MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **11**, Number **3**, June **2014**

doi:10.3934/mbe.2014.11.641

pp. **641–665**

GLOBAL STABILITY OF AN AGE-STRUCTURED CHOLERA MODEL

Jianxin Yang and Zhipeng Qiu

Department of Applied Mathematics
Nanjing University of Science and Technology
Nanjing 210094, China

Xuezhi Li

Department of Mathematics
Xinyang Normal University
Xinyang 464000, China

(Communicated by Haiyan Wang)

Abstract. In this paper, an age-structured epidemic model is formulated to describe the transmission dynamics of cholera. The PDE model incorporates direct and indirect transmission pathways, infection-age-dependent infectivity and variable periods of infectiousness. Under some suitable assumptions, the PDE model can be reduced to the multi-stage models investigated in the literature. By using the method of Lyapunov function, we established the dynamical properties of the PDE model, and the results show that the global dynamics of the model is completely determined by the basic reproduction number $\mathcal{R}_0$: if $\mathcal{R}_0 < 1$ the cholera dies out, and if $\mathcal{R}_0 > 1$ the disease will persist at the endemic equilibrium. Then the global results obtained for multi-stage models are extended to the general continuous age model.

1. **Introduction.** Several recent studies have focused on modeling the cholera dynamics ([1, 22, 23, 26, 27, 28]). In 2010, Tien et al [28] formulated a SIRW model to describe the cholera dynamics. In the SIRW model, all contaminated water or infectious individuals are assumed to be equally infectious during their periodic infectivity. Actually, laboratory studies suggest that the infectivity of *Vibrio cholera* existing outside the host decays in time [22], and the infectivity of infectious individuals is also different at the differential age of infection. The age of infection models are often formulated to describe the heterogeneity in infectious individuals. Recent studies on age of infection models show that the age of infection may play an important influence on transmission dynamics of infectious disease [13, 14, 21, 25], and consequently we should incorporate age of infection into the modeling transmission dynamics of cholera.

Variability of infectiousness in time had been described in the literature by multi-stage models if age of infection is considered as multiple discrete infection stages. Based on the existing multi-stage cholera models formulated in [22] and [28] , in

2010 *Mathematics Subject Classification.* Primary: 92B05, 92D30.

Key words and phrases. Age of infection, global stability, cholera, Lyapunov function.

The second author is supported by NSFC grant No. 11271190, and the third author is corresponding author, supported by NSFC grant No.11271314.

Table 1. Definitions of frequently used symbols

Parameter	Description
a	age of infection, i.e., the time that has lapsed since the individual became infected
θ	age of infection, i.e., the time that has lapsed since the cholera pathogen has penetrated into water
$S(t)$	the numbers of individuals in the susceptible class at time t
$i(a,t)$	the infection-age density of the infected individuals with age of infection a at time t
$w(\theta,t)$	the concentration of the cholera pathogen with age of infection θ at time t
Λ	Recruitment rate
μ	Per capita natural death rate
$\beta(a)$	transmission coefficient of the infected individuals at age of infection a
$\alpha(\theta)$	transmission coefficient for per concentration of the cholera pathogen with age of infection θ
$\xi(a)$	the pathogen production rate of an infected individual with age of infection a
$\delta(\theta)$	the rate that the cholera pathogen with age of infection θ loses the infectivity
$\gamma(a)$	the rate that an infected individual with age of infection a recovers or dies from the disease

this paper we develop more general model to study the cholera dynamics, where the age of infection is considered as a continuous variable. The model presented here takes the following form:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - (\displaystyle\int_0^\infty \beta(a)i(a,t)da + \int_0^\infty \alpha(\theta)w(\theta,t)d\theta)S - \mu S, \\ \dfrac{\partial i(a,t)}{\partial t} + \dfrac{\partial i(a,t)}{\partial a} = -(\mu+\gamma(a))i(a,t), \\ \dfrac{\partial w(\theta,t)}{\partial \theta} + \dfrac{\partial w(\theta,t)}{\partial t} = -\delta(\theta)w(\theta,t) \end{cases} \tag{1}$$

with the following boundary and initial conditions

$$\begin{aligned} & i(0,t) = (\int_0^\infty \beta(a)i(a,t)da + \int_0^\infty \alpha(\theta)w(\theta,t)d\theta)S, \\ & w(0,t) = \int_0^\infty \xi(a)i(a,t)da, \\ & S(0) = S_0, i(a,0) = i_0(a), w(\theta,0) = w_0(\theta), \end{aligned} \tag{2}$$

where $i_0(a), w_0(\theta)$ are given non-negative functions. All the parameters in the system (1) are positive. The functions $\beta(a), \alpha(\theta)$ and $\xi(a)$ belong to $L^\infty_+((0,+\infty),\mathbb{R}) \setminus \{0_{L^\infty}\}$, and $\delta(\theta), \gamma(a)$ belong to $L^1_{loc}((0,+\infty),\mathbb{R}), \delta(\theta), \gamma(a) \geq 0$ in $(0,+\infty)$. We further assume that there exists $\delta_w > 0$ such that

$$\delta(\theta) \geq \delta_w$$

for almost every $\theta \geq 0$. The definition of the different parameters in the system (1) are listed in Table 1.

Assume that the infected individuals are partitioned into n infection stages defined by the infection age intervals $[a_{i-1}, a_i]$, where $0 = a_0 < a_1 < \cdots < a_{n-1} <$

$a_n = \infty$, and the contaminated water is categorized into m levels defined by the age intervals $[\theta_{i-1}, \theta_i]$, where $0 = \theta_0 < \theta_1 < \cdots < \theta_{m-1} < \theta_m = \infty$. For $a \in [a_{i-1}, a_i)$ or $\theta \in [\theta_{i-1}, \theta_i]$, we also assume that the parameters $\beta(a)$, $\gamma(a)$, $\xi(a)$, $\delta(\theta)$, $\alpha(\theta)$ are constant with

$$\beta(a) = \mu_i, \quad \xi(a) = \xi_i, \quad \gamma(a) = \gamma_i, \quad \alpha(\theta) = \alpha_i, \quad \delta(\theta) = \delta_i. \tag{3}$$

Let $I_i(t) = \int_{a_{i-1}}^{a_i} i(a,t)da, W_i(t) = \int_{\theta_{i-1}}^{\theta_i} w(\theta,t)d\theta$. Integrating the second and third equations of system (1) on the age interval $[a_{i-1}, a_i)$ and $[\theta_{i-1}, \theta_i)$, respectively, yields

$$i(a_i,t) - i(a_{i-1},t) + \frac{dI_i(t)}{dt} = -(\mu + \gamma_i)I_i(t),$$
$$w(\theta_i,t) - w(\theta_{i-1},t) + \frac{dW_i(t)}{dt} = -\delta_i W_i(t).$$

We use c_i to denote the transfer rate constants of the infected individuals between the age groups $[a_{i-1}, a_i)$ and $[a_i, a_{i+1})$, and d_i to denote the transfer rate constants of the pathogen between age groups $[\theta_{i-1}, \theta_i)$ and $[\theta_i, \theta_{i+1})$, then it follows that $i(a_i,t) = c_i I_i(t), w(\theta_i,t) = d_i W_i(t)$, where $c_n = 0$ and $d_m = 0$. By using the boundary conditions in system (1), we can obtain the equations for $S(t), I_i(t), W_i(t)$:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - (\sum_{k=1}^{n} \beta_k I_k + \sum_{k=1}^{m} \alpha_k W_k)S - \mu S, \\ \dfrac{dI_1}{dt} = (\sum_{k=1}^{n} \beta_k I_k + \sum_{k=1}^{m} \alpha_k W_k)S - (\mu + \gamma_1 + c_1)I_1, \\ \dfrac{dI_i}{dt} = c_{i-1}I_{i-1} - (\mu + \gamma_i + c_i)I_i, \qquad i = 2,3,\cdots,n, \\ \dfrac{dW_1(t)}{dt} = \sum_{k=1}^{n} \xi_k I_k - (\delta_1 + d_1)W_1, \\ \dfrac{dW_j}{dt} = d_{j-1}W_{j-1} - (\delta_j + d_j)W_j, \qquad j = 2,3,\cdots,m. \end{cases} \tag{4}$$

Based on the above assumptions that parameters are constant within age groups, the PDE model (1) can be reduced to the ODE model investigated in paper [22] and [28]. The global properties of the ODE model (4) have been investigated in paper [22], and the results show that the global dynamics of the ODE model is completely determined by the basic reproduction number. The main objective of the paper is to prove that the global results are also true for the continuous age case. Because the continuous age model is described by first order PDEs, it is difficult to analyze the dynamics of the PDE models, particularly the global stability. In this paper, by using a class of global Lyapunov functions we prove that the dynamics of the age of infection cholera model are completely determined by the basic reproduction number $\mathcal{R}_0$: if $\mathcal{R}_0 < 1$ the disease-free equilibrium is globally asymptotically stable; if $\mathcal{R}_0 > 1$, a unique endemic equilibrium is globally asymptotically stable. Thus the global results obtained for ODE system are extended to the general continuous age model.

The paper is organized as follows. In the next section we mainly present the basic reproduction number, investigate the existence of the equilibrium, and then state the main results of the paper. In the section 3 we mainly show that the disease free equilibrium is globally asymptotically stable if the basic reproduction number is less than one. In order to prove the results on the global stability of the endemic equilibrium of the system (1), in Section 4 we present some preliminary results about uniform persistence and about the existence of global attractors. In the section 5 the proof of the results on the global stability of endemic equilibrium is strictly proved when the basic reproduction number is greater than one. We summarizes our results and outline some future work in the final Section 6.

2. **Reproduction number and main results.** Using standard methods we can verify the existence and uniqueness of solutions to the system (1) (see [11] and [30]). Moreover, we can show that all solutions with nonnegative initial conditions will remain nonnegative and bounded for all $t > 0$. In this section, we mainly present the reproduction number for system (1), investigate the existence of equilibria, and then state the main results of the paper.

In epidemiology, the reproduction number is one of the most useful threshold parameters. It is generally defined as the average number of secondary infections produced by a typical infected individual during the entire period of infection when introduced into a completely susceptible population [2, 3, 29]. As will be shown in the next section that the qualitative and quantitative behaviors of the model (1) is completely determined by the reproduction number.

First let us derive the reproduction number from the biological meanings of the model parameters. To simplify expressions, we introduce the following notations:

$$K_I(a) = e^{-\mu a - \int_0^a \gamma(s)ds},$$

$$K_W(\theta) = e^{-\int_0^\theta \delta(s)da}.$$

Notice that $\mu + \gamma(s)$ is the rate at which an infected individual of infection age s leaves the infectious class, it then follows that $K_I(a)$ represents the probability of remaining in the infected class for an infected case at the age of infection a. Then the average number of secondary cases directly produced by an infected individual can be expressed as

$$R_I = \frac{\Lambda}{\mu} \int_0^\infty \beta(a) K_I(a) da,$$

and the average concentration of the cholera pathogen produced by an infected individual can be defined as

$$\mathcal{P}_c = \int_0^\infty \xi(a) K_I(a) da.$$

Similarly, $K_W(\theta)$ represents the probability of remaining the infectivity for the unit concentration of cholera pathogen with infection age θ since $\delta(s)$ is the rate at which the cholera pathogen with age of infection s loses the infectivity. Thus, the integral of the production of $\alpha(\theta)$ and $K_W(\theta)$ over all ages,

$$\mathcal{R}_c = \frac{\Lambda}{\mu} \int_0^\infty \alpha(\theta) K_W(\theta) d\theta,$$

gives the average number of secondary cases directly produced by the unit concentration of cholera pathogen. Therefore, the average number of the secondary cases indirectly produced by an infected individual can be expressed as

$$\mathcal{R}_W = \mathcal{R}_c \times \mathcal{P}_c.$$

In summary, if a typical infected individual is introduced into a purely susceptible population, then the average number of secondary cases directly or indirectly produced during the entire period of infection can be expressed as

$$\begin{aligned}\mathcal{R}_0 : &= \mathcal{R}_I + \mathcal{R}_W \\ &= \frac{\Lambda}{\mu}\Big[\int_0^\infty \beta(a)e^{-\mu a-\int_0^a \gamma(s)ds}da + \\ &\qquad \int_0^\infty \alpha(\theta)e^{-\int_0^\theta \delta(s)ds}d\theta \int_0^\infty \xi(a)e^{-\mu a-\int_0^a \gamma(s)ds}da\Big].\end{aligned}$$

We can easily see that $\mathcal{R}_0$ is the basic reproduction number for system (1). Now we are able to state the result on the existence of equilibria for system (1).

Theorem 2.1. *The system* (1) *can have up to two equilibria. More precisely, we have*

(1) The disease free equilibrium $E_0(\frac{\Lambda}{\mu},0,0)$ *always exists.*

(2) If $\mathcal{R}_0 > 1$*, there exists a unique endemic equilibrium* $E^*(S^*, i^*(a), w^*(\theta))$*, where*

$$\begin{aligned}&S^* = \frac{\Lambda}{\mu}\frac{1}{\mathcal{R}_0}, \\ &i^*(a) = \Lambda\frac{\mathcal{R}_0 - 1}{\mathcal{R}_0}e^{-\mu a-\int_0^a \gamma(s)ds}, \\ &w^*(\theta) = \Lambda\frac{\mathcal{R}_0 - 1}{\mathcal{R}_0}\int_0^\infty \xi(a)e^{-\mu a-\int_0^a \gamma(s)ds}da \times e^{-\int_0^\theta \delta(s)ds}.\end{aligned}$$

Proof. It is obvious that the disease free equilibrium $E_0(\dfrac{\Lambda}{\mu},0,0)$ always exists and is unique. Now we prove the second case. The endemic equilibrium $E^* = (S^*, i^*(a), w^*(\theta))$ can be found by solving the following system:

$$\begin{cases}\Lambda - S(\int_0^\infty \beta(a)i(a)da + \int_0^\infty \alpha(\theta)w(\theta)d\theta) - \mu S = 0, \\ \dfrac{di(a)}{da} = -(\mu+\gamma(a))i(a), \\ \dfrac{dw(\theta)}{da} = -\delta(\theta)w(\theta), \\ i(0) = S(\int_0^\infty \beta(a)i(a)da + \int_0^\infty \alpha(\theta)w(\theta)d\theta), \\ w(0) = \int_0^\infty \xi(a)i(a)da.\end{cases} \tag{5}$$

Integrating from 0 to a or θ the second and third equations in (5) yields

$$\begin{aligned}i(a) &= i(0)e^{-\mu a-\int_0^a \gamma(s)ds}, \\ w(\theta) &= w(0)e^{-\int_0^\theta \delta(s)ds}.\end{aligned} \tag{6}$$

Substituting (6) into the fourth and fifth equation in (5) gives

$$\begin{aligned}&i(0) = S\Big(i(0)\int_0^\infty \beta(a)e^{-\mu a-\int_0^a \gamma(s)ds}da + w(0)\int_0^\infty \alpha(\theta)e^{-\int_0^\theta \delta(s)ds}d\theta\Big), \\ &w(0) = i(0)\int_0^\infty \xi(a)e^{-\mu a-\int_0^a \gamma(s)ds}da.\end{aligned} \tag{7}$$

It follows from (7) and the first equation in (5) that

$$\begin{aligned} S &= \frac{\Lambda}{\mu}\frac{1}{\mathcal{R}_0} := S^*, \\ i(0) &= \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0}\Lambda := i^*(0), \\ w(0) &= \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0}\Lambda \int_0^\infty \xi(a) e^{-\mu a - \int_0^a \gamma(s)ds} da := w^*(0). \end{aligned}$$

Thus,

$$\begin{aligned} S^* &= \frac{\Lambda}{\mu}\frac{1}{\mathcal{R}_0}, \\ i^*(a) &= \Lambda \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0} e^{-\mu a - \int_0^a \gamma(s)ds}, \\ w^*(\theta) &= \Lambda \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0} \int_0^\infty \xi(a) e^{-\mu a - \int_0^a \gamma(s)ds} da \times e^{-\int_0^\theta \delta(s)ds}. \end{aligned}$$

It can be easily see from the expressions of $S^*, i^*(a)$ and $w^*(\theta)$ that the endemic equilibrium E^* exists if and only if $\mathcal{R}_0 > 1$. This completes the proof of Theorem 2.1. □

In order to state the main results of the paper, we set

$$\begin{aligned} \tilde{a} &= \inf\{a : \int_a^\infty \beta(\sigma)d\sigma = 0 \text{ and } \int_a^\infty \xi(\sigma)d\sigma = 0\}, \\ \tilde{\theta} &= \inf\{\theta : \int_\theta^\infty \alpha(\sigma)d\sigma = 0\}. \end{aligned} \tag{8}$$

Since the functions $\beta(a), \alpha(\theta)$ and $\xi(a)$ belong to $L_+^\infty((0,+\infty),\mathbb{R}) \setminus \{0_{L^\infty}\}$, we have $\tilde{a} > 0$ and $\tilde{\theta} > 0$. Let

$$\hat{\mathcal{M}}_0 := \left\{ \begin{pmatrix} i \\ w \end{pmatrix} \in L_+((0,+\infty),\mathbb{R}^2) : \int_0^{\tilde{a}} i(a)da > 0 \text{ or } \int_0^{\tilde{\theta}} w(\theta)d\theta > 0 \right\},$$

and define

$$\begin{aligned} \mathcal{M}_0 &:= \mathbb{R}_+ \times \hat{\mathcal{M}}_0, \\ \partial\mathcal{M}_0 &:= \mathbb{R}_+ \times L_+((0,\infty),\mathbb{R}^2) \setminus \mathcal{M}_0. \end{aligned}$$

Now we are able to state the main results of the paper.

Theorem 2.2. *If $\mathcal{R}_0 < 1$, then the DFE $E_0(\frac{\Lambda}{\mu},0,0)$ is the unique equilibrium of system (1), and it is globally stable.*

Theorem 2.3. *Assume that $\mathcal{R}_0 > 1$, then the DFE $E_0(\frac{\Lambda}{\mu},0,0)$ is globally asymptotically stable in $\partial\mathcal{M}_0$, and the unique endemic equilibrium $E^*(S^*, i^*(a), w^*(\theta))$ of system (1) is globally asymptotically stable in $\mathcal{M}_0$.*

3. The proof of Theorem 2.2.

In this section, we mainly prove Theorem 2.2. First, let us consider the local stability of the DFE E_0, and we have the following Theorem 3.1.

Theorem 3.1. *The DFE $E_0(\frac{\Lambda}{\mu},0,0)$ is locally asymptotically stable if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Proof. Introducing the perturbation variables

$$x_1(t) = S(t) - \frac{\Lambda}{\mu}, x_2(a,t) = i(a,t), x_3(\theta,t) = w(\theta,t),$$

and linearizing the system (1) about E_0 we obtain the following system

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -\mu x_1(t) - \dfrac{\Lambda}{\mu}\displaystyle\int_0^\infty \beta(a)x_2(a,t)da - \dfrac{\Lambda}{\mu}\int_0^\infty \alpha(\theta)x_3(\theta,t)d\theta, \\ \dfrac{\partial x_2(a,t)}{\partial t} + \dfrac{\partial x_2(a,t)}{\partial a} = -(\mu+\gamma(a))x_2(a,t), \\ \dfrac{\partial x_3(\theta,t)}{\partial t} + \dfrac{\partial x_3(\theta,t)}{\partial \theta} = -\delta(\theta)x_3(\theta,t), \\ x_2(0,t) = \dfrac{\Lambda}{\mu}\displaystyle\int_0^\infty \beta(a)x_2(a,t)da + \dfrac{\Lambda}{\mu}\int_0^\infty \alpha(\theta)x_3(\theta,t)d\theta, \\ x_3(0,t) = \displaystyle\int_0^\infty \xi(a)x_2(a,t)da. \end{cases} \tag{9}$$

Let

$$x_1(t) = x_1^0 e^{\lambda t}, x_2(a,t) = x_2^0(a)e^{\lambda t}, x_3(\theta,t) = x_3^0(\theta)e^{\lambda t}, \tag{10}$$

where $x_1^0, x_2^0(a), x_3^0(\theta)$ are to be determined. Substituting (10) into (9), we obtain

$$\lambda x_1^0 = -\mu x_1^0 - \frac{\Lambda}{\mu}\int_0^\infty \beta(a)x_2^0(a)da - \frac{\Lambda}{\mu}\int_0^\infty \alpha(\theta)x_3^0(a)da, \tag{11a}$$

$$\begin{cases} \lambda x_2^0(a) + \dfrac{dx_2^0(a)}{da} = -(\mu+\gamma(a))x_2^0(a), \\ x_2^0(0) = \dfrac{\Lambda}{\mu}\displaystyle\int_0^\infty \beta(a)x_2^0(a)da + \dfrac{\Lambda}{\mu}\int_0^\infty \alpha(\theta)x_3^0(\theta)d\theta, \end{cases} \tag{11b}$$

$$\begin{cases} \lambda x_3^0(\theta) + \dfrac{dx_3^0(\theta)}{d\theta} = -\delta(\theta)x_3^0(\theta), \\ x_3^0(0) = \displaystyle\int_0^\infty \xi(a)x_2^0(a)da. \end{cases} \tag{11c}$$

Integrating the first equation of (11b) from 0 to a yields

$$x_2^0(a) = x_2^0(0)e^{-(\lambda+\mu)a - \int_0^a \gamma(s)ds}. \tag{12}$$

Substituting (12) into (11c) and solving (11c), we obtain

$$x_3^0(\theta) = x_2^0(0)\int_0^\infty \xi(a)e^{-(\lambda+\mu)a-\int_0^a \gamma(s)ds}da \times e^{-\lambda\theta - \int_0^\theta \delta(s)ds}. \tag{13}$$

Substituting (12) and (13) into the second equation of (11b) gives the characteristic equation

$$\begin{aligned} 1 = \frac{\Lambda}{\mu}\int_0^\infty \beta(a)e^{-(\mu+\lambda)a-\int_0^a \gamma(s)ds}da + \\ \frac{\Lambda}{\mu}\int_0^\infty \alpha(\theta)e^{-\lambda\theta-\int_0^\theta \delta(s)ds}d\theta \int_0^\infty \xi(a)e^{-(\mu+\lambda)a-\int_0^a \gamma(s)ds}da. \end{aligned} \tag{14}$$

Let $\mathscr{H}(\lambda)$ denote the right hand side of (14) . Then $\mathscr{H}(\lambda)$ is a continuously differential function with $\lim\limits_{\lambda\to+\infty} \mathscr{H}(\lambda) = 0$, $\lim\limits_{\lambda\to-\infty} \mathscr{H}(\lambda) = +\infty$. Furthermore, it can be checked that $\mathscr{H}'(\lambda) < 0$, which implies that $\mathscr{H}(\lambda)$ is a decreasing function. Thus the equation (14) has a unique real root λ^*. Noting that

$$\mathcal{R}_0 = \mathscr{H}(0)$$

we have $\lambda^* < 0$ if $\mathcal{R}_0 < 1$, and $\lambda^* > 0$ if $\mathcal{R}_0 > 1$. Let $\lambda = \xi + \eta i$ be an arbitrary complex root to equation (14). Then

$$1 = \mathscr{H}(\lambda) = |\mathscr{H}(\xi + \eta i)| \le \mathscr{H}(\xi),$$

which implies that $\lambda^* > \xi$. Thus, all the roots of the equation (14) have negative real part if and only if $\mathcal{R}_0 < 1$. Therefore we have shown that the DFE is local asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$. This completes the proof of Theorem 3.1. □

The result in Theorem 3.1 states only the local stability of the DFE E_0. Now we are able to give the proof of Theorem 2.2.

Proof. of Theorem 2.2. We know from Theorem 3.1 that the DFE E_0 is locally asymptotically stable when $\mathcal{R}_0 < 1$. It suffices to show that E_0 is a global attractor.

For convenience, let

$$\begin{aligned} \lambda(t) &= \int_0^\infty \beta(a) i(a,t) da + \int_0^\infty \alpha(\theta) w(\theta,t) d\theta, \\ \chi(t) &= \int_0^\infty \xi(a) i(a,t) da. \end{aligned} \tag{15}$$

Integrating the second equation in system (1) along characteristic lines we get

$$i(a,t) = \begin{cases} \lambda(t-a)S(t-a)e^{-\mu a - \int_0^a \gamma(s)ds}, & t > a, \\ i_0(a-t)e^{-\int_{a-t}^a (\mu+\gamma(s))ds}, & t < a. \end{cases} \tag{16}$$

Similarity, integrating the third equation in system (1) along the characteristic lines yields

$$w(\theta,t) = \begin{cases} \chi(t-\theta)e^{-\int_0^\theta \delta(s)ds}, & t > \theta, \\ w_0(\theta-t)e^{-\int_{\theta-t}^\theta (\delta(s))ds}, & t < \theta. \end{cases} \tag{17}$$

From (16) and (17) we obtain the inequality

$$\begin{aligned} & \lambda(t)S(t) \\ \le\ & \frac{\Lambda}{\mu}\Big[\int_0^t \beta(a)\lambda(t-a)S(t-a)e^{-\mu a - \int_0^a \gamma(s)ds} da + \int_t^\infty \beta(a) i(a,t) da\Big] \\ & \quad + \frac{\Lambda}{\mu}\Big[\int_0^t \alpha(\theta)\chi(t-\theta)e^{-\int_0^\theta \delta(s)ds} d\theta + \int_t^\infty \alpha(\theta) w(\theta,t) d\theta\Big] \\ =\ & \frac{\Lambda}{\mu}\Big[\int_0^t \beta(a)\lambda(t-a)S(t-a)e^{-\mu a - \int_0^a \gamma(s)ds} da + \int_t^\infty \beta(a) i(a,t) da\Big] \\ & + \frac{\Lambda}{\mu}\Big[\int_0^t \alpha(\theta)e^{-\int_0^\theta \delta(s)ds}\big(\int_0^{t-\theta} \xi(a)\lambda(t-\theta-a)S(t-\theta-a)\times \\ & e^{-\mu a - \int_0^a \gamma(s)ds} da + \int_{t-\theta}^\infty \xi(a) i(a,t-\theta) da\big) d\theta + \int_t^\infty \alpha(\theta) w(\theta,t) d\theta\Big]. \end{aligned} \tag{18}$$

Taking the lim sup when $t \to +\infty$ of both sides of inequality (18) and using the Fatou's Lemma yield

$$\begin{aligned} \tau : &= \limsup_{t\to\infty} \lambda(t)S(t) \\ &\le \frac{\Lambda}{\mu}\Big[\int_0^\infty \beta(a)\tau e^{-\mu a-\int_0^a \gamma(s)ds}da+ \\ &\qquad \int_0^\infty \alpha(\theta)e^{-\int_0^\theta \delta(s)ds}d\theta \int_0^\infty \xi(a)\tau e^{-\mu a-\int_0^a \gamma(s)ds}da\Big] \\ &= \tau\mathcal{R}_0. \end{aligned} \tag{19}$$

From (19) that $\tau = 0$ if $\mathcal{R}_0 < 1$. This implies that

$$\lim_{t\to+\infty} i(a,t) = 0, \lim_{t\to+\infty} w(\theta,t) = 0. \tag{20}$$

Then from the first equation in (1) we have $S(t) \to \frac{\Lambda}{\mu}$ as $t \to +\infty$. This implies that the DFE E_0 is global attractor. The proof of Theorem 2.2 is completed. □

4. **Preliminary results and uniform persistence.** In this section, we first reformulate the system (1) as a Volterra equation and as a non-densely defined semilinear Cauchy problem in order to apply integrated semigroup theory, and then by using the persistence theory for continuous dynamics system we present some results about uniform persistence and about the existence of global attractors.

The Volterra integral formulation of age-structured models has been used successfully in various contexts and provides explicit (or implicit) formulas for the solutions of age-structure models [19]. The system (1) with the boundary and initial conditions (2) can be rewritten as the following Volterra type equations

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - (\displaystyle\int_0^\infty \beta(a)i(a,t)da + \int_0^\infty \alpha(\theta)w(\theta,t)d\theta)S - \mu S, \\ i(a,t) = \begin{cases} \lambda(t-a)S(t-a)e^{-\mu a-\int_0^a \gamma(s)ds}, & t > a, \\ i_0(a-t)e^{-\int_{a-t}^a (\mu+\gamma(s))ds}, & t \le a. \end{cases} \\ w(\theta,t) = \begin{cases} \chi(t-\theta)e^{-\int_0^\theta \delta(s)ds}, & t > \theta, \\ w_0(\theta-t)e^{-\int_{\theta-t}^\theta \delta(s)ds}, & t \le \theta, \end{cases} \end{cases} \tag{21}$$

where $\lambda(t)$ and $\chi(t)$ are the unique solution of the following system of Volterra equations:

$$\begin{aligned} \lambda(t) = &\int_0^t \beta(a)\lambda(t-a)S(t-a)e^{-\mu a-\int_0^a \gamma(s)ds}da + \int_t^\infty \beta(a)i_0(a-t)\times \\ &e^{-\int_{a-t}^a (\mu+\gamma(s))ds}da + \int_0^t \alpha(\theta)\chi(t-\theta)e^{-\int_0^\theta \delta(s)ds}d\theta \\ &+ \int_t^\infty \alpha(\theta)w_0(\theta-t)e^{-\int_{\theta-t}^\theta (\delta(s))ds}d\theta; \\ \chi(t) = &\int_0^t \xi(a)\lambda(t-a)S(t-a)e^{-\mu a-\int_0^a \gamma(s)ds}da+ \\ &\int_t^\infty \xi(a)i_0(a-t)e^{-\int_{a-t}^a (\mu+\gamma(s))ds}da. \end{aligned}$$

We now use the approach introduced by Thieme [24] to reformulate the system (1) with the boundary and initial conditions (2) as a semilinear Cauchy problem.

In order to remove the nonlinearity from the boundary conditions, we enlarge the state space and we consider

$$\mathcal{X} = \mathbb{R} \times \mathbb{R}^2 \times L^1((0,+\infty),\mathbb{R}^2),$$

endowed with the usual product norm, and set

$$\mathcal{X}_0 = \mathbb{R} \times \{0\} \times \{0\} \times L^1((0,+\infty),\mathbb{R}^2),$$

$$\mathcal{X}_+ = \mathbb{R}_+ \times \mathbb{R}^2_+ \times L^1_+((0,+\infty),\mathbb{R}^2),$$

and

$$\mathcal{X}_{0+} = \mathcal{X}_0 \cap \mathcal{X}_+.$$

We consider the linear operator $A : \mathrm{Dom}(A) \subset \mathcal{X} \to \mathcal{X}$ defined by

$$A\begin{pmatrix} S \\ \begin{pmatrix} 0 \\ 0 \\ i \\ w \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\mu S \\ \begin{pmatrix} -i(0) \\ -w(0) \\ -i' - (\mu + \gamma(a))i \\ -w' - \delta(\theta)w \end{pmatrix} \end{pmatrix}$$

with

$$\mathrm{Dom}(A) = \mathbb{R} \times \{0\} \times \{0\} \times W^{1,1}((0,+\infty),\mathbb{R}^2),$$

where $W^{1,1}$ is a Sobolev space, and we define $F : \mathcal{X}_0 \to \mathcal{X}$ by

$$F\begin{pmatrix} S \\ \begin{pmatrix} 0 \\ 0 \\ i \\ w \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \Lambda - (\int_0^\infty \beta(a)i(a)da + \int_0^\infty \alpha(\theta)w(\theta)d\theta)S \\ \begin{pmatrix} S(\int_0^\infty \beta(a)i(a)da + \int_0^\infty \alpha(\theta)w(\theta)d\theta) \\ \int_0^\infty \xi(a)i(a)da \\ 0_{L^1} \\ 0_{L^1} \end{pmatrix} \end{pmatrix}.$$

Then by defining

$$v(t) = \begin{pmatrix} S(t) \\ \begin{pmatrix} 0 \\ 0 \\ i(\cdot,t) \\ w(\cdot,t) \end{pmatrix} \end{pmatrix},$$

we can reformulate the system (1) with the boundary and initial conditions (2) as the following abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)) \tag{22}$$

for $t \geq 0$ and $v(0) = v_0 \in \mathcal{X}_{0+}$.

By using the result in Thieme [24] and Magal [15], we derive that the existence and the uniqueness of the semiflow $\{U(t)\}_{t\geq 0}$ on $\mathcal{X}_{0+}$. By identity $(S(t), 0_{\mathbb{R}^2}, i(\cdot,t), w(\cdot,t))$ with $(S(t), i(\cdot,t), w(\cdot,t))$, it can be shown that the semiflow coincides with the one generated by using the Volterra integral formulation. By setting

$$I(t) = \int_0^\infty i(a,t)da, W(t) = \int_0^\infty w(\theta,t)d\theta,$$

and

$$N(t) = S(t) + I(t),$$

we deduce that $N(t)$ and $W(t)$ satisfy the following ordinary differential inequalities

$$\frac{dN(t)}{dt} \le \Lambda - \mu N(t),$$

and

$$\begin{aligned} \frac{dW(t)}{dt} &= \int_0^\infty \xi(a) i(a,t) da - \int_0^\infty \delta(\theta) w(\theta,t) d\theta \\ &\le \xi_{max} I(t) - \delta_w W(t), \end{aligned}$$

where $\xi_{max} = ess \sup_{\theta \in (0,\infty)} \xi(\theta)$. By using the comparison theorem, it follows that

$$N(t) \le \frac{\Lambda}{\mu}, W(t) \le \frac{\xi_{max}\Lambda}{\mu\delta_w}$$

as $t \to +\infty$. Furthermore, if

$$N(t) \le \frac{\Lambda}{\mu}, W(t) \le \frac{\xi_{max}\Lambda}{\mu\delta_w}$$

are satisfied for some $t = t_0$ then they are satisfied for all $t \ge t_0$. Thus, the system (1) leaves the set

$$\Big\{(S,i,w) \in \mathbb{R}_+ \times L^1_+((0,\infty),\mathbb{R}) \times L^1_+((0,\infty),\mathbb{R}) : \\ S + \int_0^\infty i(a)da \le \frac{\Lambda}{\mu}, \int_0^\infty w(\theta)d\theta \le \frac{\xi_{max}\Lambda}{\mu\delta_w}\Big\}$$

positively invariant. Consequently, it then follows that the set

$$B = \left\{ \left(\begin{pmatrix} S \\ 0 \\ 0 \\ i \\ w \end{pmatrix} \right) \in \mathcal{X}_{0+} : S + \int_0^\infty i(a)da \le \frac{\Lambda}{\mu}, \int_0^\infty w(\theta)d\theta \le \frac{\xi_{max}\Lambda}{\mu\delta_w} \right\}$$

is positively invariant absorbing set under the semiflow $\{U(t)\}_{t\ge0}$ on $\mathcal{X}_{0+}$, i.e., $U(t)B \subseteq B$ and for each $x \in (S_0, 0, 0, i, w) \in \mathcal{X}_{0+}$,

$$d(U(t)x, B) := \inf_{y\in B} \|U(t)x - y\| \to 0$$

as $t \to +\infty$. This means that the semiflow $\{U(t)\}_{t\ge0}$ is bound dissipative on $\mathcal{X}_{0+}$ (see Hale [7]). Furthermore, it follows from [16, 30] that the semiflow $\{U(t)\}_{t\ge0}$ is asymptotically smooth. As a consequence of the results on the existence of global attractors in Hale [7], we have the following theorem.

Theorem 4.1. *The system* (22) *generates a unique continuous semiflow* $\{U(t)\}_{t\ge0}$ *on* $\mathcal{X}_{0+}$ *that is asymptotically smooth and bounded dissipative. Furthermore, the semiflow* $\{U(t)\}_{t\ge0}$ *has a global attractor* $\mathcal{A}$ *in* $\mathcal{X}_{0+}$ *which attracts the bound sets of* $\mathcal{X}_{0+}$.

In order to define the invariant sets for the uniform persistence analysis, we define

$$\tilde{\mathcal{M}}_0 := \mathbb{R}_+ \times \{0\} \times \{0\} \times \hat{\mathcal{M}}_0,$$

$$\partial\tilde{\mathcal{M}}_0 := \mathcal{X}_{0+} \setminus \tilde{\mathcal{M}}_0.$$

Theorem 4.2. *$\partial\tilde{\mathcal{M}}_0$ is positively invariant under the semiflow $\{U(t)\}_{t\geq 0}$ generated by (22). Moreover, the DFE $E_0(\frac{\Lambda}{\mu},0,0,0_{L^1((0,\infty),\mathbb{R}^1)},0_{L^1((0,\infty),\mathbb{R}^1)})$ is globally asymptotically stable for the semiflow $\{U(t)\}_{t\geq 0}$ restricted to $\partial\tilde{\mathcal{M}}_0$.*

Proof. Let $(S_0,0,0,i_0,w_0)\in\partial\tilde{\mathcal{M}}_0$. Then $(i_0,w_0)\in L_+((0,+\infty),\mathbb{R}^2)\setminus\hat{\mathcal{M}}_0$ and we have

$$\begin{cases}\dfrac{\partial i(a,t)}{\partial t}+\dfrac{\partial i(a,t)}{\partial a}=-(\mu+\gamma(a))i(a,t),\\ \dfrac{\partial w(\theta,t)}{\partial\theta}+\dfrac{\partial w(\theta,t)}{\partial t}=-\delta(\theta)w(\theta,t),\\ i(0,t)=(\displaystyle\int_0^\infty\beta(a)i(a,t)da+\int_0^\infty\alpha(\theta)w(\theta,t)d\theta)S(t),\\ w(0,t)=\displaystyle\int_0^\infty\xi(a)i(a,t)da,\\ i(a,0)=i_0(a),w(\theta,0)=w_0(\theta).\end{cases}$$

Since $S(t)\leq\frac{\Lambda}{\mu}$, it follows that

$$i(a,t)\leq\hat{i}(a,t),w(\theta,t)\leq\hat{w}(\theta,t),\tag{23}$$

where

$$\begin{cases}\dfrac{\partial\hat{i}(a,t)}{\partial t}+\dfrac{\partial\hat{i}(a,t)}{\partial a}=-(\mu+\gamma(a))\hat{i}(a,t),\\ \dfrac{\partial\hat{w}(\theta,t)}{\partial\theta}+\dfrac{\partial\hat{w}(\theta,t)}{\partial t}=-\delta(\theta)\hat{w}(\theta,t),\\ \hat{i}(0,t)=\dfrac{\Lambda}{\mu}(\displaystyle\int_0^\infty\beta(a)\hat{i}(a,t)da+\int_0^\infty\alpha(\theta)\hat{w}(\theta,t)d\theta),\\ \hat{w}(0,t)=\displaystyle\int_0^\infty\xi(a)\hat{i}(a,t)da,\\ \hat{i}(a,0)=i_0(a),\hat{w}(\theta,0)=w_0(\theta).\end{cases}\tag{24}$$

The equation (24) can be rewritten as the following Volterra type equation

$$\begin{cases}\hat{i}(a,t)=\begin{cases}\dfrac{\Lambda}{\mu}\hat{\lambda}(t-a)e^{-\mu a-\int_0^a\gamma(s)ds}, & t>a,\\ i_0(a-t)e^{-\int_{a-t}^a(\mu+\gamma(s))ds}, & t\leq a.\end{cases}\\ \hat{w}(\theta,t)=\begin{cases}\hat{\chi}(t-\theta)e^{-\int_0^\theta\delta(s)ds}, & t>\theta,\\ w_0(\theta-t)e^{-\int_{\theta-t}^\theta\delta(s)ds}, & t\leq\theta,\end{cases}\end{cases}\tag{25}$$

where $\hat{\lambda}(t)$ and $\hat{\chi}(t)$ are the unique solution of the following system of Volterra equations:

$$\begin{aligned}\hat{\lambda}(t)=&\frac{\Lambda}{\mu}\int_0^t\beta(a)\hat{\lambda}(t-a)e^{-\mu a-\int_0^a\gamma(s)ds}da+\int_t^\infty\beta(a)i_0(a-t)e^{-\int_{a-t}^a(\mu+\gamma(s))ds}da+\\&\int_0^t\alpha(\theta)\hat{\chi}(t-\theta)e^{-\int_0^\theta\delta(s)ds}d\theta+\int_t^\infty\alpha(\theta)w_0(\theta-t)e^{-\int_{\theta-t}^\theta(\delta(s))ds}d\theta;\\ \hat{\chi}(t)=&\frac{\Lambda}{\mu}\int_0^t\xi(a)\hat{\lambda}(t-a)e^{-\mu a-\int_0^a\gamma(s)ds}da+\int_t^\infty\xi(a)i_0(a-t)e^{-\int_{a-t}^a(\mu+\gamma(s))ds}da.\end{aligned}\tag{26}$$

Since $(i_0, w_0) \in L_+((0,+\infty),\mathbb{R}^2) \setminus \hat{\mathcal{M}}_0$, we can easily deduce that

$$\int_t^\infty \beta(a)i_0(a-t)e^{-\int_{a-t}^a(\mu+\gamma(s))ds}da + \int_t^\infty \alpha(\theta)w_0(\theta-t)e^{-\int_{\theta-t}^\theta(\delta(s))ds}d\theta = 0;$$
$$\int_t^\infty \xi(a)i_0(a-t)e^{-\int_{a-t}^a(\mu+\gamma(s))ds}da = 0.$$

Then the Volterra equation (26) becomes

$$\begin{aligned}\hat{\lambda}(t) &= \frac{\Lambda}{\mu}\int_0^t \beta(a)\hat{\lambda}(t-a)e^{-\mu a-\int_0^a\gamma(s)ds}da + \int_0^t \alpha(\theta)\hat{\chi}(t-\theta)e^{-\int_0^\theta\delta(s)ds}d\theta;\\ \hat{\chi}(t) &= \frac{\Lambda}{\mu}\int_0^t \xi(a)\hat{\lambda}(t-a)e^{-\mu a-\int_0^a\gamma(s)ds}da.\end{aligned} \tag{27}$$

which has a unique solution

$$\hat{\lambda}(t) = 0, \hat{\chi}(t) = 0, \ \forall\, t \geq 0.$$

It now follows that $i(a,t) = 0, w(\theta,t) = 0$ for $0 \leq a, \theta \leq t$. For $t > a$ or $t > \theta$, we have

$$\|\hat{i}(a,t)\|_{L^1} = \|i_0(a-t)e^{-\int_{a-t}^a(\mu+\gamma(s))ds}\|_{L^1} \leq e^{-\mu t}\|i_0\|_{L^1},$$

$$\|\hat{w}(\theta,t)\|_{L^1} = \|w_0(\theta-t)e^{-\int_{\theta-t}^\theta\delta(s)ds}\|_{L^1} \leq e^{-\delta_w t}\|w_0\|_{L^1}$$

for all $t \geq 0$. We can easily see that $\hat{i}(a,t) \to 0, \hat{w}(\theta,t) \to 0$ as $t \to +\infty$. By using (23) the results follows. This completes the proof of Theorem 4.2. ☐

By combining Theorem 4.2 in Hale and Waltman [8], and Theorem 3.7 in Magal and Zhao [17], we are able to prove the following theorem

Theorem 4.3. *If $\mathcal{R}_0 > 1$, the semiflow $\{U(t)\}_{t\geq 0}$ generated by (22) is unifomly persistent in $\tilde{\mathcal{M}}_0$ with respect to the decomposition $(\partial\tilde{\mathcal{M}}_0, \tilde{\mathcal{M}}_0)$, i.e., there exists a $\varepsilon > 0$ such that for each $(S, 0, 0, i_0, w_0) \in \tilde{\mathcal{M}}_0$,*

$$\liminf_{t\to+\infty} S(t) \geq \varepsilon, \liminf_{t\to+\infty}\|i(\cdot,t)\| \geq \varepsilon, \liminf_{t\to+\infty}\|w(\cdot,t)\| \geq \varepsilon.$$

Furthermore, the semiflow $\{U(t)\}_{t\geq 0}$ has a compact global attractor $\mathcal{A}_0$ in $\tilde{\mathcal{M}}_0$, and there exists a $\epsilon > 0$ such that for each $(S, 0, 0, i_0, w_0) \in \mathcal{A}_0$,

$$S \geq \epsilon, \int_0^\infty \beta(a)i(a)da + \int_0^\infty \alpha(\theta)w(\theta)d\theta > \epsilon, \int_0^\infty \xi(a)i(a)da > \epsilon.$$

Proof. From Theorem 4.2 we know that the DFE $E_0(\frac{\Lambda}{\mu}, 0, 0, 0_{L^1((0,\infty),\mathbb{R}^1)}, 0_{L^1((0,\infty),\mathbb{R}^1)})$ is globally asymptotically stable in $\partial\tilde{\mathcal{M}}_0$. First let us show that $W^s(E_0) \cap \tilde{\mathcal{M}}_0 = \emptyset$. Since $\mathcal{R}_0 > 1$, there exists $\varsigma > 0$ such that

$$\begin{aligned}(\frac{\Lambda}{\mu} - \varsigma)\Big[&\int_0^\infty \beta(a)e^{-\mu a-\int_0^a\gamma(s)ds}da +\\ &\int_0^\infty \alpha(\theta)e^{-\int_0^\theta\delta(s)ds}d\theta\int_0^\infty \xi(a)e^{-\mu a-\int_0^a\gamma(s)ds}da\Big] > 1.\end{aligned} \tag{28}$$

Assume that there exists $(S_0, 0, 0, i_0, w_0) \in \tilde{\mathcal{M}}_0$ such that

$$\|(S(t), 0, 0, i(\cdot,t), w(\cdot,t)) - (\frac{\Lambda}{\mu}, 0, 0, 0_{L^1((0,\infty),\mathbb{R}^1)}, 0_{L^1((0,\infty),\mathbb{R}^1)})\| \leq \varsigma \tag{29}$$

for all $t \geq 0$. Then we have

$$S(t) \geq \frac{\Lambda}{\mu} - \varsigma$$

for all $t \geq 0$, and hence

$$\begin{cases} \dfrac{\partial i(a,t)}{\partial t} + \dfrac{\partial i(a,t)}{\partial a} = -(\mu + \gamma(a))i(a,t), \\ \dfrac{\partial w(\theta,t)}{\partial \theta} + \dfrac{\partial w(\theta,t)}{\partial t} = -\delta(\theta)w(\theta,t), \\ i(0,t) \geq (\dfrac{\Lambda}{\mu} - \varsigma)(\displaystyle\int_0^\infty \beta(a)i(a,t)da + \int_0^\infty \alpha(\theta)w(\theta,t)d\theta), \\ w(0,t) = \displaystyle\int_0^\infty \xi(a)i(a,t)da, \\ i(\cdot,0) = i_0, w(\cdot,0) = w_0, (i_0,w_0) \in \hat{\mathcal{M}}_0. \end{cases}$$

By the comparison principle we have

$$i(a,t) \geq \tilde{i}(a,t), w(\theta,t) \geq \tilde{w}(\theta,t),$$

where $(\tilde{i}(a,t), \tilde{w}(\theta,t))$ satisfies the following auxiliary system

$$\begin{cases} \dfrac{\partial \tilde{i}(a,t)}{\partial t} + \dfrac{\partial \tilde{i}(a,t)}{\partial a} = -(\mu + \gamma(a))\tilde{i}(a,t), \\ \dfrac{\partial \tilde{w}(\theta,t)}{\partial \theta} + \dfrac{\partial \tilde{w}(\theta,t)}{\partial t} = -\delta(\theta)\tilde{w}(\theta,t), \\ \tilde{i}(0,t) = (\dfrac{\Lambda}{\mu} - \varsigma)(\displaystyle\int_0^\infty \beta(a)\tilde{i}(a,t)da + \int_0^\infty \alpha(\theta)\tilde{w}(\theta,t)d\theta), \\ \tilde{w}(0,t) = \displaystyle\int_0^\infty \xi(a)\tilde{i}(a,t)da, \\ \tilde{i}(\cdot,0) = i_0, \tilde{w}(\cdot,0) = w_0, (i_0,w_0) \in \hat{\mathcal{M}}_0. \end{cases} \tag{30}$$

Similarly to the proof of Theorem 3.1 we can derive the characteristic equation for system (30)

$$1 = \tilde{\mathcal{H}}(\lambda), \tag{31}$$

where

$$\tilde{\mathcal{H}}(\lambda) := (\frac{\Lambda}{\mu} - \varsigma)\int_0^\infty \beta(a)e^{-(\mu+\lambda)a - \int_0^a \gamma(s)ds}da + (\frac{\Lambda}{\mu} - \varsigma)\int_0^\infty \alpha(\theta)e^{-\lambda\theta - \int_0^\theta \delta(s)ds}d\theta \int_0^\infty \xi(a)e^{-(\mu+\lambda)a - \int_0^a \gamma(s)ds}da,$$

and the equation (31) has a unique real root $\tilde{\lambda}^* > 0$ since the inequality (28) holds. Moreover, if $\tilde{\lambda} = \xi + \eta i$ is an arbitrary complex root to equation (31), then we have $\xi < \tilde{\lambda}^*$.

For ease of notation, let us rewrite the linear system (30) as the following form

$$\frac{du(t)}{dt} = \mathcal{A}u(t) \tag{32}$$

where $u(t) = (\tilde{i}(\cdot,t), \tilde{w}(\cdot,t))^T$ and

$$\mathcal{A}u = \begin{pmatrix} -\frac{d\tilde{i}}{da} - (\mu+\gamma(a))\tilde{i} \\ -\frac{d\tilde{w}}{d\theta} - \delta(\theta)\tilde{w} \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \Big\{ \begin{pmatrix} \tilde{i} \\ \tilde{w} \end{pmatrix} \in W^{1,1}((0,+\infty),\mathbb{R}^2) : \tilde{i}(0) = (\frac{\Lambda}{\mu} - \varsigma)\times (\int_0^\infty \beta(a)\tilde{i}(a)da + \int_0^\infty \alpha(\theta)\tilde{w}(\theta)d\theta), \tilde{w}(0) = \int_0^\infty \xi(a)\tilde{i}(a)da \Big\}.$$

By using the Hille-Yosida theorem, the operator $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ generates a strongly continuous semigroup, denoted by $(\tilde{T}(t))_{t\geq 0}$. Let $\Pi_{\tilde{\lambda}^*}$ be the eigenprojection corresponding to $\tilde{\lambda}^*$, then by using the Proposition 3.1 in paper [12] and after extensive algebraic calculations, we can obtain

$$\begin{aligned} \Pi_{\tilde{\lambda}^*}\tilde{T}(t)\begin{pmatrix} i_0 \\ w_0 \end{pmatrix} &= e^{\tilde{\lambda}^* t}\Pi_{\tilde{\lambda}^*}\begin{pmatrix} i_0 \\ w_0 \end{pmatrix} \\ &= e^{\tilde{\lambda}^* t}\lim_{\lambda\to\tilde{\lambda}^*}(\lambda-\tilde{\lambda}^*)(\lambda I - \mathcal{A})^{-1}\begin{pmatrix} i_0 \\ w_0 \end{pmatrix} \\ &= e^{\tilde{\lambda}^* t}\begin{pmatrix} \psi \\ \phi \end{pmatrix}, \end{aligned}$$

where $\psi(a), \phi(\theta)$ can be expressed as

$$\begin{aligned} \psi(a) = \Big[-(\frac{\Lambda}{\mu} - \varsigma)\frac{1}{\frac{d\tilde{H}(\lambda)}{d\lambda}|_{\lambda=\tilde{\lambda}^*}}\Big(\int_0^\infty \beta(a)\int_0^a e^{-\int_\tau^a(\tilde{\lambda}^*+\mu+\gamma(s))ds} i_0(\tau)d\tau da + \\ \int_0^{+\infty}\alpha(\theta)\int_0^\theta e^{-\int_\tau^\theta(\tilde{\lambda}^*+\delta(s))ds} w_0(\tau)d\tau d\theta + \int_0^{+\infty}\alpha(\theta)e^{-\tilde{\lambda}^*\theta - \int_0^\theta \delta(s)ds}d\theta \\ \times \int_0^{+\infty}\xi(a)\int_0^a e^{-\int_\tau^a(\tilde{\lambda}^*+\mu+\gamma(s))ds} i_0(\tau)d\tau da\Big)\Big] e^{-(\tilde{\lambda}^*+\mu)a - \int_0^a \gamma(s)ds}; \end{aligned}$$

$$\begin{aligned} \phi(\theta) = -\frac{1}{\frac{d\tilde{H}(\lambda)}{d\lambda}|_{\lambda=\tilde{\lambda}^*}}\Big[\Big(1 - (\frac{\lambda}{\mu} - \varsigma)\big(\int_0^\infty \beta(a)\int_0^a e^{-\int_\tau^a(\tilde{\lambda}^*+\mu+\gamma(s))ds} i_0(\tau)d\tau da + \\ \int_0^\infty \alpha(\theta)\int_0^\theta e^{-\int_\tau^\theta(\tilde{\lambda}^*+\delta(s))ds} w_0(\tau)d\tau d\theta\big)\Big)\int_0^\infty \xi(a)\int_0^a e^{-\int_\tau^a(\tilde{\lambda}^*+\mu+\gamma(s))ds} i_0(\tau) \\ d\tau da + (\frac{\lambda}{\mu} - \varsigma)\int_0^\infty \xi(a)e^{-\int_0^a(\tilde{\lambda}^*+\mu+\gamma(s))ds}da\Big(\int_0^\infty \beta(a)\int_0^a e^{-\int_\tau^a(\tilde{\lambda}^*+\mu+\gamma(s))ds}\times \\ i_0(\tau)d\tau da + \int_0^\infty \alpha(\theta)\int_0^\theta e^{-\int_\tau^\theta(\tilde{\lambda}^*+\delta(s))ds} w_0(\tau)d\tau d\theta\Big)\Big] e^{-\tilde{\lambda}^*\theta - \int_0^\theta \delta(s)ds}. \end{aligned}$$

Since $(i_0, w_0) \in \hat{\mathcal{M}}_0$, after extensive calculations we can obtain that $\psi(a) > 0, \phi(\theta) > 0$ for all $a > 0, \theta > 0$. Note that $\tilde{\lambda}^* > 0$, it then follows that

$$\lim_{t\to+\infty}\|\Pi_{\tilde{\lambda}^*}\tilde{T}(t)\begin{pmatrix} i_0 \\ w_0 \end{pmatrix}\|_{L^1} = +\infty.$$

Therefore, $\|(\tilde{i}(\cdot,t), \tilde{w}(\cdot,t))^T\|_{L^1} \to +\infty$ as $t \to +\infty$. This contradicits (29) since $i(a,t) \geq \tilde{i}(a,t), w(\theta,t) \geq \tilde{w}(\theta,t)$. Thus, $W^s(E_0) \cap \hat{\mathcal{M}}_0 = \emptyset$.

From Theorem 4.1, it then follows that the semiflow $\{U(t)\}_{t\geq 0}$ is asymptotically smooth, point dissipative and that the forward trajectory of a bound set is bounded.

Furthermore, the DFE E_0 is globally asymptotically stable in $\partial\tilde{\mathcal{M}}_0$. Thus, Theorem 4.2 of Hale and Waltman [8] implies the semiflow $\{U(t)\}_{t\geq 0}$ is uniformly persistent with respect to $(\partial\tilde{\mathcal{M}}_0, \tilde{\mathcal{M}}_0)$. This completes the proof of the first result.

Next, let us prove the second result. Since

$$\int_0^\infty i(a,t)da \leq \frac{\Lambda}{\mu}, \int_0^\infty w(\theta)d\theta \leq \frac{\xi_{max}\Lambda}{\mu\delta_w},$$

it then follows that

$$\int_0^\infty \beta(a)i(a,t)da \leq \parallel \beta(a) \parallel_{L^\infty} \frac{\Lambda}{\mu}, \int_0^\infty \alpha(\theta)w(\theta)d\theta \leq \parallel \alpha(\theta) \parallel_{L^\infty} \frac{\xi_{max}\Lambda}{\mu\delta_w}.$$

Therefore,

$$\frac{dS(t)}{dt} \geq \Lambda - \bar{\mu}S(t),$$

where

$$\bar{\mu} := \mu + \parallel \beta(a) \parallel_{L^\infty} \frac{\Lambda}{\mu} + \parallel \alpha(\theta) \parallel_{L^\infty} \frac{\xi_{max}\Lambda}{\mu\delta_w}.$$

Then for sufficiently large t we have

$$S(t) \geq \frac{\Lambda}{\bar{\mu}}.$$

Let $(S(t), 0, 0, i(\cdot,t), w(\cdot,t))$ be the solution to the system (22) with initial condition $(S_0, 0, 0, i_0, w_0) \in \mathcal{A}_0$. Then for sufficiently small $\epsilon > 0$, we have $S(t) > \delta$ for all $t \geq 0$. Furthermore, since the solution $(S(t), 0, 0, i(\cdot,t), w(\cdot,t))$ is in the attractor $\mathcal{A}_0$, we have

$$i(\cdot,t) \geq \hat{i}(\cdot,t), w(\cdot,t) \geq \hat{w}(\cdot,t)$$

for all $t \geq 0$, where $(\hat{i}(a,t), \hat{w}(\theta,t))$ satisfies the following auxiliary system

$$\begin{cases} \dfrac{\partial \hat{i}(a,t)}{\partial t} + \dfrac{\partial \hat{i}(a,t)}{\partial a} = -(\mu + \gamma(a))\hat{i}(a,t), \\ \dfrac{\partial \hat{w}(\theta,t)}{\partial \theta} + \dfrac{\partial \hat{w}(\theta,t)}{\partial t} = -\delta(\theta)\hat{w}(\theta,t), \\ \hat{i}(0,t) = \dfrac{\Lambda}{\bar{\mu}}(\displaystyle\int_0^\infty \beta(a)\hat{i}(a,t)da + \int_0^\infty \alpha(\theta)\hat{w}(\theta,t)d\theta), \\ \hat{w}(0,t) = \displaystyle\int_0^\infty \xi(a)\hat{i}(a,t)da, \\ \hat{i}(\cdot,0) = i_0, \hat{w}(\cdot,0) = w_0. \end{cases} \tag{33}$$

Similarly to the discussion of system (30), we obtain

$$\begin{aligned} \lim_{t\to+\infty} e^{-\hat{\lambda}^* t}\hat{T}(t) \begin{pmatrix} i_0 \\ w_0 \end{pmatrix} &= \Pi_{\hat{\lambda}^*} \begin{pmatrix} i_0 \\ w_0 \end{pmatrix} \\ &:= \begin{pmatrix} \hat{\psi} \\ \hat{\phi} \end{pmatrix}, \end{aligned}$$

where $\hat{T}(t)$ is the semigroup generated by the linear system (33) and $\hat{\lambda}^*$ is the unique real eigenvalue of the system (33), and $\hat{\psi}(a) > 0, \hat{\phi}(\theta) > 0$ for all $a, \theta > 0$

since $(i_0, w_0) \in \hat{\mathcal{M}}_0$. Thus there exists $\hat{T} > 0$ such that $i(a,t) > 0, w(\theta,t) > 0$ for all $a, \theta > 0$ and $t > \hat{T}$. It then follows that

$$\int_0^\infty \beta(a)i(a,t)da + \int_0^\infty \alpha(\theta)w(\theta,t)d\theta$$
$$\geq \int_0^\infty \beta(a)\hat{i}(a,t)da + \int_0^\infty \alpha(\theta)\hat{w}(\theta,t)d\theta > 0,$$
$$\int_0^\infty \xi(a)i(a,t)da \geq \int_0^\infty \xi(a)\hat{i}(a,t)da > 0$$

for all $t > \hat{T}, a, \theta > 0$.

Recall that the mapping $(t, (S_0, 0, 0, i_0, w_0)) \mapsto U(t)(S_0, 0, 0, i_0, w_0)$ is continuous. Also, the mapping

$$\begin{pmatrix} i_0 \\ w_0 \end{pmatrix} \longmapsto \begin{pmatrix} \int_0^\infty \beta(a)i_0(a)da + \int_0^\infty \alpha(\theta)w_0(\theta)d\theta \\ \int_0^\infty \xi(a)i_0(a)da \end{pmatrix} \tag{34}$$

is continuous in the L^1 norm. Using the compactness of $\mathcal{A}_0$ and $\mathcal{A}_0$ is invariant under the semiflow $\{u(t)\}_{t\geq 0}$, it follows that for each $(S_0, 0, 0, i_0, w_0) \in \mathcal{A}_0$ we have

$$\int_0^\infty \beta(a)i_0(a)da + \int_0^\infty \alpha(\theta)w_0(\theta)d\theta > 0, \int_0^\infty \xi(a)i_0(a)da > 0.$$

Using the continuity of the map (34) again and the compactness of $\mathcal{A}_0$, the second result follows. This complete the proof of Theorem 4.3. ☐

5. Proof of Theorem 2.3.

Theorem 5.1. *Assume that $\mathcal{R}_0 > 1$, then the unique endemic equilibrium $E^*(S^*, i^*(a), w^*(\theta))$ of system (1) is locally asymptotically stable.*

Proof. Introducing the perturbation variables

$$y_1(t) = S(t) - S^*, y_2(a,t) = i(a,t) - i^*(a), y_3(\theta,t) = w(\theta,t) - w^*(\theta),$$

and linearizing the system (1) about E^* we obtain the following system

$$\begin{cases} \dfrac{dy_1(t)}{dt} = -\mu\mathcal{R}_0 y_1(t) - (\displaystyle\int_0^\infty \beta(a)y_2(a,t) + \int_0^\infty \alpha(\theta)y_3(\theta,t)d\theta)S^*, \\ \dfrac{\partial y_2(a,t)}{\partial t} + \dfrac{\partial y_2(a,t)}{\partial a} = -(\mu + \gamma(a))y_2(a,t), \\ \dfrac{\partial y_3(\theta,t)}{\partial t} + \dfrac{\partial y_3(\theta,t)}{\partial \theta} = -\delta(\theta)y_3(\theta,t), \\ y_2(0,t) = \mu(\mathcal{R}_0 - 1)y_1(t) + (\displaystyle\int_0^\infty \beta(a)y_2(a,t)da + \int_0^\infty \alpha(\theta)y_3(\theta,t)d\theta)S^*, \\ y_3(0,t) = \displaystyle\int_0^\infty \xi(a)y_2(a,t)da. \end{cases} \tag{35}$$

Let

$$y_1(t) = y_1^0 e^{\lambda t}, y_2(a,t) = y_2^0(a)e^{\lambda t}, y_3(\theta,t) = y_3^0(\theta)e^{\lambda t}, \tag{36}$$

where $y_1^0, y_2^0(a), y_3^0(\theta)$ are to be determined. Substituting (36) into (35), we obtain

$$\lambda y_1^0 = -\mu\mathcal{R}_0 y_1^0 - (\int_0^\infty \beta(a)y_2^0(a)da + \int_0^\infty \alpha(\theta)y_3^0(\theta)d\theta)S^*, \tag{37a}$$

$$\begin{cases} \lambda y_2^0(a) + \dfrac{dy_2^0(a)}{da} = -(\mu+\gamma(a))y_2^0(a), \\ y_2^0(0) = \mu(\mathcal{R}_0 - 1)y_1^0 + (\int_0^\infty \beta(a)y_2^0(a)da + \int_0^\infty \alpha(\theta)y_3^0(\theta)d\theta)S^*, \end{cases} \tag{37b}$$

$$\begin{cases} \lambda y_3^0(\theta) + \dfrac{dy_3^0(\theta)}{d\theta} = -\delta(\theta)y_3^0(\theta), \\ y_3^0(0) = \int_0^\infty \xi(a)y_2^0(a)da. \end{cases} \tag{37c}$$

Integrating the first equation of (37b) from 0 to a yields that

$$y_2^0(a) = y_2^0(0)e^{-(\lambda+\mu)a - \int_0^a \gamma(s)ds}. \tag{38}$$

Substituting (38) into (37c) and solving (37c), we obtain

$$y_3^0(\theta) = y_2^0(0)\int_0^\infty \xi(a)e^{-(\lambda+\mu)a-\int_0^a \gamma(s)ds}da \times e^{-\lambda\theta - \int_0^\theta \delta(s)ds}. \tag{39}$$

Substituting (38) and (39) into the (37a) and the second equation of (37b) gives the following characteristic equation

$$\det\begin{pmatrix} \mu(\mathcal{R}_0 - 1) & H_1(\lambda) - 1 \\ \lambda + \mu\mathcal{R}_0 & H_1(\lambda) \end{pmatrix} = 0,$$

i.e.,

$$H(\lambda) := (\lambda+\mu)H_1(\lambda) - \lambda - \mu\mathcal{R}_0 = 0, \tag{40}$$

where

$$\begin{aligned} H_1(\lambda) \quad = \quad & S^*(\int_0^\infty \beta(a)e^{-(\lambda+\mu)a-\int_0^a \gamma(s)ds}da + \\ & \int_0^\infty \xi(a)e^{-(\lambda+\mu)a-\int_0^a \gamma(s)ds}da \int_0^\infty \alpha(\theta)e^{-\lambda\theta-\int_0^\theta \delta(s)ds}d\theta). \end{aligned}$$

It can be easily check that $H_1(\lambda)$ is continuously differential function and $H_1'(\lambda) < 0$, which implies that $H_1(\lambda)$ is a decreasing function.

We claim that the equation (40) has no root with non-negative real part. Suppose not. Then the equation (40) has a root $\lambda = x + iy$ with $x \geq 0$. It then follows that

$$(x+\mu+iy)H_1(x+iy) - x - iy - \mu\mathcal{R}_0 = 0. \tag{41}$$

Separating the real part of the expression in (41) we get

$$\Re H_1(x+iy) = \frac{(x+\mu\mathcal{R}_0)(x+\mu)+y^2}{(x+\mu)^2+y^2} > 1. \tag{42}$$

Notice that

$$\begin{aligned} H_1(0) &= S^*\Big(\int_0^\infty \beta(a)e^{-\mu a-\int_0^a \gamma(s)ds}da+ \\ &\qquad \int_0^\infty \xi(a)e^{-\mu a-\int_0^a \gamma(s)ds}da \times \int_0^\infty \alpha(\theta)e^{-\int_0^\theta \delta(s)da}d\theta\Big) \\ &= S^*\frac{\mu\mathcal{R}_0}{\lambda} \\ &= 1 \end{aligned}$$

and

$$\Re H_1(x+iy) \le |H_1(x)| = H_1(x) \le H_1(0) = 1.$$

This contradicts the equation in (42), implying that (40) cannot have a root with nonnegative real part. There we have shown that the unique endemic equilibrium E^* is locally asymptotically stable. This completes the proof of Theorem 5.1. □

Proof of Theorem 2.3. We only need to prove the second results because the first result can be easily seen from Theorem 4.2. If $\mathcal{R}_0 > 1$, it then follows from Theorem 2.1 that the system (1) has a unique positive equilibrium $E^*(S^*, i^*(a), w^*(\theta))$, and $S^*, i^*(a), w^*(\theta)$ satisfy the following equations:

$$\begin{aligned} &\Lambda = \mu S^* + i^*(0), \\ &\frac{di^*(a)}{da} = -(\mu+\gamma(a))i^*(a), \\ &\frac{dw^*(\theta)}{d\theta} = -\delta(\theta)w^*(\theta), \end{aligned} \tag{43}$$

and

$$\begin{aligned} &i^*(0) = S^*\int_0^\infty \beta(a)i^*(a)da + S^*\int_0^\infty \alpha(\theta)w^*(\theta)d\theta, \\ &w^*(0) = \int_0^\infty \xi(a)i^*(a)da. \end{aligned} \tag{44}$$

From Theorem 5.1 we know that the unique endemic equilibrium $E^*(S^*, i^*(a), w^*(\theta))$ is locally asymptotically stable. In the following, we only need to show that the unique endemic equilibrium $E^*(S^*, i^*(a), w^*(\theta))$ is global attractor in $\mathbb{R}_+ \times L^1_+((0,\infty),\mathbb{R}^2) \setminus \partial\mathcal{M}_0$, i.e.,

$$\mathcal{A}_0 = \{E^*\}.$$

For ease of presentation, let us define

$$\begin{aligned} \mathcal{G}(x) &:= x - 1 - \ln x; \\ \rho(\theta) &:= e^{\int_0^\theta \delta(s)ds}\int_\theta^\infty \alpha(s)e^{-\int_0^s \delta(\varsigma)d\varsigma}ds; \\ \eta(a) &:= e^{\mu a+\int_0^a \gamma(s)ds}\Big(\int_a^\infty \beta(s)e^{-\mu s-\int_0^s \gamma(\varsigma)d\varsigma}ds \\ &\qquad +\rho(0)\textstyle\int_a^\infty \xi(s)e^{-\mu s-\int_0^s \gamma(\varsigma)d\varsigma}ds\Big). \end{aligned}$$

We can easily check that

$$\eta(0) = \frac{1}{S^*}, \rho(0) = \int_0^\infty \alpha(\theta)e^{-\int_0^\theta \delta(s)ds}d\theta, \tag{45}$$

and the function $\mathcal{G}(x)$ has the global minimum at $x = 1$ and $\mathcal{G}(1) = 0$.

Let $u(t) = (S(t), i(\cdot,t), w(\cdot,t))$ ba a complete solution to system (1) that lies in the attractor $\mathcal{A}_0$. From Theorem 4.3, we know there exist $\delta_1, \delta_2 > 0$ such that

$$\delta_1 \leq \frac{S(t)}{S^*} \leq \delta_2, \delta_1 \leq \frac{i(a,t)}{i^*(a)} \leq \delta_2, \delta_1 \leq \frac{w(\theta,t)}{w^*(\theta)} \leq \delta_2 \tag{46}$$

for all $t \in \mathbb{R}$ and $a, \theta \geq 0$. Now let us define the following Lyapunov function

$$\begin{aligned} V(t) &= S^*\mathcal{G}\Big(\frac{S(t)}{S^*}\Big) + S^*\int_0^\infty \eta(a)i^*(a)\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)da \\ &\qquad + S^*\int_0^\infty \rho(\theta)w^*(\theta)\mathcal{G}\Big(\frac{w(\theta,t)}{w^*(\theta)}\Big)d\theta. \end{aligned} \tag{47}$$

From (46) we can easily see that the function V is bounded when restricted to $\mathcal{A}_0$. Since the function $\mathcal{G}(x)$ is nonnegative for all $x > 0$, and has the global minimum at $x = 1$, it then follows that the function $V(t)$ is nonnegative and the point E^* is the global minimum point. We can also easily see that the function $V(t)$ is continuously differentiable. Differentiating $V(t)$ along solutions to (1) in $\mathcal{A}_0$, using (43),(44) and collecting terms, we obtain

$$\begin{aligned} \frac{dV(t)}{dt}\Big|_{(1)} &= -\frac{\mu}{S(t)}(S(t)-S^*)^2 + (1-\frac{S^*}{S(t)})(i^*(0)-i(0,t)) - \\ &\quad S^*\int_0^\infty \eta(a)\big(1-\frac{i^*(a)}{i(a,t)}\big)\big(\frac{\partial i(a,t)}{\partial a} + (\mu+\gamma(a))i(a,t)\big)da - \\ &\qquad S^*\int_0^\infty \rho(\theta)\big(1-\frac{w^*(\theta)}{w(\theta,t)}\big)\big(\frac{\partial w(\theta,t)}{\partial \theta} + \delta w(\theta,t)\big)d\theta. \end{aligned} \tag{48}$$

Noting that

$$\frac{\partial}{\partial a}\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big) = \Big(1-\frac{i^*(a)}{i(a,t)}\Big)\Big(\frac{i_a(a,t)}{i^*(a)} - \frac{i(a,t)i_a^*(a)}{[i^*(a)]^2}\Big),$$

where $i_a(a,t) = \dfrac{\partial i(a,t)}{\partial a}, i_a^*(a) = \dfrac{di^*(a)}{da}$, by using (43) it then follows that

$$\Big(1-\frac{i^*(a)}{i(a,t)}\Big)\frac{\partial i(a,t)}{\partial a} = i^*(a)\frac{\partial}{\partial a}\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big) + (\mu+\gamma(a))(i^*(a)-i(a,t)).$$

Thus, by using integration by parts we have

$$\begin{aligned} &\int_0^\infty \eta(a)\Big(1-\frac{i^*(a)}{i(a,t)}\Big)\frac{\partial i(a,t)}{\partial a}da \\ &= \int_0^\infty \eta(a)i^*(a)\frac{\partial}{\partial a}\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)da + \int_0^\infty \eta(a)(\mu+\gamma(a))(i^*(a)-i(a,t))da \\ &= \eta(a)i^*(a)\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big|_{a=0}^{a=\infty} - \int_0^\infty \mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big(\frac{d\eta(a)}{da}i^*(a) \\ &\qquad + \eta(a)\frac{di^*(a)}{da}\Big)da + \int_0^\infty \eta(a)(\mu+\gamma(a))(i^*(a)-i(a,t))da \\ &= \eta(a)i^*(a)\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big|_{a=0}^{a=\infty} + \int_0^\infty \mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big(\beta(a)+\rho(0)\xi(a)\Big)i^*(a)da \\ &\qquad + \int_0^\infty \eta(a)(\mu+\gamma(a))(i^*(a)-i(a,t))da. \end{aligned} \tag{49}$$

Here, we have used the second equation in (43) and the equation that

$$\frac{d\eta(a)}{da} = (\mu + \gamma(a))\eta(a) - (\beta(a) + \rho(0)\xi(a)).$$

Substituting the equation (49) into the second term in equation (48) gives

$$\begin{aligned} &S^* \int_0^\infty \eta(a)\big(1 - \frac{i^*(a)}{i(a,t)}\big)\big(\frac{\partial i(a,t)}{\partial a} + (\mu + \gamma(a))i(a,t)\big)da = S^*\eta(a)\times \\ &i^*(a)\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big|_{a=0}^{a=\infty} + S^* \int_0^\infty \mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big(\beta(a) + \rho(0)\xi(a)\Big)i^*(a)da. \end{aligned} \tag{50}$$

Similarly, by using the third equation in (43), the second equation in (44) and the equation that

$$\frac{d\rho(\theta)}{d\theta} = \delta(\theta)\rho(\theta) - \alpha(\theta),$$

the third term in equation (48) can be expressed as

$$\begin{aligned} &S^* \int_0^\infty \rho(\theta)\big(1 - \frac{w^*(\theta)}{w(\theta,t)}\big)\big(\frac{\partial w(\theta,t)}{\partial \theta} + \delta(\theta)w(\theta,t)\big)d\theta \\ &= S^*\rho(\theta)w^*(\theta)\mathcal{G}\Big(\frac{w(\theta,t)}{w^*(\theta)}\Big)\Big|_{\theta=0}^{\theta=\infty} + S^* \int_0^\infty \mathcal{G}\Big(\frac{w(\theta,t)}{w^*(\theta)}\Big)w^*(\theta)\alpha(\theta)d\theta. \end{aligned} \tag{51}$$

Substituting (50) and (51) into (48) yields that

$$\begin{aligned} &\frac{dV(t)}{dt}\Big|_{(1)} \\ &= -\frac{\mu}{S(t)}(S(t) - S^*)^2 + (1 - \frac{S^*}{S(t)})(i^*(0) - i(0,t)) - S^*\eta(a)i^*(a)\mathcal{G}\times \\ &\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big|_{a=0}^{a=\infty} + S^* \int_0^\infty \mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big(\beta(a) + \rho(0)\xi(a)\Big)i^*(a)da - \\ &S^*\rho(\theta)w^*(\theta)\mathcal{G}\Big(\frac{w(\theta,t)}{w^*(\theta)}\Big)\Big|_{\theta=0}^{\theta=\infty} + S^* \int_0^\infty \mathcal{G}\Big(\frac{w(\theta,t)}{w^*(\theta)}\Big)w^*(\theta)\alpha(\theta)d\theta. \end{aligned} \tag{52}$$

Putting the equations (45) into (52), using (2) and (44), and rearranging the terms, we have

$$\begin{aligned} &\frac{dV(t)}{dt}\Big|_{(1)} = -\frac{\mu}{S(t)}(S(t) - S^*)^2 - S^*\rho(\theta)w^*(\theta)\mathcal{G}\Big(\frac{w(\theta,t)}{w^*(\theta)}\Big)\Big|_{\theta=\infty} \\ &-S^*\eta(a)i^*(a)\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big|_{a=\infty} + S^* \int_0^\infty \Big(1 - \frac{S^*}{S(t)} - \ln\frac{i(0,t)}{i^*(0)} \\ &+ \ln\frac{i(a,t)}{i^*(a)}\Big)\beta(a)i^*(a)da + S^* \int_0^\infty \Big(1 - \frac{S^*}{S(t)} - \ln\frac{i(0,t)}{i^*(0)} + \ln\frac{w(\theta,t)}{w^*(\theta)}\Big)\times \\ &w^*(\theta)\alpha(\theta)d\theta - S^*\rho(0)w^*(0)\ln\frac{w(0,t)}{w^*(0)} + S^*\rho(0)\int_0^\infty \ln\frac{i(a,t)}{i^*(a)}\xi(a)i^*(a)da. \end{aligned} \tag{53}$$

Notice that

$$\rho(0)w^*(0)\ln\frac{w(0,t)}{w^*(0)} = \rho(0)\int_0^\infty \xi(a)i^*(a)\ln\frac{w(0,t)}{w^*(0)}da.$$

Straight forward computation yields that

$$\begin{aligned} &-S^*\rho(0)w^*(0)\ln\frac{w(0,t)}{w^*(0)} + S^*\rho(0)\int_0^\infty \ln\frac{i(a,t)}{i^*(a)}\xi(a)i^*(a)da \\ &\qquad = -S^*\rho(0)\int_0^\infty \xi(a)i^*(a)\mathcal{G}\Big(\frac{i(a,t)w^*(0)}{i^*(a)w(0,t)}\Big)da. \end{aligned} \tag{54}$$

Observe also that

$$\begin{aligned}\frac{i^*(0)}{S^*} &= \frac{i^*(0)S(t)}{S^*i(0,t)}\Big(\int_0^\infty \beta(a)i(a,t)da + \int_0^\infty \alpha(\theta)w(\theta,t)d\theta\Big)\\ &= \int_0^\infty \beta(a)i^*(a)\frac{i^*(0)S(t)i(a,t)}{S^*i(0,t)i^*(a)}da + \int_0^\infty \alpha(\theta)w^*(\theta)\frac{i^*(0)S(t)w(\theta,t)}{S^*i(0,t)w^*(\theta)}d\theta.\end{aligned}$$

By using the first equation in (44), we have

$$\frac{i^*(0)}{S^*} = \int_0^\infty \beta(a)i^*(a)da + \int_0^\infty \alpha(\theta)w^*(\theta)d\theta.$$

It then follows that

$$\begin{aligned}&\int_0^\infty \beta(a)i^*(a)\Big(1-\frac{i^*(0)S(t)i(a,t)}{S^*i(0,t)i^*(a)}\Big)da+\\ &\qquad\int_0^\infty \alpha(\theta)w^*(\theta)\Big(1-\frac{i^*(0)S(t)w(\theta,t)}{S^*i(0,t)w^*(\theta)}\Big)d\theta = 0.\end{aligned} \tag{55}$$

Adding equation (55) to the right side of equation (53), the sum of the fourth term and the fifth term in equation (53) can be expressed as

$$\begin{aligned}&S^*\int_0^\infty \Big(1-\frac{S^*}{S(t)}-\ln\frac{i(0,t)}{i^*(0)}+\ln\frac{i(a,t)}{i^*(a)}\Big)\beta(a)i^*(a)da+\\ &\qquad S^*\int_0^\infty \Big(1-\frac{S^*}{S(t)}-\ln\frac{i(0,t)}{i^*(0)}+\ln\frac{w(\theta,t)}{w^*(\theta)}\Big)w^*(\theta)\alpha(\theta)d\theta\\ &= S^*\int_0^\infty \Big(2-\frac{S^*}{S(t)}-\ln\frac{i(0,t)}{i^*(0)}+\ln\frac{i(a,t)}{i^*(a)}-\frac{i^*(0)S(t)i(a,t)}{S^*i(0,t)i^*(a)}\Big)\beta(a)i^*(a)da\\ &+S^*\int_0^\infty \Big(2-\frac{S^*}{S(t)}-\ln\frac{i(0,t)}{i^*(0)}+\ln\frac{w(\theta,t)}{w^*(\theta)}-\frac{i^*(0)S(t)w(\theta,t)}{S^*i(0,t)w^*(\theta)}\Big)w^*(\theta)\alpha(\theta)d\theta\\ &= -S^*\int_0^\infty \beta(a)i^*(a)\Big[\mathcal{G}\Big(\frac{S^*}{S(t)}\Big)+\mathcal{G}\Big(\frac{i^*(0)S(t)i(a,t)}{S^*i(0,t)i^*(a)}\Big)\Big]da-\\ &\qquad S^*\int_0^\infty w^*(\theta)\alpha(\theta)\Big[\mathcal{G}\Big(\frac{S^*}{S(t)}\Big)+\mathcal{G}\Big(\frac{i^*(0)S(t)w(\theta,t)}{S^*i(0,t)w^*(\theta)}\Big)\Big]d\theta.\end{aligned} \tag{56}$$

Substituting equations (54) and (56) into (53) gives

$$\begin{aligned}\frac{dV(t)}{dt}\Big|_{(1)} &= -\frac{\mu}{S(t)}(S(t)-S^*)^2 - S^*\rho(\theta)w^*(\theta)\mathcal{G}\Big(\frac{w(\theta,t)}{w^*(\theta)}\Big)\Big|_{\theta=\infty}\\ &-S^*\eta(a)i^*(a)\mathcal{G}\Big(\frac{i(a,t)}{i^*(a)}\Big)\Big|_{a=\infty} - S^*\int_0^\infty \beta(a)i^*(a)\Big[\mathcal{G}\Big(\frac{S^*}{S(t)}\Big)+\\ &\mathcal{G}\Big(\frac{i^*(0)S(t)i(a,t)}{S^*i(0,t)i^*(a)}\Big)\Big]da - S^*\int_0^\infty w^*(\theta)\alpha(\theta)\Big[\mathcal{G}\Big(\frac{S^*}{S(t)}\Big)+\\ &\mathcal{G}\Big(\frac{i^*(0)S(t)w(\theta,t)}{S^*i(0,t)w^*(\theta)}\Big)\Big]d\theta - S^*\rho(0)\int_0^\infty \mathcal{G}\Big(\frac{i(a,t)w^*(0)}{i^*(a)w(0,t)}\Big)\xi(a)i^*(a)da\\ &\le 0\end{aligned} \tag{57}$$

since the function $\mathcal{G}(x)$ is nonnegative for all $x > 0$. Furthermore, we can easily see that $\frac{dV}{dt}|_{(1)} = 0$ if and only if

$$S(t) = S^*, i(a,t) = i^*(a), w(\theta,t) = w^*(\theta). \tag{58}$$

We have shown that the function V is non-increasing along any complete solution $u(t)$ in the attractor $\mathcal{A}_0$. Consider a point $P(S^P, i^P, w^p)$ in the alpha limit set of $u(t)$. We deduce that V is constant along any complete orbit $u^P(t) = (S^P(t), i^P(\cdot,t), w(\cdot,t))$ passing through $P \in \mathcal{A}_0$. By (58), applied to $v^P(t)$, we have

$$S^P = S^*, i^P(a,t) = i^*(a), w^P(\theta,t) = w^*(\theta).$$

Therefore the alpha limit set of $u(t)$ is simply $\{E^*\}$. Similarly, the omega limit set of $u(t)$ is also $\{E^*\}$. Since $t \to V(u(t))$ is non-increasing function , it then follows that

$$V(u(t)) = V(E^*)$$

for all $t \in \mathbb{R}$.. By (58), we can obtain that $u(t) = E^*$ for all $t \in \mathbb{R}$. We have shown that the arbitrary complete solution $u(t)$ in the attractor $\mathcal{A}_0$ must be the endemic equilibrium solution. Thus we have shown that

$$\mathcal{A}_0 = \{E^*\}.$$

This completes the proof of Theorem 2.3. □

6. **Discussion.** In this section, we mainly summarize our results and make further remarks.

Based on the existing multi-stage cholera models formulated in [28] and [22], we first formulated more general model to describe the transmission dynamics of cholera. In this model we include two pathways of infection, namely direct and indirect via contaminated water, infection-age-dependent infectivity and variable periods of infectiousness. The model can be not only used for modeling waterborne disease transmission, such as *Giardia, Cryptosporidium, and Campylobacter*, but also regarded as a general virus dynamics model to describe the in vivo infection process of many viruses [22], such as HIV, HBV. Since in the paper age is considered as a continuous variable, the more general model is described by first order partial differential equations with nonlocal boundary conditions. We further discussed the explicit relevance of model formulations between age-structured model and the multi-stage ODE models, i.e., under some suitable conditions the age-structured model can be reduced into the multi-stage ODE models. This means that the age-structured model is a generalization of the multi-stage ODE models, and has greater flexibility that may better represent the underlying biology of the infection process [10].

We then derived the basic reproduction number $\mathcal{R}_0$ from the biological meanings of the model parameters. We showed that the qualitative behaviors of the system (1) are completely determined by the magnitudes of the basic reproduction $\mathcal{R}_0$: if $\mathcal{R}_0 < 1$ the disease-free equilibrium is globally asymptotically stable in the feasible region and the disease always dies out; if $\mathcal{R}_0 > 1$, a unique endemic equilibrium is globally asymptotically stable in the interior of the feasible region and the disease will persist at the endemic equilibrium if it is initially present. The same results are also obtained from analyzing the multi-stage models in many papers, such as [22] [27] and references therein. Thus the results obtained from the multi-stage ODE models are extended to the general age-structured model.

Because the age-structured model in this paper is described by partial differential equations and the tools used for the ODE models can not be used for analyzing the dynamics of PDE models, it is difficult to analyze the dynamics, particularly the global stability, of the PDE models due to the lack of practical tools. As far as we know, global studies of age-structured models are very limited due to the lack of applicable theories [4]. The method of Lyapunov functions is most commonly used to prove the global stability of nonlinear dynamical systems. In this paper, by constructed a class of global Lyapunov functions, and proved that the dynamics of the age of infection cholera model are completely determined by the basic reproduction number $\mathcal{R}_0$. Lyapunov functions of this type has been widely used for analyzing the ODE models in the literature (e.g., [5],[6],[9]) and was recently rediscovered ([10],[19]) to study the global stability of endemic equilibrium for the epidemic models with age of infection. We should point out that the techniques used for the PDE models are quite different from the techniques used for the ODE models, thus the process that we prove the global stability of the age-structured model is not the trivial extension used for analyzing the multi-stage ODE models in [22, 27] despite the conclusions are the same.

Finally, in this paper the modeling of cholera transmission dynamics is relatively simple compared with the modeling of many other infectious diseases. Thus it is necessary to improve the mathematical modeling for describing the transmission dynamics of waterborne diseases, and analyze the qualitative behavior of the improved models so that we can have a better understanding of the waterborne disease dynamics. In addition, as we all know that the top priority of global public safety is to prevent and contain the spread of infectious diseases, therefore we are also interested in exploring the effectiveness and implication of various preventive and control strategies on the transmission dynamics of waterborne diseases. We leave these for future investigations.

Acknowledgments. We would like to the reviewers for their comments and suggestions that helped to improve this paper greatly.

REFERENCES

[1] J. R. Andrews and S. Basu, Transmission dynamics and control of cholera in Haiti: An epidemic model, *Lancet*, **377** (2011), 1248–1255.

[2] O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, On the definition and computation of the basic reproduction ratio R_0 in models for infections diseases in hetereogeneous populations, *J. Math. Biol.*, **28** (1998), 365–382.

[3] O. Diekmann and J. A. P. Heesterbeek, *Mathematical Epidemiology of Infections Disease: Model Building, Analysis and Interpretation*, Wiley, New York, 2000.

[4] Z. L. Feng, W. Z. Huang and C. Castillo-Chavez, Global behavior of a multi-group SIS epidemic model with age structure, *J. Diff. Equs.*, **218** (2005), 292–324.

[5] H. I. Freedman and J. W. H. So, Global stability and persistence of simple food chains, *Math. Biosci.*, **76** (1985), 69–86.

[6] B. S. Goh, Global stability in many-species systems, *Amer. Natur.*, **111** (1977), 135–143.

[7] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs 25, American Mathematical Society, Providence, RI, 1988.

[8] J. K. Hale and P. Waltman, Persistence in infinite dimensional systems, *SIAM J. Math. Anal.*, **20** (1989), 388–395.

[9] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems: Mathematical Aspects of Selection*, Cambridge University Press, Cambridge, 1988.

[10] G. Huang, X. N. Liu and Y. Takeuchi, Lyapunov function and global stability for age-structured HIV infection model, *SIAM J. Appl. Math.*, **72** (2012), 25–38.

[11] M. Iannelli, *Mathematical Theory of Age-structured Population Dynamics*, Applied Mathematics Monographs 7, comitato nazionale per le scienze matematiche, Consiglio Nazionale delle Ricerche (C. N. R), Giardini, Pisa, 1995.

[12] H. Inaba, A semigroup approach to the strong ergodic theorem of the multistate stable population process, *Math. Popul. Studi.*, **17** (1988), 47–77.

[13] A. L. Lloyd, Destabilization of epidemic models with the inclusion of realistic distributions of infectious periods, *Proc. Roy. Soc. Lond. B*, **268** (2001), 985–993.

[14] A. L. Lloyd, Realistic distributions of infectious periods in epidemic models: Changing patterns of persistence and dynamics, *Theor. Popul. Biol.*, **60** (2001), 59–71.

[15] P. Magal, Compact attractors for time periodic age-structured population models, *Electron. J. Diff. Equs.*, **65** (2001), 1–35.

[16] P. Magal and H. R. Thieme, Eventual compactness for a semiflow generated by an age-structured models, *Communications on Pure and Applied Analysis*, **3** (2004), 695–727.

[17] P. Magal and X. Q. Zhao, Global attracotor in uniformly persistence dynamical systems, *SIAM J. Math. Anal.*, **37** (2005), 251–275.

[18] E. D'Agata, P. Magal, S. Ruan and G. F. Webb, Asymptotical behavior in nosocomial epidemic model with antibiotic resistance, *Diff. Integr. Equs.*, **19** (2006), 573–600.

[19] P. Magal, C. C. McCluskey and G. Webb, Lyapunov functional and global asymptotic stability for an infection-age model, *Appl. Anal.*, **89** (2010), 1109–1140.

[20] P. Magal and C. McCluskey, Two group infection age model: an application to nosocomial infection, *SIAM J. Appl. Math.*, **73** (2013), 1058–1095.

[21] F. A. Milner and A. Pugliese, Periodic solutions: a robust numerical method for an SIR model of epidemics, *J. Math. Biol.*, **39** (1999), 471–492.

[22] Z. S. Shuai and P. Van den Driessche, Global dynamics of cholera models with differential infectivity, *Math. Biosci.*, **234** (2011), 118–126.

[23] Z. S. Shuai, J. H. Tien and P. van den Driessche, Cholera models with hyperinfectivity and temporary immunity, *Bull. Math. Biol.*, **74** (2010), 2423–2445.

[24] H. R. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, *Diff. Integr. Equs.*, **3** (1990), 1035–1066.

[25] H. R. Thieme and C. Castillo-Chavez, How may infection-age-dependent infectivity affect the dynamics if HIV/AIDs?, *SIAM J. Appl. Math.*, **53** (1993), 1447–1479.

[26] J. P. Tian, S. Liao and J. Wang, *Dynamical Analysis and Control Strategies in Modeling Cholera*, 2010. Available from: `http://www.math.wm.edu/~jptian/preprints/pr-7-ode-cholera.pdf`.

[27] J. P. Tian and J. Wang, Global stability for cholera epidemic model, *Math. Biosci.*, **232** (2011), 31–41.

[28] J. H. Tien and D. J. D. Earn, Multiple transmission pathways and disease dynamics in a waterborne pathogen model, *Bull. Math. Biol.*, **72** (2010), 1506–1533.

[29] P. van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, **180** (2002), 29–48.

[30] G. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker, New York, 1985.

Received November 11, 2012; Accepted August 13, 2013.

E-mail address: `yjxin46501@sina.com`
E-mail address: `nustqzp@njust.edu.cn`
E-mail address: `xzli66@126.com`