

STOCHASTIC DYNAMICS OF SIRS EPIDEMIC MODELS WITH RANDOM PERTURBATION

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ABSTRACT. In this paper, we consider a stochastic SIRS model with parameter perturbation, which is a standard technique in modeling population dynamics. In our model, the disease transmission coefficient and the removal rates are all affected by noise. We show that the stochastic model has a unique positive solution as is essential in any population model. Then we establish conditions for extinction or persistence of the infectious disease. When the infective part is forced to expire, the susceptible part converges weakly to an inverse-gamma distribution with explicit shape and scale parameters. In case of persistence, by new stochastic Lyapunov functions, we show the ergodic property and positive recurrence of the stochastic model. We also derive the an estimate for the mean of the stationary distribution. The analytical results are all verified by computer simulations, including examples based on experiments in laboratory populations of mice.

1. Introduction. Numerous health agencies frequently use mathematical models to analyze the spread and the control of infectious diseases in host populations. There is an intensive literature on the mathematical epidemiology, for examples, [11, 15, 16, 18] and the references therein. In particular, [3, 22] are excellent books in this area.

One of classic epidemic models is the SIR model, which subdivides a homogeneous host population into three epidemiologically distinct types of individuals, the susceptible, the infective, and the removed, with their population sizes denoted by S, I and R , respectively. It is suitable for some infectious diseases of permanent or long immunity, such as chickenpox, smallpox, measles, etc. For some diseases, see e.g. influenza and sexual diseases, the removed or recovered individuals finally go

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back to the susceptible state, called the SIRS model, which can be characterized by the following differential equations

$$\begin{cases} dS = (\lambda - \beta SI - d_S S + \gamma R)dt, \\ dI = (\beta SI - (d_I + v)I)dt, \\ dR = (vI - (d_R + \gamma)R)dt. \end{cases} \quad (1)$$

Recall that the parameter $\lambda > 0$ is the rate of susceptible individuals recruited into the population (either by birth or immigration) per unit time; $\beta > 0$ is some transmission coefficient and it is assumed that the rate at which the susceptible individuals acquire the infection is proportional to the number of encounters between the susceptible and infective individuals per unit time, being βSI ; $d_S > 0$ is the natural mortality rate or the removal rate of the susceptible individual; $d_I > 0$ is the removal rate of infectious individual and usually be the plus of the natural mortality rate and the mortality rate caused by the disease; $v > 0$ is the recovery rate of infective individual; $d_R > 0$ is the removal rate of the recovered individual and $\gamma > 0$ is the rate at which the recovered individual loses immunity. It is well known that if the reproductive number $R_0 = \frac{\lambda\beta}{d_S(d_I+v)} \leq 1$ ([3], [6], [13], etc), Eq. (1) has a globally stable disease free equilibrium $E_0 = (\frac{\lambda}{d_S}, 0, 0)$ whilst $R_0 > 1$ there exists a unique globally stable endemic equilibrium $E^* = (S^*, I^*, R^*)$.

There is variability in the spread of the disease and this is incorporated in a model via assumptions about stochasticity in the transmission coefficient β and the removal rates d_S, d_I and d_R which is one of the standard ways for the stochastic model posed (see e.g. [9], [17], [12]). By stochastic Lyapunov functions, we obtain some analytical results for stochastic model posed in this paper. In particular we establish conditions for extinction or persistence of the infective population. In case of persistence, we show the existence and the uniqueness of the stationary distribution. Furthermore, we derive an estimate for the mean of the stationary distribution.

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while \mathcal{F}_0 contains all \mathbb{P} -null sets), and let $B(t)$ be a n -dimensional standard Brownian motion with independent individuals $B_i(t), 1 \leq i \leq n$. In practice we usually estimate a parameter by an average value plus an error term. In this case, the parameters β, d_S, d_I and d_R in Eq. (1) change to random variables $\tilde{\beta}, \tilde{d}_S, \tilde{d}_I$ and \tilde{d}_R respectively such that

$$\tilde{\beta} = \beta + \text{error}_0, \tilde{d}_S = d_S + \text{error}_1, \tilde{d}_I = d_I + \text{error}_2, \tilde{d}_R = d_R + \text{error}_3.$$

Accordingly, Eq. (1) becomes

$$\begin{cases} dS = (\lambda - \beta SI - d_S S + \gamma R)dt - S \text{error}_0 dt - S \text{error}_1 dt, \\ dI = (\beta SI - (d_I + v)I)dt + S \text{error}_0 dt - I \text{error}_2 dt, \\ dR = (vI - (d_R + \gamma)R)dt - R \text{error}_3 dt. \end{cases} \quad (2)$$

By the central limit theorem, the error term $\text{error}_i dt, 0 \leq i \leq 3$ may be approximated by a normal distribution with zero mean and variance $\sigma_i^2 dt, 0 \leq i \leq 3$ respectively. That is, $\text{error}_i dt = \tilde{N}(0, \sigma_i^2 dt)$. Since these $\text{error}_i dt, 0 \leq i \leq 3$ may correlate to each other, we represent them by N -dimensional Brownian motion

$B(t) = (B_1(t), \dots, B_N(t))$ as follows

$$\text{error}_i dt = \sum_{j=1}^N \sigma_{ij} dB_j(t), \quad 0 \leq i \leq 3,$$

where $dB_j(t) = B_j(t + dt) - B_j(t)$, σ_{ij} are all real numbers such that

$$\sigma_i^2 := \sum_{j=1}^N \sigma_{ij}^2, \quad 0 \leq i \leq 3, \quad \sigma^2 := \sum_{i=0}^3 \sigma_i^2.$$

Thus Eq. (2) is characterized by the following Itô SDE

$$\begin{cases} dS = (\lambda - \beta SI - d_S S + \gamma R)dt - SI \sum_{j=1}^N \sigma_{0j} dB_j(t) - S \sum_{j=1}^N \sigma_{1j} dB_j(t), \\ dI = (\beta SI - (d_I + \nu)I)dt + SI \sum_{j=1}^N \sigma_{0j} dB_j(t) - I \sum_{j=1}^N \sigma_{2j} dB_j(t), \\ dR = (\nu I - (d_R + \gamma)R)dt - R \sum_{j=1}^N \sigma_{3j} dB_j(t). \end{cases} \quad (3)$$

Obviously when $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0$, $d(S + I + R) = (\lambda - (d_S S + d_I I + d_R R))dt$ and thus $S + I + R$ is bounded. There are lots of papers in such a case. For example, Tornatore, Buccellato and Vetro [24] discuss the asymptotic stability of the disease free equilibrium of SDE SIR model; Chen and Li [7] study another SDE version of the SIR model both with and without delay where they introduce stochastic noise in a way different from ours and that of Tornatore, Buccellato and Vetro [24]; Lu [17] extends the results of [24] by including the possibility of temporary immunity and improving the analytical bound on the sufficient condition of the stability of the disease free equilibrium. Recently, Gray et. al [9] establish the conditions of extinction and persistence of the SDE SIS model. But few papers consider the both perturbation of the disease transmission coefficient and the removal rates, which may happen in the real world. In our model, $S + I + R$ is not bounded in case of $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \neq 0$ even if it is sufficiently small. Here we use the limit results of Chow [8] and moments methods to derive conditions on the asymptotic stability of the disease free equilibrium. In the case of persistence, by new stochastic Lyapunov functions to counteract high-order terms, we establish conditions for persistence of $I(t)$, where we show the existence of its stationary distribution and ergodic property; We also give an estimate of the mean by ergodic theorem.

The paper is organized as follows. In Section 2, we introduce some preliminaries to be used in later sections. In Section 3, we prove the positivity of the solution which is essential in stochastic population dynamics. In Section 4, we establish the conditions for extinction of infectious diseases whist the susceptible population converges weakly to an inverse-gamma distribution with explicit shape and scale parameters, where the mean and the variance of the susceptible are also expressed. In Section 5, we discuss the ergodicity of SDE model under mild conditions. The means of the stationary distribution are also estimated by ergodic theorem. In Section 6, we make a concluding remark to end the paper.

2. Examples. In this paper we will use the law of large numbers for martingales and criteria on ergodicity of SDE, so we recall some classic results on them.

Assume that $(s_n, \mathcal{F}_n, n \geq 1)$ is a martingale with $E|s_n| < \infty$, and $x_1 = s_1$, $x_n = s_n - s_{n-1}$ for $n \geq 2$. The following result is given by Chow (Theorem 5(a) in [8]), and we introduce it as a lemma.

Lemma 2.1. *Let $(y_n, \mathcal{F}_{n-1}, n \geq 2)$ be strictly positive stochastic sequence such that $E x_n y_n^{-1} < \infty$. If $1 \leq p \leq 2$, then*

$$\lim_{n \rightarrow \infty} s_n y_n^{-1} = 0$$

a.e. where

$$\sum_{n=2}^{\infty} E(|x_n|^p | \mathcal{F}_{n-1}) y_n^{-p} < \infty, \quad y_n \uparrow \infty.$$

Next, we give some criteria on the ergodic property. Denote

$$R_+^l = \{x \in R^l : x_i > 0 \text{ for all } 1 \leq i \leq l\}.$$

In general, let X be a regular temporally homogeneous Markov process in $E_l \subset R_+^l$ described by the SDE

$$dX(t) = b(X(t)) dt + \sum_{r=1}^d \sigma_r(X(t)) dB_r(t), \quad (4)$$

with initial value $X(t_0) = x_0 \in E_l$ and $B_r(t), 1 \leq r \leq d$, are standard Brownian motions defined on the above probability space. The diffusion matrix is defined as follows

$$A(x) = (A_{ij}(x))_{1 \leq i, j \leq l}, \quad A_{ij}(x) = \sum_{r=1}^d \sigma_r^i(x) \sigma_r^j(x).$$

Define the differential operator L associated with equation (4) by

$$L = \sum_{i=1}^l b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i, j=1}^l A_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(E_l \times R_+; R)$, then

$$LV(x) = \sum_{i=1}^l b_i(x) \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i, j=1}^l A_{ij}(x) \frac{\partial^2 V}{\partial x_i \partial x_j},$$

where $V_x = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_l})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{l \times l}$. By Itô's formula, we have

$$dV(X(t)) = LV(X(t))dt + \sum_{r=1}^d V_x(X(t)) \sigma_r(X(t)) dB_r(t).$$

Lemma 2.2. ([10]) *We assume that there exists a bounded domain $U \subset E_l$ with regular boundary, having the following properties:*

(B.1) *In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.*

(B.2) *If $x \in E_l \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x \in K} E_x \tau < \infty$ for every compact subset $K \subset E_l$.*

Then, the Markov process $X(t)$ has a stationary distribution $v(\cdot)$ with density in E_l such that for any Borel set $B \subset E_l$

$$\lim_{t \rightarrow \infty} P(t, x, B) = v(B),$$

and

$$P_x \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x(t)) dt = \int_{E_l} f(x) v(dx) \right\} = 1,$$

for all $x \in E_l$ and $f(x)$ being a function integrable with respect to the probability measure v .

Remark 1. (i) The existence of the stationary distribution with density is shown to Theorem 4.1 on page 119 and Lemma 9.4 on page 138 in [10] while the ergodicity and the weak convergence are shown to Theorem 5.1 on page 121 and Theorem 7.1 on page 130 in [10].

(ii) To verify Assumptions (B.1) and (B.2), it suffices to show that there exists a bounded domain U with regular boundary and a non-negative C^2 -function V such that $A(x)$ is uniformly elliptical in U and for any $x \in E_l \setminus U$, $LV(x) \leq -C$ for some $C > 0$ (See e.g. [25], page 1163).

3. Existence and uniqueness of positive solution. In order for the SDE SIRS model to make sense, we must show that this model has a unique global positive solution. Since the SDE SIRS model is a special SDE, the existing general existence-uniqueness theorem on the SDE (see e.g., [19], [20]) is not applicable. We need to establish the following theorem.

Theorem 3.1. For any given initial value $(S(0), I(0), R(0))' \in R_+^3$, the SDE (3) has a unique global positive solution with probability one, namely,

$$\mathbb{P}\{(S(t), I(t), R(t))' \in R_+^3, \forall t \geq 0\} = 1.$$

Proof. Note that the coefficients of SDE (3) are locally Lipschitz continuous. By well known results, there is a unique local solution on $[0, \tau_e)$, where τ_e is the explosion time.

Assume that $m_0 \geq 0$ is sufficiently large such that $S(0), I(0), R(0)$, all lie in the interval $[m_0^{-1}, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\tau_m = \inf\{t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \leq m^{-1} \text{ or } \max\{S(t), I(t), R(t)\} \geq m\}.$$

As usual, we set $\inf \emptyset = \infty$. Clearly, τ_m is increasing. Set $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$, where $0 \leq \tau_\infty \leq \tau_e$ a.e. If we show that $\tau_\infty = \infty$ a.e., then $\tau_e = \infty$ and the solution remains in R_+^3 for all $t \geq 0$, a.e. If this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \epsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \epsilon \text{ for all } m \geq m_1. \tag{5}$$

Let $x = (S, I, R)$, we define the C^3 -function $V_1 : R_+^3 \rightarrow R_+$ as

$$V_1(x) = (S - a - a \log \frac{S}{a}) + (I - 1 - \log I) + (R - 1 - \log R),$$

where a is a positive constant to be determined later.

By Itô's formula, we see

$$\begin{aligned}
 & LV_1(x) \\
 &= \lambda - d_S S - d_I I - d_R R + a \left(\beta I + d_S - \frac{\lambda}{S} - \frac{\gamma R}{S} \right) - \beta S - \frac{vI}{R} \\
 &\quad + d_I + v + d_R + \gamma + a \sum_{j=1}^N (I\sigma_{0j} + \sigma_{1j})^2 + \sum_{j=1}^N (S\sigma_{0j} - \sigma_{2j})^2 + \sigma_3^2.
 \end{aligned} \tag{6}$$

Let $a\beta \leq d_I$, then there exists a constant C_1 such that

$$LV_1(x) \leq C_1 + a \sum_{j=1}^N (I\sigma_{0j} + \sigma_{1j})^2 + \sum_{j=1}^N (S\sigma_{0j} - \sigma_{2j})^2.$$

Define a function $V_2 : R_+^3 \rightarrow R_+$ by

$$V_2(x) = (S + I + R)^2, \quad x = (S, I, R).$$

Then

$$\begin{aligned}
 & LV_2(x) \\
 &= 2(S + I + R)(\lambda - d_S S - d_I I - d_R R) + \sum_{j=1}^N (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})^2 \\
 &\leq (S + I + R)^2 + \lambda^2 + (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)(S + I + R)^2 \\
 &\leq C_2 + C_3(S + I + R)^2,
 \end{aligned} \tag{7}$$

where C_2, C_3 are positive constants.

Let $V(x) = V_1(x) + V_2(x)$. By (6) and (7), then there exists a positive constant C such that

$$LV(x) \leq C + CV(x).$$

Let $\tilde{V}(x, t) = e^{-Ct}(1 + V(x))$, then

$$L\tilde{V}(x) = -Ce^{-Ct}(1 + V(x)) + e^{-Ct}LV(x) \leq 0.$$

Let $x(t) = (S(t), I(t), R(t)), t \geq 0$, by Itô's formula, we have for any $m \geq m_1$

$$E\tilde{V}(x(t \wedge \tau_m), t \wedge \tau_m) = \tilde{V}(x(0)) + E \int_0^{t \wedge \tau_m} L\tilde{V}(x(u), u) ds \leq \tilde{V}(x(0)). \tag{8}$$

Set $\Omega_m = \{\tau_m \leq T\}$ and by (5), $\mathbb{P}\{\Omega_m\} \geq \varepsilon$. Note that for every $\omega \in \Omega_m$, $V(x(\tau_m, \omega)) \geq b_m := \min\{V(y)|y \text{ has a individual as } m^{-1} \text{ or } m\} \rightarrow \infty$, as $m \rightarrow \infty$. It then follows from (8) that

$$\varepsilon b_m \leq E[V(x(\tau_m, \omega)I_{\Omega_m})] \leq e^{CT}\tilde{V}(x(0)).$$

Letting $m \rightarrow \infty$ leads to the contradiction

$$\infty > e^{CT}\tilde{V}(x(0)) \geq \infty.$$

Therefore $\tau_\infty = \infty$, a.e., whence the proof is complete. □

4. Disease extinction. In the preceding section, positive solutions have already been discussed. In an epidemic model, disease extinction or persistence are two of the most interesting topics. In this section, we will establish conditions for disease extinction in the SDE SIRS model (3) and discuss disease persistence in the next section.

Lemma 4.1. *Let $N(t) = S(t) + I(t) + R(t)$, then there exists some $p_0 > 1$ such that for any $p \in (1, p_0)$,*

$$\sup_n E \left(\max_{t \in [n, n+1]} N^p(t) \right) < +\infty. \tag{9}$$

Proof. Obviously, there exists some $p_0 > 1$ such that for any $p \in (1, p_0)$,

$$C_p = p\check{d} - \frac{p(p-1)\sigma^2}{2} > 0.$$

Note that

$$dN(t) = (\lambda - (d_S S(t) + d_I I(t) + d_R R(t)))dt - \sum_{j=1}^N (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})dB_j(t).$$

For any $p > 1$, we have

$$\begin{aligned} & d(e^{0.5C_p t} N^p(t)) \\ &= e^{0.5C_p t} [\lambda p N^{p-1}(t) - p N^{p-1}(t)(d_S S(t) + d_I I(t) + d_R R(t)) \\ &+ \frac{p(p-1)}{2} N^{p-2}(t) \sum_{j=1}^N (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})^2 + 0.5C_p N^p(t)] dt + dM(t), \end{aligned} \tag{10}$$

where $M(t) = -\int_0^t p N^{p-1}(u) e^{0.5C_p u} \sum_{j=1}^N (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j}) dB_j(t)$, $t \geq 0$, is a local martingale.

Denote $\max\{d_S, d_I, d_R\}$, $\min\{d_S, d_I, d_R\}$ by \hat{d} , \check{d} respectively, then

$$\begin{aligned} & E(e^{0.5C_p t} N^p(t)) \\ &\leq N^p(0) + E \int_0^t e^{0.5C_p u} [\lambda p N^{p-1}(u) - p\check{d} N^p(u) \\ &+ \frac{p(p-1)\sigma^2}{2} N^p(u) + 0.5C_p N^p(u)] du \\ &\leq N^p(0) + E \int_0^t e^{0.5C_p u} [\lambda p N^{p-1}(u) - 0.5C_p N^p(u)] dt. \end{aligned}$$

Hence,

$$e^{0.5C_p t} E N^p(t) \leq N^p(0) + \frac{K}{0.5C_p} e^{0.5C_p t},$$

where $K = \sup_{x>0} (\lambda p x^{p-1} - 0.5C_p x^p)$. This implies for any $p \in (1, p_0)$,

$$\sup_{t \geq 0} E N^p(t) \leq N^p(0) + \frac{K}{0.5C_p} < +\infty. \tag{11}$$

By (10), we have

$$\max_{t \in [n, n+1]} N^p(t) \leq N^p(n) + \int_n^{n+1} [\lambda p N^{p-1}(u) - C_p N^p(u)] dt + \max_{t \in [n, n+1]} |\widetilde{M}(t)|,$$

where $\widetilde{M}(t) = \int_n^t p N^{p-1}(u) \sum_{j=1}^N (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j}) dB_j(t)$, $t \in [n, n+1]$.

Therefore,

$$\begin{aligned} & E \left(\max_{t \in [n, n+1]} N^p(t) \right) \\ & \leq EN^p(n) + \int_n^{n+1} [\lambda p EN^{p-1}(u) + C_p EN^p(u)] dt + E \left(\max_{t \in [n, n+1]} |\widetilde{M}(t)| \right). \end{aligned}$$

By (11), there exists a positive constant C_1 independent of n such that

$$E \left(\max_{t \in [n, n+1]} N^p(t) \right) \leq C_1 + E \left(\max_{t \in [n, n+1]} |\widetilde{M}(t)| \right). \quad (12)$$

Since

$$\begin{aligned} & \widetilde{M}(t) \\ & = N^p(t) - N^p(n) - \int_n^t [\lambda p N^{p-1}(u) - p N^{p-1}(u)(d_S S(u) + d_I I(u) + d_R R(u)) \\ & \quad + \frac{p(p-1)}{2} N^{p-2}(u) \sum_{j=1}^N (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})^2] du. \end{aligned}$$

Hence for some $C_2 > 0$, we have

$$|\widetilde{M}(t)| \leq N^p(t) + N^p(n) + C_2 \int_n^t [N^{p-1}(u) + N^p(u)] dt. \quad (13)$$

For any $p \in (1, p_0)$, there exists $p' > 1$ such that $pp' \in (1, p_0)$ and the constants C_3, C_4 independent of n , we have

$$\begin{aligned} & E \left(\max_{t \in [n, n+1]} |M(t)|^{p'} \right) \\ & \leq C_3 \max_{t \in [n, n+1]} E |M(t)|^{p'} \\ & \leq C_3 E \left[N^{pp'}(t) + N^{pp'}(n) + \int_n^{n+1} [N^{(p-1)p'}(u) + N^{pp'}(u)] dt \right] \\ & \leq C_4, \end{aligned}$$

where the first inequality is derived from the maximal inequality for martingales, the second by (13) and Jensen's inequality, and the last by (11).

By (12), we have

$$E \left(\max_{t \in [n, n+1]} N^p(t) \right) \leq C_1 + \left(E \left(\max_{t \in [n, n+1]} |M(t)|^{p'} \right) \right)^{1/p'} < +\infty,$$

whence the proof is complete. \square

Lemma 4.2. For any $1 \leq j \leq N$,

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S(t) dB_j(t) &= 0, & \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T I(t) dB_j(t) &= 0, \\ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t) dB_j(t) &= 0, & a.e. \end{aligned}$$

Proof. Note that for any $T \in [n, n + 1)$,

$$\frac{1}{T} \int_0^T S(t)dB_j(t) = \frac{n}{T} \cdot \frac{1}{n} \int_0^n S(t)dB_j(t) + \frac{1}{T} \int_n^T S(t)dB_j(t).$$

Let $x_i = \int_{i-1}^i S(t)dB_j(t)$, $1 \leq i \leq n$. For some $p \in (1, p_0)$, by B-D-G inequality, there exists some $C < +\infty$ such that

$$\begin{aligned} E|x_i|^p &\leq CE \left(\int_{i-1}^i S^2(t)dt \right)^{p/2} \leq CE \left(\max_{t \in [i-1, i]} S^p(t) \right) \\ &\leq C \sup_n E \left(\max_{t \in [n, n+1]} N^p(t) \right) < +\infty, \end{aligned}$$

where the last inequality is derived from Lemma 4.1.

Then

$$E \left(\sum_{i=1}^{\infty} \frac{(|x_i|^p | \mathcal{F}_{i-1})}{i^p} \right) = \sum_{i=1}^{\infty} \frac{E|x_i|^p}{i^p} < \infty,$$

which implies

$$\sum_{i=1}^{\infty} \frac{(|x_i|^p | \mathcal{F}_{i-1})}{i^p} < \infty, \quad a.e.,$$

and thus by Lemma 2.1

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n S(t)dB_j(t) = 0, \quad a.e. \tag{14}$$

On the other hand, by Lemma 4.1 again we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathbb{P} \left\{ \max_{T \in [n, n+1]} \frac{1}{T} \left| \int_n^T S(t)dB_j(t) \right| > \varepsilon \right\} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{\varepsilon^p n^p} E \left| \max_{T \in [n, n+1]} \int_n^T S(t)dB_j(t) \right|^p \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{n^p} E \left(\max_{t \in [n, n+1]} S^p(t) \right) \\ &\leq C \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \right) \sup_n E \left(\max_{t \in [n, n+1]} N^p(t) \right) < +\infty. \end{aligned}$$

Applying Borel-Cantelli lemma, we see

$$\mathbb{P} \left\{ \lim_{n \rightarrow \infty} \max_{T \in [n, n+1]} \frac{1}{T} \left| \int_n^T S(t)dB_j(t) \right| = 0 \right\} = 1. \tag{15}$$

Taking (14) and (15) into account, we get

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S(t)dB_j(t) = 0, \quad a.e.$$

In the similar way, we can prove the other results and hence the proof is complete. \square

Lemma 4.3.

$$\lim_{T \rightarrow \infty} \frac{N(T)}{T} = 0, \quad a.e.$$

Proof. For any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \max_{T \in [n, n+1]} \frac{N(T)}{T} > \varepsilon \right\} \leq \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n n^n} E \left(\max_{t \in [n, n+1]} N^p(t) \right) < +\infty,$$

where the last inequality is derived from Lemma 4.1.

Applying Borel-Cantelli lemma, we see

$$\mathbb{P} \left\{ \lim_{n \rightarrow \infty} \max_{T \in [n, n+1]} \frac{N(T)}{T} = 0 \right\} = 1.$$

The proof is hence complete. □

Theorem 4.4. *If*

$$\frac{\lambda \left(\beta + \sum_{j=1}^n \sigma_{0j} \sigma_{2j} \right)}{d_S \left(d_I + v + \frac{\sigma_0^2}{2} \right)} - \frac{\lambda^2 \sigma_0^2}{2d_S^2 \left(d_I + v + \frac{\sigma_0^2}{2} \right)} < 1 \text{ and } \sigma_0^2 \leq \frac{d_S}{\lambda} \left(\beta + \sum_{j=1}^n \sigma_{0j} \sigma_{2j} \right),$$

then for any initial value, SDE (3) obeys

$$\lim_{T \rightarrow \infty} \frac{\log I(T)}{T} \leq \frac{\lambda}{d_S} \left(\beta + \sum_{j=1}^n \sigma_{0j} \sigma_{2j} \right) - \left(d_I + v + \frac{\sigma_0^2}{2} \right) - \frac{\lambda^2 \sigma_0^2}{2d_S^2} < 0, \text{ a.e.} \quad (16)$$

In other words, the disease decays exponentially with probability one.

Proof. By Itô's formula, we have

$$\log(I(T)) = \log(I(0)) + \int_0^T f(x(t)) dt + \sum_{j=1}^N \sigma_{0j} \int_0^T S(t) dB_j(t) - \sum_{j=1}^N \sigma_{2j} B_j(T), \quad (17)$$

where $f(x) = \beta S - (d_I + v) - \frac{1}{2} \sum_{j=1}^N (S \sigma_{0j} - \sigma_{2j})^2$ for $x = (S, I) \in R_+^2$.

Define

$$\tilde{N}(t) = S(t) + \frac{d_I}{d_S} I(t) + \frac{d_R}{d_S} R(t), \quad x(t) = (S(t), I(t)) \text{ and } y(t) = \frac{d_I}{d_S} I(t) + \frac{d_R}{d_S} R(t),$$

then

$$\begin{aligned} & f(x(t)) \\ &= \beta S(t) - (d_I + v) - \frac{1}{2} \sigma_0^2 S^2(t) + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} S(t) - \frac{\sigma_0^2}{2} \\ &= \beta (\tilde{N}(t) - y(t)) - \left(d_I + v + \frac{\sigma_0^2}{2} \right) - \frac{1}{2} \sigma_0^2 (\tilde{N}(t) - y(t))^2 + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} (\tilde{N}(t) - y(t)) \\ &= \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} \right) \tilde{N}(t) - \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} \right) y(t) - \left(d_I + v + \frac{\sigma_0^2}{2} \right) \\ &\quad - \frac{\sigma_0^2}{2} \tilde{N}^2(t) + \sigma_0^2 \tilde{N}(t) y(t) - \frac{\sigma_0^2}{2} y^2(t) \\ &= \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} \right) \tilde{N}(t) - \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} - \varepsilon \sigma_0^2 \right) y(t) - \left(d_I + v + \frac{\sigma_0^2}{2} \right) \\ &\quad - \frac{\sigma_0^2}{2} \tilde{N}^2(t) + \sigma_0^2 (\tilde{N}(t) - \varepsilon) y(t) - \frac{\sigma_0^2}{2} y^2(t), \end{aligned}$$

where $\varepsilon > 0$ is a constant to be determined later.

Assume that $\varepsilon\sigma_0^2 \leq \beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j}$, then

$$\begin{aligned} f(x(t)) &\leq \left(\beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j} \right) \tilde{N}(t) - \left(d_I + v + \frac{\sigma_2^2}{2} \right) - \frac{\sigma_0^2}{2} \tilde{N}^2(t) \\ &\quad + \frac{\sigma_0^2}{2} (\tilde{N}(t) - \varepsilon)^2 - \frac{\sigma_0^2}{2} (\tilde{N}(t) - \varepsilon - y(t))^2 \\ &\leq \left(\beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j} - \sigma_0^2\varepsilon \right) \tilde{N}(t) - \left(d_I + v + \frac{\sigma_2^2}{2} \right) + \frac{\varepsilon^2\sigma_0^2}{2}. \end{aligned} \tag{18}$$

Recall that $N(t) = S(t) + I(t) + R(t)$, then

$$dN(t) = (\lambda - d_S \tilde{N}(t))dt + \sum_{j=1}^N (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})dB_j(t).$$

Thus, we see

$$\frac{N(T) - N(0)}{T} = \lambda - \frac{d_S}{T} \int_0^T \tilde{N}(t)dt + \frac{1}{T} \sum_{j=1}^N \int_0^T (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})dB_j(t).$$

Let $T \rightarrow \infty$, Lemma 4.2 and Lemma 4.3 yields

$$\frac{1}{T} \int_0^T \tilde{N}(t)dt = \frac{\lambda}{d_S}, \quad a.e.$$

Since Lemma 4.2 and Lemma 4.3 implies that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S(t)dB_j(t) = \lim_{T \rightarrow \infty} \frac{B_j(T)}{T} = 0,$$

a.e., by (17), (18) and let $\varepsilon = \frac{\lambda}{d_S}$, we have

$$\lim_{T \rightarrow \infty} \frac{\log I(T)}{T} \leq \frac{\lambda}{d_S} \left(\beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j} \right) - \left(d_I + v + \frac{\sigma_2^2}{2} \right) - \frac{\lambda^2\sigma_0^2}{2d_S^2} < 0.$$

The proof is hence complete. □

Remark 2. In the deterministic model (1.1), λ is the rate of susceptible individuals recruited into the population (either by birth or immigration) per unit time; β is some transmission coefficient and it is assumed that the rate at which the susceptible individuals acquire the infection is proportional to the number of encounters between the susceptible and infective individuals per unit time, being βSI ; d_S is the natural mortality rate or the removal rate of the susceptible individual; d_I is the removal rate of infectious individual and may be the sum of the natural mortality rate and the disease mortality rate; v is the recovery rate of infective individual; d_R is the removal rate of the recovered individual and γ is the rate at which the recovered individual loses immunity. Thus, from the biological point of view, $\frac{1}{d_S}$ is the average death age of the susceptible individual, that is, the average lifespan of the susceptible individual; $\frac{\lambda}{d_S}$ is the number of the susceptible population without infection during the life span; $\frac{1}{d_I+v}$ is the mean infective period, or the mean course of infection. It is well-known that the basic reproductive number R_0 of the deterministic model (1.1) is just the product of the transmission coefficient

β per unit time, the number $\frac{\lambda}{d_S}$ of when all of the individuals in the population are initially susceptible and the mean infective period $\frac{1}{d_I+v}$, which is actually the average number of secondary infections produced by one infected individual during the mean course of infection in a completely susceptible population and thus determines whether a disease persists or goes extinct.

Because of the existence of random fluctuations in the environment, we consider the stochastic model (1.3) to investigate how the randomness affects the behavior of the disease transmission. By Theorem 4.4, under some mild condition and

$$\begin{aligned} R_{0;\sigma_{ij}} &:= \frac{\lambda}{d_S} \cdot \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} - \frac{\lambda \sigma_0^2}{2d_S} \right) \cdot \frac{1}{d_I + v + \frac{\sigma_2^2}{2}} \\ &= \frac{\lambda \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} \right)}{d_S \left(d_I + v + \frac{\sigma_2^2}{2} \right)} - \frac{\lambda^2 \sigma_0^2}{2d_S^2 \left(d_I + v + \frac{\sigma_2^2}{2} \right)} \\ &< 1, \end{aligned}$$

we obtain

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{\log I(T)}{T} &\leq \frac{\lambda}{d_S} \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} \right) - \left(d_I + v + \frac{\sigma_2^2}{2} \right) - \frac{\sigma_0^2}{2} \left(\frac{\lambda}{d_S} \right)^2 \\ &= \left(d_I + v + \frac{\sigma_2^2}{2} \right) (R_{0;\sigma_{ij}} - 1) \\ &< 0, \text{ a.e.} \end{aligned}$$

It seems that in the stochastic model (1.3), the random fluctuations impose some effect on the transmission of infectious diseases. Note that under the condition of Theorem 4.1, the average number of secondary infections in the stochastic model is less than 1 which implies that the disease dies out eventually.

Theorem 4.5. *Under the conditions of Theorem 4.4, the susceptible class $S(t)$ of the SDE (3) converges weakly to an inverse-gamma distribution ν , which is the distribution of the reciprocal of a gamma distribution with shape parameter $\frac{2d_S}{\sigma_1^2} + 1$ and scale parameter $\frac{\sigma_1^2}{2\lambda}$. Furthermore, we have*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S(t) dt = \frac{\lambda}{d_S}, \text{ a.e.}$$

and if $d_S > \frac{\sigma_1^2}{2}$, then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(S(t) - \frac{\lambda}{d_S} \right)^2 dt = \frac{\sigma_1^2}{2d_S - \sigma_1^2} \left(\frac{\lambda}{d_S} \right)^2, \text{ a.e.}$$

Proof. In the same way as Theorem 4.4 was proved, we can show that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log R(T) \leq - \left(d_R + \gamma + \frac{\sigma_3^2}{2} \right) < 0, \text{ a.e.} \quad (19)$$

Recall that $N(t) = S(t) + I(t) + R(t)$, $t \geq 0$ and

$$dN(t) = (\lambda - d_S N(t) + f(t)) dt + \sum_{j=1}^N (\sigma_{1j} N(t) + g_j(t)) dB_j(t),$$

where $f(t) = (d_S - d_I)I(t) + (d_S - d_R)R(t)$ and $g_j(t) = (\sigma_{2j} - \sigma_{1j})I(t) + (\sigma_{3j} - \sigma_{1j})R(t)$ for any $t \geq 0, 1 \leq j \leq N$.

Applying Theorem 3.1 in [21] again,

$$\begin{aligned} &N(T) \\ &= \tilde{\Phi}(T) \left(N(0) + \int_0^T \tilde{\Phi}^{-1}(t) \left(\lambda + f(t) - \sum_{j=1}^N \sigma_{1j} g_j(t) \right) dt \right. \\ &\quad \left. + \sum_{j=1}^N \int_0^T \tilde{\Phi}^{-1}(t) g_j(t) dB_j(t) \right), \end{aligned}$$

where $\tilde{\Phi}(t) = \exp \left\{ - \left(d_S + \frac{\sigma_1^2}{2} \right) t - \sum_{j=1}^N \sigma_{1j} B_j(t) \right\}, t \geq 0$.

Similar to the proof of $R(t)$, we get by (16) and (19)

$$\lim_{T \rightarrow \infty} \tilde{\Phi}(T) \int_0^T \tilde{\Phi}^{-1}(t) \left(f(t) - \sum_{j=1}^N \sigma_{1j} g_j(t) \right) dt = 0.$$

On the other hand, note that $\lim_{T \rightarrow \infty} \tilde{\Phi}^{-1}(T) = \infty$ and by (16), (19) again,

$$\int_0^\infty \frac{(\tilde{\Phi}^{-1}(t))^2 g_j^2(t)}{(1 + \tilde{\Phi}^{-1}(t))^2} dt < \infty, \text{ a.e., } 1 \leq j \leq N.$$

Thus Theorem 3.4 in [21] yields

$$\lim_{T \rightarrow \infty} \tilde{\Phi}(T) \left(\sum_{j=1}^N \int_0^T \tilde{\Phi}^{-1}(t) g_j(t) dB_j(t) \right) = 0, \text{ a.e.}$$

In all,

$$N(T) = \tilde{\Phi}(T) \left(N(0) + \int_0^T \lambda \tilde{\Phi}^{-1}(t) dt \right) + o(1), \text{ a.e.,}$$

where $o(1) \rightarrow 0$, a.e. as $T \rightarrow \infty$. On the other hand, since $\lim_{T \rightarrow \infty} I(t) = \lim_{T \rightarrow \infty} R(t) = 0$, a.e.,

$$S(T) = \tilde{\Phi}(T) \left(N(0) + \int_0^T \lambda \tilde{\Phi}^{-1}(t) dt \right) + o(1), \text{ a.e.} \tag{20}$$

Let $X(t)$ be the solution characterized by the SDE

$$dX(t) = (\lambda - d_S X(t)) dt + X(t) \sum_{j=1}^N \sigma_{1j} dB_j(t), \quad X(0) = N(0). \tag{21}$$

By Theorem 3.1 in [21], $X(t)$ of the SDE (3) can be expressed as

$$X(t) = \tilde{\Phi}(T) \left(N(0) + \int_0^T \lambda \tilde{\Phi}^{-1}(t) dt \right).$$

Thus (20) yields

$$\lim_{T \rightarrow \infty} (S(T) - X(T)) = 0, \text{ a.e.} \tag{22}$$

Assume that $\{B(t), t \geq 0\}$ is a standard Brownian motion. Since the processes $\{\sum_{j=1}^N \sigma_{1j} B_j(t), t \geq 0\}$ and $\{\sigma_1 B(t), t \geq 0\}$ are equivalent in distribution, we may

replace $\sum_{j=1}^N \sigma_{1j} dB_j(t)$ by $\sigma_1 dB(t)$ in the SDE (21). Let $Y(t) = X(t) - \frac{\lambda}{d_S}$, then $Y(t)$ satisfies

$$dY = -d_S Y dt + \sigma_1 \left(Y + \frac{\lambda}{d_S} \right) dB(t). \quad (23)$$

Theorem 2.1 (a) in [4] with $C = 1$ implies that $Y(t)$ is stable in distribution, so does $X(t)$. Let $q(x) = \exp\left(-2 \int_1^x \frac{\lambda - d_S y}{\sigma_1^2 y^2} dy\right)$. We have

$$\int_1^\infty q(x) dx = \infty, \quad \int_0^1 q(x) dx = \infty, \quad \int_0^\infty \frac{dx}{\sigma_1^2 q(x) x^2} < \infty.$$

So $X(t)$ is ergodic (Theorem 1.16 in [14]), and its unique invariant distribution ν has density $(Mx^2 p(x))^{-1}$, where $p(x) = \exp\left(\frac{2\lambda}{\sigma_1^2 x} + \frac{2d_S}{\sigma_1^2} \ln x\right)$, $x > 0$, M is a normal constant. By computation, ν is an inverse-gamma distribution, which is the reciprocal of a gamma distribution with shape parameter $\frac{2d_S}{\sigma_1^2} + 1$ and scale parameter $\frac{\sigma_1^2}{2\lambda}$. It is clear that the stability in distribution implies that the limiting distribution is just the invariant distribution. Therefore, $X(t)$ converges weakly to ν as $t \rightarrow \infty$. By (22), we conclude that $S(t)$ converges weakly to ν , too.

Note that

$$\begin{aligned} \frac{N(T)}{T} &= \frac{N(0)}{T} + \lambda - \frac{d_S}{T} \int_0^T S(t) dt - \frac{d_I}{T} \int_0^T I(t) dt - \frac{d_R}{T} \int_0^T R(t) dt \\ &\quad - \sum_{j=1}^N \int_0^T (\sigma_{ij} S(t) + \sigma_{2j} I(t) + \sigma_{3j} R(t)) dB_j(t). \end{aligned}$$

Under conditions of Theorem 4.4, we get

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T I(t) dt = 0, \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t) dt = 0, \quad a.e.,$$

which yields, together with Lemma 4.2 and Lemma 4.3,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S(t) dt = \frac{\lambda}{d_S}, \quad a.e.$$

After computation, we get

$$\begin{aligned} &\frac{1}{T} \int_0^T S^2(t) dt \\ &= \frac{1}{T} \int_0^T (S(t) - X(t))^2 dt + \frac{2}{T} \int_0^T (S(t) - X(t))X(t) dt + \frac{1}{T} \int_0^T X^2(t) dt. \end{aligned}$$

By (22) and $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t) dt = \frac{\lambda}{d_S}$, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (S(t) - X(t))^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (S(t) - X(t))X(t) dt = 0, \quad a.e.,$$

which implies

$$\lim_{T \rightarrow \infty} \left(\frac{1}{T} \int_0^T S^2(t) dt - \frac{1}{T} \int_0^T X^2(t) dt \right) = 0, \quad a.e.$$

Therefore,

$$\lim_{T \rightarrow \infty} \left(\frac{1}{T} \int_0^T \left(S(t) - \frac{\lambda}{d_S} \right)^2 dt - \frac{1}{T} \int_0^T \left(X(t) - \frac{\lambda}{d_S} \right)^2 dt \right) = 0, \text{ a.e.}$$

Applying Itô's formula to $\left(X(t) - \frac{\lambda}{d_S} \right)^2$, we get

$$\begin{aligned} & \frac{1}{2T} \left(X(T) - \frac{\lambda}{d_S} \right)^2 \\ &= \frac{\sigma_1^2}{2} \left(\frac{\lambda}{d_S} \right)^2 + \frac{\lambda \sigma_1^2}{d_S T} \int_0^T \left(X(t) - \frac{\lambda}{d_S} \right) dt \\ & - \left(d_S - \frac{\sigma_1^2}{2} \right) \frac{1}{T} \int_0^T \left(X(t) - \frac{\lambda}{d_S} \right)^2 dt + \frac{\sigma_1}{T} \int_0^T X(t) \left(X(t) - \frac{\lambda}{d_S} \right) dB(t), \end{aligned} \tag{24}$$

If $d_S > \frac{\sigma_1^2}{2}$, similar to the way in the proof of Lemma 4.1, 4.2 and 4.3, we derive

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(X(T) - \frac{\lambda}{d_S} \right)^2 = 0, \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t) \left(X(t) - \frac{\lambda}{d_S} \right) dB(t) = 0, \text{ a.e.}$$

Let $T \rightarrow \infty$ in (24), we get

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(X(t) - \frac{\lambda}{d_S} \right)^2 dt = \frac{\sigma_1^2}{2d_S - \sigma_1^2} \left(\frac{\lambda}{d_S} \right)^2, \text{ a.e.}$$

This completes the proof. □

Example 4.1. To illustrate our analytical results, we introduce a SIRS model which is used to investigate the dynamics of *Pasteurella muris* in colonies of laboratory mice (see [2]). We choose parameters in p362 and from p363 in [2]. That is to say, $\lambda = 0.33$, and the other parameters are as following

Parameter	β	d_S	d_R	d_I	ν	γ
Value/day	0.0056	0.006	0.006	0.0066	0.04	0.0021

To see the effect of random fluctuations, we consider the following diffusion coefficient

$$\sigma = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}.$$

Note that in the deterministic model, $R_0 = 2.9057$. Hence, the infectious disease will persist in the deterministic model. For the stochastic model, it is easy to check that the conditions of Theorem 4.4 are satisfied and the infectious disease will eventually extinctive in the effect of random fluctuations. In Fig.1, the red, blue and green lines represent the susceptible, the infective and the recovered individuals, respectively. It is seen that the infective and the recovered individuals are forced to expire. The following table shows sample means and variances at different time which are very close to our theoretical results in Theorem 4.5.

Time point	20	40	60	80	100
Sample mean	57.1999	56.1885	55.1778	55.2005	55.1588
Sample variance	0.0356	0.0318	0.0257	0.0285	0.0233

To verify the density function of the stationary distribution ν , we use software R to get an estimate for the kernel densities of ν and $S(t)$, respectively. In Fig.2 and Fig.3, these histograms of kernel densities look alike, and thus confirm our analytical results.

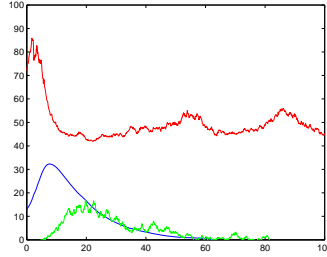


FIGURE 1. The simulation of SDE

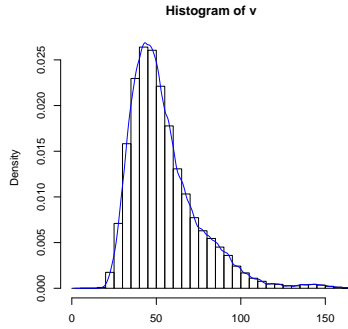


FIGURE 2. The histogram of the limit distribution

5. Ergodicity. In this section, we discuss the persistence of SDE (3) by ergodic property of Markovian processes.

Theorem 5.1. *If matrix (σ_{kj}) is full row bank, $R_0 > 1$, $\check{d} := \min\{d_S, d_I, d_R\} > 6\sigma^2$ and*

$$\begin{aligned}
 & \left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1\right) S^{*2} \wedge \left(\frac{\gamma d_I}{d_S + d_R} - C_2\right) I^{*2} \\
 & \wedge \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{v(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3\right) R^{*2} \\
 & > \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta}\right) \sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2} + \frac{\lambda^4 \sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)},
 \end{aligned} \tag{25}$$

where (S^*, I^*, R^*) is the unique endemic equilibrium of the deterministic model (1), $C_1 = 2\left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta}\right) I^* \sigma_0^2 + \frac{\gamma \sigma^2}{d_S + d_R} + 2\sigma_1^2$, $C_2 = \frac{\gamma \sigma^2}{d_S + d_R}$, $C_3 = \frac{\gamma \sigma^2}{d_S + d_R} + \frac{\gamma(d_I + d_R)}{v(d_S + d_R)} \sigma_3^2$, then the SDE (3) is ergodic and positive recurrent Markovian process.

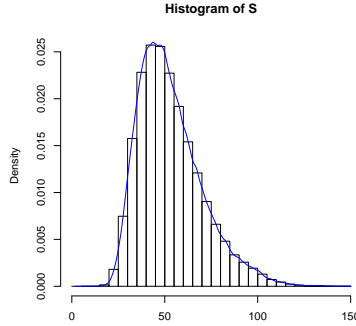


FIGURE 3. The histogram of the susceptible individuals

Proof. Note that the diffusion coefficient $g(x)$ of the SDE (3) is defined as

$$g(x) = (g_{ij}(x)), \quad 1 \leq i \leq 3, \quad 1 \leq j \leq N, \quad x = (S, I, R)^\tau \in R_+^3,$$

where $g_{1j}(x) = -SI\sigma_{0j} - S\sigma_{1j}$, $g_{2j}(x) = SI\sigma_{0j} - I\sigma_{2j}$, $g_{3j}(x) = -R\sigma_{3j}$, $1 \leq j \leq N$.

If matrix $(\sigma_{kj})_{0 \leq k \leq 3, 1 \leq j \leq N}$ is full row bank, then $\text{rank}(g(x)) = 3$ and thus $A(x) := g(x)g^\tau(x)$ is positive definite in R_+^3 . Since $g(x)$ is continuous in x , $A(x)$ is uniformly elliptical in any compact set $K \subset R_+^3$.

By Lemma 2.2 and its remark, it suffices to find a positively stochastic Lyapunov function $V(x)$ and compact set $K \subset R_+^3$ such that $LV(x) \leq -C$ for some $C > 0$ and $x \in R_+^3/K$.

When $R_0 > 1$, there exists unique positive equilibrium (S^*, I^*, R^*) of (1) such that

$$\lambda + \gamma R^* = \beta S^* I^* + d_S S^*, \quad \beta S^* = d_I + v, \quad v I^* = (d_R + \gamma) R^*. \quad (26)$$

For any $x = (S, I, R)^\tau \in R_+^3$, we consider a stochastic Lyapunov function V_1 as follows

$$V_1(x) = I - I^* - I^* \log \frac{I}{I^*} = I^* \left(\frac{I}{I^*} - 1 - \log \frac{I}{I^*} \right) > 0.$$

Itô's formula and (26) yield

$$\begin{aligned} LV_1(x) &= (I - I^*)(\beta S - d_I - v) + \frac{I^*}{2} \sum_{j=1}^N (\sigma_{0j} S - \sigma_{2j})^2 \\ &= \beta(I - I^*)(S - S^*) + \frac{\sigma_0^2 I^*}{2} S^2 - SI^* \sum_{j=1}^N \sigma_{0j} \sigma_{2j} + \frac{\sigma_2^2 I^*}{2} \\ &\leq \beta(I - I^*)(S - S^*) + \sigma_0^2 I^* S^2 + \sigma_2^2 I^*. \end{aligned}$$

Define V_2 as follows

$$V_2(x) = \frac{1}{2}(R - R^*)^2.$$

Then by (26) we compute

$$\begin{aligned} LV_2(x) &= (R - R^*)(vI - (d_R + \gamma)R) + \frac{\sigma_3^2}{2}R^2 \\ &= v(R - R^*)(I - I^*) - (d_R + \gamma)(R - R^*)^2 + \frac{\sigma_3^2}{2}R^2. \end{aligned}$$

Define V_3 as follows

$$V_3(x) = \frac{1}{2}(S - S^* + I - I^* + R - R^*)^2.$$

Then applying Itô's formula and (26) again, we get

$$\begin{aligned} &LV_3(x) \\ &= (S - S^* + I - I^* + R - R^*)(\lambda - d_S S - d_I I - d_R R) \\ &\quad + \frac{1}{2} \sum_{j=1}^N (\sigma_{1j} S + \sigma_{2j} I + \sigma_{3j} R)^2 \\ &\leq (S - S^* + I - I^* + R - R^*)(\lambda - d_S S - d_I I - d_R R) \\ &\quad + \frac{1}{2} (S^2 + I^2 + R^2)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &\leq -d_S (S - S^*)^2 - d_I (I - I^*)^2 - d_R (R - R^*)^2 + \frac{\sigma^2}{2} (S^2 + I^2 + R^2) \\ &\quad - (d_S + d_I)(S - S^*)(I - I^*) - (d_S + d_R)(S - S^*)(R - R^*) \\ &\quad - (d_I + d_R)(I - I^*)(R - R^*). \end{aligned}$$

Define V_4 as follows

$$V_4(x) = \frac{1}{2}(S - S^*)^2.$$

Direct computation leads to

$$\begin{aligned} &LV_4(x) \\ &= (S - S^*)(\lambda - \beta SI - d_S S + \gamma R) + \frac{1}{2} \sum_{j=1}^N (\sigma_{0j} SI + \sigma_{1j} S)^2 \\ &= -d_S (S - S^*)^2 - \beta (S - S^*)(SI - S^* I^*) + \gamma (S - S^*)(R - R^*) \\ &\quad + \frac{1}{2} \sum_{j=1}^N (\sigma_{0j} SI + \sigma_{1j} S)^2 \\ &\leq -d_S (S - S^*)^2 - \beta S^* (S - S^*)(I - I^*) - \beta (S - S^*)^2 I \\ &\quad + \gamma (S - S^*)(R - R^*) + \sigma_0^2 S^2 I^2 + \sigma_1^2 S^2 \\ &\leq -d_S (S - S^*)^2 - \beta S^* (S - S^*)(I - I^*) + \gamma (S - S^*)(R - R^*) \\ &\quad + \sigma_0^2 S^2 I^2 + \sigma_1^2 S^2 \end{aligned}$$

Define V_5 as follows

$$V_5(x) = \frac{1}{4}(S + I + R)^4.$$

We compute

$$\begin{aligned} & LV_5(x) \\ &= (S + I + R)^3(\lambda - d_S S - d_I I - d_R R) \\ &\quad + \frac{3}{2}(S + I + R)^2 \sum_{j=1}^N (\sigma_{1j} S + \sigma_{2j} I + \sigma_{3j} R)^2 \\ &\leq \lambda(S + I + R)^3 - \check{d}(S + I + R)^4 + \frac{3\sigma^2}{2}(S + I + R)^4, \end{aligned}$$

where $\check{d} = \min\{d_S, d_I, d_R\}$.

Applying Young's inequality, we get

$$\lambda(S + I + R)^3 \leq \frac{\lambda^4}{4\check{d}^3} + \frac{3\check{d}}{4}(S + I + R)^4,$$

which yields

$$LV_5(x) \leq \frac{\lambda^4}{4\check{d}^3} - \left(\frac{\check{d}}{4} - \frac{3\sigma^2}{2}\right)(S + I + R)^4.$$

At last, we consider

$$\begin{aligned} V(x) &= \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta}\right)V_1 + \frac{\gamma(d_I + d_R)}{v(d_S + d_R)}V_2 \\ &\quad + \frac{\gamma}{d_S + d_R}V_3 + V_4 + \frac{4\sigma_0^2}{\check{d} - 6\sigma^2}V_5. \end{aligned}$$

Applying Itô's formula and (26), yield

$$\begin{aligned} & LV(x) \\ &\leq -\left(d_S + \frac{\gamma d_S}{d_S + d_R}\right)(S - S^*)^2 - \frac{\gamma d_I}{d_S + d_R}(I - I^*)^2 \\ &\quad - \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{v(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R}\right)(R - R^*)^2 \\ &\quad + \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta}\right)\sigma_0^2 I^* S^2 + \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta}\right)\sigma_2^2 I^* \\ &\quad + \frac{\sigma_3^2}{2} \frac{\gamma(d_I + d_R)}{v(d_S + d_R)} R^2 + \frac{\gamma\sigma^2}{2(d_S + d_R)}(S^2 + I^2 + R^2) + \sigma_1^2 S^2 + \sigma_0^2 S^2 I^2 \\ &\quad + \frac{\lambda^4 \sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)} - \sigma_0^2(S + I + R)^4 \\ &\leq -\left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1\right)(S - S^*)^2 - \left(\frac{\gamma d_I}{d_S + d_R} - C_2\right)(I - I^*)^2 \\ &\quad - \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{v(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3\right)(R - R^*)^2 + \frac{\lambda^4 \sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)} \\ &\quad + \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta}\right)\sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2}. \end{aligned}$$

If (25) holds, then the expression

$$\begin{aligned} & \left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1 \right) (S - S^*)^2 + \left(\frac{\gamma d_I}{d_S + d_R} - C_2 \right) (I - I^*)^2 \\ & + \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{v(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3 \right) (R - R^*)^2 \\ & = \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta} \right) \sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2} + \frac{\lambda^4 \sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)} \end{aligned}$$

lies in the positive zone of R^3 and thus there exists a positive constant $\varepsilon > 0$ and a compact set K of R_+^3 such that for any $x \in R_+^3/K$,

$$\begin{aligned} & \left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1 \right) (S - S^*)^2 + \left(\frac{\gamma d_I}{d_S + d_R} - C_2 \right) (I - I^*)^2 \\ & + \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{v(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3 \right) (R - R^*)^2 \\ & \geq \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + v}{\beta} \right) \sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2} + \frac{\lambda^4 \sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)} + \varepsilon. \end{aligned}$$

Hence for any $x \in R_+^3/K$,

$$LV(x) \leq -\varepsilon < 0.$$

Applying Lemma 2.2, we prove the ergodic property and the positive persistence of the SDE (3) whence the proof is complete. \square

Remark 3. In fact, the variances of errors usually should be small enough to justify their validity of real data, otherwise, the data may not be considered as a good one. When σ_{ij} are very small, condition (25) is satisfied.

Theorem 5.2. Under the conditions of Theorem 5.1, let μ denote the stationary distribution of the SDE (3). Then we have

$$\begin{aligned} & d_S \int_{R_+^3} x\mu(dx, dy, dz) + \left(d_I + \frac{vd_R}{d_R + \gamma} \right) \int_{R_+^3} y\mu(dx, dy, dz) = \lambda, \\ & v \int_{R_+^3} y\mu(dx, dy, dz) = (d_R + \gamma) \int_{R_+^3} z\mu(dx, dy, dz), \\ & \left(\beta + \sum_{j=1}^N \sigma_{0j} \sigma_{2j} \right) \int_{R_+^3} x\mu(dx, dy, dz) - \frac{\sigma_0^2}{2} \int_{R_+^3} x^2\mu(dx, dy, dz) = d_I + v + \frac{\sigma_2^2}{2}. \end{aligned}$$

Proof. Consider

$$\begin{aligned} \frac{N(T) - N(0)}{T} &= \lambda - \frac{d_S}{T} \int_0^T S(t)dt - \frac{d_I}{T} \int_0^T I(t)dt - \frac{d_R}{T} \int_0^T R(t)dt \\ &\quad + \frac{1}{T} \sum_{j=1}^N \int_0^T (\sigma_{0j} S(t) + \sigma_{1j} I(t) + \sigma_{3j} R(t)) dB_j(t). \end{aligned}$$

Let $T \rightarrow \infty$, by ergodic theorem and Lemma 4.2, 4.3, we have

$$d_S \int_{R_+^3} x\mu(dx, dy, dz) + d_I \int_{R_+^3} y\mu(dx, dy, dz) + d_R \int_{R_+^3} z\mu(dx, dy, dz) = \lambda. \quad (27)$$

Similarly, by

$$\frac{R(T) - R(0)}{T} = \frac{v}{T} \int_0^T I(t)dt - \frac{d_R + \gamma}{T} \int_0^T R(t)dt + \frac{1}{T} \sum_{j=1}^N \sigma_{3j} \int_0^T R(t)dB_j(t),$$

we have

$$v \int_{R_+^3} y\mu(dx, dy, dz) = (d_R + \gamma) \int_{R_+^3} z\mu(dx, dy, dz), \tag{28}$$

which, together with (27), yields,

$$d_S \int_{R_+^3} x\mu(dx, dy, dz) + \left(d_I + \frac{vd_R}{d_R + \gamma} \right) \int_{R_+^3} y\mu(dx, dy, dz) = \lambda.$$

Applying Itô's formula to $\log I(t)$, yields

$$\begin{aligned} & \frac{\log I(T) - \log I(0)}{T} \\ &= \left(\beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j} \right) \frac{1}{T} \int_0^T S(t)dt - \frac{\sigma_0^2}{2T} \int_0^T S^2(t)dt \\ & \quad - \left(d_I + v + \frac{\sigma_2^2}{2} \right) + \frac{1}{T} \sum_{j=1}^N \int_0^T (\sigma_{0j}S(t) - \sigma_{1j})dB_j(t). \end{aligned}$$

By the ergodic theorem again, we get $\lim_{T \rightarrow \infty} \frac{\log I(T)}{T}$ exists, a.e. Now, we claim that $\lim_{T \rightarrow \infty} \frac{\log I(T)}{T} = 0$, a.e. If this statement is false, then $\lim_{T \rightarrow \infty} \frac{\log I(T)}{T} > 0$ or $\lim_{T \rightarrow \infty} \frac{\log I(T)}{T} < 0$, a.e. This means that $\lim_{T \rightarrow \infty} I(T) = \infty$ or 0 , a.e., which contradicts the conclusion of the weak convergence to the invariant distribution μ lying in R_+^3 . Therefore, we have

$$\lim_{T \rightarrow \infty} \frac{\log I(T)}{T} = 0, \text{ a.e.}$$

and hence

$$\left(\beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j} \right) \int_{R_+^3} x\mu(dx, dy, dz) - \frac{\sigma_0^2}{2} \int_{R_+^3} x^2\mu(dx, dy, dz) = d_I + v + \frac{\sigma_2^2}{2},$$

whence the proof is complete. □

6. Discussion. Stochastic epidemic models have been studied by many authors, see e.g., [1, 5, 23]. In this paper, we impose the stochasticity on the disease transmission coefficient β and the removal rates d_S, d_I, d_R of deterministic model (1). If we only consider perturbation of β , then $N(t) = S(t) + I(t) + R(t)$ is uniformly bounded, and some papers, see e.g. [7], [17] and their references, mainly study its extinction, but there are few papers concerned with its ergodic property. If the removal rates are also affected by noise, which may happen in the real world, $N(t)$ is unbounded even if the noise is small. In such a case, we adapt Chow's approach ([8]) and the moment estimate to establish the conditions for extinction of the infective population. Furthermore, we construct new stochastic Lyapunov functions to study the ergodic property of the SDE (3). When the deterministic model has an endemic equilibrium, under some mild conditions, we prove that the SDE (1.3) is ergodic. In such a case, by ergodic theorem, the average of the solution converges to the mean of the stationary distribution as the time increases.

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