particular, this is in contrast to the instability results obtained for age-structured SIR epidemic models (see, for instance, [1, 7, 17]). Here we note that in this paper we use the SIS epidemic model as a case study.

The organization of this paper is as follows. In Section 2, we formulate the model and normalize it. In Section 3, we show the well-posedness of our problem. In Section 4, we show that if the spectral radius $\rho(\mathcal{F})$ of a linear operator $\mathcal{F}$ is greater than unity, then the normalized system has an endemic periodic solution. Moreover, applying the results obtained in [15], we obtain uniqueness and global stability results for the periodic solution . In Section 5, we show that if the spectral radius $\rho(\mathcal{F})$ is less than unity, then the disease-free of the normalized system is globally asymptotically stable. In Section 6, we investigate the relation between our threshold value $\rho(\mathcal{F})$ and the basic reproduction number $\mathcal{R}_0$.

2. A basic model.

Let $p(a,t)$ be the age-density of the host population at time t ($a \in [0, a_\dagger]$ and $t \geq 0$, where $a_\dagger \in (0, +\infty)$ is the maximum age for the population). Let $\mu(a)$ be the age-specific mortality rate and $\beta(a)$ be the age-specific birth rate. Let us assume that the host population dynamics is described by the following von Foerster equation with initial and boundary conditions:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial t} + \dfrac{\partial p(a,t)}{\partial a} + \mu(a)\, p(a,t) = 0, \\ p(a,0) = p_0(a), \quad p(0,t) = \displaystyle\int_0^{a_\dagger} \beta(\sigma)\, p(\sigma,t)\, \mathrm{d}\sigma, \end{cases} \tag{1}$$

where $p_0(a)$ is the given initial age distribution. μ and β are assumed to be nonnegative and measurable. In addition, β is assumed to be uniformly bounded above, and the (demographic) basic reproduction number of the host population is assumed to satisfy

$$R := \int_0^{a_\dagger} \beta(a)\, \mathrm{e}^{-\int_0^a \mu(\sigma)\mathrm{d}\sigma} \mathrm{d}a = 1.$$

Thus, the demographic steady state

$$p^*(a) := b_0 \mathrm{e}^{-\int_0^a \mu(\sigma)\mathrm{d}\sigma} \tag{2}$$

of system (1) exists, where

$$b_0 = \int_0^{a_\dagger} \beta(\sigma) p^*(\sigma) \mathrm{d}\sigma.$$

Since we can scale the size of b_0 arbitrary, we set

$$b_0 = \frac{1}{\int_0^{a_\dagger} \mathrm{e}^{-\int_0^a \mu(\sigma)\mathrm{d}\sigma} \mathrm{d}a}, \tag{3}$$

and obtain $\int_0^{a_\dagger} p^*(a) \mathrm{d}a = 1$ from (2). Note that $p^*(a)$ is the solution of

$$\begin{cases} \dfrac{\mathrm{d}p^*(a)}{\mathrm{d}a} + \mu(a) p^*(a) = 0, \\ p^*(0) = \displaystyle\int_0^{a_\dagger} \beta(\sigma) p^*(\sigma) \mathrm{d}\sigma = b_0. \end{cases} \tag{4}$$

In what follows, we assume that the density of the host population has reached the steady state, that is, $p(a,t) \equiv p^*(a)$.

As far as the epidemics is concerned, the host population is divided into the two epidemiological subclasses of susceptibles $s(a,t)$ and infectives $i(a,t)$. That is, we have

$$p^*(a) = s(a,t) + i(a,t). \tag{5}$$

For each $t \geq 0$, let $\gamma(a,t)$ be the age-specific recovery rate, $\lambda(a,t)$ be the force of infection to susceptible individuals aged a and $k(\sigma,a,t)$ be the transmission coefficient between susceptible individuals aged a and infective individuals aged σ. In order to model the vertical transmission process of the disease, we introduce a coefficient $q \in (0,1)$ which is the proportion of newborn offspring of infective parents who are themselves infective. Under this setting, the SIS epidemic model we consider in this paper is formulated as follows:

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial t} + \dfrac{\partial s(a,t)}{\partial a} + \mu(a) s(a,t) = -\lambda(a,t) s(a,t) + \gamma(a,t) i(a,t), \\ \dfrac{\partial i(a,t)}{\partial t} + \dfrac{\partial i(a,t)}{\partial a} + \mu(a) i(a,t) = \lambda(a,t) s(a,t) - \gamma(a,t) i(a,t), \\ \lambda(a,t) = \displaystyle\int_0^{a_\dagger} k(\sigma,a,t) i(\sigma,t) \mathrm{d}\sigma, \\ s(0,t) = \displaystyle\int_0^{a_\dagger} \beta(\sigma) \{s(\sigma,t) + (1-q) i(\sigma,t)\} \mathrm{d}\sigma, \\ i(0,t) = q \displaystyle\int_0^{a_\dagger} \beta(\sigma) i(\sigma,t) \mathrm{d}\sigma, \quad s(a,0) = s_0(a), \quad i(a,0) = i_0(a), \end{cases} \tag{6}$$

where $s_0(a)$ and $i_0(a)$ are given initial distributions. γ and k are assumed to be nonnegative, measurable and uniformly bounded above by positive constants $\gamma^+ < +\infty$ and $k^+ < +\infty$, respectively. In addition, in order to model the seasonally fluctuating process of the disease, γ and k are assumed to be time-periodic with common period $T > 0$, that is,

$$\gamma(a,t) = \gamma(a,t+T), \quad k(\sigma,a,t) = k(\sigma,a,t+T)$$

for all a, t and σ.

From (5), we have $s(a,t) = p^*(a) - i(a,t)$. Hence, substituting this into (6), we can obtain the following single equation for $i(a,t)$:

$$\begin{cases} \dfrac{\partial i(a,t)}{\partial t} + \dfrac{\partial i(a,t)}{\partial a} + \mu(a)\, i(a,t) = \lambda(a,t)\{p^*(a) - i(a,t)\} - \gamma(a,t)\, i(a,t), \\ \lambda(a,t) = \displaystyle\int_0^{a_\dagger} k(\sigma,a,t)\, i(\sigma,t)\, \mathrm{d}\sigma, \\ i(0,t) = q\displaystyle\int_0^{a_\dagger} \beta(\sigma)\, i(\sigma,t)\, \mathrm{d}\sigma, \quad i(a,0) = i_0(a). \end{cases}$$

Using $u(a,t) := i(a,t)/p^*(a)$ and equation (4), the system is normalized as

$$\begin{cases} \dfrac{\partial u(a,t)}{\partial t} + \dfrac{\partial u(a,t)}{\partial a} = \lambda(a,t)\{1 - u(a,t)\} - \gamma(a,t)\, u(a,t), \\ \lambda(a,t) = \displaystyle\int_0^{a_\dagger} \kappa(\sigma,a,t)\, u(\sigma,t)\, \mathrm{d}\sigma, \\ u(0,t) = q\displaystyle\int_0^{a_\dagger} g(\sigma)\, u(\sigma,t)\, \mathrm{d}\sigma, \quad u(a,0) = u_0(a), \end{cases} \tag{7}$$

where

$$\kappa(\sigma,a,t) := k(\sigma,a,t)\, p^*(\sigma), \quad g(\sigma) := \frac{\beta(\sigma)\, p^*(\sigma)}{\int_0^{a_\dagger} \beta(\sigma)\, p^*(\sigma)\, \mathrm{d}\sigma}. \tag{8}$$

and the solution must satisfy

$$0 \le u(a,t) \le 1\ .$$

Note that, from the definition, κ is also T-periodic with respect to time t.

The global behavior of system (7) is the main problem we consider in the following sections.

3. **Abstract formulation.**

Let $E := L^1(0,a_\dagger)$. System (7) can be formulated as the abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) + F(t,u(t)), \quad u(0) = u_0 \tag{9}$$

in E, where A is a linear operator on E defined as

$$\begin{cases} (A\varphi)(a) := -\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi(a), \\ D(A) := \left\{\varphi \in E : \ \varphi \in W^{1,1}(0,a_\dagger),\ \varphi(0) = q\displaystyle\int_0^{a_\dagger} g(\sigma)\,\varphi(\sigma)\,\mathrm{d}\sigma\right\} \end{cases} \tag{10}$$

and $\{F(t,\cdot)\}_{t\in\mathbb{R}_+}$ is a family of nonlinear operators on E

$$(F(t,\varphi))(a) := \lambda[a,t|\varphi]\{1 - \varphi(a)\} - \gamma(a,t)\,\varphi(a), \tag{11}$$

where

$$\lambda[a,t|\varphi] := \int_0^{a_\dagger} \kappa(\sigma,a,t)\,\varphi(\sigma,t)\,\mathrm{d}\sigma. \tag{12}$$

It is easy to see that the convex feasible region

$$C := \{\varphi \in E : \ 0 \le \varphi(a) \le 1 \text{ a.e.}\} \tag{13}$$

is positively invariant under the strongly continuous semigroup $\left\{\mathrm{e}^{tA}\right\}_{t\in\mathbb{R}_+}$ defined by

$$\left(\mathrm{e}^{tA}\varphi\right)(a) := \begin{cases} \varphi(a-t), & a > t, \\ b(t-a), & t > a, \end{cases} \tag{14}$$

where b is the solution of integral equation

$$b(t) = \begin{cases} q\int_0^t g(\sigma)\, b(t-\sigma)\, \mathrm{d}\sigma + q\int_t^{a_\dagger} g(\sigma)\, \varphi(\sigma - t)\, \mathrm{d}\sigma, & t < a_\dagger, \\ q\int_0^{a_\dagger} g(\sigma)\, b(t-\sigma)\, \mathrm{d}\sigma, & t > a_\dagger. \end{cases} \tag{15}$$

That is, $\mathrm{e}^{tA}(C) \subset C$. We assume that the domain of $\{F(t,\cdot)\}_{t\in\mathbb{R}_+}$ is limited to $C \subset E$. As in [4], we can prove the following Lemma:

Lemma 3.1. *(i) $F(t,\cdot) : C \subset E \to E$ is Lipschitz continuous for any fixed $t \in \mathbb{R}_+$.*
(ii) There exists a positive constant $\alpha \in (0,1)$ such that $\varphi + \alpha F(t,\varphi) \in C$ for all $\varphi \in C$ and $t \in \mathbb{R}_+$.

The proof is omitted here (see the proof of Proposition 3.1 of [4]). Using α in (ii) of Lemma 3.1, we rewrite problem (9) as

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \left(A - \frac{1}{\alpha}\right)u(t) + \frac{1}{\alpha}\{u(t) + \alpha F(t, u(t))\}, \quad u(0) = u_0. \tag{16}$$

The mild solution of (16) is given by the solution of the integral equation

$$u(t) = \mathrm{e}^{-\frac{1}{\alpha}t}\mathrm{e}^{tA}u_0 + \frac{1}{\alpha}\int_0^t \mathrm{e}^{-\frac{1}{\alpha}(t-\sigma)}\mathrm{e}^{(t-\sigma)A}\{u(\sigma) + \alpha F(\sigma, u(\sigma))\}\,\mathrm{d}\sigma. \tag{17}$$

Consider the scheme

$$\begin{cases} u^0(t) := u_0, \\ u^{n+1}(t) := \mathrm{e}^{-\frac{1}{\alpha}t}\mathrm{e}^{tA}u_0 + \frac{1}{\alpha}\int_0^t \mathrm{e}^{-\frac{1}{\alpha}(t-\sigma)}\mathrm{e}^{(t-\sigma)A}\{u^n(\sigma) + \alpha F(\sigma, u^n(\sigma))\}\,\mathrm{d}\sigma, \\ n = 0, 1, 2, \ldots \end{cases} \tag{18}$$

for the standard iterative procedure. From Lemma 3.1, we see that $u^{n+1} \in C$ if $u^n \in C$. Hence, according to the argument in [4], we can prove the following proposition:

Proposition 1. *For $u_0 \in C$, problem (9) has a unique mild solution in C, which defines a flow $S(t)u_0$ satisfying $S(t)(C) \subset C$.*

4. **Existence of an endemic periodic solution.** In this section, we investigate the existence of an endemic periodic solution u^* to system (7). Such a solution must satisfy

$$\begin{cases} \dfrac{\partial u^*(a,t)}{\partial t} + \dfrac{\partial u^*(a,t)}{\partial a} = \lambda^*(a,t)\{1 - u^*(a,t)\} - \gamma(a,t)\,u^*(a,t), \\ \lambda^*(a,t) = \displaystyle\int_0^{a_\dagger} \kappa(\sigma,a,t)\, u^*(\sigma,t)\,\mathrm{d}\sigma, \\ u^*(0,t) = q\displaystyle\int_0^{a_\dagger} g(\sigma)\, u^*(\sigma,t)\,\mathrm{d}\sigma, \quad u^*(a,t) = u^*(a,t+T). \end{cases} \tag{19}$$

Integrating the first equation of (19) along the characteristic lines, we have

$$\begin{aligned} u^*(a,t) &= \int_0^a \lambda^*(\sigma, t-a+\sigma)\, \mathrm{e}^{-\int_\sigma^a \{\lambda^*(\rho,t-a+\rho)+\gamma(\rho,t-a+\rho)\}\mathrm{d}\rho}\mathrm{d}\sigma \\ &\quad + \mathrm{e}^{-\int_0^a \{\lambda^*(\rho,t-a+\rho)+\gamma(\rho,t-a+\rho)\}\mathrm{d}\rho}\; u^*(0,t-a) \\ &= \int_0^a \lambda^*(a-\tau, t-\tau)\, \mathrm{e}^{-\int_0^\tau \{\lambda^*(a-\eta,t-\eta)+\gamma(a-\eta,t-\eta)\}\mathrm{d}\eta}\mathrm{d}\tau \\ &\quad + \mathrm{e}^{-\int_0^a \{\lambda^*(a-\eta,t-\eta)+\gamma(a-\eta,t-\eta)\}\mathrm{d}\eta}\; u^*(0,t-a)\,. \end{aligned} \tag{20}$$

Substituting (20) into the second and third equations of (19), we have

$$\begin{aligned} &\lambda^*(a,t) = \\ &\quad \int_0^{a_\dagger} \kappa(\sigma,a,t) \int_0^\sigma \lambda^*(\sigma-\tau,t-\tau)\, \mathrm{e}^{-\int_0^\tau \{\lambda^*(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma \\ &\quad + \int_0^{a_\dagger} \kappa(\sigma,a,t)\, \mathrm{e}^{-\int_0^\sigma \{\lambda^*(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\; u^*(0,t-\sigma)\,\mathrm{d}\sigma \end{aligned} \tag{21}$$

and

$$\begin{aligned} &u^*(0,t) = \\ &\quad q\int_0^{a_\dagger} g(\sigma) \int_0^\sigma \lambda^*(\sigma-\tau,t-\tau)\, \mathrm{e}^{-\int_0^\tau \{\lambda^*(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma \\ &\quad +q\int_0^{a_\dagger} g(\sigma)\, \mathrm{e}^{-\int_0^\sigma \{\lambda^*(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\; u^*(0,t-\sigma)\,\mathrm{d}\sigma, \end{aligned} \tag{22}$$

respectively. Thus, if we find nontrivial, positive, T-periodic λ^* and $u^*(0,\cdot)$, satisfying (21) and (22), by (20) we also obtain a nontrivial positive T-periodic u^*, because so are λ^* and $u^*(0,\cdot)$. Such u^* can be regarded as the desired endemic periodic solution of system (7) in a weak sense, namely in (7) the differential operator $\partial_t + \partial_a$ is interpreted as the directional derivative D

$$(D\varphi)(a,t) := \lim_{h\to 0} \frac{\varphi(a+h,t+h) - \varphi(a,t)}{h}.$$

Thus, in what follows, we look for nontrivial positive periodic λ^* and $u^*(0,\cdot)$ satisfying (21) and (22).

Let

- X_T be the set of locally integrable T-periodic E-valued functions;
- $X_{T,+}$ be the positive cone of X_T;
- Y_T be the set of locally integrable T-periodic real-valued functions;
- $Y_{T,+}$ be the positive cone of Y_T;
- $\tilde{Y}_{T,+} := \{\varphi \in Y_{T,+} :\ 0 \le \varphi(t) \le 1 \text{ a.e.}\}$.

The sets X_T and Y_T are actually Banach spaces when respectively endowed with the norms

$$||\varphi||_{X_T} := \int_0^T ||\varphi(t)||_E\, \mathrm{d}t = \int_0^T \int_0^{a_\dagger} |\varphi(a,t)|\,\mathrm{d}a\mathrm{d}t$$

and

$$||\varphi||_{Y_T} := \int_0^T |\varphi(t)|\,\mathrm{d}t\ .$$

Let us define a nonlinear positive operator $\Phi : X_T \times Y_T \to X_T \times Y_T$ as

$$\Phi(\varphi_1,\varphi_2) := (\Phi_1(\varphi_1,\varphi_2), \Phi_2(\varphi_1,\varphi_2)), \tag{23}$$

where

$$
\begin{aligned}
&(\Phi_1(\varphi_1,\varphi_2))(a,t) = \\
&\quad \int_0^{a_\dagger} \kappa(\sigma,a,t) \int_0^{\sigma} \varphi_1(\sigma-\tau,t-\tau)\, \mathrm{e}^{-\int_0^{\tau}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \mathrm{d}\tau \mathrm{d}\sigma \\
&\quad + \int_0^{a_\dagger} \kappa(\sigma,a,t)\, \mathrm{e}^{-\int_0^{\sigma}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\ \varphi_2(t-\sigma)\,\mathrm{d}\sigma
\end{aligned} \tag{24}
$$

and

$$
\begin{aligned}
&(\Phi_2(\varphi_1,\varphi_2))(t) = \\
&\quad q\int_0^{a_\dagger} g(\sigma) \int_0^{\sigma} \varphi_1(\sigma-\tau,t-\tau)\, \mathrm{e}^{-\int_0^{\tau}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \mathrm{d}\tau \mathrm{d}\sigma \\
&\quad + q\int_0^{a_\dagger} g(\sigma)\, \mathrm{e}^{-\int_0^{\sigma}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\ \varphi_2(t-\sigma)\,\mathrm{d}\sigma.
\end{aligned} \tag{25}
$$

Then, from (21) and (22), we see that a nontrivial positive fixed point $(\varphi_1^*,\varphi_2^*) \in (X_{T,+}\setminus\{0\}) \times \left(\tilde{Y}_{T,+}\setminus\{0\}\right)$ of the operator Φ corresponds to the desired λ^* and $u^*(0,\cdot)$. Therefore, in what follows, we look for such a fixed point $(\varphi_1^*,\varphi_2^*)$ of Φ.

First, we investigate the mathematical properties of the operator Φ. We have the following lemma:

Lemma 4.1. *The operator $\Phi : X_T \times Y_T \to X_T \times Y_T$, defined in (23) has the following properties*

(i) $X_{T,+}\times\tilde{Y}_{T,+}$ is positively invariant for Φ, that is,

$$\Phi\left(X_{T,+}\times\tilde{Y}_{T,+}\right) \subset X_{T,+}\times\tilde{Y}_{T,+};$$

(ii) Φ is uniformly bounded on $X_{T,+}\times\tilde{Y}_{T,+}$;

(iii) Φ is monotone nondecreasing on $X_{T,+}\times\tilde{Y}_{T,+}$.

Proof. First we prove (i). Since the inclusion $\Phi_1\left(X_{T,+}\times\tilde{Y}_{T,+}\right) \subset X_{T,+}$ is obvious from (24), it suffices to show that $\Phi_2\left(X_{T,+}\times\tilde{Y}_{T,+}\right) \subset \tilde{Y}_{T,+}$. From (25), we have

$$
\begin{aligned}
\Phi_2(\varphi_1,\varphi_2) &= q\int_0^{a_\dagger} g(\sigma)\,\mathrm{d}\sigma \\
&\quad -q\int_0^{a_\dagger} g(\sigma)\int_0^{\sigma} \gamma(\sigma-\tau,t-\tau)\, \mathrm{e}^{-\int_0^{\tau}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \mathrm{d}\tau \mathrm{d}\sigma \\
&\quad +q\int_0^{a_\dagger} g(\sigma)\, \mathrm{e}^{-\int_0^{\sigma}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\ \{\varphi_2(t-\sigma)-1\}\,\mathrm{d}\sigma,
\end{aligned} \tag{26}
$$

where we have used

$$
\begin{aligned}
&-\int_0^{\sigma} \varphi_1(\sigma-\tau,t-\tau)\, \mathrm{e}^{-\int_0^{\tau}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \mathrm{d}\tau \\
&\qquad = \int_0^{\sigma} \gamma(\sigma-\tau,t-\tau)\, \mathrm{e}^{-\int_0^{\tau}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \mathrm{d}\tau \\
&\qquad\quad + \int_0^{\sigma} \frac{d}{d\,\tau} \mathrm{e}^{-\int_0^{\tau}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \mathrm{d}\tau.
\end{aligned}
$$

Since $\int_0^{a_\dagger} g(\sigma)\,\mathrm{d}\sigma = 1$ and $\varphi_2 \le 1$ a.e., we have $\Phi_2(\varphi_1,\varphi_2) \le q \le 1$ a.e.. Since $\Phi_2(\varphi_1,\varphi_2) \ge 0$ a.e. is obvious from (25), the claim is proved.

Next we prove (ii). From the above proof, we have $\Phi_2\left(X_{T,+}\times\tilde{Y}_{T,+}\right)\leq 1$, hence, we only have to show that there exists a positive constant $M>0$ such that $\Phi_1\left(X_{T,+}\times\tilde{Y}_{T,+}\right)\leq M$. From (24), we have

$$\begin{aligned}\Phi_1\left(\varphi_1,\varphi_2\right)=&\int_0^{a_\dagger}\kappa\left(\sigma,a,t\right)\mathrm{d}\sigma\\&-\int_0^{a_\dagger}\kappa\left(\sigma,a,t\right)\int_0^{\sigma}\gamma\left(\sigma-\tau,t-\tau\right)\mathrm{e}^{-\int_0^{\tau}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma\\&+\int_0^{a_\dagger}\kappa\left(\sigma,a,t\right)\mathrm{e}^{-\int_0^{\sigma}\{\varphi_1(\sigma-\eta,t-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\left\{\varphi_2\left(t-\sigma\right)-1\right\}\mathrm{d}\sigma.\end{aligned}\tag{27}$$

Hence, since $\varphi_2\leq 1$ a.e. because $\varphi_2\in\tilde{Y}_{T,+}$, we have

$$\Phi_1\left(X_{T,+}\times\tilde{Y}_{T,+}\right)\leq\int_0^{a_\dagger}\kappa\left(\sigma,a,t\right)\mathrm{d}\sigma\leq k^+\int_0^{a_\dagger}p^*\left(\sigma\right)\mathrm{d}\sigma=:M$$

and the claim is proved.

(iii) is obvious from (26) and (27) (note that each of their last terms is nonpositive for $\varphi_2\in\tilde{Y}_{T,+}$). □

As in the previous studies [4, 5, 9, 10], we can expect that the spectral radius of the Fréchet derivative $D\Phi[0]$ of Φ at zero plays the role of a threshold for the existence of the desired fixed point $(\varphi_1^*,\varphi_2^*)$ of Φ. Thus, we consider the positive linear operator

$$\mathcal{F}\left(\varphi_1,\varphi_2\right):=\left(\mathcal{F}_1\left(\varphi_1,\varphi_2\right),\mathcal{F}_2\left(\varphi_1,\varphi_2\right)\right)\ \left(=D\Phi\left[0\right]\left(\varphi_1,\varphi_2\right)\right),\tag{28}$$

on $X_T\times Y_T$, where

$$\begin{aligned}&\left(\mathcal{F}_1\left(\varphi_1,\varphi_2\right)\right)\left(a,t\right)=\\&\quad\int_0^{a_\dagger}\kappa\left(\sigma,a,t\right)\int_0^{\sigma}\varphi_1\left(\sigma-\tau,t-\tau\right)\mathrm{e}^{-\int_0^{\tau}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma\\&\quad+\int_0^{a_\dagger}\kappa\left(\sigma,a,t\right)\mathrm{e}^{-\int_0^{\sigma}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\ \varphi_2\left(t-\sigma\right)\mathrm{d}\sigma\end{aligned}\tag{29}$$

and

$$\begin{aligned}&\left(\mathcal{F}_2\left(\varphi_1,\varphi_2\right)\right)\left(t\right)=\\&\quad q\int_0^{a_\dagger}g\left(\sigma\right)\int_0^{\sigma}\varphi_1\left(\sigma-\tau,t-\tau\right)\mathrm{e}^{-\int_0^{\tau}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma\\&\quad+q\int_0^{a_\dagger}g\left(\sigma\right)\mathrm{e}^{-\int_0^{\sigma}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\ \varphi_2\left(t-\sigma\right)\mathrm{d}\sigma,\end{aligned}\tag{30}$$

and show that the spectral radius $\rho\left(\mathcal{F}\right)$ of $\mathcal{F}$ plays the role of such a threshold.

In order to establish the main theorem of this section, we need some additional assumptions on the parameters. First we assume

Assumption 1. *$k\left(\sigma,a,t\right)=0$, $\beta\left(\sigma\right)=0$ and $\gamma\left(\sigma,t\right)=0$ for all $\sigma\in(-\infty,0)\cup(a_\dagger,+\infty)$.*

Note that this assumption implies $\kappa\left(\sigma,a,t\right)=0$ and $g\left(\sigma\right)=0$ for all $\sigma\in(-\infty,0)\cup(a_\dagger,+\infty)$.

Next, setting

$$\begin{cases} \Psi_{1,1}(x,z,a,t) := \kappa(z+x,a,t)\,\mathrm{e}^{-\int_0^z \gamma(z+x-\eta,t-\eta)\mathrm{d}\eta}, \\ \Psi_{1,2}(z,a,t) := \kappa(z,a,t)\,\mathrm{e}^{-\int_0^z \gamma(z-\eta,t-\eta)\mathrm{d}\eta}, \\ \Psi_{2,1}(x,z,t) := qg(z+x)\,\mathrm{e}^{-\int_0^z \gamma(z+x-\eta,t-\eta)\mathrm{d}\eta}, \\ \Psi_{2,2}(z,t) := qg(z)\,\mathrm{e}^{-\int_0^z \gamma(z-\eta,t-\eta)\mathrm{d}\eta}, \quad x \geq 0,\ z \geq 0 \end{cases} \tag{31}$$

and

$$\begin{cases} \hat{\Psi}_{1,1}(x,s,a,t) := \begin{cases} \sum_{n=0}^{+\infty} \Psi_{1,1}(x,t-s+nT,a,t), & t-s>0, \\ \sum_{n=1}^{+\infty} \Psi_{1,1}(x,t-s+nT,a,t), & t-s<0, \end{cases} \\ \hat{\Psi}_{1,2}(s,a,t) := \begin{cases} \sum_{n=0}^{+\infty} \Psi_{1,2}(t-s+nT,a,t), & t-s>0, \\ \sum_{n=1}^{+\infty} \Psi_{1,2}(t-s+nT,a,t), & t-s<0, \end{cases} \\ \hat{\Psi}_{2,1}(x,s,t) := \begin{cases} \sum_{n=0}^{+\infty} \Psi_{2,1}(x,t-s+nT,t), & t-s>0, \\ \sum_{n=1}^{+\infty} \Psi_{2,1}(x,t-s+nT,t), & t-s<0, \end{cases} \\ \hat{\Psi}_{2,2}(s,t) := \begin{cases} \sum_{n=0}^{+\infty} \Psi_{2,2}(t-s+nT,t), & t-s>0, \\ \sum_{n=1}^{+\infty} \Psi_{2,2}(t-s+nT,t), & t-s<0 \end{cases} \end{cases} \tag{32}$$

(note that each of the series is well-defined by Assumption 1), we make the following assumption:

Assumption 2. *The following equations hold uniformly for $x \in [0,T]$ and $s \in [0,a_\dagger]$:*

$$\begin{cases} \lim_{h\to 0} \int_0^T \int_0^{a_\dagger} \left| \hat{\Psi}_{1,1}(x,s,a+h,t+h) - \hat{\Psi}_{1,1}(x,s,a,t) \right| \mathrm{d}a\mathrm{d}t = 0, \\ \lim_{h\to 0} \int_0^T \int_0^{a_\dagger} \left| \hat{\Psi}_{1,2}(s,a+h,t+h) - \hat{\Psi}_{1,2}(s,a,t) \right| \mathrm{d}a\mathrm{d}t = 0, \\ \lim_{h\to 0} \int_0^T \left| \hat{\Psi}_{2,1}(x,s,t+h) - \hat{\Psi}_{2,1}(x,s,t) \right| \mathrm{d}t = 0, \\ \lim_{h\to 0} \int_0^T \left| \hat{\Psi}_{2,2}(s,t+h) - \hat{\Psi}_{2,2}(s,t) \right| \mathrm{d}t = 0. \end{cases} \tag{33}$$

These assumptions are required for proving the compactness of operator $\mathcal{F}$, in view of the use of the Krein-Rutman theorem (see [14]). Thus, we proceed to prove the following:

Lemma 4.2. *$\mathcal{F}$ is compact.*

Proof. Let

$$\begin{cases} (\mathcal{F}_{1,1}\varphi)(a,t) := \int_0^{a_\dagger} \kappa(\sigma,a,t) \int_0^\sigma \varphi(\sigma-\tau,t-\tau)\,\mathrm{e}^{-\int_0^\tau \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma, \\ (\mathcal{F}_{1,2}\psi)(a,t) := \int_0^{a_\dagger} \kappa(\sigma,a,t)\,\mathrm{e}^{-\int_0^\sigma \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\, \psi(t-\sigma)\,\mathrm{d}\sigma, \\ (\mathcal{F}_{2,1}\varphi)(t) := q\int_0^{a_\dagger} g(\sigma) \int_0^\sigma \varphi(\sigma-\tau,t-\tau)\,\mathrm{e}^{-\int_0^\tau \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma, \\ (\mathcal{F}_{2,2}\psi)(t) := q\int_0^{a_\dagger} g(\sigma)\,\mathrm{e}^{-\int_0^\sigma \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\, \psi(t-\sigma)\,\mathrm{d}\sigma, \quad \varphi \in X_T,\ \psi \in Y_T. \end{cases} \tag{34}$$

These are linear operators defined on X_T or on Y_T, precisely

$$\mathcal{F}_{1,1}: X_T \to X_T\ , \quad \mathcal{F}_{1,2}: Y_T \to X_T,$$
$$\mathcal{F}_{2,1}: X_T \to Y_T\ , \quad \mathcal{F}_{2,2}: Y_T \to Y_T.$$

Then, from (28)-(30), we have

$$\mathcal{F}(\varphi_1, \varphi_2) = (\mathcal{F}_{1,1}\varphi_1 + \mathcal{F}_{1,2}\varphi_2,\ \mathcal{F}_{2,1}\varphi_1 + \mathcal{F}_{2,2}\varphi_2).$$

and, to complete the proof, it suffices to show the compactness of each $\mathcal{F}_{i,j}$ $(i, j = 1, 2)$.

First we consider $\mathcal{F}_{1,1}$. From Assumption 1, we have

$$\begin{aligned}(\mathcal{F}_{1,1}\varphi)(a,t) &= \int_0^{+\infty} \kappa(\sigma, a, t) \int_0^{\sigma} \varphi(\sigma - \tau, t - \tau)\, \mathrm{e}^{-\int_0^{\tau} \gamma(\sigma-\eta, t-\eta)\mathrm{d}\eta} \mathrm{d}\tau \mathrm{d}\sigma \\ &= \int_0^{+\infty} \int_{\tau}^{+\infty} \kappa(\sigma, a, t)\, \mathrm{e}^{-\int_0^{\tau} \gamma(\sigma-\eta, t-\eta)\mathrm{d}\eta} \varphi(\sigma - \tau, t - \tau)\, \mathrm{d}\sigma \mathrm{d}\tau \\ &= \int_0^{+\infty} \int_0^{+\infty} \kappa(\tau + x, a, t)\, \mathrm{e}^{-\int_0^{\tau} \gamma(\tau+x-\eta, t-\eta)\mathrm{d}\eta} \varphi(x, t - \tau)\, \mathrm{d}x \mathrm{d}\tau \\ &= \int_{-\infty}^{t} \int_0^{+\infty} \kappa(t - s + x, a, t)\, \mathrm{e}^{-\int_0^{t-s} \gamma(t-s+x-\eta, t-\eta)\mathrm{d}\eta} \varphi(x, s)\, \mathrm{d}x \mathrm{d}s. \quad (35)\end{aligned}$$

Note that $\int_{-\infty}^{t} = \int_0^t + \sum_{n=0}^{+\infty} \int_{-(n+1)T}^{-nT}$ and

$$\begin{aligned}&\int_{-(n+1)T}^{-nT} \int_0^{+\infty} \kappa(t - s + x, a, t)\, \mathrm{e}^{-\int_0^{t-s} \gamma(t-s+x-\eta, t-\eta)\mathrm{d}\eta} \varphi(x, s)\, \mathrm{d}x \mathrm{d}s \\ =\ & \int_0^{T} \int_0^{+\infty} \kappa(t - s + (n+1)T + x, a, t)\, \mathrm{e}^{-\int_0^{t-s+(n+1)T} \gamma(t-s+(n+1)T+x-\eta, t-\eta)\mathrm{d}\eta} \\ & \qquad \varphi(x, s - (n+1)T)\, \mathrm{d}x \mathrm{d}s \\ =\ & \int_0^{T} \int_0^{+\infty} \Psi_{1,1}(x, t - s + (n+1)T, a, t)\, \varphi(x, s)\, \mathrm{d}x \mathrm{d}s. \quad (36)\end{aligned}$$

From (35)-(36) and Assumption 1, we have

$$\begin{aligned}(\mathcal{F}_{1,1}\varphi)(a,t) &= \int_0^{T} \int_0^{+\infty} \hat{\Psi}_{1,1}(x, s, a, t)\, \varphi(x, s)\, \mathrm{d}x \mathrm{d}s \\ &= \int_0^{T} \int_0^{a_\dagger} \hat{\Psi}_{1,1}(x, s, a, t)\, \varphi(x, s)\, \mathrm{d}x \mathrm{d}s\ . \quad (37)\end{aligned}$$

Then, regarding $\mathcal{F}_{1,1}$ as an operator on $L^1([0, a_\dagger] \times [0, T])$, from Assumption 2 and the well-known compactness criteria in L^1 (see, for instance, [20], p.275), we see that $\mathcal{F}_{1,1}$ is compact. Of course $\mathcal{F}_{1,1}$ is compact also regarded as an operator in X_T.

Similarly, we have

$$\begin{aligned}(\mathcal{F}_{1,2}\psi)(a,t) &= \int_{-\infty}^{t} \kappa(t - s, a, t)\, \mathrm{e}^{-\int_0^{t-s} \gamma(t-s-\eta, t-\eta)\mathrm{d}\eta} \psi(s)\, \mathrm{d}s, \\ (\mathcal{F}_{2,1}\varphi)(t) &= \int_{-\infty}^{t} \int_0^{+\infty} qg(t - s + x)\, \mathrm{e}^{-\int_0^{t-s} \gamma(t-s+x-\eta, t-\eta)\mathrm{d}\eta} \varphi(x, s)\, \mathrm{d}x \mathrm{d}s, \\ (\mathcal{F}_{2,2}\psi)(t) &= \int_{-\infty}^{t} qg(t - s)\, \mathrm{e}^{-\int_0^{t-s} \gamma(t-s-\eta, t-\eta)\mathrm{d}\eta} \psi(s)\, \mathrm{d}s, \quad \varphi \in X_T,\ \psi \in Y_T\end{aligned}$$

and

$$\left(\mathcal{F}_{1,2}\psi\right)(a,t) = \int_0^T \hat{\Psi}_{1,2}(x,a,t)\,\psi(x)\,\mathrm{d}x, \tag{38}$$
$$\left(\mathcal{F}_{2,1}\varphi\right)(t) = \int_0^T \int_0^{a_\dagger} \hat{\Psi}_{2,1}(x,s,t)\,\varphi(x,s)\,\mathrm{d}x\mathrm{d}s,$$
$$\left(\mathcal{F}_{2,2}\psi\right)(t) = \int_0^T \hat{\Psi}_{2,2}(s,t)\,\psi(s)\,\mathrm{d}s.$$

Thus, as in the above case of $\mathcal{F}_{1,1}$, we see that $\mathcal{F}_{1,2}$, $\mathcal{F}_{2,1}$ and $\mathcal{F}_{2,2}$ are compact. □

As we mentioned above, the Krein-Rutman theorem can be applied to $\mathcal{F}$. In fact, since $\mathcal{F}$ is compact, linear and positive, the Krein-Rutman theorem (see [14]) guarantees that if $\rho(\mathcal{F})$ is strictly positive, then it is an eigenvalue associated with a nonzero nonnegative eigenvector $\mathbf{v} = (v_1, v_2) \in X_{T,+} \times Y_{T,+}$. Using this fact, we prove the following proposition, which is the main result of this section:

Proposition 2. *If $\rho(\mathcal{F}) > 1$, then there exists a nontrivial positive fixed point $\varphi^* = (\varphi_1^*, \varphi_2^*)$ of Φ in $(X_{T,+} \setminus \{0\}) \times (Y_{T,+} \setminus \{0\})$, that is, $\varphi^* = \Phi(\varphi^*)$.*

Proof. As we noticed before, there exists a non trivial $\mathbf{v} = (v_1, v_2) \in X_{T,+} \times Y_{T,+}$ such that we have

$$\rho(\mathcal{F})\,\mathbf{v} = \mathcal{F}\mathbf{v}.$$

Hence, from (37) and (38), we have

$$\begin{aligned} \rho(\mathcal{F})\,v_1 &= \mathcal{F}_{1,1}v_1 + \mathcal{F}_{1,2}v_2 \\ &= \int_0^T \int_0^{a_\dagger} \hat{\Psi}_{1,1}(x,s,a,t)\,v_1(x,s)\,\mathrm{d}x\mathrm{d}s + \int_0^T \hat{\Psi}_{1,2}(x,a,t)\,v_2(x)\,\mathrm{d}x \\ &\le \hat{\Psi}^+_{1,1}\,\|v_1\|_{X_T} + \hat{\Psi}^+_{1,2}\,\|v_2\|_{Y_T}, \end{aligned} \tag{39}$$

where $\hat{\Psi}^+_{1,1} := \sup \hat{\Psi}_{1,1}(x,s,a,t) < +\infty$ and $\hat{\Psi}^+_{1,2} := \sup \hat{\Psi}_{1,2}(x,a,t) < +\infty$ (note that such suprema exist because of Assumption 1). Let

$$\lambda_0 := (\lambda_{0,1}, \lambda_{0,2}) = \frac{\rho(\mathcal{F}) \log \rho(\mathcal{F})}{\left(\hat{\Psi}^+_{1,1}\,\|v_1\|_{X_T} + \hat{\Psi}^+_{1,2}\,\|v_2\|_{Y_T}\right) c a_\dagger}\,(v_1, v_2), \tag{40}$$

where $c > 1$ is a sufficiently large constant such that $\lambda_{0,2} \le 1$ *a.e.*. Since $\rho(\mathcal{F}) > 1$, we have $\log \rho(\mathcal{F}) > 0$, hence $\lambda_0 = (\lambda_{0,1}, \lambda_{0,2}) \in X_{T,+} \times \tilde{Y}_{T,+}$. From (24), we have

$$\begin{aligned} &\Phi_1\lambda_0 = \Phi_1(\lambda_{0,1}, \lambda_{0,2}) \ge \\ &\int_0^{a_\dagger} \kappa(\sigma,a,t) \int_0^{\sigma} \lambda_{0,1}(\sigma-\tau, t-\tau)\, \mathrm{e}^{-\int_0^{ca_\dagger} \lambda_{0,1}(\sigma-\eta,t-\eta)\mathrm{d}\eta} \mathrm{e}^{-\int_0^{\tau} \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta} \mathrm{d}\tau\mathrm{d}\sigma \\ &\quad + \int_0^{a_\dagger} \kappa(\sigma,a,t)\, \mathrm{e}^{-\int_0^{ca_\dagger} \lambda_{0,1}(\sigma-\eta,t-\eta)\mathrm{d}\eta} \mathrm{e}^{-\int_0^{\sigma} \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\, \lambda_{0,2}(t-\sigma)\,\mathrm{d}\sigma. \end{aligned}$$

Hence, from (39) and (40),

$$\begin{aligned}\Phi_1\lambda_0 \quad &\geq \quad \frac{1}{\rho(\mathcal{F})}\left\{\int_0^{a_\dagger}\kappa(\sigma,a,t)\int_0^{\sigma}\lambda_{0,1}(\sigma-\tau,t-\tau)\,\mathrm{e}^{-\int_0^{\tau}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma\right.\\ &\qquad\left.+\int_0^{a_\dagger}\kappa(\sigma,a,t)\,\mathrm{e}^{-\int_0^{\sigma}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\;\lambda_{0,2}(t-\sigma)\,\mathrm{d}\sigma\right\}\\ &= \quad \frac{1}{\rho(\mathcal{F})}\mathcal{F}_1(\lambda_{0,1},\lambda_{0,2})=\lambda_{0,1}. \qquad (41)\end{aligned}$$

Moreover, from (25),

$$\begin{aligned}&\Phi_2\lambda_0=\Phi_2(\lambda_{0,1},\lambda_{0,2})\geq\\ &\quad q\int_0^{a_\dagger}g(\sigma)\int_0^{\sigma}\lambda_{0,1}(\sigma-\tau,t-\tau)\,\mathrm{e}^{-\int_0^{ca_\dagger}\lambda_{0,1}(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{e}^{-\int_0^{\tau}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma\\ &\quad +q\int_0^{a_\dagger}g(\sigma)\,\mathrm{e}^{-\int_0^{ca_\dagger}\lambda_{0,1}(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{e}^{-\int_0^{\sigma}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\;\lambda_{0,2}(t-\sigma)\,\mathrm{d}\sigma.\end{aligned}$$

Hence, again from (39) and (40), we have

$$\begin{aligned}\Phi_2\lambda_0 \quad &\geq \quad \frac{1}{\rho(\mathcal{F})}\left\{q\int_0^{a_\dagger}g(\sigma)\int_0^{\sigma}\lambda_{0,1}(\sigma-\tau,t-\tau)\,\mathrm{e}^{-\int_0^{\tau}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma\right.\\ &\qquad\left.+q\int_0^{a_\dagger}g(\sigma)\,\mathrm{e}^{-\int_0^{\sigma}\gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\;\lambda_{0,2}(t-\sigma)\,\mathrm{d}\sigma\right\}\\ &= \quad \frac{1}{\rho(\mathcal{F})}\mathcal{F}_2(\lambda_{0,1},\lambda_{0,2})=\lambda_{0,2}. \qquad (42)\end{aligned}$$

From (41) and (42), we have $\Phi\lambda_0 \geq \lambda_0$. Since the monotonicity of Φ on $X_{T,+}\times\tilde{Y}_{T,+}$ is guaranteed by Lemma 4.1, we can construct a monotone sequence

$$\lambda_n := \Phi(\lambda_{n-1}), \quad n = 1, 2, \ldots$$

satisfying $\lambda_n \geq \lambda_{n-1}$ for all n. Since the positive invariance of $X_{T,+}\times\tilde{Y}_{T,+}$ for Φ and the uniform boundedness of Φ on the space is guaranteed by Lemma 4.1, it follows from B. Levi's theorem that $\lambda^* := \lim_{n\to+\infty}\lambda_n^* \in X_{T,+}\times\tilde{Y}_{T,+}$ exists and satisfies $\lambda^* = \Phi(\lambda^*)$. Thus λ^* is the fixed point φ^* of Φ we look for. □

The previous result can be implemented with the results obtained in [15]. In fact, once existence of an endemic periodic solution u^* of system (7) is proved, we can resort to [15] to get uniqueness and global asymptotic stability. To this aim we consider the following assumption

Assumption 3. *(i)* $\beta(a)\neq 0$ *and* $a_0 := \max\operatorname{supp}\beta < a_\dagger$.

(ii) There exist a positive constant $\epsilon > 0$ *and nonnegative measurable functions* $\kappa_1(a)$ *and* $\kappa_2(a)$ *such that*

$$\epsilon\kappa_1(\sigma)\kappa_2(a) \leq \kappa(\sigma,a,t) \leq \kappa_1(\sigma)\kappa_2(a)$$

and $\kappa_1(a)\neq 0$ *on* $(0,a_\dagger)$ *and* $\kappa_2(a)\neq 0$ *on* $(0,a_0)$.

Then, from Theorem 5.5 in [15], we have the following proposition.

Proposition 3. *Under the assumptions 1-3, system (7) has at most one endemic (nontrivial) periodic solution u^* in $\Omega_T := \{\varphi \in X_{T,+} : 0 \le \varphi \le 1 \text{ a.e.}\}$. Furthermore, if the initial condition $0 \le u_0(a) \le 1$ satisfies*

$$\text{supp } u_0 \cap [0, a_0) \ne \emptyset$$

or

$$\text{supp } u_0 \cap [0, \max \text{supp } \kappa_1) \ne \emptyset,$$

then u^* is globally asymptotically stable.

5. **Global stability of the disease-free steady state.** The global stability of the disease-free steady state $u \equiv 0$ of system (7), for $\rho(\mathcal{F}) < 1$, can also be proved by using Theorem 5.6 in [15]. The theorem is applied to system (7) as the following lemma:

Lemma 5.1. *If system (7) has no endemic (nontrivial) periodic solution u^* in Ω_T, then the disease-free steady state $u \equiv 0$ of the system is globally asymptotically stable.*

In order to use this lemma, we claim that system (7) has no endemic periodic solution $u^* \in \Omega_T \setminus \{0\}$ if $\rho(\mathcal{F}) < 1$. In fact, if such a periodic solution u^* exists, then the nonlinear operator Φ defined above by (23) has a nontrivial fixed point $\varphi^* = \Phi(\varphi^*)$ in $(X_{T,+} \setminus \{0\}) \times (Y_{T,+} \setminus \{0\})$. However, since $\varphi^* = \Phi(\varphi^*) \le \mathcal{F}\varphi^*$, we have that $\rho(\mathcal{F}) \ge 1$ as a contradiction. In conclusion, we obtain the following proposition:

Proposition 4. *If $\rho(\mathcal{F}) < 1$, then system (7) has no endemic (nontrivial) periodic solution u^* in Ω_T, and the disease-free steady state $u \equiv 0$ of the system is globally asymptotically stable.*

6. **The basic reproduction number $\mathcal{R}_0$.** Finally we investigate the relation between our threshold value $\rho(\mathcal{F})$ and the basic reproduction number $\mathcal{R}_0$ [8] which is a well-known epidemiological threshold value.

According to its epidemiological definition, $\mathcal{R}_0$ is the average number of secondary cases produced by a typical infected individual, introduced into a completely susceptible population, during its entire period of infectiousness. From the mathematical viewpoint $\mathcal{R}_0$ is the spectral radius of an integral operator called the next generation operator and, recently, its definition has been extended to the case of time periodic environments [2, 3, 12, 18, 19].

Linearizing system (6) around the disease-free steady state $(p^*(a), 0)$, we have

$$\begin{cases} \dfrac{\partial \tilde{i}(a,t)}{\partial t} + \dfrac{\partial \tilde{i}(a,t)}{\partial a} + \mu(a)\,\tilde{i}(a,t) = \tilde{\lambda}(a,t) - \gamma(a,t)\,\tilde{i}(a,t), \\ \tilde{\lambda}(a,t) = p^*(a) \displaystyle\int_0^{a_\dagger} k(\sigma,a,t)\,\tilde{i}(\sigma,t)\,\mathrm{d}\sigma, \\ \tilde{i}(0,t) = q \displaystyle\int_0^{a_\dagger} \beta(\sigma)\,\tilde{i}(\sigma,t)\,\mathrm{d}\sigma, \end{cases} \tag{43}$$

where $\tilde{i}$ denotes the perturbation from the disease-free steady state $i \equiv 0$. Integrating the first equation of (43) along the characteristic lines, we have

$$\begin{aligned}\tilde{i}(a,t) &= \int_0^a \tilde{\lambda}(\sigma, t-a+\sigma)\, \mathrm{e}^{-\int_\sigma^a \{\mu(\rho)+\gamma(\rho,t-a+\rho)\}\mathrm{d}\rho}\mathrm{d}\sigma \\ &\quad + \mathrm{e}^{-\int_0^a \{\mu(\rho)+\gamma(\rho,t-a+\rho)\}\mathrm{d}\rho}\, \tilde{i}(0,t-a) \\ &= \int_0^a \tilde{\lambda}(a-\tau, t-\tau)\, \mathrm{e}^{-\int_0^\tau \{\mu(a-\eta)+\gamma(a-\eta,t-\eta)\}\mathrm{d}\eta}\mathrm{d}\tau \\ &\quad + \mathrm{e}^{-\int_0^a \{\mu(a-\eta)+\gamma(a-\eta,t-\eta)\}\mathrm{d}\eta}\, \tilde{i}(0,t-a). \end{aligned} \tag{44}$$

Substituting (44) into the second equation of (43), and using Assumption 1, we have

$$\begin{aligned}\tilde{\lambda}(a,t) &= \\ &p^*(a)\int_0^{a_\dagger} k(\sigma,a,t)\int_0^\sigma \tilde{\lambda}(\sigma-\tau,t-\tau)\,\mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\mathrm{d}\tau\mathrm{d}\sigma \\ &+p^*(a)\int_0^{a_\dagger} k(\sigma,a,t)\,\mathrm{e}^{-\int_0^\sigma \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\,\tilde{i}(0,t-\sigma)\,\mathrm{d}\sigma \\ =\ &p^*(a)\int_0^{+\infty}\int_\tau^{+\infty} k(\sigma,a,t)\,\mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\tilde{\lambda}(\sigma-\tau,t-\tau)\,\mathrm{d}\sigma\mathrm{d}\tau \\ &+p^*(a)\int_0^{+\infty} k(\sigma,a,t)\,\mathrm{e}^{-\int_0^\sigma \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\,\tilde{i}(0,t-\sigma)\,\mathrm{d}\sigma\ . \end{aligned} \tag{45}$$

Similarly, substituting (44) into the third equation of (43), we have

$$\begin{aligned}\tilde{i}(0,t) = q&\int_0^{+\infty}\int_\tau^{+\infty}\beta(\sigma)\,\mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\tilde{\lambda}(\sigma-\tau,t-\tau)\,\mathrm{d}\sigma\mathrm{d}\tau \\ &+q\int_0^{+\infty}\beta(\sigma)\,\mathrm{e}^{-\int_0^\sigma \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\,\tilde{i}(0,t-\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{46}$$

Let us define the linear operator $A(t,\tau)$ from $L^1(0,+\infty)\times\mathbb{R}$ into itself as

$$(A(t,\tau)\varphi)(a) := \begin{pmatrix} (A_{1,1}(t,\tau)\varphi_1)(a) + (A_{1,2}(t,\tau)\varphi_2)(a) \\ A_{2,1}(t,\tau)\varphi_1 + A_{2,2}(t,\tau)\varphi_2 \end{pmatrix},\ \varphi = \begin{pmatrix}\varphi_1(a)\\ \varphi_2\end{pmatrix}, \tag{47}$$

where

$$\begin{aligned} (A_{1,1}(t,\tau)\varphi_1)(a) &:= p^*(a)\int_\tau^{+\infty} k(\sigma,a,t)\,\mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\varphi_1(\sigma-\tau)\,\mathrm{d}\sigma, \\ (A_{1,2}(t,\tau)\varphi_2)(a) &:= p^*(a)\,k(\tau,a,t)\,\mathrm{e}^{-\int_0^\tau \{\mu(\tau-\eta)+\gamma(\tau-\eta,t-\eta)\}\mathrm{d}\eta}\ \varphi_2, \\ A_{2,1}(t,\tau)\varphi_1 &:= q\int_\tau^{+\infty}\beta(\sigma)\,\mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta}\varphi_1(\sigma-\tau)\,\mathrm{d}\sigma, \\ A_{2,2}(t,\tau)\varphi_2 &:= q\beta(\tau)\,\mathrm{e}^{-\int_0^\tau \{\mu(\tau-\eta)+\gamma(\tau-\eta,t-\eta)\}\mathrm{d}\eta}\varphi_2,\quad \varphi_1\in L^1(0,+\infty),\ \varphi_2\in\mathbb{R}. \end{aligned}$$

Then, following the arguments in [2, 3, 12, 18, 19], we see that the basic reproduction number $\mathcal{R}_0$ is obtained as the spectral radius of the next generation operator

$$(K\varphi)(t) := \int_0^{+\infty} A(t,\tau)\,\varphi(t-\tau)\,\mathrm{d}\tau,\quad \varphi\in V_T, \tag{48}$$

where V_T denotes the space of T-periodic vector-valued functions $\varphi = (\varphi_1, \varphi_2)$ such that $\varphi_1(t) \in L^1(0, +\infty)$ and $\varphi_2(t) \in \mathbb{R}$ for each t. Concerning the relation between this $\mathcal{R}_0 = \rho(K)$ and the threshold value $\rho(\mathcal{F})$, we have the following proposition.

Proposition 5. *Let $\mathcal{F}$ and K be defined by (28) and (48), respectively. Then, $\rho(\mathcal{F}) = \rho(K) = \mathcal{R}_0$.*

Proof. The next generation operator K can be regarded as a linear operator on $X_{T,+} \times Y_{T,+}$ such that

$$(K(\varphi_1, \varphi_2))(a,t) := \begin{pmatrix} (K_{1,1}\varphi_1)(a,t) + (K_{1,2}\varphi_2)(a,t) \\ (K_{2,1}\varphi_1)(t) + (K_{2,2}\varphi_2)(t) \end{pmatrix}, \tag{49}$$

where

$$(K_{1,1}\varphi_1)(a,t) := p^*(a) \int_0^{+\infty} \int_\tau^{+\infty} k(\sigma, a, t)\, \mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \varphi_1(\sigma-\tau, t-\tau)\, \mathrm{d}\sigma \mathrm{d}\tau,$$

$$(K_{1,2}\varphi_2)(a,t) := p^*(a) \int_0^{+\infty} k(\sigma, a, t)\, \mathrm{e}^{-\int_0^\sigma \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \varphi_2(t-\sigma)\mathrm{d}\sigma,$$

$$(K_{2,1}\varphi_1)(t) := q \int_0^{+\infty} \int_\tau^{+\infty} \beta(\sigma)\, \mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \varphi_1(\sigma-\tau, t-\tau)\, \mathrm{d}\sigma \mathrm{d}\tau,$$

$$(K_{2,2}\varphi_2)(t) := q \int_0^{+\infty} \beta(\sigma)\, \mathrm{e}^{-\int_0^\sigma \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \varphi_2(t-\sigma)\, \mathrm{d}\sigma,$$

$$\varphi_1 \in X_{T,+}, \ \varphi_2 \in Y_{T,+}.$$

Let us define an operator L on $X_{T,+} \times Y_{T,+}$ by

$$(L(\varphi_1, \varphi_2))(a,t) = \begin{pmatrix} p^*(a)\varphi_1(a,t) \\ b_0\varphi_2(t) \end{pmatrix}, \tag{50}$$

where $p^*(a)$ and b_0 are the demographic parameters defined by (2) and (3), respectively. We claim that $KL = L\mathcal{F}$. In fact, we have

$$(KL(\varphi_1, \varphi_2))(a,t) := \begin{pmatrix} (K_{1,1}p^*\varphi_1)(a,t) + (K_{1,2}\ b_0\varphi_2)(a,t) \\ (K_{2,1}p^*\varphi_1)(t) + (K_{2,2}\ b_0\varphi_2)(t) \end{pmatrix}, \tag{51}$$

and

$$\begin{aligned}
(K_{1,1}p^*\varphi_1)(a,t) &= p^*(a) \int_0^{+\infty} \int_\tau^{+\infty} k(\sigma, a, t)\, \mathrm{e}^{-\int_0^\tau \{\mu(\sigma-\eta)+\gamma(\sigma-\eta,t-\eta)\}\mathrm{d}\eta} \\
&\qquad p^*(\sigma-\tau)\varphi_1(\sigma-\tau, t-\tau)\, \mathrm{d}\sigma\mathrm{d}\tau \\
&= p^*(a) \int_0^{a_\dagger} \int_0^\sigma \kappa(\sigma, a, t)\, \mathrm{e}^{-\int_0^\tau \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta} \varphi_1(\sigma-\tau, t-\tau)\, \mathrm{d}\tau\mathrm{d}\sigma \\
&= (p^*\mathcal{F}_{1,1}\varphi_1)(a,t),
\end{aligned}$$

$$\begin{aligned}
(K_{1,2}b_0\varphi_2)(a,t) &= p^*(a) \int_0^{+\infty} k(\sigma, a, t)\, b_0 \mathrm{e}^{-\int_0^\sigma \mu(\sigma-\eta)\mathrm{d}\eta} \mathrm{e}^{-\int_0^\sigma \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta} \\
&\qquad \varphi_2(t-\sigma)\, \mathrm{d}\sigma \\
&= p^*(a) \int_0^{a_\dagger} \kappa(\sigma, a, t)\, \mathrm{e}^{-\int_0^\sigma \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta} \varphi_2(t-\sigma)\, \mathrm{d}\sigma \\
&= (p^*\mathcal{F}_{1,2}\varphi_2)(a,t),
\end{aligned}$$

$$\begin{aligned}(K_{2,1}p^*\varphi_1)(t) &= q\int_0^{+\infty}\int_\tau^{+\infty}\beta(\sigma)\,b_0\mathrm{e}^{-\int_0^\sigma \mu(\rho)\mathrm{d}\rho}\mathrm{e}^{-\int_0^\tau \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\\ &\qquad\qquad \varphi_1(\sigma-\tau,t-\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\\ &= b_0 q\int_0^{a_\dagger}\int_0^\sigma g(\sigma)\,\mathrm{e}^{-\int_0^\tau \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\varphi_1(\sigma-\tau,t-\tau)\,\mathrm{d}\tau\mathrm{d}\sigma\\ &= (b_0\mathcal{F}_{2,1}\varphi_1)(t)\end{aligned}$$

and

$$\begin{aligned}(K_{2,2}b_0\varphi_2)(t) &= q\int_0^{+\infty}\beta(\sigma)\,p^*(\sigma)\,\mathrm{e}^{-\int_0^\sigma \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\varphi_2(t-\sigma)\,\mathrm{d}\sigma\\ &= b_0 q\int_0^{a_\dagger} g(\sigma)\,\mathrm{e}^{-\int_0^\sigma \gamma(\sigma-\eta,t-\eta)\mathrm{d}\eta}\varphi_2(t-\sigma)\,\mathrm{d}\sigma\\ &= (b_0\mathcal{F}_{2,2}\varphi_2)(t)\,.\end{aligned}$$

Thus, we have

$$KL(\varphi_1,\varphi_2) = \begin{pmatrix} p^*(\mathcal{F}_{1,1}\varphi_1+\mathcal{F}_{1,2}\varphi_2)\\ b_0(\mathcal{F}_{2,1}\varphi_1+\mathcal{F}_{2,2}\varphi_2)\end{pmatrix} = L\mathcal{F}(\varphi_1,\varphi_2)\,,$$

hence $L^{-1}KL=\mathcal{F}$. Since $\rho\left(L^{-1}KL\right)=\rho(K)=\mathcal{R}_0$, we arrive at the conclusion. □

In conclusion, from Propositions 2-5, we obtain the following main theorem of this paper.

Theorem 6.1. *Let K be the next generation operator defined by (48). Then*

(i) If $\mathcal{R}_0=\rho(K)<1$, then system (7) has no endemic (nontrivial) periodic solution in Ω_T, and the disease-free steady state $u\equiv 0$ of the system is globally asymptotically stable.

(ii) If $\mathcal{R}_0=\rho(K)>1$, then system (7) has a unique endemic (nontrivial) periodic solution $u^*\in\Omega_T\setminus\{0\}$. Furthermore, it is globally asymptotically stable if the initial condition $0\le u_0(a)\le 1$ satisfies $\operatorname{supp} u_0\cap[0,a_0)\neq\emptyset$ or $\operatorname{supp} u_0\cap[0,\max\operatorname{supp}\kappa_1)\neq\emptyset$.

7. Discussion.

We have formulated an age-structured SIS epidemic model (6) with periodicity and vertical transmission. The system was normalized to system (7), and the existence of an endemic (nontrivial) periodic solution u^* of (7) was investigated. We have shown that the spectral radius $\rho(\mathcal{F})$ of the Fréchet derivative $\mathcal{F}$ of a nonlinear operator Φ at 0 plays the role of a threshold for the existence of such u^*, that is, if $\rho(\mathcal{F})>1$, then u^* is obtained as a nontrivial fixed point of Φ. The uniqueness and global stability results obtained in [15] were directly applied to our case, and thus, we have shown that $\rho(\mathcal{F})$ is a threshold for the global behavior of system (7). Furthermore, we have shown that if $\rho(\mathcal{F})<1$, then the disease-free steady state $u\equiv 0$ of system (7) is globally asymptotically stable. The relation between $\rho(\mathcal{F})$ and the basic reproduction number $\mathcal{R}_0$ was also investigated. We have shown that the two threshold values coincide. Consequently, this study is regarded as the first one showing that $\mathcal{R}_0$ plays the role of a threshold for the global behavior of an age-structured SIS epidemic model with periodicity and vertical transmission.

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