MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **11**, Number **3**, June **2014**

doi:10.3934/mbe.2014.11.449

pp. **449–469**

THE GLOBAL STABILITY OF AN SIRS MODEL WITH INFECTION AGE

Yuming Chen

Department of Applied Mathematics, Yuncheng University
Yuncheng 044000, Shanxi, China
and
Department of Mathematics, Wilfrid Laurier University
Waterloo, Ontario N2L 3C5, Canada

Junyuan Yang and Fengqin Zhang

Department of Applied Mathematics, Yuncheng University
Yuncheng 044000, Shanxi, China

(Communicated by Pierre Magal)

Abstract. Infection age is an important factor affecting the transmission of infectious diseases. In this paper, we consider an SIRS model with infection age, which is described by a mixed system of ordinary differential equations and partial differential equations. The expression of the basic reproduction number $\mathscr{R}_0$ is obtained. If $\mathscr{R}_0 \leq 1$ then the model only has the disease-free equilibrium, while if $\mathscr{R}_0 > 1$ then besides the disease-free equilibrium the model also has an endemic equilibrium. Moreover, if $\mathscr{R}_0 < 1$ then the disease-free equilibrium is globally asymptotically stable otherwise it is unstable; if $\mathscr{R}_0 > 1$ then the endemic equilibrium is globally asymptotically stable under additional conditions. The local stability is established through linearization. The global stability of the disease-free equilibrium is shown by applying the fluctuation lemma and that of the endemic equilibrium is proved by employing Lyapunov functionals. The theoretical results are illustrated with numerical simulations.

1. Introduction.

In most epidemiological models for the transmission of infectious diseases, the infectious individuals are assumed to have the same infectivity. This assumption is reasonable in modeling communicable diseases such as influenza [2] and sexually transmitted diseases such as gonorrhea [7]. However, in the study of the HIV/AIDS epidemic, early infectivity experiments and the measurements of antigen and antibody titers suggest the possibility of an early infectivity peak (a few weeks after exposure) [6] and a late infectivity plateau (one year or so before the onset of "full-blown" AIDS) for HIV-infected individuals [12, 20, 21]. Therefore,

2010 *Mathematics Subject Classification.* Primary: 34D23; Secondary: 34G20, 35B35, 92D30.

Key words and phrases. SIRS model, infection age, global stability, persistence.

Research is supported partially by the NSF of China (61203228,11371313), Mathematical Tianyuan Foundation (11226258), the Young Sciences Foundation of Shanxi (2011021001-1), the Foundation of University (YQ-2011045, JY-2011036), Program for the Outstanding Innovative Teams of Higher Learning Institutions of Shanxi, the Natural Science and Engineering Research Council of Canada (NSERC), the Early Researchers Award (ERA) program of Ontario, the One Hundred Talents Project of Shanxi Province, and the Program of Key Disciplines in Shanxi Province (20111028).

it is necessary to incorporate the infection age (that is, the time that has passed since infection) into modeling.

By aggregating individuals with certain infection-ages into groups and by assuming homogeneity in each infection-age group, some researchers formulate infection age models governed by systems of ordinary differential equations or systems of difference equations. To name a few, see [9, 19, 28]. Then, to unify such models with an arbitrary number of stages, models described by systems of ordinary and differential equations are proposed (see, for example, [1, 4, 5, 10, 16, 18, 24, 25, 26, 27] and the references therein).

Though epidemic models with infection age have been studied extensively, most of the works only focus on the local stability of equilibria. To the best of our knowledge, results on global stability of equilibria are established only in [16, 25, 26]. In [16], Magal et al. studied an SI model; while in [25], Yang et al. considered an DI-DS model. As we know, in some infectious diseases, the infected can recover and be infected again. Therefore, it is worthy to study SIS and SIRS models. The purpose of this paper is to study the global stability of an SIRS model with infection age. This model is a special case of the one proposed in [27]. However, to be self-contained, we formulate the model in Section 2. For some recent results on the stability of SIRS models described by systems of ordinary differential equations, we refer readers to [3, 11, 13].

The remaining of this paper is organized as follows. After formulating the model in Section 2, we study the existence of equilibria in Section 3. We obtain the expression of the basic reproduction number $\mathscr{R}_0$. If $\mathscr{R}_0 \leq 1$ then the model only has the disease-free equilibrium; while if $\mathscr{R}_0 > 1$ then the model also has an endemic equilibrium besides the disease-free equilibrium. Section 4 is the main part of this paper, which is devoted to the stability of equilibria. We first develop the results on the linearized system around an equilibrium and its characteristic equation. These results are then applied to establish the local stability of the equilibria. The significant contribution of this paper is the global stability of equilibria. The global stability of the disease-free equilibrium is established by applying the fluctuation lemma and that of the endemic equilibrium is obtained by employing Lyapunov functionals. The paper concludes with a brief discussion together with numerical simulations to illustrate the main theoretical results.

2. **The model formulation.**

In the classic SIRS models, the total population is divided into three epidemiological classes: susceptible, infected, and recovered (or removed). The numbers of individuals in these classes at time t are denoted respectively by $S(t)$, $I(t)$, and $R(t)$. Our model is based on the following SIRS model,

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta S(t)I(t) - \mu S(t) + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + k)I(t), \\ \dfrac{dR(t)}{dt} = kI(t) - (\mu + \delta)R(t). \end{cases}$$

Here Λ is the recruitment rate of susceptibles, β is the transmission coefficient, μ is the natural death rate, $1/k$ is the infectious period, and δ is the progression rate of the removed.

To incorporate the age of infection (for simplicity, referred to age in the remaining of this paper), let a denote the age and let $i(t, a)$ denote the age density of infected individuals at time t. Then we obtain the following model described by a system of ordinary and partial differential equations,

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - S(t)\displaystyle\int_0^\infty \beta(a)i(t,a)da - \mu S(t) + \delta R(t), \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -(\mu + k(a))i(t,a), \\ \dfrac{dR(t)}{dt} = \displaystyle\int_0^\infty k(a)i(t,a)da - (\mu+\delta)R(t), \end{cases} \tag{1}$$

with the boundary and initial conditions

$$\begin{aligned} &i(t,0) = S(t)\int_0^\infty \beta(a)i(t,a)da, \qquad t > 0, \\ &S(0) = S_0 \geq 0, \qquad R(0) = R_0 \geq 0, \\ &i(0,a) = i_0(a) \in L_+^1(\mathbb{R}_+, \mathbb{R}), \end{aligned}$$

where $\beta(a)$ is the transmission coefficient with age a, $1/k(a)$ is the infectious period of infected individuals with age a, and $L_+^1(\mathbb{R}_+, \mathbb{R}) = \{f : \mathbb{R}_+ \to \mathbb{R} | f(x) \geq 0 \text{ for } x \in \mathbb{R}_+ \text{ and } \int_0^\infty f(x)dx < \infty\}$. From the perspective of biology, $\beta(a)$, $k(a) \in C_{BU}(\mathbb{R}_+, \mathbb{R}_+)$, where $C_{BU}(\mathbb{R}_+, \mathbb{R}_+)$ is the set of all bounded and uniformly continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$.

Notice that the total population at time t is

$$N(t) = S(t) + \int_0^\infty i(t,a)da + R(t).$$

We can easily see that $N(t)$ satisfies the following ordinary differential equation,

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t).$$

It follows that

$$\lim_{t\to\infty} N(t) = \frac{\Lambda}{\mu}. \tag{2}$$

For convenience, we rewrite (1) as follows,

$$\begin{cases} \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -(\mu + k(a))i(t,a), \\ \dfrac{dV(t)}{dt} = G(i(t,a), V(t)) - CV(t), \\ i(t,0) = S(t)\int_0^\infty \beta(a)i(t,a)da, \qquad t > 0, \\ i(0,a) = i_0(a) \in L_+^1(\mathbb{R}_+, \mathbb{R}), \\ V(0) = V_0 \in \mathbb{R}_+^2, \end{cases} \tag{3}$$

where

$$V(t) = \begin{pmatrix} S(t) \\ R(t) \end{pmatrix},$$

$$C = \begin{pmatrix} \mu & 0 \\ 0 & \mu+\delta \end{pmatrix},$$

$$G(i(t,a),V(t))=\begin{pmatrix}\Lambda-S(t)\int_0^\infty\beta(a)i(t,a)da+\delta R(t)\\ \int_0^\infty k(a)i(t,a)da\end{pmatrix}.$$

Set $X=Y\times\mathbb{R}^2$, where $Y=\mathbb{R}\times L^1(\mathbb{R}_+,\mathbb{R})$ with $\left\|\begin{pmatrix}\alpha\\ \phi\end{pmatrix}\right\|=|\alpha|+\|\phi\|_{L^1(\mathbb{R}_+,\mathbb{R})}$.
Furthermore, we define

$$X_+=Y_+\times\mathbb{R}^2_+,\qquad X_0=Y_0\times\mathbb{R}^2,\qquad X_{0+}=X_0\cap X_+,$$

where

$$Y_+=\mathbb{R}_+\times L^1_+(\mathbb{R}_+,\mathbb{R}),\qquad Y_0=\{0\}\times L^1(\mathbb{R}_+,\mathbb{R}).$$

Define $\mathcal{A}_1:D(\mathcal{A}_1)\subset Y\to Y$ by

$$\mathcal{A}_1\begin{pmatrix}0\\ \phi\end{pmatrix}=\begin{pmatrix}-\phi(0)\\ -\phi'-(\mu+k(a))\phi\end{pmatrix}$$

with $D(\mathcal{A}_1)=\{0\}\times W^{1,1}(\mathbb{R}_+,\mathbb{R})$. If $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda>-\mu$, then $\lambda\in\rho(\mathcal{A}_1)$, where $\rho(\mathcal{A}_1)$ is the resolvent set of $\mathcal{A}_1$. Moreover, if $\lambda\in\rho(\mathcal{A}_1)$ and

$$(\lambda I-\mathcal{A}_1)^{-1}\begin{pmatrix}\theta\\ \psi\end{pmatrix}=\begin{pmatrix}0\\ \phi\end{pmatrix},$$

then

$$\phi(a)=e^{-(\lambda+\mu)a}\theta+\int_0^a e^{-\int_s^a k(l)dl}e^{-(\lambda+\mu)(a-s)}\psi(s)ds.$$

Now we can rewrite (3) as

$$\begin{cases}\dfrac{d}{dt}\begin{pmatrix}0\\ i(t,.)\end{pmatrix}=\mathcal{A}_1\begin{pmatrix}0\\ i(t,.)\end{pmatrix}+F_1\left(\begin{pmatrix}0\\ i(t,.)\end{pmatrix},V(t)\right),\\ \dfrac{dV(t)}{dt}=-CV(t)+F_2\left(\begin{pmatrix}0\\ i(t,.)\end{pmatrix},V(t)\right),\\ i(0,a)=i_0(a)\in L^1_+(\mathbb{R}_+,\mathbb{R}),\\ V(0)=V_0\in\mathbb{R}^2_+,\end{cases}\tag{4}$$

where

$$\begin{aligned}F_1\left(\begin{pmatrix}0\\ i(t,.)\end{pmatrix},V(t)\right)&=\begin{pmatrix}B(i(t,.),V(t))\\ 0\end{pmatrix},\\ B(i(t,.),V(t))&=S(t)\int_0^\infty\beta(a)i(t,a)da,\\ F_2\left(\begin{pmatrix}0\\ i(t,.)\end{pmatrix},V(t)\right)&=\begin{pmatrix}\Lambda-S(t)\int_0^\infty\beta(a)i(t,a)da+\delta R(t)\\ \int_0^\infty k(a)i(t,a)da\end{pmatrix}.\end{aligned}$$

To express (4) as an ordinary differential equation on a Banach space, we write

$$u(t)=\left(\begin{pmatrix}0\\ i(t,.)\end{pmatrix},V(t)\right).$$

Let $L : D(L) \subset X \to X$ be the linear operator defined by

$$L(u(t)) = \left(\mathcal{A}_1 \begin{pmatrix} 0 \\ i(t,.) \end{pmatrix}, -CV(t) \right),$$

where $D(L) = Z \times \mathbb{R}^2$ with $Z = \{0\} \times W^{1,1}(\mathbb{R}_+, \mathbb{R})$. It follows that $X_0 = \overline{D(L)}$ and $X_{0+} = \overline{D(L)} \cap X_+$. So $\overline{D(L)} = X_0$ is not dense in X. We consider the nonlinear map $F : \overline{D(L)} \to X$ defined by $F(u(t)) = \begin{pmatrix} F_1(u(t)) \\ F_2(u(t)) \end{pmatrix}$. Then (4) can be rewritten as

$$\begin{cases} \frac{du(t)}{dt} = Lu(t) + F(u(t)), \\ u(0) = \left(\begin{pmatrix} 0 \\ i_0(\cdot) \end{pmatrix}, V_0 \right) \in \overline{D(L)}. \end{cases} \tag{5}$$

The following result can be obtained by using the fact that the nonlinearities are Lipschitz continuous on bounded sets, by using (2), and by applying the results in [15].

Proposition 2.1. *There exists a uniquely determined semiflow $\{U(t)\}_{t\geq 0}$ on X_{0+} such that, for each $u = \left(\begin{pmatrix} 0 \\ i(t,.) \end{pmatrix}, V(t) \right) \in X_{0+}$, there exists a unique continuous map $U \in C(\mathbb{R}_+, X_{0+})$ which is an integral solution of the Cauchy problem (5), that is, for all $t \geq 0$,*

$$\int_0^t U(s)u ds \in D(L) \qquad \text{and} \qquad U(t)u = u + L\int_0^t U(s)u ds + \int_0^t F(U(s)u)ds.$$

Since $k(a)$ is uniformly continuous, we can deduce that the semiflow $\{U(t)\}_{t\geq 0}$ is asymptotically smooth. Moreover, $\{U(t)\}_{t\geq 0}$ is bounded dissipative by using (2).

Define $\hat{\mu} = \max\limits_{0\leq a<\infty} \{\mu, \mu + k(a), \mu + \delta\}$ and $\Omega := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\hat{\mu}\}$. By using the results in [22], we obtain the following lemma.

Lemma 2.2. *If $\lambda \in \Omega$ then $\lambda \in \rho(\lambda)$. More precisely, for any $\lambda > -\hat{\mu}$, any $\left(\begin{pmatrix} \alpha \\ \psi \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right) \in X$, and $\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right) \in D(L)$, we have*

$$(\lambda I - L)^{-1} \left(\begin{pmatrix} \alpha \\ \psi \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right)$$

if and only if

$$\begin{aligned} \varphi(a) &= e^{-\int_0^a (\lambda+\mu+k(l))dl}\alpha + \int_0^a e^{-\int_s^a (\lambda+\mu+k(l))dl}\psi(s)ds, \qquad (6) \\ \varphi_1 &= \frac{\psi_1}{\lambda+\mu}, \\ \varphi_2 &= \frac{\psi_2}{\lambda+\mu+\delta}. \end{aligned}$$

Moreover, L is a Hille-Yosida operator and

$$\|(\lambda I - L)^{-n}\| \le \frac{M}{(Re(\lambda)+\hat{\mu})^n} \qquad \text{for } \mathrm{Re}(\lambda) > -\hat{\mu} \text{ and } n \ge 1.$$

Proof. For any $\begin{pmatrix} \alpha \\ \psi \end{pmatrix} \in Y$ and $\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(\mathcal{A}_1)$, we have

$$(\lambda I - \mathcal{A}_1)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix},$$

that is,

$$\begin{aligned} \varphi(0) &= \alpha, \\ \varphi'(a) &= -(\lambda+\mu+k(a))\varphi + \psi. \end{aligned}$$

Then (6) follows easily. From the definition of L, one directly gets

$$\begin{aligned} (\lambda+\mu)\varphi_1 = \psi_1, \\ (\lambda+\mu+\delta)\varphi_2 = \psi_2. \end{aligned}$$

Thus

$$\begin{aligned} \varphi_1 &= \frac{\psi_1}{\lambda+\mu}, \\ \varphi_2 &= \frac{\psi_2}{\lambda+\mu+\delta}. \end{aligned}$$

For any $\lambda > -\hat{\mu}$, we have

$$\begin{aligned} &\left\| (\lambda - L)^{-1} \left(\begin{pmatrix} \alpha \\ \psi \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right) \right\|_X \\ = &\left\| \left(\begin{pmatrix} 0 \\ e^{-\int_0^a (\lambda+\mu+k(l))dl}\alpha + \int_0^a e^{-\int_s^a(\lambda+\mu+k(l))dl}\psi(s)ds \end{pmatrix}, \begin{pmatrix} \frac{\psi_1}{\lambda+\mu} \\ \frac{\psi_2}{\lambda+\mu+\delta} \end{pmatrix} \right) \right\|_X \\ \le &\int_0^\infty \left| e^{-\int_0^a(\lambda+\mu+k(l))dl} \right| da|\alpha| + \int_0^\infty \left| \int_0^a e^{-\int_s^a(\lambda+\mu+k(l))dl}\psi(s)ds \right| da \\ &+ \frac{|\psi_1|}{|\lambda+\mu|} + \frac{|\psi_2|}{|\lambda+\mu+\delta|} \\ \le &\frac{|\alpha|}{|\lambda+\hat{\mu}|} + \int_0^\infty e^{(\lambda+\hat{\mu})\tau}|\psi(\tau)| \int_\tau^\infty |e^{-(\lambda+\hat{\mu})a}| da d\tau + \frac{|\psi_1|}{|\lambda+\mu|} + \frac{|\psi_2|}{|\lambda+\mu+\delta|} \\ \le &\frac{1}{|\mathrm{Re}(\lambda)+\hat{\mu}|}(|\alpha| + \|\psi\|_{L^1} + |\psi_1| + |\psi_2|). \end{aligned}$$

The results immediately follow. □

3. **Existence of equilibria.** In this section, we consider the existence of equilibria of (5). Denote

$$\begin{aligned} \pi(a) &= e^{-\int_0^a k(s)ds}, \qquad a \ge 0, \\ K &= \int_0^\infty \beta(a) e^{-\mu a}\pi(a) da, \end{aligned}$$

$$K_1 = \int_0^\infty k(a)e^{-\mu a}\pi(a)da.$$

Note that $K_1 \le \int_0^\infty k(a)\pi(a)da = 1$.

Theorem 3.1. *Let $\mathscr{R}_0 = \Lambda K/\mu$. Then the following statements are true.*

(i) *Equation (5) always has a disease-free equilibrium* $P_0 = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\Lambda}{\mu} \\ 0 \end{pmatrix}\right)$.

(ii) *If $\mathscr{R}_0 \le 1$ then P_0 is the only equilibrium of (5) while if $\mathscr{R}_0 > 1$ then it also has an endemic equilibrium P^* given by*

$$P^* = \left(\begin{pmatrix} 0 \\ \bar{i}(0)e^{-\mu a}\pi(a) \end{pmatrix}, \begin{pmatrix} \frac{1}{K} \\ \frac{\bar{i}(0)K_1}{\mu+\delta} \end{pmatrix}\right), \tag{7}$$

where

$$\bar{i}(0) = \frac{\Lambda(1 - \frac{1}{\mathscr{R}_0})}{1 - \frac{\delta}{\mu+\delta}K_1}. \tag{8}$$

Proof. Suppose that $\overline{u} = \left(\begin{pmatrix} 0 \\ \bar{i}(a) \end{pmatrix}, \begin{pmatrix} \overline{S} \\ \overline{R} \end{pmatrix}\right) \in X_0$ is an equilibrium of (5). Then we have

$$\begin{aligned} -\bar{i}(0) + \overline{S}\int_0^\infty \beta(a)\bar{i}(a)da &= 0, & (9)\\ -\bar{i}'(a) - (\mu + k(a))\bar{i}(a) &= 0, & (10)\\ \Lambda - \overline{S}\int_0^\infty \beta(a)\bar{i}(a)da - \mu\overline{S} + \delta\overline{R} &= 0, & (11)\\ \int_0^\infty k(a)\bar{i}(a)da - (\mu+\delta)\overline{R} &= 0. & (12)\end{aligned}$$

It follows from (10) that $\bar{i}(a) = \bar{i}(0)e^{-\mu a}\pi(a)$. If $\bar{i}(0) = 0$ then we easily get the disease-free equilibrium P_0. Now, suppose that $\bar{i}(0) \ne 0$. It follows from (9) and (12) respectively that $\overline{S} = 1/K$ and $\overline{R} = \bar{i}(0)K_1/(\mu+\delta)$. These, combined with (11), tell us that $\bar{i}(0)$ is given by (8). Noting $K_1 < 1$, we see that $\bar{i}(0) > 0$ if and only if $\mathscr{R}_0 > 1$. Thus if $\mathscr{R}_0 \le 1$, then (5) only has the disease-free equilibrium P_0; if $\mathscr{R}_0 > 1$, then, besides P_0, (5) also has an endemic equilibrium P^* given by (7). This completes the proof. $\square$

Notice that $\beta(a)e^{-\mu a}\pi(a)$ is the probability of an infected person with age a infecting a susceptible and still staying in the infected compartment due to death and treatment. So the biological meaning of the basic reproduction number $\mathscr{R}_0$ is the total expected number of the infected susceptibles.

4. **Stability of equilibria.** In the previous section, we obtained the existence of equilibria. In this section, we study their stability. The local stability is obtained by analyzing the spectrum of the linearized operator around each equilibrium.

4.1. **Linearized systems and their characteristic equations.** Let $\overline{u}$ be an equilibrium of (5). Linearize (5) about $\overline{u}$ to obtain the linearized system,

$$\begin{cases} \frac{dy}{dt} = (L + DF(\overline{u}))y(t), \\ y(0) = y_0 \in \overline{D(L)}. \end{cases}$$

Denote the part of L in $\overline{D(L)}$ by L_0, that is, $D(L_0) = \{x \in D(L) : Lx \in \overline{D(L)}\}$ and $L_0 u = Lu$ for $u \in D(L_0)$. It follows that, for $\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) \in D(L_0)$, we have

$$L_0 \left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 \\ \widehat{\mathcal{A}}_{10}\varphi \end{pmatrix}, \begin{pmatrix} -\mu\varphi_1 \\ -(\mu+\delta)\varphi_2 \end{pmatrix}\right),$$

where $\widehat{\mathcal{A}}_{10}\varphi = -\varphi' - (\mu + k(a))\varphi$ with $D(\widehat{\mathcal{A}}_{10}) = \{\varphi \in W^{1,1}(\mathbb{R}_+, \mathbb{R}) : \varphi(0) = 0\}$. Then we can easily get the following result.

Lemma 4.1. *The linear operator L_0 is the infinitesimal generator of the strongly continuous semigroup $\{T_{L_0}(t)\}_{t\geq 0}$ of bounded linear operators on $\overline{D(L)}$ and for each $t \geq 0$ the linear operator $T_{L_0}(t)$ is defined by*

$$T_{L_0}(t) \left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 \\ \widehat{T}_{\mathcal{A}_{10}}(t)\varphi \end{pmatrix}, \begin{pmatrix} e^{-\mu t}\varphi_1 \\ e^{-(\mu+\delta)t}\varphi_2 \end{pmatrix}\right),$$

where

$$\widehat{T}_{\mathcal{A}_{10}}(t)\varphi(a) = \begin{cases} e^{-\int_{a-t}^{a}(\mu+k(l))dl}\varphi(a-t), & a \geq t, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 4.2 ([14])**.** Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}_{t\geq 0}$ on a Banach space X. We define the growth bound $w_0(L) \in [-\infty, \infty)$ of L by

$$w_0(L) = \lim_{t\to\infty} \frac{\ln(\|T_L(t)\|_X)}{t}.$$

The essential growth bound $w_{0,ess}(L) \in [-\infty, \infty)$ of L is defined by

$$w_{0,ess}(L) = \lim_{t\to\infty} \frac{\ln(\|T_L(t)\|_{ess})}{t},$$

where $\|T_L(t)\|_{ess}$ is the essential norm of $T_L(t)$ defined by

$$\|T_L(t)\|_{ess} = \kappa(T_L(t)B_X(0,1)).$$

Here $B_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$ and, for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius} \leq \varepsilon\}$$

is the Kuratowski measure of non-compactness.

Now we estimate the essential growth bound of the strongly semigroup generated by $(L + DF(\overline{u}))_0$, which is part of $L + DF(\overline{u})$ in $\overline{D(L)}$. Note that for any

$\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) \in D(L)$, we have

$$
\begin{aligned}
& DF(\overline{u})\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) \\
= & \left(\begin{pmatrix} DB(\bar{i}(\cdot), \overline{V})(\varphi, \varphi_1) \\ 0 \end{pmatrix}, \begin{pmatrix} DF_2(\overline{u})(\varphi, \varphi_1, \varphi_2) \end{pmatrix}\right) \\
= & \left(\begin{pmatrix} \varphi_1 \int_0^\infty \beta(a)\bar{i}(a)da + \overline{S}\int_0^\infty \beta(a)\varphi(a)da \\ 0 \end{pmatrix}, \right. \\
& \left. \begin{pmatrix} -\varphi_1 \int_0^\infty \beta(a)\bar{i}(a)da - \overline{S}\int_0^\infty \beta(a)\varphi(a)da + \delta\varphi_2 \\ \int_0^\infty k(a)\varphi(a)da \end{pmatrix}\right).
\end{aligned}
$$

We can easily see that $DF(\overline{u})$ is a Fredholm operator. It follows from Lemma 2.2 and Definition 4.2 that

$$w_{0,ess}(L_0) \leq w_0(L_0) \leq -\hat{\mu}.$$

Due to Thieme [23], we obtain

$$w_{0,ess}(L + DF(\overline{u}))_0 \leq -\hat{\mu} < 0.$$

In summary, we have proved the following result.

Theorem 4.3. *The essential growth rate of the strongly continuous semigroup generated by $(L + DF(\overline{u}))_0$ is strictly negative, that is, $\omega_{0,ess}((L + DF(\overline{u}))_0) < 0$.*

To study the stability of the equilibria, we denote $Q := DF(\overline{u})$ for simplicity of notation. Let $\lambda \in \Omega$. Since $\lambda I - L$ is invertible, it follows that $\lambda I - (L + DF(\overline{u})) = \lambda I - (L + Q)$ is invertible if and only if $I - Q(\lambda I - L)^{-1}$ is invertible. In addition,

$$(\lambda I - (L + Q))^{-1} = (\lambda I - L)^{-1}[I - Q(\lambda I - L)^{-1}]^{-1}.$$

Consider

$$(I - Q(\lambda I - L)^{-1})\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) = \left(\begin{pmatrix} \gamma \\ \psi \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right).$$

By Lemma 2.2, we obtain

$$\begin{pmatrix} \alpha \\ \varphi \\ \varphi_1 \\ \varphi_2 \end{pmatrix} - Q\begin{pmatrix} 0 \\ e^{-\int_0^a (\lambda+\mu+k(l))dl}\alpha + \int_0^a e^{-\int_s^a (\lambda+\mu+k(l))dl}\psi(s)ds \\ \frac{\varphi_1}{\lambda+\mu} \\ \frac{\varphi_2}{\lambda+\mu+\delta} \end{pmatrix} = \begin{pmatrix} \gamma \\ \psi \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

Noting the definitions of DF and DB, we obtain

$$\begin{cases} \Delta(\lambda)\begin{pmatrix} \alpha \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \Psi(\lambda,\psi) + \begin{pmatrix} \gamma \\ \psi_1 \\ \psi_2 \end{pmatrix}, \\ \varphi = \psi, \end{cases}$$

where

$$\Delta(\lambda) = I - \begin{pmatrix} \overline{S}\widehat{K}(\lambda) & \frac{\int_0^\infty \beta(a)\bar{i}(a)da}{\lambda+\mu} & 0 \\ -\overline{S}\widehat{K}(\lambda) & -\frac{\int_0^\infty \beta(a)\bar{i}(a)da}{\lambda+\mu} & \frac{\delta}{\lambda+\mu+\delta} \\ \widehat{K_1}(\lambda) & 0 & 0 \end{pmatrix}, \tag{13}$$

$$\widehat{K}(\lambda) = \int_0^\infty \beta(a)\pi(a)e^{-(\lambda+\mu)a}da,$$

$$\widehat{K_1}(\lambda) = \int_0^\infty k(a)\pi(a)e^{-(\lambda+\mu)a}da,$$

$$\Psi(\lambda,\psi) = \begin{pmatrix} \overline{S}\int_0^\infty \beta(a)\int_0^a e^{-\int_s^a(\lambda+\mu+k(l))dl}\psi(s)dsda \\ -\overline{S}\int_0^\infty \beta(a)\int_0^a e^{-\int_s^a(\lambda+\mu+k(l))dl}\psi(s)dsda \\ \int_0^\infty k(a)\int_0^a e^{-\int_s^a(\lambda+\mu+k(l))dl}\psi(s)dsda \end{pmatrix}. \tag{14}$$

Lemma 4.4. *The following statements are true.*

(i) $\sigma(L+Q)\cap\Omega = \sigma_p(L+Q)\cap\Omega = \{\lambda\in\Omega : \det\Delta(\lambda) = 0\}$.

(ii) *Suppose* $\lambda\in\rho(L+Q)\cap\Omega$. *Then*

$$(\lambda I-(L+Q))^{-1}\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 \\ \psi \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right)$$

if and only

$$\begin{aligned} \psi(a) &= e^{-(\lambda+\mu)a}\pi(a)(\Delta(\lambda)^{-1}((\alpha,\varphi_1,\varphi_2)^T + \Psi(\lambda,\varphi))_1 \\ &\quad + \int_0^a e^{-\int_s^a(\lambda+\mu+k(l))dl}\varphi(s)ds, \\ \psi_1 &= \frac{(\Delta(\lambda)^{-1}((\alpha,\varphi_1,\varphi_2)^T + \Psi(\lambda,\varphi))_2}{\lambda+\mu}, \\ \psi_2 &= \frac{(\Delta(\lambda)^{-1}((\alpha,\varphi_1,\varphi_2)^T + \Psi(\lambda,\varphi))_3}{\lambda+\mu+\delta}, \end{aligned}$$

where $\Delta(\lambda)$ *and* $\Psi(\lambda,\varphi)$ *are defined respectively by* (13) *and* (14), *and* $()_i$ *denotes the ith element of a vector* $()$.

Proof. Assume that $\lambda\in\Omega$ and $\det(\Delta(\lambda))\neq 0$. Then we have

$$\begin{aligned} &(\lambda I-(L+Q))^{-1}\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) \\ = \; &(\lambda I-L)^{-1}(I-Q(\lambda I-L)^{-1})^{-1}\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right) \end{aligned}$$

$$= \left(\begin{pmatrix} 0 \\ \psi \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right).$$

Thus (ii) follows from Lemma 2.2. We also see that $\{\lambda \in \Omega : \det \Delta(\lambda) \neq 0\} \subset \rho(L+Q)$ and hence $\sigma(L+Q) \cap \Omega \subset \{\lambda \in \Omega : \det \Delta(\lambda) = 0\}$. Now, assume that $\lambda \in \Omega$ and $\det \Delta(\lambda) = 0$. We want to show that $\lambda \in \sigma_p(L+Q)$, that is, we want to find a nonzero solution to $(L+Q)u = \lambda u$. In fact, for $u = \left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right) \in D(L)$, we have

$$\begin{aligned} (L+Q)u &= \left(\begin{pmatrix} -\varphi(0) \\ -\varphi' - (\mu + k(a))\varphi \end{pmatrix}, \begin{pmatrix} -\mu\varphi_1 \\ -(\mu+\delta)\varphi_2 \end{pmatrix} \right) \\ &\quad + \left(\begin{pmatrix} B(\overline{u})u \\ 0 \end{pmatrix}, DF_2(\overline{u})u \right). \end{aligned}$$

Then $(L+Q)u = \lambda u$ gives

$$\begin{cases} -\varphi(0) + B(\overline{u})u = 0, \\ -\varphi' - (\mu + k(a))\varphi = \lambda\varphi, \\ \begin{pmatrix} -(\lambda+\mu) & 0 \\ 0 & -(\lambda+\mu+\delta) \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} + DF_2(\overline{u})u = 0. \end{cases} \tag{15}$$

From the second equation of (15) we get $\varphi(a) = \varphi(0)e^{-(\lambda+\mu)a}\pi(a)$. Substituting it into the other equations of (15) yields $\Delta(\lambda)(\varphi(0), \varphi_1, \varphi_2)^T = 0$. This implies that $(L+Q)u = \lambda u$ has a nonzero solution if and only if we can find $(\varphi(0), \varphi_1, \varphi_2)^T \neq 0$ such that $\Delta(\lambda)(\varphi(0), \varphi_1, \varphi_2)^T = 0$, which is true since $\det \Delta(\lambda) = 0$. This proves that $\lambda \in \sigma_p(L+Q)$ and hence the proof is complete. □

With the above preparation, we are ready to study the stability of equilibria.

4.2. Stability of the disease-free equilibrium P_0.

Theorem 4.5. *The disease-free equilibrium P_0 is locally asymptotically stable if $\mathscr{R}_0 < 1$ and it is unstable if $\mathscr{R}_0 > 1$.*

Proof. At P_0, we have

$$\Delta(\lambda) = I - \begin{pmatrix} \frac{\Lambda}{\mu}\hat{K}(\lambda) & 0 & 0 \\ -\frac{\Lambda}{\mu}\hat{K}(\lambda) & 0 & \frac{\delta}{\lambda+\mu+\delta} \\ \widehat{K_1}(\lambda) & 0 & 0 \end{pmatrix}.$$

Obviously, $\det(\Delta(\lambda)) = 0$ if and only if $1 = \frac{\Lambda}{\mu}\hat{K}(\lambda)$. Notice that $|\frac{\Lambda}{\mu}\hat{K}(\lambda)| \leq \mathscr{R}_0$ for all $\lambda \in \mathbb{C}$ with nonnegative real parts. Therefore, if $\mathscr{R}_0 < 1$ then $\frac{\Lambda}{\mu}\hat{K}(\lambda) < 1$. This implies that all solutions of $\det(\Delta(\lambda)) = 0$ have negative real parts if $\mathscr{R}_0 < 1$ and hence P_0 is locally asymptotically stable if $\mathscr{R}_0 < 1$. Now suppose that $\mathscr{R}_0 > 1$. Note that $\frac{\Lambda}{\mu}\hat{K}(\lambda)$ is a decreasing function with $\frac{\Lambda}{\mu}\hat{K}(0) = \mathscr{R}_0$ and $\frac{\Lambda}{\mu}\hat{K}(\lambda) \to 0$ as $\lambda \to \infty$. Therefore, if $\mathscr{R}_0 > 1$ then there always exists a positive solution to $\det(\Delta(\lambda)) = 0$. This means that P_0 is unstable if $\mathscr{R}_0 > 1$ and the proof is complete. □

To deal with the global stability of P_0, we need some notations. For a function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, we denote

$$\varphi_\infty = \liminf_{t\to\infty} \varphi(t) \qquad \text{and} \qquad \varphi^\infty = \limsup_{t\to\infty} \varphi(t).$$

Lemma 4.6 (Fluctuation Lemma [8])**.** *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a bounded and continuously differentiable function. Then there exist sequences $\{s_n\}$ and $\{t_n\}$ such that $s_n \to \infty$, $t_n \to \infty$, $\varphi(s_n) \to \varphi_\infty$, $\varphi'(s_n) \to 0$, $\varphi(t_n) \to \varphi^\infty$, and $\varphi'(t_n) \to 0$ as $n \to \infty$.*

Theorem 4.7. *Suppose that $\mathscr{R}_0 < 1$. Then the disease-free equilibrium P_0 is a global attractor, i.e., $\lim_{t\to\infty}(i, S, R) = (0, \frac{\Lambda}{\mu}, 0)$.*

Proof. By Proposition 2.1, any solution interested is nonnegative. It follows from (2) that $N^\infty = \frac{\Lambda}{\mu}$ and hence $S^\infty \le \frac{\Lambda}{\mu}$. Integrating the equation for i in (1) along the characteristic lines, $t - a =$ constant, yields

$$i(t,a) = \begin{cases} B(t-a)e^{-\mu a}\pi(a), & a < t, \\ i_0(a-t)e^{-\mu t}\frac{\pi(a)}{\pi(a-t)}, & a \ge t, \end{cases} \tag{16}$$

where $B(t) = S(t)\int_0^\infty \beta(a)i(t,a)da$. Substitute $i(t,a)$ into $B(t)$ to obtain

$$B(t) = S(t)\int_0^t \beta(a)B(t-a)e^{-\mu a}\pi(a)da + F(t), \tag{17}$$

where $F(t) = S(t)\int_t^\infty i_0(a-t)e^{-\mu t}\frac{\pi(a)}{\pi(a-t)}da$. By Lemma 4.6, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $B(t_n) \to B^\infty$ as $n \to \infty$. Then in (17) letting $t = t_n$ and $n \to \infty$ gives us $B^\infty \le S^\infty K B^\infty$ since $\lim_{t\to\infty} F(t) = 0$. This implies that $B^\infty = 0$ since $B^\infty \ge 0$ and $S^\infty K \le \Lambda K/\mu = \mathscr{R}_0 < 1$. So $\lim_{t\to\infty} B(t) = 0$. This, combined with (16), gives $\lim_{t\to\infty} i(t,a) = 0$ for all $a \in \mathbb{R}_+$.

Now, we get from the equation on $R(t)$ in (1) that

$$\begin{aligned}\frac{dR(t)}{dt} &= \int_0^t k(a)B(t-a)e^{-\mu a}\pi(a)da - (\mu+\delta)R(t) \\ &\quad + \int_t^\infty k(a)i_0(a-t)e^{-\mu t}\frac{\pi(a)}{\pi(a-t)}da.\end{aligned}$$

With the help of Lemma 4.6 again, there is a sequence $\{v_n\}$ such that $v_n \to \infty$, $R(v_n) \to R^\infty$, and $\frac{d}{dt}R(v_n) \to 0$. Then we can get $R^\infty \le \frac{K_1 B^\infty}{\mu+\delta} = 0$.

Finally, we apply Lemma 4.6 to S_∞. It follows from the first equation of (1) that $S_\infty \ge \Lambda/\mu$. This, combined with $S^\infty \le \Lambda/\mu$, gives $\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu}$ and hence the proof is complete. $\square$

4.3. **Persistence.** When $\mathscr{R}_0 > 1$, P_0 is unstable. We show that in this case the system is persistent and hence the disease will establish.

Let $\Pi : X_0 \to L^1(\mathbb{R}_+, \mathbb{R})$ be the Poincare projector defined by

$$\Pi(v) = v_1(t,a) \qquad \text{for } v = \left(\begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} \right) \in X_0.$$

Set

$$M = X_{0+}, \qquad M_0 = \{v \in M : \Pi(v) \ne 0\}, \qquad \partial M_0 = M \setminus M_0.$$

Following [15], we have the following lemma.

Lemma 4.8. *The subsets M_0 and ∂M_0 are both positively invariant under the semiflow $\{U(t)\}_{t\geq 0}$, namely, $U(t)M_0 \subset M_0$ and $U(t)\partial M_0 \subset \partial M_0$ for $t \geq 0$.*

The persistence of (5) is established by using the results in Magal and Zhao [17]. For this purpose, since P_0 is globally asymptotically stable in ∂M_0, it is sufficient to prove that there exists $\varepsilon > 0$ with the property that for each $v \in \{y \in M_0 : \|P_0 - y\| \leq \varepsilon\}$ there exists $t_0 \geq 0$ such that

$$\|P_0 - U(t)v\| > \varepsilon \qquad \text{for } t > t_0.$$

This shows that P_0 is the largest invariant set for U in the neighborhood of P_0 and also leads to

$$W^s(P_0) \cap M_0 = \emptyset,$$

where

$$W^s(P_0) = \{v \in X_{0+} : \lim_{t\to\infty} U(t)v = P_0\}.$$

Theorem 4.9. *Assume that $\mathscr{R}_0 > 1$. The semiflow $\{U(t)\}_{t\geq 0}$ is uniformly persistent with respect to the pair $(\partial M_0, M_0)$, that is, there exists $\varepsilon > 0$ such that $\underline{\lim}_{t\to\infty} \|\Pi v(t)\| \geq \varepsilon$ for $x \in M_0$. Moreover, there exists a compact subset $\mathscr{A}_0$ of M_0 which is a global attractor for $\{U(t)\}_{t\geq 0}$ in M_0.*

Proof. By way of contradiction, assume that for every $n \in \mathbb{N}$ there exists $v_n \in \{y \in M_0 : \|P_0 - y\| \leq \frac{1}{n+1}\}$ such that

$$\|P_0 - U(t)v_n\| \leq \frac{1}{n+1} \qquad \text{for } t \geq 0. \tag{18}$$

Write $U(t)v_n$ as $\left(\begin{pmatrix} 0 \\ v_1^n \end{pmatrix}, \begin{pmatrix} v_2^n \\ v_3^n \end{pmatrix}\right)$. Then we have

$$|v_2^n(t) - S^0| = |S^n(t) - S^0| \leq \frac{1}{n+1} \qquad \text{for } t \geq 0.$$

Moreover, the map $t \mapsto \begin{pmatrix} 0 \\ v_1^n \end{pmatrix}$ is an integral solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} 0 \\ v_1^n(t,\cdot) \end{pmatrix} = \mathcal{A}_1 \begin{pmatrix} 0 \\ v_1^n \end{pmatrix} + \begin{pmatrix} v_2^n(t)\int_0^\infty \beta(a) v_1^n(t,a)da \\ 0 \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geq 0, \\ \begin{pmatrix} 0 \\ v_1^n(0,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ v_{10}^n \end{pmatrix}. \end{cases}$$

Since $v_2^n(t) \geq S^0 - \frac{1}{n+1}$, by the comparison principle, we deduce that

$$v_1^n(t,\cdot) \geq \widehat{v_1^n}(t,\cdot), \tag{19}$$

where $\widehat{v_1^n}(t,\cdot)$ is a solution of the linear abstract ordinary differential equation

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} 0 \\ v_1^n(t,\cdot) \end{pmatrix} = \mathcal{A}_1 \begin{pmatrix} 0 \\ v_1^n \end{pmatrix} + \begin{pmatrix} \left(S^0 - \frac{1}{n+1}\right)\int_0^\infty \beta(a)\widehat{v_1^n}(t,a)da \\ 0 \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geq 0, \\ \begin{pmatrix} 0 \\ v_1^n(0,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ v_{10}^n \end{pmatrix}. \end{cases} \tag{20}$$

We can easily see that, for all n large enough, the dominated eigenvalue of the linear equation (20) satisfies the characteristic equation

$$\widehat{K}(\lambda_{0n})\left(S^0 - \frac{1}{n+1}\right) = 1.$$

Since $\mathscr{R}_0 > 1$, using similar arguments as in the proof of Theorem 4.5, we can easily show that $\lambda_{0n} > 0$ for all n large enough. Since $v_0^n \in M_0$, we have $\Pi(v_0^n) = v_{10}^n(a) \neq 0$. Thus

$$\lim_{t\to\infty} \|\widehat{v_1^n}(t,\cdot)\| = \lim_{t\to\infty} \|v_{10}^n e^{\lambda_{0n}t}\| = \infty$$

and hence it follows from (19) that

$$\lim_{t\to\infty} \|v_1^n(t,\cdot)\| \geq \lim_{t\to\infty} \|\widehat{v_1^n}(t,\cdot)\| = \infty.$$

This contradicts with (18) and the proof is complete. $\square$

4.4. **Stability of the endemic equilibrium P^*.** As noted earlier, P^* exists if and only if $\mathscr{R}_0 > 1$.

Theorem 4.10. *The endemic equilibrium P^* is locally asymptotically stable if $\mathscr{R}_0 > 1$ and $\mu > \delta$.*

Proof. The characteristic equation of (1) at the endemic equilibrium is

$$\overline{S}\widehat{K}(\lambda) = 1 + \frac{\int_0^\infty \beta(a)\bar{i}(a)da}{\lambda + \mu}\left(1 - \frac{\delta\widehat{K_1}(\lambda)}{\lambda + \mu + \delta}\right)$$

or

$$\frac{\widehat{K}(\lambda)}{K} = 1 + \frac{\int_0^\infty \beta(a)\bar{i}(a)da}{\lambda + \mu}\left(1 - \frac{\delta\widehat{K_1}(\lambda)}{\lambda + \mu + \delta}\right) \tag{21}$$

as $\overline{S} = 1/K$. To show that P^* is locally asymptotically stable if $\mathscr{R}_0 > 1$, it suffices to show that all roots of (21) have negative real parts. The proof is based on the observation that roots of (21) depend continuously on δ. We complete the proof in three steps.

Step 1. Show that (21) with $\delta = 0$ has no roots with nonnegative real parts. When $\delta = 0$, (21) reduces to

$$\frac{\widehat{K}(\lambda)}{K} = 1 + \frac{\int_0^\infty \beta(a)\bar{i}(a)da}{\lambda + \mu}. \tag{22}$$

By way of contradiction, assume that (22) has a root $\lambda_0 = u_0 + iv_0$ with $u_0 \geq 0$. Then substitute λ_0 into (22) and equate the real parts to get

$$\frac{\int_0^\infty \beta(a)\pi(a)e^{-(\mu+u_0)a}\cos(v_0 a)da}{K} = 1 + \frac{(\mu + u_0)\int_0^\infty \beta(a)\bar{i}(a)da}{(u_0 + \mu)^2 + v_0^2},$$

which is clearly impossible as the left hand side is less than 1 while the right hand side is larger than 1.

Step 2. Show that (21) has no roots on the imaginary axis. Again, by way of contradiction, suppose that iw is a root of (23). For the simplicity of notation, denote $\int_0^\infty \beta(a)\bar{i}(a)da$ by C. Then we have

$$\lambda + \mu + C = (\lambda + \mu)\bar{S}\widehat{K}(\lambda) + C\frac{\delta\widehat{K_1}(\lambda)}{\lambda + \mu + \delta}. \tag{23}$$

Then

$$\begin{aligned} \mu + C + iw = \quad & (iw + \mu)\bar{S}\int_0^\infty \beta(a)\pi(a)e^{-\mu a}e^{-iwa}da \\ & + \frac{C\delta\int_0^\infty k(a)\pi(a)e^{-\mu a}e^{-iwa}da}{iw + \mu + \delta}. \end{aligned} \tag{24}$$

Note that the modulus of the left hand side of (24) is $\sqrt{(\mu + C)^2 + w^2}$ while the modulus of the right hand side of it is

$$\begin{aligned} & \left|(iw + \mu)\bar{S}\int_0^\infty \beta(a)\pi(a)e^{-\mu a}e^{-iwa}da + \frac{C\delta\int_0^\infty k(a)\pi(a)e^{-\mu a}e^{-iwa}da}{iw + \mu + \delta}\right| \\ \le \quad & \sqrt{\mu^2 + w^2} + \frac{C\delta}{\sqrt{(\mu + \delta)^2 + w^2}} \\ \le \quad & \sqrt{\mu^2 + w^2} + \frac{C\delta}{\sqrt{\mu^2 + w^2}} \\ = \quad & \sqrt{\mu^2 + w^2 + \frac{C^2\delta^2}{\mu^2 + w^2} + 2C\delta} \\ < \quad & \sqrt{(\mu + C)^2 + w^2} \qquad \text{when } \mu > \delta. \end{aligned}$$

Thus we have a contradiction and this proves that (21) has no roots on the imaginary axis.

Step 3. Show all roots of (21) has negative real parts. By way of contradiction, suppose that (21) has a root with nonnegative real part. Then the real part of it must be positive by the result of Step 2. By the continuous dependence of roots of (21) on δ, there exists a $\delta_0 \in (0, \delta)$ such that (21) with $\delta = \delta_0$ has a root on the imaginary axis, a contradiction to the result in Step 2 again. This completes the proof. $\square$

Finally, we study the global stability of P^*. The following result can be easily proved by applying Theorem 4.9 and Lemma 3.1 of [16].

Lemma 4.11. *There exist constants $M > \varepsilon > 0$ such that for each complete orbit* $\left\{\left(\begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}\right)\right\}_{t\in\mathbb{R}}$ *of U in $\mathscr{A}_0$ we have*

$$\varepsilon \le S(t), R(t), \int_0^\infty \beta(a)i(t,a)da \le M \qquad \textit{for } t \in \mathbb{R}.$$

By using Volterra's formulation of the solution, we have

$$i(t, a) = B(t - a)e^{-\mu a}\pi(a),$$

where $B(t) = S(t)\int_0^\infty \beta(a)i(t,a)da$. Furthermore,

$$\frac{i(t,a)}{\bar{i}(a)} = \frac{B(t-a)}{\bar{i}(0)} = \frac{S(t-a)\int_0^\infty \beta(\tau)i(t-\tau,\tau)d\tau}{\bar{i}(a)}$$

and hence

$$\frac{\varepsilon^2}{\bar{i}(0)} \le \frac{i(t,a)}{\bar{i}(a)} \le \frac{M^2}{\bar{i}(0)}.$$

The following result tells us that P^* is globally asymptotically stable under some additional conditions.

Theorem 4.12. *Let $\mathscr{R}_0 > 1$ and $\delta < \mu$. If $\mu\overline{S} > \delta\overline{R}$ and $k(a) = k$, then the endemic equilibrium P^* is globally asymptotically stable.*

Proof. By Theorem 4.10, it suffices to show that P^* is globally attractive. This is achieved by using the Lyapunov functional approach. To construct the Lyapunov functional, we first introduce a function $g := (0,\infty) \ni u \mapsto u - 1 - \ln u$. It is well known that g is nonnegative and attains its minimum value 0 only at $u = 1$. Also, for the simplicity of notation, we denote $x = S/\overline{S}$ and $z = R/\overline{R}$. Then we construct the Lyapunov functional as follows,

$$U(t) = U_S(t) + U_i(t) + U_R(t),$$

where $U_S(t) = g(x)$, $U_i(t) = \int_0^\infty \Theta(a) g(\frac{i(t,a)}{\bar{i}(a)})da$, $\Theta(a) = \int_a^\infty \beta(l)\bar{i}(l)dl$, $U_R(t) = \delta(R(t) - \overline{R})^2/(2k\overline{S}^2)$.

First, we have

$$\begin{aligned}
\frac{dU_S(t)}{dt} &= \left(1 - \frac{1}{x}\right)\frac{1}{\overline{S}}\frac{dS}{dt} \\
&= \left(1 - \frac{1}{x}\right)\frac{1}{\overline{S}}\left(\Lambda - \mu S - S(t)\int_0^\infty \beta(a)i(t,a)da + \delta R\right) \\
&= \left(1 - \frac{1}{x}\right)\frac{1}{\overline{S}}\Bigg[\left(\mu\overline{S} + \overline{S}\int_0^\infty \beta(a)\bar{i}(t,a)da - \delta\overline{R}\right) \\
&\quad -\mu S - S(t)\int_0^\infty \beta(a)i(t,a)da + \delta R\Bigg] \\
&= -\mu\left(1 - \frac{1}{x}\right)(x-1) + \frac{\delta\overline{R}}{\overline{S}}\left(1 - \frac{1}{x}\right)(z-1) \\
&\quad + \int_0^\infty \beta(a)\bar{i}(a)\left(1 - \frac{1}{x} - x\frac{i(t,a)}{\bar{i}(a)} + \frac{i(t,a)}{\bar{i}(a)}\right)da.
\end{aligned}$$

Then due to the integral transformation and character of $\Theta(a)$ we have

$$\frac{dU_i}{dt} = \int_0^\infty \beta(a)\bar{i}(a)\left[\frac{i(t,0)}{\bar{i}(0)} - \ln\frac{i(t,0)}{\bar{i}(0)} - \frac{i(t,a)}{\bar{i}(a)} + \ln\frac{i(t,a)}{\bar{i}(a)}\right]da.$$

Note that

$$\begin{aligned}
\int_0^\infty \beta(a)\bar{i}(a)\left(\tfrac{i(t,0)}{\bar{i}(0)} - x\tfrac{i(t,a)}{\bar{i}(a)}\right)da &= \tfrac{i(t,0)}{\bar{i}(0)}\int_0^\infty \beta(a)\bar{i}(a)da - x\int_0^\infty \beta(a)i(t,a)da \\
&= \frac{i(t,0)}{\overline{S}} - \frac{i(t,0)}{\overline{S}} \\
&= 0.
\end{aligned}$$

Thus

$$\begin{aligned}\frac{d(U_S(t)+U_i(t))}{dt} &= -\mu\left(1-\frac{1}{x}\right)(x-1)+\frac{\delta\overline{R}}{\overline{S}}\left(1-\frac{1}{x}\right)(z-1)\\ &\quad+\int_0^\infty \beta(a)\bar{i}(a)\left[1-\frac{1}{x}-\ln\frac{i(t,0)}{\bar{i}(0)}+\ln\frac{i(t,a)}{\bar{i}(a)}\right]da\\ &= -\mu\left(1-\frac{1}{x}\right)(x-1)+\frac{\delta\overline{R}}{\overline{S}}\left(1-\frac{1}{x}\right)(z-1)\\ &\quad-\int_0^\infty \beta(a)\bar{i}(a)\left[g\left(\frac{1}{x}\right)+g\left(x\frac{xi(t,a)\bar{i}(0)}{\bar{i}(a)i(t,0)}\right)\right]da.\end{aligned}$$

Moreover, it follows from $k(a)=k$, $\frac{\Lambda}{\mu}=\overline{S}+\frac{\mu+\delta+k}{k}\overline{R}$, and $I(t)=\int_0^\infty i(t,a)da=\frac{\Lambda}{\mu}-S(t)-R(t)$ that

$$\begin{aligned}\frac{dU_R(t)}{dt} &= \frac{\delta}{k\overline{S}^2}(R(t)-\overline{R})\left[k\left(\frac{\Lambda}{\mu}-S(t)-R(t)\right)-(\mu+\delta)R(t)\right]\\ &= \frac{\delta}{k\overline{S}^2}(R(t)-\overline{R})[-k(S(t)-\overline{S})-(\mu+\delta+k)(R(t)-\overline{R})]\\ &= -\frac{\delta\overline{R}}{\overline{S}}(z-1)(x-1)-\frac{\delta\overline{R}^2(\mu+\delta+k)}{k\overline{S}}(z-1)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dU(t)}{dt} &= -\mu\left(1-\frac{1}{x}\right)(x-1)+\frac{\delta\overline{R}}{\overline{S}}(z-1)\left(2-x-\frac{1}{x}\right)\\ &\quad-\frac{\delta\overline{R}^2}{k\overline{S}}(\mu+k+\delta)(z-1)^2\\ &\quad-\int_0^\infty \beta(a)\bar{i}(a)\left[g\left(\frac{1}{x}\right)+g\left(x\frac{i(t,a)}{\bar{i}(a)}\frac{\bar{i}(0)}{i(t,0)}\right)\right]da\\ &= \left[\mu+\frac{\delta\overline{R}}{\overline{S}}(z-1)\right]\left(2-x-\frac{1}{x}\right)-\frac{\delta\overline{R}^2}{k\overline{S}}(\mu+k+\delta)(z-1)^2\\ &\quad-\int_0^\infty \beta(a)\bar{i}(a)\left[g\left(\frac{1}{x}\right)+g\left(x\frac{i(t,a)}{\bar{i}(a)}\frac{\bar{i}(0)}{i(t,0)}\right)\right]da\\ &\le \left[\mu-\frac{\delta\overline{R}}{\overline{S}}\right]\left(2-x-\frac{1}{x}\right)-\frac{\delta\overline{R}^2}{k\overline{S}}(\mu+k+\delta)(z-1)^2\\ &\quad-\int_0^\infty \beta(a)\bar{i}(a)\left[g\left(\frac{1}{x}\right)+g\left(x\frac{i(t,a)}{\bar{i}(a)}\frac{\bar{i}(0)}{i(t,0)}\right)\right]da\\ &\le 0.\end{aligned}$$

Let M be the largest invariant set of $\{(S(t),i(t,a),R(t)):\frac{dU(t)}{dt}=0\}$. We show that $M=\{P^*\}$. Obviously, $\{P^*\}\subseteq M$. Now, since $g(u)=0$ if and only if $u=1$, we have that $\frac{dU(t)}{dt}=0$ if and only if

$$S(t)=\overline{S},\qquad R(t)=\overline{R},\qquad \text{and}\qquad \frac{i(t,a)}{\bar{i}(a)}\frac{\bar{i}(0)}{i(t,0)}=1 \text{ for } a\in(0,\infty). \tag{25}$$

Then $\frac{dS(t)}{dt} = 0$. This, combined with (25), yields

$$\begin{aligned} 0 &= \frac{dS}{dt} = \Lambda - \mu\overline{S} - \overline{S}\int_0^\infty \beta(a)i(t,a)da + \delta\overline{R} \\ &= \overline{S}\int_0^\infty \beta(a)(\overline{i}(a) - i(t,a))da \\ &= \overline{S}\int_0^\infty \beta(a)(\overline{i}(a) - i(t,0)e^{-\mu a}\pi(a))da \\ &= \overline{S}\int_0^\infty \beta(a)\overline{i}(a)da - i(t,0)\overline{S}\int_0^\infty \beta(a)e^{-\mu a}\pi(a)da \\ &= \overline{i}(0) - i(t,0), \end{aligned}$$

or $i(t,0) = \overline{i}(0)$. It follows from (25) that $i(t,a) = \overline{i}(a)$ for $a \in (0,\infty)$. This proves that $M \subseteq \{P^*\}$. Therefore, $M = \{P^*\}$ and it follows that P^* is globally attractive. This completes the proof. □

5. **Discussion.** In the study of the global behavior of diseases, prevalence has played a vital role on predicting the dynamics of the disease transmission in the long run and taking more efficient control measures such as vaccination and curetment for immunization in the communicable diseases. In particular, the global stability of the epidemic model with infection age becomes much more interesting from the realistic views to theoretical views.

In this paper, we obtain the global asymptotic stability of the disease free equilibrium P_0 by using integral semigroup theory and fluctuation lemma, that is, the disease will die out when the basic reproduction number $\mathscr{R}_0 < 1$. When $\mathscr{R}_0 > 1$, the disease is persistent and the endemic equilibrium is also globally asymptotically stable under some additional conditions.

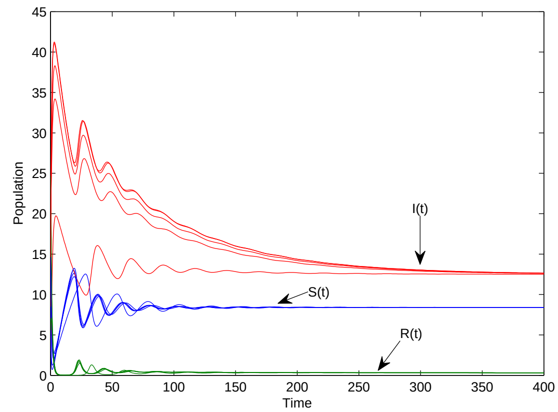
Let's reexamine the global stability conditions on the endemic equilibrium. It is easy to see from the expressions of $\overline{S}$, $\overline{i}(0)$, and $\overline{R}$ that

$$\begin{aligned} \mu - \frac{\delta\overline{R}}{\overline{S}} &= \mu - \frac{\delta\overline{i}(0)K_1}{(\mu+\delta)\overline{S}} = \mu\left(1 - \frac{\delta K_1}{\mu+\delta}\frac{\mathscr{R}_0 - 1}{1 - \frac{\delta K_1}{\mu+\delta}}\right) \\ &= \mu\frac{\mu+\delta-\delta K_1\mathscr{R}_0}{\mu+\delta(1-K_1)} = \frac{\mu}{\mu+\delta(1-K_1)}[\mu - \delta(K_1\mathscr{R}_0 - 1)]. \end{aligned}$$

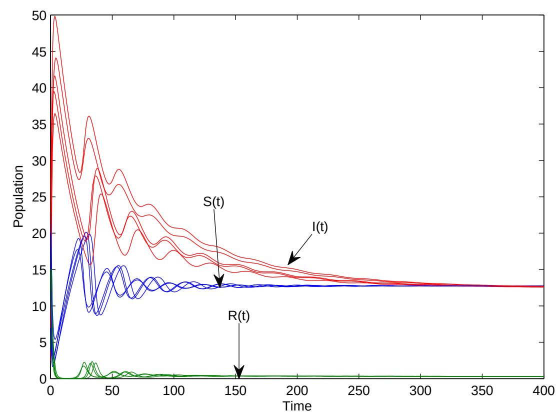
Hence $\mu - \frac{\delta\overline{R}}{\overline{S}} > 0$ is equivalent to $\delta \in \mathbb{R}_+$ if $1 < \mathscr{R}_0 \leq \frac{1}{K_1}$ or $\delta \in [0,\overline{\delta}] \triangleq [0, \frac{\mu}{K_1\mathscr{R}_0 - 1}]$ if $\mathscr{R}_0 > \frac{1}{K_1}$. In particular, if $k(a) = k$, then $K_1 = \frac{k}{\mu+k}$. It follows that the disease will eventually tend to the endemic equilibrium if the basic reproduction number lies in the interval determined by the death rate and the cure rate for any rate of immunity loss and any duration of the infection; if the basic reproduction number is larger than $\frac{1}{K_1}$ then there exists a maximal rate of immunity loss which blends the stability of the endemic equilibrium.

To illustrate our theoretical results, we choose $\Lambda = 0.16$, $k(a) = 1.3$, $\mu = 0.0125$, $\delta = 0.02$. First, let

$$\beta(a) = \begin{cases} 0.17, & a \geq 0.065 \\ 0, & 0 \leq a \leq 0.065. \end{cases}$$



(a)



(b)

FIGURE 1. The global stability of the endemic equilibrium with stochastic initial data

Then we obtain $\mathscr{R}_0 = 1.5223 > 1/K_1 = 1.0096$ and $\delta < \overline{\delta} = 0.026$. From Theorem 4.12, the endemic equilibrium P_* is globally asymptotically stable (see Fig. 1(a)).

Next, let

$$\beta(a) = \begin{cases} 0.112, & a \geq 0.065, \\ 0, & 0 \leq a \leq 0.065. \end{cases}$$

It is easy to obtain $\mathscr{R}_0 = 1.0029 < 1/K_1 = 1.0096$. By Theorem 4.12 again, the endemic equilibrium is globally stable (see Fig. 1(b)).

Acknowledgments. We would like to thank the anonymous reviewers for their constructive comments, which greatly improve the presentation of this paper. This work was done when Dr. Yang was a postdoctoral fellow at the Department of

Mathematics, Wilfrid Laurier University. He would like to thank the Department for the hospitality.

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Received November 03, 2012; Accepted October 24, 2013.

E-mail address: ychen@wlu.ca (YC)
E-mail address: yangjunyuan00@126.com (JY)
E-mail address: zhafq@263.net (FZ)