

A NON-AUTONOMOUS STOCHASTIC PREDATOR-PREY MODEL

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Dedicated to the memory of our “Maestro” Professor Luigi M. Ricciardi

ABSTRACT. The aim of this paper is to consider a non-autonomous predator-prey-like system, with a Gompertz growth law for the prey. By introducing random variations in both prey birth and predator death rates, a stochastic model for the predator-prey-like system in a random environment is proposed and investigated. The corresponding Fokker-Planck equation is solved to obtain the joint probability density for the prey and predator populations and the marginal probability densities. The asymptotic behavior of the predator-prey stochastic model is also analyzed.

1. Introduction. During the past four decades, several deterministic and stochastic predator-prey models have been proposed and applied in a wide range of fields, including tumor cells (virus)-immune system, susceptible-infectious diseases, parasite-host interactions, plant-herbivore systems (cf., for instance, [1], [2], [5]–[13], [16], [19], [22]–[24]). In particular, a non linear two dimensional dynamical system is considered in [8] and [13]:

$$\begin{aligned}\frac{dx}{dt} &= \gamma x - x \left(\varrho \frac{x^\xi - 1}{\xi} + \alpha \frac{y^\xi - 1}{\xi} \right), \\ \frac{dy}{dt} &= -\eta y + \beta y \frac{x^\xi - 1}{\xi},\end{aligned}\tag{1}$$

where $x(t)$ and $y(t)$ are the population densities of preys and predators, respectively, with $x(0) = x_0$, $y(0) = y_0$, and $\alpha, \beta, \eta, \gamma, \xi \in \mathbb{R}^+$, $\varrho \geq 0$.

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For $\xi = 1$ the system (1) becomes:

$$\begin{aligned}\frac{dx}{dt} &= x(r - \varrho x) - \alpha xy, \\ \frac{dy}{dt} &= -sy + \beta xy,\end{aligned}\tag{2}$$

with $r = \gamma + \varrho + \alpha$ and $s = \eta + \beta$. By setting $\varrho = 0$ in (2), we obtain the Lotka-Volterra model (cf. [9], [20]) in which the prey population will grow without limit in absence of the predators (malthusian growth of prey population), contrary to what is expected in a more realistic predator-prey ecosystem. The constant α tells how rapidly the prey population would die out through encounters with predators, and β is the constant increasing rate for the predators due to encounters with the preys. This model has no asymptotic stability and the equations admit periodic solutions oscillating around its equilibrium values $x = s/\beta$ and $y = r/\alpha$. Differently from Lotka-Volterra model, for $\varrho > 0$ a self-regulation term $-\varrho x^2$ is added to the prey equation, so that system (2) admits an asymptotic stable equilibrium at $x = s/\beta$ and $y = (\beta r - \varrho s)/(\alpha\beta)$.

Note that system (1) extends the Lotka-Volterra predator-prey system. The case $\xi < 1$ in (1) represents a situation in which the prey population adapts themselves somewhat to the growing menace of predator population so that it is affected to a lesser degree compared to Lotka-Volterra case. The case $\xi > 1$ in (1) represents a situation in which the prey population becomes exhausted by the predators so that they suffer to a greater degree from the increase in predator population compared to Lotka-Volterra case.

For $\xi \rightarrow 0$ the system (1) leads to:

$$\begin{aligned}\frac{dx}{dt} &= (\gamma - \varrho \ln x) x - \alpha x \ln y, \\ \frac{dy}{dt} &= -\eta y + \beta y \ln x,\end{aligned}\tag{3}$$

with $x(0) = x_0$, $y(0) = y_0$. Such a predator-prey-like system shows that the prey population has the property of self-regulation; it is similar to (2), but the associated set of differential equations can be solved exactly. The prey population grows with a Gompertz law in the absence of the predators

$$x(t) = \exp\left\{\frac{\gamma}{\varrho} + \left(\ln x_0 - \frac{\gamma}{\varrho}\right) e^{-\varrho t}\right\},$$

where γ denotes the Gompertz intrinsic growth rate of the prey and $\exp(\gamma/\varrho)$ is the carrying capacity. The population of the predators, instead, decreases exponentially without the preys, i.e.

$$y(t) = y_0 e^{-\eta t},$$

where η is the death rate of the predator in the absence of the prey. Similarly to Lotka-Volterra system, for $\varrho = 0$ the model described by (3) has no asymptotic stability and the equations have periodic solutions oscillating around its equilibrium values $x = \exp\{\eta/\beta\}$ and $y = \exp\{\gamma/\alpha\}$. When $\varrho > 0$, system (3) admits an asymptotic stable equilibrium at $x = \exp\{\eta/\beta\}$ and $y = \exp\{(\beta\gamma - \varrho\eta)/(\alpha\beta)\}$.

The aim of this paper is to consider a non-autonomous predator-prey-like system described by the differential equations

$$\begin{aligned}\frac{dx}{dt} &= k(t) [(\gamma - \varrho \ln x)x - \alpha x \ln y], \\ \frac{dy}{dt} &= k(t) (-\eta y + \beta y \ln x),\end{aligned}\tag{4}$$

where $x(t)$ and $y(t)$ describe the densities of the prey and predator populations, respectively, with $x(0) = x_0$, $y(0) = y_0$. We assume that $\alpha, \beta, \eta, \gamma \in \mathbb{R}^+$, $\varrho \geq 0$ and $k(t)$ is a continuous positive function such that $\int_0^{+\infty} k(\tau) d\tau = +\infty$. System (4) provides a generalization of system (3); indeed, when $k(t) = 1$ we note that (4) identifies with (3). In (4) we suppose that the growth rate of prey, the death rate of the predator, the self-regulation rate and the interactive terms are proportional to $k(t)$. For instance, the choice of $k(t)$ as a time periodic function is equivalent to assuming that the rates and the interaction terms oscillate between a minimum and a maximum value.

In Section 2 the explicit solutions of (4) are obtained; furthermore, the state of the system at time $t + \Delta t$ is determined exactly in terms of the state of the system at time t . Similarly to the deterministic model (4), in Section 3 a stochastic model is proposed. Under suitable assumptions of random environment, we obtain a two-dimensional diffusion process $\{X(t), Y(t), t \geq 0\}$, where $X(t)$ and $Y(t)$ are two correlated stochastic processes, describing the prey and predator population densities, respectively. Furthermore, the joint probability density function $f(x, y, t | x_0, y_0)$, solution of a two-dimensional Fokker-Planck equation, is explicitly determined. In Section 4 the marginal probability densities for the prey and predator populations are obtained. Finally, the asymptotic behavior of the predator-prey stochastic model is analyzed in Section 5.

We want to dedicate the remainder of this paper to the memory of our late mentor, colleague and unforgettable friend, Luigi M. Ricciardi.

2. Deterministic time evolution. In this section we write the explicit solutions of deterministic system (4), by distinguishing the following cases: (i) $\varrho^2 - 4\alpha\beta > 0$, (ii) $\varrho^2 - 4\alpha\beta < 0$ and (iii) $\varrho^2 - 4\alpha\beta = 0$.

Case (i). For $\varrho^2 - 4\alpha\beta > 0$, the solutions of system (4) are:

$$\begin{aligned}x(t) &= \exp \left\{ \frac{\eta}{\beta} + \left[\frac{\lambda_1}{\lambda_1 - \lambda_2} \left(\ln x_0 - \frac{\eta}{\beta} \right) - \frac{\alpha}{\lambda_1 - \lambda_2} \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \right] e^{\lambda_1 \psi(t)} \right. \\ &\quad \left. - \left[\frac{\lambda_2}{\lambda_1 - \lambda_2} \left(\ln x_0 - \frac{\eta}{\beta} \right) - \frac{\alpha}{\lambda_1 - \lambda_2} \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \right] e^{\lambda_2 \psi(t)} \right\},\end{aligned}\tag{5}$$

$$\begin{aligned}y(t) &= \exp \left\{ \frac{\beta\gamma - \varrho\eta}{\alpha\beta} + \left[\frac{\beta}{\lambda_1 - \lambda_2} \left(\ln x_0 - \frac{\eta}{\beta} \right) - \frac{\lambda_2}{\lambda_1 - \lambda_2} \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \right] e^{\lambda_1 \psi(t)} \right. \\ &\quad \left. - \left[\frac{\beta}{\lambda_1 - \lambda_2} \left(\ln x_0 - \frac{\eta}{\beta} \right) - \frac{\lambda_1}{\lambda_1 - \lambda_2} \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \right] e^{\lambda_2 \psi(t)} \right\},\end{aligned}$$

where

$$\psi(t) = \int_0^t k(\tau) d\tau, \quad \lambda_1 = \frac{-\varrho + \sqrt{\varrho^2 - 4\alpha\beta}}{2}, \quad \lambda_2 = \frac{-\varrho - \sqrt{\varrho^2 - 4\alpha\beta}}{2}.\tag{6}$$

Since $\int_0^{+\infty} k(\tau) d\tau = +\infty$, prey and predator populations admit an asymptotic behavior:

$$\lim_{t \rightarrow +\infty} x(t) = \exp\left\{\frac{\eta}{\beta}\right\}, \quad \lim_{t \rightarrow +\infty} y(t) = \exp\left\{\frac{\beta\gamma - \varrho\eta}{\alpha\beta}\right\}. \quad (7)$$

Case (ii). For $\varrho^2 - 4\alpha\beta < 0$, by setting

$$z(t) = \frac{1}{2}\psi(t)\sqrt{4\alpha\beta - \varrho^2}, \quad (8)$$

the solutions of (4) are:

$$\begin{aligned} x(t) = \exp\left\{\frac{\eta}{\beta} + e^{-\varrho\psi(t)/2} \left[\left(\ln x_0 - \frac{\eta}{\beta}\right) \cos[z(t)] \right. \right. \\ \left. \left. - \frac{1}{\sqrt{4\alpha\beta - \varrho^2}} \left(\varrho \left(\ln x_0 - \frac{\eta}{\beta}\right) + 2\alpha \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta}\right) \right) \sin[z(t)] \right] \right\}, \end{aligned} \quad (9)$$

$$\begin{aligned} y(t) = \exp\left\{\frac{\beta\gamma - \varrho\eta}{\alpha\beta} + e^{-\varrho\psi(t)/2} \left[\left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta}\right) \cos[z(t)] \right. \right. \\ \left. \left. + \frac{1}{\sqrt{4\alpha\beta - \varrho^2}} \left(2\beta \left(\ln x_0 - \frac{\eta}{\beta}\right) + \varrho \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta}\right) \right) \sin[z(t)] \right] \right\}. \end{aligned}$$

For $\varrho > 0$, the system moves around the stable equilibrium state with a decreasing amplitude, and finally reaches the equilibrium state. When $\varrho = 0$ (malthusian growth of prey population) (9) become:

$$x(t) = \exp\left\{\frac{\eta}{\beta} + \left(\ln x_0 - \frac{\eta}{\beta}\right) \cos[\sqrt{\alpha\beta}\psi(t)] - \sqrt{\frac{\alpha}{\beta}} \left(\ln y_0 - \frac{\gamma}{\alpha}\right) \sin[\sqrt{\alpha\beta}\psi(t)]\right\}, \quad (10)$$

$$y(t) = \exp\left\{\frac{\gamma}{\alpha} + \left(\ln y_0 - \frac{\gamma}{\alpha}\right) \cos[\sqrt{\alpha\beta}\psi(t)] + \sqrt{\frac{\beta}{\alpha}} \left(\ln x_0 - \frac{\eta}{\beta}\right) \sin[\sqrt{\alpha\beta}\psi(t)]\right\},$$

showing that $x(t)$ and $y(t)$ become periodic functions when t increases.

Case (iii). For $\varrho^2 - 4\alpha\beta = 0$, i.e. $\varrho = 2\sqrt{\alpha\beta}$, the solutions of (4) are:

$$\begin{aligned} x(t) = \exp\left\{\frac{\eta}{\beta} + \left[\left(1 - \sqrt{\alpha\beta}\psi(t)\right) \left(\ln x_0 - \frac{\eta}{\beta}\right) \right. \right. \\ \left. \left. - \alpha\psi(t) \left(\ln y_0 - \frac{\beta\gamma - 2\eta\sqrt{\alpha\beta}}{\alpha\beta}\right) \right] e^{-\sqrt{\alpha\beta}\psi(t)}\right\}, \end{aligned} \quad (11)$$

$$\begin{aligned} y(t) = \exp\left\{\frac{\beta\gamma - 2\eta\sqrt{\alpha\beta}}{\alpha\beta} + \left[\beta\psi(t) \left(\ln x_0 - \frac{\eta}{\beta}\right) \right. \right. \\ \left. \left. + \left(1 + \sqrt{\alpha\beta}\psi(t)\right) \left(\ln y_0 - \frac{\beta\gamma - 2\eta\sqrt{\alpha\beta}}{\alpha\beta}\right) \right] e^{-\sqrt{\alpha\beta}\psi(t)}\right\}. \end{aligned}$$

Since $\int_0^{+\infty} k(\tau) d\tau = +\infty$, the prey and predator populations admit the asymptotic behavior (7).

3. Stochastic Model: Joint probability density function. The deterministic approach has some limitations in biology, in the sense that it is always difficult to predict the future of the system accurately. One of the reasons to this difficulty is that biological systems are subject to random fluctuations, that partially result from the environment factors such as epidemics and nature disasters. From this viewpoint, we denote by $\{X(t), Y(t), t \geq 0\}$ a two-dimensional stochastic process, where $X(t)$ and $Y(t)$ represent the prey and predator population densities, respectively. Under the assumption of random environment, we interpret the increments of prey birth rate ($\gamma \Delta t$) and of predator death rate ($\eta \Delta t$) in the time generation Δt as the components of a two-dimensional correlated Wiener process $\{W_1(t), W_2(t), t \geq 0\}$ such that $E[W_1(t)] = \gamma t$, $E[W_2(t)] = \eta t$, $\text{Var}[W_1(t)] = \sigma_1^2 t$, $\text{Var}[W_2(t)] = \sigma_2^2 t$ and $\text{Cov}[W_1(t), W_2(t)] = -\sigma_{12} t$, with $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_{12} \in \mathbb{R}$ and $\sigma_1^2 \sigma_2^2 - \sigma_{12}^2 > 0$. The stochastic processes $W_1(t)$ and $W_2(t)$ are respectively referred as the fluctuations of the prey birth rate and of the predator death rate, and they are correlated. The constants σ_1^2 and σ_2^2 are the intensities of noises and $\zeta = -\sigma_{12}/(\sigma_1 \sigma_2)$ denotes the correlation coefficient between $W_1(t)$ and $W_2(t)$.

Our approach follows the lines indicated in [3], [4], [14], [15], [17], [18] to describe the evolution of a single species in the random environment. In particular, in [4], [14], [15] and [18], starting from the solution of the differential equation (malthusian, logistic, Gompertz or other growth models), one-dimensional diffusion processes have been constructed, analyzed and compared.

For $i, j = 0, 1, \dots$, let

$$B_{ij}(x, y, t) = \lim_{\Delta t \rightarrow 0} \frac{E\{[X(t+\Delta t) - X(t)]^i [Y(t+\Delta t) - Y(t)]^j | X(t) = x, Y(t) = y\}}{\Delta t}$$

be the infinitesimal moments of a two-dimensional process $\{X(t), Y(t), t \geq 0\}$. In Appendix A, we prove that $\{X(t), Y(t), t \geq 0\}$ is a diffusion process characterized by the following infinitesimal moments:

$$\begin{aligned} B_{10}(x, y, t) &= xk(t) \left\{ \gamma - \rho \ln x - \alpha \ln y + \frac{1}{2} \sigma_1^2 k^2(t) \right\}, \\ B_{01}(x, y, t) &= yk(t) \left\{ -\eta + \beta \ln x + \frac{1}{2} \sigma_2^2 k^2(t) \right\}, \\ B_{20}(x, y, t) &= \sigma_1^2 k^2(t) x^2, \\ B_{02}(x, y, t) &= \sigma_2^2 k^2(t) y^2, \\ B_{11}(x, y, t) &= \sigma_{12} k^2(t) xy, \\ B_{ij}(x, y, t) &= 0 \quad (i, j = 0, 1, \dots; i + j > 2). \end{aligned} \tag{12}$$

We note that the drifts $B_{10}(x, y, t)$ and $B_{01}(x, y, t)$ and the infinitesimal variances $B_{20}(x, y, t)$ and $B_{02}(x, y, t)$ satisfy the relations:

$$\begin{aligned} B_{10}(x, y, t) &= h_1(x, y, t) + \frac{1}{4} \frac{\partial B_{20}(x, y, t)}{\partial x}, \\ B_{01}(x, y, t) &= h_2(x, y, t) + \frac{1}{4} \frac{\partial B_{02}(x, y, t)}{\partial y}, \end{aligned} \tag{13}$$

where $h_1(x, y, t)$ and $h_2(x, y, t)$ identify with the right-hand side of the equations of system (4), respectively. Then, the joint probability density function $f(x, y, t | x_0, y_0)$

of $\{X(t), Y(t), t \geq 0\}$ is solution of the Fokker-Planck equation:

$$\begin{aligned} \frac{\partial f}{\partial t} &= k(t) \frac{\partial}{\partial x} \left[\left\{ \gamma - \varrho \ln x - \alpha \ln y + \frac{\sigma_1^2}{2} k(t) \right\} x f \right] \\ &\quad - k(t) \frac{\partial}{\partial y} \left[\left\{ -\eta + \beta \ln x + \frac{\sigma_2^2}{2} k(t) \right\} y f \right] + \frac{1}{2} \sigma_1^2 k^2(t) \frac{\partial^2}{\partial x^2} \{x^2 f\} \\ &\quad + \frac{1}{2} \sigma_2^2 k^2(t) \frac{\partial^2}{\partial y^2} \{y^2 f\} + \sigma_{12} k^2(t) \frac{\partial^2}{\partial x \partial y} \{x y f\}, \end{aligned} \quad (14)$$

with the delta initial condition:

$$\lim_{t \rightarrow 0} f(x, y, t | x_0, y_0) = \delta(x - x_0) \delta(y - y_0). \quad (15)$$

In order to determine $f(x, y, t | x_0, y_0)$, we first of all carry out the positions:

$$u = \ln x - \frac{\eta}{\beta}, \quad v = \ln y - \frac{\beta\gamma - \varrho\eta}{\alpha\beta}, \quad u_0 = \ln x_0 - \frac{\eta}{\beta}, \quad v_0 = \ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta}. \quad (16)$$

Next, we consider the transformations

$$\begin{aligned} z &= c_1 \left(u - \frac{\lambda_2}{\beta} v \right), \quad w = c_2 \left(u - \frac{\lambda_1}{\beta} v \right), \quad z_0 = c_1 \left(u_0 - \frac{\lambda_2}{\beta} v_0 \right), \quad w_0 = c_2 \left(u_0 - \frac{\lambda_1}{\beta} v_0 \right), \\ f(x, y, t | x_0, y_0) &= \frac{|c_1 c_2 (\lambda_2 - \lambda_1)|}{\beta x y} \widehat{f}(z, w, t | z_0, w_0), \end{aligned} \quad (17)$$

with $c_1 c_2 (\lambda_2 - \lambda_1) \neq 0$. These allow you to change the Fokker-Planck equation (14) into the following:

$$\begin{aligned} \frac{\partial \widehat{f}}{\partial t} &= -\frac{\partial}{\partial z} \{ \lambda_1 k(t) z \widehat{f} \} - \frac{\partial}{\partial w} \{ \lambda_2 k(t) w \widehat{f} \} \\ &\quad + \frac{1}{2} c_1^2 \frac{D_1}{\beta^2} k^2(t) \frac{\partial^2 \widehat{f}}{\partial z^2} + \frac{1}{2} c_2^2 \frac{D_2}{\beta^2} k^2(t) \frac{\partial^2 \widehat{f}}{\partial w^2} + c_1 c_2 \frac{D_3}{\beta^2} k^2(t) \frac{\partial^2 \widehat{f}}{\partial z \partial w}, \end{aligned} \quad (18)$$

where we have set

$$\begin{aligned} D_1 &= \beta^2 \sigma_1^2 + \lambda_2^2 \sigma_2^2 - 2\beta \lambda_2 \sigma_{12}, \\ D_2 &= \beta^2 \sigma_1^2 + \lambda_1^2 \sigma_2^2 - 2\beta \lambda_1 \sigma_{12}, \\ D_3 &= \beta^2 \sigma_1^2 + \alpha \beta \sigma_2^2 + \varrho \beta \sigma_{12}. \end{aligned} \quad (19)$$

The delta condition (15) becomes:

$$\lim_{t \rightarrow 0} \widehat{f}(z, w, t | z_0, w_0) = \delta(z - z_0) \delta(w - w_0). \quad (20)$$

Equation (18), with condition (20), is the Fokker-Planck equation of a non-homogeneous Ornstein-Uhlenbeck process, whose solution $\widehat{f}(z, w, t | z_0, w_0)$ is a two-dimensional normal density. Multidimensional time-homogeneous Ornstein-Uhlenbeck process are take in account in [21].

In the sequel we set:

$$E_1(t) = \frac{e^{2\lambda_1 \psi(t)} - 1}{2\lambda_1}, \quad E_2(t) = \frac{e^{2\lambda_2 \psi(t)} - 1}{2\lambda_2}, \quad E_3(t) = \begin{cases} \frac{1 - e^{-\varrho \psi(t)}}{\varrho}, & \varrho > 0 \\ \psi(t), & \varrho = 0. \end{cases} \quad (21)$$

Proposition 1. *When $\varrho^2 - 4\alpha\beta > 0$, the joint probability density function of the diffusion process $\{X(t), Y(t), t \geq 0\}$ is:*

$$f(x, y, t|x_0, y_0) = \frac{\beta}{2\pi xy} \sqrt{\frac{\varrho^2 - 4\alpha\beta}{\varphi(t)}} \exp\left\{-\frac{\beta^2}{2\varphi(t)} \left[P_2(t)u^2 + Q_2(t)v^2 + P_1(t)u + Q_1(t)v + P_{12}(t)uv + P_0(t) \right]\right\}, \quad (22)$$

where we have set:

$$\varphi(t) = D_1 D_2 E_1(t) E_2(t) - D_3^2 E_3^2(t), \quad (23)$$

$$\begin{aligned} P_2(t) &= D_2 E_2(t) + D_1 E_1(t) - 2D_3 E_3(t), \\ Q_2(t) &= D_2 E_2(t) \left(\frac{\lambda_2}{\beta}\right)^2 + D_1 E_1(t) \left(\frac{\lambda_1}{\beta}\right)^2 - \frac{2\alpha}{\beta} D_3 E_3(t), \\ P_1(t) &= -2D_2 E_2(t) m_1(t|x_0, y_0) - 2D_1 E_1(t) m_2(t|x_0, y_0) \\ &\quad + 2D_3 E_3(t) m_2(t|x_0, y_0) + 2D_3 E_3(t) m_1(t|x_0, y_0), \\ Q_1(t) &= 2D_2 E_2(t) \frac{\lambda_2}{\beta} m_1(t|x_0, y_0) + 2D_1 E_1(t) \frac{\lambda_1}{\beta} m_2(t|x_0, y_0) \\ &\quad - 2D_3 E_3(t) \frac{\lambda_2}{\beta} m_2(t|x_0, y_0) - 2D_3 E_3(t) \frac{\lambda_1}{\beta} m_1(t|x_0, y_0), \\ P_{12}(t) &= -2D_2 E_2(t) \frac{\lambda_2}{\beta} - 2D_1 E_1(t) \frac{\lambda_1}{\beta} - 2D_3 \frac{\varrho}{\beta} E_3(t), \\ P_0(t) &= D_2 E_2(t) m_1^2(t|x_0, y_0) + D_1 E_1(t) m_2^2(t|x_0, y_0) \\ &\quad - 2D_3 E_3(t) m_1(t|x_0, y_0) m_2(t|x_0, y_0), \end{aligned} \quad (24)$$

and

$$\begin{aligned} m_1(t|x_0, y_0) &= \left[(\ln x_0 - \frac{\eta}{\beta}) - \frac{\lambda_2}{\beta} \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \right] e^{\lambda_1 \psi(t)}, \\ m_2(t|x_0, y_0) &= \left[(\ln x_0 - \frac{\eta}{\beta}) - \frac{\lambda_1}{\beta} \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \right] e^{\lambda_2 \psi(t)}. \end{aligned} \quad (25)$$

Proof. The proof is given in Appendix B. \square

Proposition 2. *When $\varrho^2 - 4\alpha\beta < 0$ the joint probability density function of the diffusion process $\{X(t), Y(t), t \geq 0\}$ is:*

$$f(x, y, t|x_0, y_0) = \frac{\beta}{2\pi xy} \sqrt{\frac{4\alpha\beta - \varrho^2}{\tilde{\varphi}(t)}} \exp\left\{-\frac{\beta^2}{2\tilde{\varphi}(t)} \left[\tilde{P}_2(t)u^2 + \tilde{Q}_2(t)v^2 + \tilde{P}_1(t)u + \tilde{Q}_1(t)v + \tilde{P}_{12}(t)uv + \tilde{P}_0(t) \right]\right\}, \quad (26)$$

where

$$\begin{aligned} \tilde{\varphi}(t) &= D_3^2 E_3^2(t) - D_1 D_2 E_1(t) E_2(t) \\ &= D_3^2 E_3^2(t) - \frac{D_1 D_2}{4\alpha\beta} \left\{ 1 + e^{-2\varrho\psi(t)} - 2e^{-\varrho\psi(t)} \cos[2z(t)] \right\}, \end{aligned} \quad (27)$$

$$\tilde{P}_j(t) = -P_j(t) \quad (j = 0, 1, 2), \quad \tilde{Q}_j(t) = -Q_j(t) \quad (j = 1, 2), \quad \tilde{P}_{12}(t) = -P_{12}(t). \quad (28)$$

Proof. The proof is given in Appendix C. \square

When $\varrho^2 - 4\alpha\beta = 0$, i.e. $\varrho = 2\sqrt{\alpha\beta}$, the joint probability density function of the diffusion process $\{X(t), Y(t), t \geq 0\}$ can be obtained taking the limit as $\varrho \rightarrow 2\sqrt{\alpha\beta}$ in (22) or, equivalently, in (26).

Proposition 3. For $\varrho = 2\sqrt{\alpha\beta}$ one has:

$$f(x, y, t|x_0, y_0) = \frac{\beta}{2\pi xy} \sqrt{\frac{1}{\phi(t)}} \exp\left\{-\frac{\beta^2}{2\phi(t)} \left[\Lambda_2(t)u^2 + \Omega_2(t)v^2 + \Lambda_1(t)u + \Omega_1(t)v + \Lambda_{12}(t)uv + \Lambda_0(t) \right]\right\}, \quad (29)$$

where:

$$\begin{aligned} \phi(t) &= \lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \frac{\varphi(t)}{\varrho^2 - 4\alpha\beta}, & \Lambda_j(t) &= \lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \frac{P_j(t)}{\varrho^2 - 4\alpha\beta}, \quad (j = 0, 1, 2), \\ \Omega_j(t) &= \lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \frac{Q_j(t)}{\varrho^2 - 4\alpha\beta}, \quad (j = 1, 2), & \Lambda_{12}(t) &= \lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \frac{P_{12}(t)}{\varrho^2 - 4\alpha\beta}. \end{aligned} \quad (30)$$

In particular, from (30) one has

$$\begin{aligned} \phi(t) &= \frac{(\beta\sigma_1^2 + \alpha\sigma_2^2)(\beta\sigma_1^2 + \alpha\sigma_2^2 + 4\sqrt{\alpha\beta}\sigma_{12}) + 4\alpha\beta\sigma_1^2\sigma_2^2}{16\alpha^2} \\ &\quad \times \left(1 - e^{-2\sqrt{\alpha\beta}\psi(t)}\right)^2 - \frac{\beta\psi^2(t) (\beta\sigma_1^2 + \alpha\sigma_2^2 + 2\sigma_{12}\sqrt{\alpha\beta})^2}{4\alpha} e^{-2\sqrt{\alpha\beta}\psi(t)}, \\ \Lambda_2(t) &= \frac{\beta\sigma_1^2 + 5\alpha\sigma_2^2 + 4\sigma_{12}\sqrt{\alpha\beta}}{4\alpha\sqrt{\alpha\beta}} \left(1 - e^{-2\sqrt{\alpha\beta}\psi(t)}\right) - e^{-2\sqrt{\alpha\beta}\psi(t)} \\ &\quad \times \left\{ \psi(t) \left(\frac{\beta}{2\alpha}\sigma_1^2 + \frac{3}{2}\sigma_2^2 + 2\sqrt{\frac{\beta}{\alpha}}\sigma_{12} \right) + \beta\psi^2(t) \frac{\beta\sigma_1^2 + \alpha\sigma_2^2 + 2\sqrt{\alpha\beta}\sigma_{12}}{2} \right\}, \\ \Omega_2(t) &= \frac{\beta\sigma_1^2 + \alpha\sigma_2^2}{4\beta\sqrt{\alpha\beta}} \left(1 - e^{-2\sqrt{\alpha\beta}\psi(t)}\right) + e^{-2\sqrt{\alpha\beta}\psi(t)} \\ &\quad \times \left\{ \psi(t) \frac{\beta\sigma_1^2 - \alpha\sigma_2^2}{2\beta} - \sqrt{\frac{\alpha}{\beta}}\psi^2(t) \frac{\beta\sigma_1^2 + \alpha\sigma_2^2 + 2\sqrt{\alpha\beta}\sigma_{12}}{2} \right\}. \end{aligned} \quad (31)$$

From (31), it follows that $\phi(t) > 0$, $\Lambda_2(t) > 0$ and $\Omega_2(t) > 0$ for $t > 0$.

In the next section, in order to determine the probability densities of prey and predator populations, the densities (22), (26) and (29) are utilized.

4. Stochastic Model: Marginal probability densities. Let $f_X(x, t|x_0, y_0)$ and $f_Y(y, t|x_0, y_0)$ the probability densities of prey and predator populations:

$$f_X(x, t|x_0, y_0) = \int_0^{+\infty} f(x, y, t|x_0, t_0) dy, \quad f_Y(y, t|x_0, y_0) = \int_0^{+\infty} f(x, y, t|x_0, t_0) dx. \quad (32)$$

In the sequel, we determine the marginal densities, the averages, the medians and the coefficients of variation of prey and predator populations.

Proposition 4. For $\varrho^2 - 4\alpha\beta > 0$, one has

$$f_X(x, t|x_0, y_0) = \frac{1}{x\sqrt{2\pi}} \sqrt{\frac{\varrho^2 - 4\alpha\beta}{Q_2(t)}} \exp\left\{-\frac{\varrho^2 - 4\alpha\beta}{2Q_2(t)} \left[\ln x - \mu_X(t|x_0, y_0)\right]^2\right\}, \quad (33)$$

$$f_Y(y, t|x_0, y_0) = \frac{1}{y\sqrt{2\pi}} \sqrt{\frac{\varrho^2 - 4\alpha\beta}{P_2(t)}} \exp\left\{-\frac{\varrho^2 - 4\alpha\beta}{2P_2(t)} \left[\ln y - \mu_Y(t|x_0, y_0)\right]^2\right\},$$

where $P_2(t)$, $Q_2(t)$ are defined in (24) and

$$\mu_X(t|x_0, y_0) \equiv \ln x(t), \quad \mu_Y(t|x_0, y_0) \equiv \ln y(t), \quad (34)$$

with $x(t)$ and $y(t)$ given in (5).

Proof. Making use of (22) in (32), one obtains:

$$f_X(x, t|x_0, y_0) = \frac{1}{x\sqrt{2\pi}} \sqrt{\frac{\varrho^2 - 4\alpha\beta}{Q_2(t)}} \exp\left\{-\frac{\beta^2}{2\varphi(t)} \left[H_1(t)u^2 + H_2(t)u + H_3(t)\right]\right\}, \quad (35)$$

$$f_Y(y, t|x_0, y_0) = \frac{1}{y\sqrt{2\pi}} \sqrt{\frac{\varrho^2 - 4\alpha\beta}{P_2(t)}} \exp\left\{-\frac{\beta^2}{2\varphi(t)} \left[K_1(t)v^2 + K_2(t)v + K_3(t)\right]\right\},$$

with u, v are defined in (16) and

$$\begin{aligned} H_1(t) &= P_2(t) - \frac{P_{12}^2(t)}{4Q_2(t)} = \frac{\varphi(t)(\varrho^2 - 4\alpha\beta)}{\beta^2 Q_2(t)}, \\ H_2(t) &= P_1(t) - \frac{Q_1(t)P_{12}(t)}{2Q_2(t)} = -\frac{2\varphi(t)\sqrt{\varrho^2 - 4\alpha\beta}}{\beta^2 Q_2(t)} [\lambda_1 m_1(t|x_0, y_0) - \lambda_2 m_2(t|x_0, y_0)], \\ H_3(t) &= P_0(t) - \frac{Q_1^2(t)}{4Q_2(t)} = \frac{\varphi(t)}{\beta^2 Q_2(t)} [\lambda_1 m_1(t|x_0, y_0) - \lambda_2 m_2(t|x_0, y_0)]^2, \end{aligned} \quad (36)$$

$$\begin{aligned} K_1(t) &= Q_2(t) - \frac{P_{12}^2(t)}{4P_2(t)} = \frac{\varphi(t)(\varrho^2 - 4\alpha\beta)}{\beta^2 P_2(t)}, \\ K_2(t) &= Q_1(t) - \frac{P_1(t)P_{12}(t)}{2P_2(t)} = -\frac{2\varphi(t)\sqrt{\varrho^2 - 4\alpha\beta}}{\beta P_2(t)} [m_1(t|x_0, y_0) - m_2(t|x_0, y_0)], \\ K_3(t) &= P_0(t) - \frac{P_1^2(t)}{4P_2(t)} = \frac{\varphi(t)}{P_2(t)} [m_1(t|x_0, y_0) - m_2(t|x_0, y_0)]^2. \end{aligned}$$

The right-hand sides of (36) are obtained making use of (19), (21), (23) and (24). Substituting (36) in (35) one obtains the marginal densities (33), with

$$\mu_X(t|x_0, y_0) = \frac{\eta}{\beta} + \frac{\lambda_1 m_1(t|x_0, y_0) - \lambda_2 m_2(t|x_0, y_0)}{\sqrt{\varrho^2 - 4\alpha\beta}}, \quad (37)$$

$$\mu_Y(t|x_0, y_0) = \frac{\beta\gamma - \varrho\eta}{\alpha\beta} + \frac{\beta [m_1(t|x_0, y_0) - m_2(t|x_0, y_0)]}{\sqrt{\varrho^2 - 4\alpha\beta}}.$$

Hence, by virtue of (25), the right-hand sides of (37) are identified with (34), with $x(t)$ and $y(t)$ given in (5). \square

Proposition 4 shows that the marginal densities of prey and predator populations are lognormal densities. Furthermore, the conditional medians of $X(t)$ and $Y(t)$ coincide with the solutions $x(t)$ and $y(t)$ of the deterministic system (4), i.e.

$$M[X(t)|x_0, y_0] = e^{\mu_X(t|x_0, y_0)} \equiv x(t), \quad M[Y(t)|x_0, y_0] = e^{\mu_Y(t|x_0, y_0)} \equiv y(t), \quad (38)$$

with $x(t), y(t)$ given in (5). The conditional means and the conditional coefficients of variation of prey and predator populations are:

$$E[X(t)|x_0, y_0] = x(t) \exp\left\{\frac{Q_2(t)}{2(\varrho^2 - 4\alpha\beta)}\right\}, \quad E[Y(t)|x_0, y_0] = y(t) \exp\left\{\frac{P_2(t)}{2(\varrho^2 - 4\alpha\beta)}\right\},$$

$$C[X(t)|x_0, y_0] = \sqrt{\exp\left\{\frac{Q_2(t)}{\varrho^2 - 4\alpha\beta}\right\} - 1}, \quad C[Y(t)|x_0, y_0] = \sqrt{\exp\left\{\frac{P_2(t)}{\varrho^2 - 4\alpha\beta}\right\} - 1},$$

with $P_2(t), Q_2(t)$ as in (24).

Figures 1 and 2 show the medians and the averages of prey and predator populations in the case $\varrho^2 - 4\alpha\beta > 0$ for $k(t) = 1$ and for $k(t) = 1 + 0.8 \sin(t)$, respectively, with the same choices of parameters. In these figures, the averages and the medians are moving toward the stable equilibrium state, and finally reach the equilibrium values represented by the dashed lines.

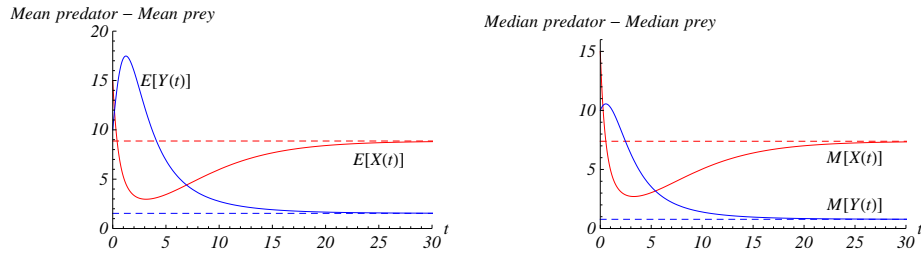


FIGURE 1. Averages and medians of prey and predator populations for $x_0 = 15, y_0 = 10, \alpha = 0.4, \beta = 0.3, \gamma = 1.5, \eta = 0.6, \varrho = 0.8, \sigma_1 = 0.5, \sigma_2 = 0.5, \sigma_{12} = 0.1, k(t) = 1$.

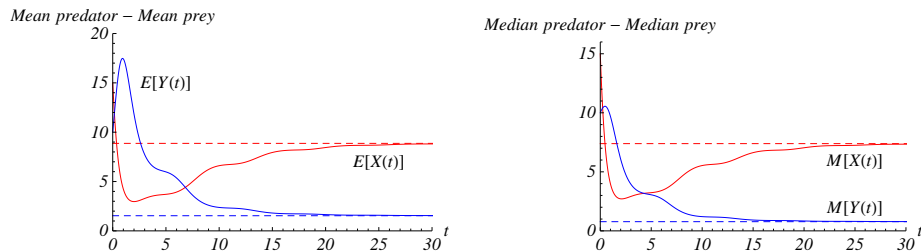


FIGURE 2. As in Figure 1 with $k(t) = 1 + 0.8 \sin(t)$.

Proposition 5. For $\varrho^2 - 4\alpha\beta < 0$, one has

$$f_X(x, t|x_0, y_0) = \frac{1}{x\sqrt{2\pi}} \sqrt{\frac{4\alpha\beta - \varrho^2}{\tilde{Q}_2(t)}} \exp\left\{-\frac{4\alpha\beta - \varrho^2}{2\tilde{Q}_2(t)} \left[\ln x - \mu_X(t|x_0, y_0)\right]^2\right\}, \quad (39)$$

$$f_Y(y, t|x_0, y_0) = \frac{1}{y\sqrt{2\pi}} \sqrt{\frac{4\alpha\beta - \varrho^2}{\tilde{P}_2(t)}} \exp\left\{-\frac{4\alpha\beta - \varrho^2}{2\tilde{P}_2(t)} \left[\ln y - \mu_Y(t|x_0, y_0)\right]^2\right\},$$

where $\tilde{P}_2(t)$, $\tilde{Q}_2(t)$ are given in (28), and where $\mu_X(t|x_0, y_0) = \ln x(t)$ and $\mu_Y(t|x_0, y_0) = \ln y(t)$, with $x(t)$ and $y(t)$ given in (9).

Proof. By virtue of (26), (32) lead to:

$$f_X(x, t|x_0, y_0) = \frac{1}{x\sqrt{2\pi}} \sqrt{\frac{4\alpha\beta - \varrho^2}{\tilde{Q}_2(t)}} \exp\left\{-\frac{\beta^2}{2\tilde{\varphi}(t)} \left[\tilde{H}_1(t)u^2 + \tilde{H}_2(t)u + \tilde{H}_3(t)\right]\right\}, \quad (40)$$

$$f_Y(y, t|x_0, y_0) = \frac{1}{y\sqrt{2\pi}} \sqrt{\frac{4\alpha\beta - \varrho^2}{\tilde{P}_2(t)}} \exp\left\{-\frac{\beta^2}{2\tilde{\varphi}(t)} \left[\tilde{K}_1(t)v^2 + \tilde{K}_2(t)v + \tilde{K}_3(t)\right]\right\},$$

with u, v defined in (16) and

$$\begin{aligned} \tilde{H}_1(t) &= \tilde{P}_2(t) - \frac{\tilde{P}_{12}^2(t)}{4\tilde{Q}_2(t)} = \frac{\tilde{\varphi}(t)(4\alpha\beta - \varrho^2)}{\beta^2\tilde{Q}_2(t)}, \\ \tilde{H}_2(t) &= \tilde{P}_1(t) - \frac{\tilde{Q}_1(t)\tilde{P}_{12}(t)}{2\tilde{Q}_2(t)} = \frac{2i\tilde{\varphi}(t)\sqrt{4\alpha\beta - \varrho^2}}{\beta^2\tilde{Q}_2(t)} [\lambda_1 m_1(t|x_0, y_0) - \lambda_2 m_2(t|x_0, y_0)], \\ \tilde{H}_3(t) &= \tilde{P}_0(t) - \frac{\tilde{Q}_1^2(t)}{4\tilde{Q}_2(t)} = -\frac{\tilde{\varphi}(t)}{\beta^2\tilde{Q}_2(t)} [\lambda_1 m_1(t|x_0, y_0) - \lambda_2 m_2(t|x_0, y_0)]^2, \end{aligned} \quad (41)$$

$$\begin{aligned} \tilde{K}_1(t) &= \tilde{Q}_2(t) - \frac{\tilde{P}_{12}^2(t)}{4\tilde{P}_2(t)} = \frac{\tilde{\varphi}(t)(4\alpha\beta - \varrho^2)}{\beta^2\tilde{P}_2(t)}, \\ \tilde{K}_2(t) &= \tilde{Q}_1(t) - \frac{\tilde{P}_1(t)\tilde{P}_{12}(t)}{2\tilde{P}_2(t)} = \frac{2i\tilde{\varphi}(t)\sqrt{4\alpha\beta - \varrho^2}}{\beta\tilde{P}_2(t)} [m_1(t|x_0, y_0) - m_2(t|x_0, y_0)], \\ \tilde{K}_3(t) &= \tilde{P}_0(t) - \frac{\tilde{P}_1^2(t)}{4\tilde{P}_2(t)} = -\frac{\tilde{\varphi}(t)}{\tilde{P}_2(t)} [m_1(t|x_0, y_0) - m_2(t|x_0, y_0)]^2. \end{aligned}$$

The right-hand sides of (41) are obtained making use of (19), (21), (27) and (28). Furthermore, by virtue of (25), we note that the following identities hold:

$$\begin{aligned} m_1(t|x_0, y_0) - m_2(t|x_0, y_0) &= i e^{-\varrho\psi(t)/2} \left\{ 2 \left(\ln x_0 - \frac{\eta}{\beta} \right) \sin[z(t)] \right. \\ &\quad \left. + \frac{1}{\beta} \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \left[\sqrt{4\alpha\beta - \varrho^2} \cos[z(t)] + \varrho \sin[z(t)] \right] \right\}, \end{aligned} \quad (42)$$

$$\begin{aligned} \lambda_1 m_1(t|x_0, y_0) - \lambda_2 m_2(t|x_0, y_0) &= i e^{-\varrho\psi(t)/2} \left\{ \left(\ln x_0 - \frac{\eta}{\beta} \right) \right. \\ &\quad \left. \times \left(\sqrt{4\alpha\beta - \varrho^2} \cos[z(t)] - \varrho \sin[z(t)] \right) - 2\alpha \left(\ln y_0 - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \sin[z(t)] \right\}, \end{aligned}$$

where $z(t)$ is defined in (8). Hence, making use of (41) and (42) in (40) we obtain (39), where $\mu_X(t|x_0, y_0) = \ln x(t)$ and $\mu_Y(t|x_0, y_0) = \ln y(t)$, with $x(t)$ and $y(t)$ given in (9). \square

Since (39) are lognormal densities, the conditional medians of $X(t)$ and $Y(t)$ coincide with the solutions $x(t)$ and $y(t)$ given in (9). Here, the conditional means and the conditional coefficients of variation of prey and predator populations are:

$$E[X(t)|x_0, y_0] = x(t) \exp\left\{\frac{\tilde{Q}_2(t)}{2(4\alpha\beta - \varrho^2)}\right\}, \quad E[Y(t)|x_0, y_0] = y(t) \exp\left\{\frac{\tilde{P}_2(t)}{2(4\alpha\beta - \varrho^2)}\right\},$$

$$C[X(t)|x_0, y_0] = \sqrt{\exp\left\{\frac{\tilde{Q}_2(t)}{4\alpha\beta - \varrho^2}\right\} - 1}, \quad C[Y(t)|x_0, y_0] = \sqrt{\exp\left\{\frac{\tilde{P}_2(t)}{4\alpha\beta - \varrho^2}\right\} - 1},$$

with $\tilde{P}_2(t), \tilde{Q}_2(t)$ as in (28).

Figures 3 and 4 show the averages and the medians of prey and predator populations in the case $\varrho = 0$, for $k(t) = 1$ and for $k(t) = 1 + t$, respectively, with the same choices of parameters. The choice of $\varrho = 0$ corresponds to the case of malthusian growth of prey population. The stochastic model does not have asymptotic stability and averages and medians exhibit different behaviors. Indeed, in Figure 3 the medians of prey and predator populations oscillate around their equilibrium values, whereas the averages increase with t . Instead, in Figure 4 the medians become periodic functions when t increases, whereas the averages increase with t .

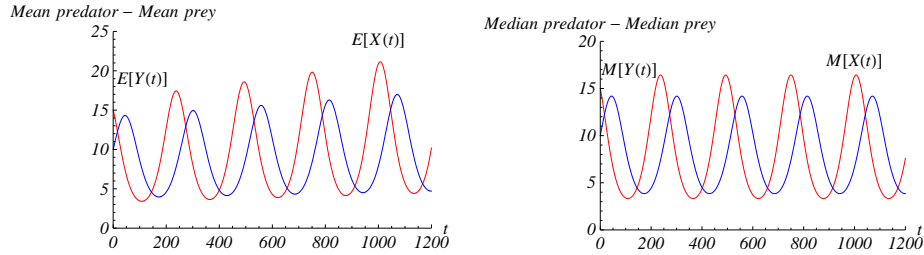


FIGURE 3. Averages and medians of prey and predator populations for $x_0 = 15, y_0 = 10, \alpha = 0.03, \beta = 0.02, \gamma = 0.06, \eta = 0.04, \varrho = 0, \sigma_1 = 0.02, \sigma_2 = 0.02, \sigma_{12} = 0.0001, k(t) = 1$.

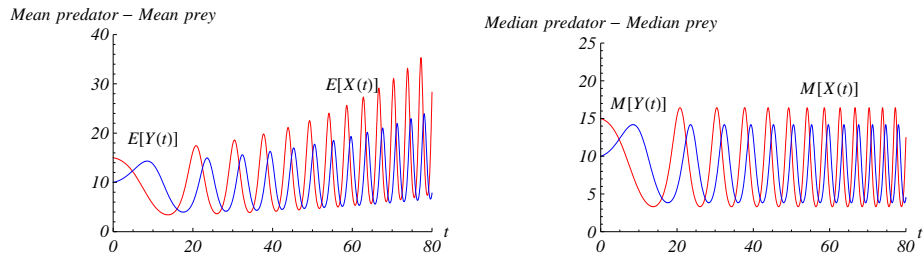


FIGURE 4. As in Figure 3 with $k(t) = 1 + t$.

Figures 5 and 6 show the averages and medians of prey and predator populations in the case $\varrho > 0$ and $\varrho^2 - 4\alpha\beta < 0$ for $k(t) = 1$ and for $k(t) = 1 + 0.8 \sin(t)$,

respectively, with the same choices of parameters. The averages and the medians are moving around the stable equilibrium state with a decreasing amplitude, and finally reach the equilibrium values represented by the dashed lines.

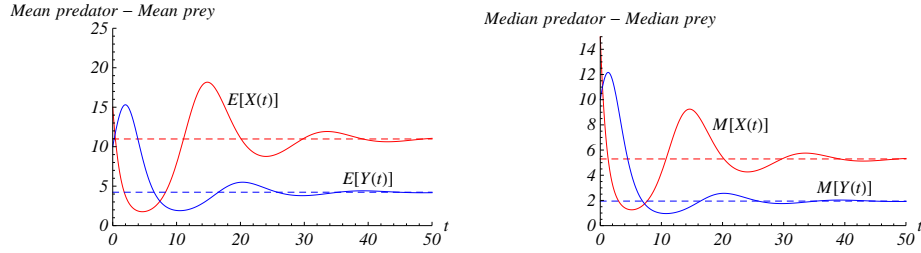


FIGURE 5. Averages and medians of prey and predator populations for $x_0 = 15$, $y_0 = 10$, $\alpha = 0.4$, $\beta = 0.3$, $\gamma = 0.6$, $\eta = 0.5$, $\varrho = 0.2$, $\sigma_1 = 0.5$, $\sigma_2 = 0.5$, $\sigma_{12} = 0.1$, $k(t) = 1$.

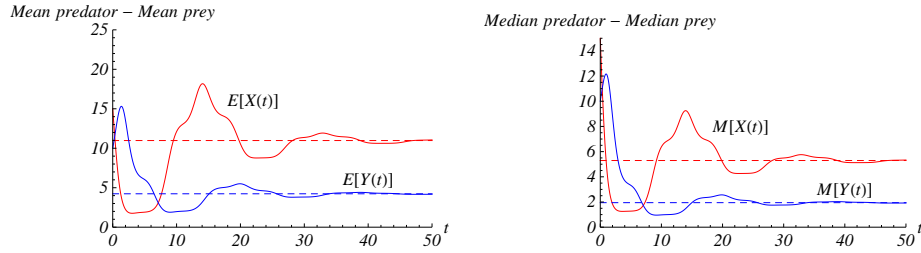


FIGURE 6. As in Figure 5 with $k(t) = 1 + 0.8 \sin(t)$.

Proposition 6. For $\varrho = 2\sqrt{\alpha\beta}$, one has

$$f_X(x, t|x_0, y_0) = \frac{1}{x\sqrt{2\pi}} \sqrt{\frac{1}{\Omega_2(t)}} \exp\left\{-\frac{1}{2\Omega_2(t)} [\ln x - \mu_X(t|x_0, y_0)]^2\right\}, \tag{43}$$

$$f_Y(y, t|x_0, y_0) = \frac{1}{y\sqrt{2\pi}} \sqrt{\frac{1}{\Lambda_2(t)}} \exp\left\{-\frac{1}{2\Lambda_2(t)} [\ln y - \mu_Y(t|x_0, y_0)]^2\right\},$$

where $\Omega_2(t)$, $\Lambda_2(t)$ are given in (31), and where $\mu_X(t|x_0, y_0) = \ln x(t)$ and $\mu_Y(t|x_0, y_0) = \ln y(t)$, with $x(t)$ and $y(t)$ given in (11).

Proof. When $\varrho = 2\sqrt{\alpha\beta}$, the marginal density functions of prey and predator populations can be obtained by taking the limit as $\varrho \rightarrow 2\sqrt{\alpha\beta}$ in (33) or, equivalently,

in (39). Indeed, making use of (25) and (37), one can prove that

$$\begin{aligned} \lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \mu_X(t|x_0, y_0) &= \frac{\eta}{\beta} + e^{-\sqrt{\alpha\beta}\psi(t)} \left[\left(1 - \sqrt{\alpha\beta}\psi(t)\right) \left(\ln x_0 - \frac{\eta}{\beta}\right) \right. \\ &\quad \left. - \alpha\psi(t) \left(\ln y_0 - \frac{\beta\gamma - 2\eta\sqrt{\alpha\beta}}{\alpha\beta}\right) \right] = \ln x(t), \end{aligned} \quad (44)$$

$$\begin{aligned} \lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \mu_Y(t|x_0, y_0) &= \frac{\beta\gamma - 2\eta\sqrt{\alpha\beta}}{\alpha\beta} + e^{-\sqrt{\alpha\beta}\psi(t)} \left[\beta\psi(t) \left(\ln x_0 - \frac{\eta}{\beta}\right) \right. \\ &\quad \left. + \left(1 + \sqrt{\alpha\beta}\psi(t)\right) \left(\ln y_0 - \frac{\beta\gamma - 2\eta\sqrt{\alpha\beta}}{\alpha\beta}\right) \right] = \ln y(t), \end{aligned}$$

with $x(t)$ and $y(t)$ given in (11). Furthermore, recalling (30) and (31), one has

$$\lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \frac{Q_2(t)}{\varrho^2 - 4\alpha\beta} = \Omega_2(t), \quad \lim_{\varrho \rightarrow 2\sqrt{\alpha\beta}} \frac{P_2(t)}{\varrho^2 - 4\alpha\beta} = \Lambda_2(t). \quad (45)$$

Hence, taking the limit as $\varrho \rightarrow 2\sqrt{\alpha\beta}$ in (33) and making use of (44) and (45), one is led to (43). \square

From Proposition 6, the conditional medians of $X(t)$ and $Y(t)$ coincide with the solutions $x(t)$ and $y(t)$ given in (11). Furthermore, the conditional means and the conditional coefficients of variation of prey and predator populations are:

$$\begin{aligned} E[X(t)|x_0, y_0] &= x(t)e^{\Omega_2(t)/2}, & E[Y(t)|x_0, y_0] &= y(t)e^{\Lambda_2(t)/2}, \\ C[X(t)|x_0, y_0] &= \sqrt{e^{\Omega_2(t)} - 1}, & C[Y(t)|x_0, y_0] &= \sqrt{e^{\Lambda_2(t)} - 1}. \end{aligned}$$

Figures 7 and 8 show the averages and medians of prey and predator populations in the case $\varrho = 2\sqrt{\alpha\beta}$ for $k(t) = 1$ and for $k(t) = 1 + 0.8 \sin(t)$, respectively, with the same choices of parameters. In these figures, the averages and the medians are moving toward the stable equilibrium state, and finally reach the equilibrium values represented by the dashed lines.

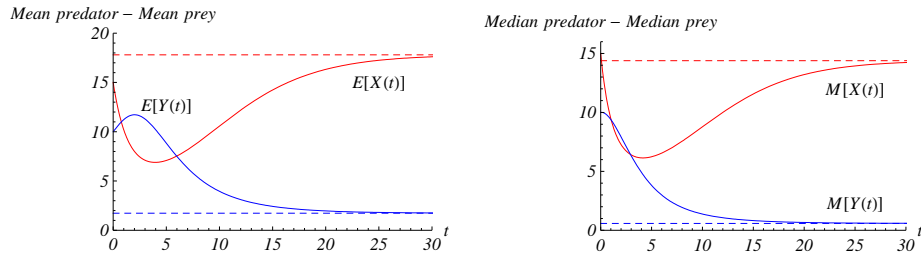
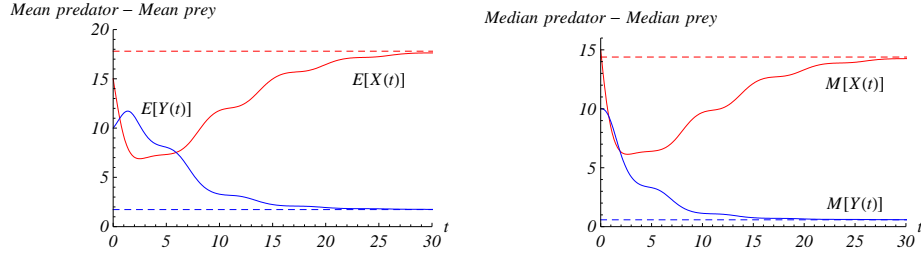


FIGURE 7. Averages and medians of prey and predator populations for $x_0 = 15$, $y_0 = 10$, $\alpha = 0.2$, $\beta = 0.3$, $\gamma = 1.2$, $\eta = 0.8$, $\varrho = 2\sqrt{0.06}$, $\sigma_1 = 0.5$, $\sigma_2 = 0.5$, $\sigma_{12} = 0.1$, $k(t) = 1$.

As shown in Figures 1–8 the choice of the function $k(t)$ plays an important role in the description of the transient phase of the evolution of prey and predator populations. For $\varrho > 0$, on the contrary, the behaviors of prey and predator populations are insensitive to the functional form of $k(t)$ when t increases.

FIGURE 8. As in Figure 7 with $k(t) = 1 + 0.8 \sin(t)$.

5. Steady-state probability densities. Under the assumption that $k(t)$ is a continuous positive function, such that $\int_0^{+\infty} k(\tau) d\tau = +\infty$, the predator-prey system in random environment reaches a situation of equilibrium when $\varrho > 0$. We denote by

$$W(x, y) = \lim_{t \rightarrow +\infty} f(x, y, t | x_0, y_0) \quad (46)$$

the steady-state joint density and by

$$W_X(x) = \lim_{t \rightarrow +\infty} f_X(x, t | x_0, y_0), \quad W_Y(y) = \lim_{t \rightarrow +\infty} f_Y(y, t | x_0, y_0) \quad (47)$$

the steady-state marginal densities of prey and predator populations.

Proposition 7. For $\varrho > 0$ the steady-state joint density is

$$W(x, y) = \frac{\varrho}{\pi xy} \sqrt{\frac{\alpha\beta}{(\beta\sigma_1^2 + \alpha\sigma_2^2)^2 + \varrho^2\sigma_1^2\sigma_2^2 + 2\varrho(\beta\sigma_1^2 + \alpha\sigma_2^2)\sigma_{12}}} \\ \times \exp\left\{-\frac{\varrho[(\beta^2\sigma_1^2 + \varrho^2\sigma_2^2 + \alpha\beta\sigma_2^2 + 2\beta\varrho\sigma_{12})u^2 + \alpha(\beta\sigma_1^2 + \alpha\sigma_2^2)v^2 + 2\alpha\varrho\sigma_2^2uv]}{(\beta\sigma_1^2 + \alpha\sigma_2^2)^2 + \varrho^2\sigma_1^2\sigma_2^2 + 2\varrho\sigma_{12}(\beta\sigma_1^2 + \alpha\sigma_2^2)}\right\}, \quad (48)$$

with u, v defined in (16).

Proof. When $\varrho > 0$ and $\varrho^2 - 4\alpha\beta > 0$, from (23) and (24) one has

$$\lim_{t \rightarrow +\infty} \frac{\varphi(t)}{\varrho^2 - 4\alpha\beta} = \frac{\beta[(\beta\sigma_1^2 + \alpha\sigma_2^2)^2 + \varrho^2\sigma_1^2\sigma_2^2 + 2\varrho\sigma_{12}(\beta\sigma_1^2 + \alpha\sigma_2^2)]}{4\alpha\varrho^2}, \\ \lim_{t \rightarrow +\infty} \frac{P_2(t)}{\varrho^2 - 4\alpha\beta} = \frac{\beta^2\sigma_1^2 + (\varrho^2 + \alpha\beta)\sigma_2^2 + 2\beta\varrho\sigma_{12}}{2\alpha\beta\varrho}, \quad (49)$$

$$\lim_{t \rightarrow +\infty} \frac{Q_2(t)}{\varrho^2 - 4\alpha\beta} = \frac{\beta\sigma_1^2 + \alpha\sigma_2^2}{2\beta\varrho}, \quad \lim_{t \rightarrow +\infty} \frac{P_{12}(t)}{\varrho^2 - 4\alpha\beta} = \frac{\sigma_2^2}{\beta}, \\ \lim_{t \rightarrow +\infty} \frac{P_1(t)}{\varrho^2 - 4\alpha\beta} = 0, \quad \lim_{t \rightarrow +\infty} \frac{Q_1(t)}{\varrho^2 - 4\alpha\beta} = 0, \quad \lim_{t \rightarrow +\infty} \frac{P_0(t)}{\varrho^2 - 4\alpha\beta} = 0.$$

Hence, for $\varrho > 0$ and $\varrho^2 - 4\alpha\beta > 0$ the steady-state joint density (48) follows taking the limit as t increases in (22) and making use of (49). For $\varrho > 0$ and $\varrho^2 - 4\alpha\beta \leq 0$ we proceed in a similar way, obtaining again (48). \square

Proposition 8. For $\varrho > 0$, the steady-state marginal densities of prey and predator populations are

$$W_X(x) = \frac{1}{x\sqrt{\pi}} \sqrt{\frac{\beta\varrho}{\beta\sigma_1^2 + \alpha\sigma_2^2}} \exp\left\{-\frac{\beta\varrho}{\beta\sigma_1^2 + \alpha\sigma_2^2} \left(\ln x - \frac{\eta}{\beta}\right)^2\right\},$$

$$W_Y(y) = \frac{1}{y\sqrt{\pi}} \sqrt{\frac{\alpha\beta\varrho}{\beta^2\sigma_1^2 + (\alpha\beta + \varrho^2)\sigma_2^2 + 2\beta\varrho\sigma_{12}}} \times \exp\left\{-\frac{\alpha\beta\varrho}{\beta^2\sigma_1^2 + (\alpha\beta + \varrho^2)\sigma_2^2 + 2\beta\varrho\sigma_{12}} \left(\ln y - \frac{\beta\gamma - \varrho\eta}{\alpha\beta}\right)^2\right\}.$$
(50)

Proof. Relations (50) immediately follow from (33), (39) and (43). \square

We note that the asymptotic densities of prey and predator populations are both sensitive to the noise intensities σ_1^2 and σ_2^2 , while the noise coefficient σ_{12} has a crucial impact only on the asymptotic density of predator population.

For $\varrho > 0$ the asymptotic medians and averages are:

$$M(X) = \exp\left\{\frac{\eta}{\beta}\right\}, \quad E(X) = \exp\left\{\frac{\eta}{\beta} + \frac{\beta\sigma_1^2 + \alpha\sigma_2^2}{4\beta\varrho}\right\},$$
(51)

$$M(Y) = \exp\left\{\frac{\beta\gamma - \varrho\eta}{\alpha\beta}\right\}, \quad E(Y) = \exp\left\{\frac{\beta\gamma - \varrho\eta}{\alpha\beta} + \frac{\beta^2\sigma_1^2 + (\alpha\beta + \varrho^2)\sigma_2^2 + 2\beta\varrho\sigma_{12}}{4\alpha\beta\varrho}\right\}.$$

Figures 9 and 10 show the steady-state densities for prey and predator populations for different values of the noise intensities σ_1^2 and σ_2^2 , respectively. We note that an increase in the noise intensities induce a leftward shift of the peaks of the densities.

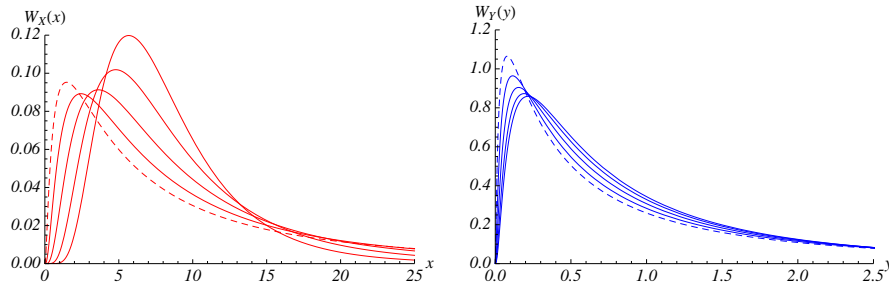


FIGURE 9. Asymptotic marginal densities of prey and predator populations for $\alpha = 0.4$, $\beta = 0.3$, $\gamma = 1.5$, $\eta = 0.6$, $\varrho = 0.8$. Here, $\sigma_2 = 0.5$, $\sigma_{12} = 0.1$ and for $\sigma_1 = 0.3, 0.6, 0.9, 1.2, 1.5$ (from the right to the left).

Figure 11 show the steady-state densities for prey and predator populations for different values of the noise coefficient σ_{12} . The peaks' height of the steady-state densities of the predator population increases when σ_{12} increases positively or decreases negatively, whereas the steady-state density of prey population has the same fitting curve for different values of σ_{12} .

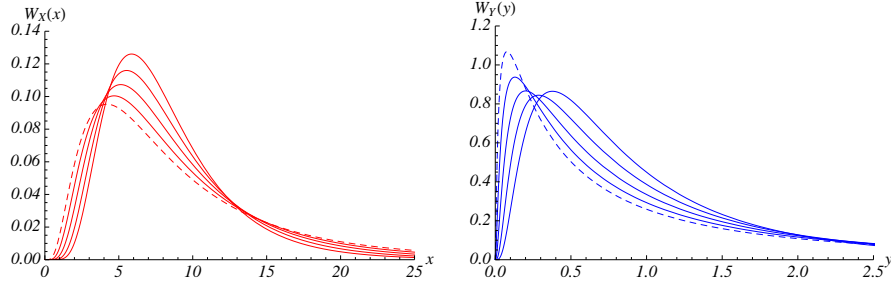


FIGURE 10. As in Figure 9 with $\sigma_1 = 0.5$, $\sigma_{12} = 0.1$ and for $\sigma_2 = 0.3, 0.4, 0.5, 0.6, 0.7$ (from the right to the left).

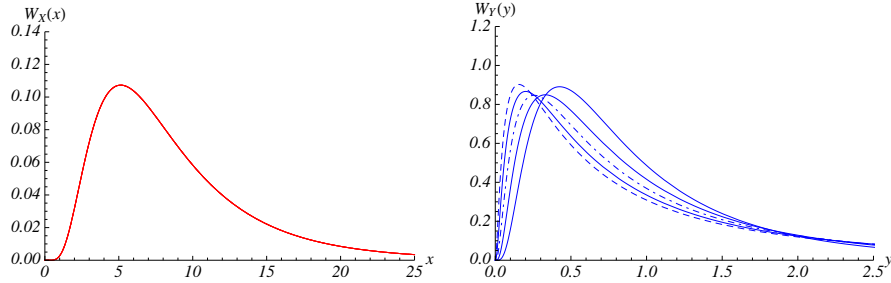


FIGURE 11. As in Figure 9 with $\sigma_1 = 0.5$, $\sigma_2 = 0.5$ and $\sigma_{12} = -0.2, -0.1, 0, 0.1, 0.2$ (from the right to the left).

6. Concluding remarks. A non-autonomous deterministic system to describe the evolution of prey and predator system is considered. A self-regulation term is included in the prey equation and the effects of prey-predator interactions are taken in account. Under the assumption of random environment, a stochastic model is constructed to investigate the interactions between prey and predator populations. The joint probability density and the marginal probability densities for the prey and predator populations are explicitly obtained. We note that the medians of prey and predator processes coincide with the solution of the deterministic system, where the averages depend on the intensities of noise and on the noise correlation coefficient. The asymptotic behavior of the predator-prey stochastic model is also analyzed. We prove that the steady-state density of the predator population is sensitive not only to the noise intensities, but also to the noise correlation coefficient.

Appendix A. Infinitesimal moments. Under the assumption of random environment, we express the increments $X(t + \Delta t) - X(t)$ and $Y(t + \Delta t) - Y(t)$ for the prey and predator populations in the interval $(t, t + \Delta t)$ in terms of the state of the system at time t . In order to derive the infinitesimal moments (12), we consider the following cases: (i) $\varrho^2 - 4\alpha\beta > 0$, (ii) $\varrho^2 - 4\alpha\beta < 0$ and (iii) $\varrho^2 - 4\alpha\beta = 0$.

Case (i). For $\varrho^2 - 4\alpha\beta > 0$, recalling (5), the state of the system at time $t + \Delta t$ can be expressed as:

$$\begin{aligned}
x(t + \Delta t) = x(t) \exp \left\{ \left(\ln x(t) - \frac{\eta}{\beta} \right) \left[-1 + \frac{\lambda_1 e^{\lambda_1 \psi_t(\Delta t)} - \lambda_2 e^{\lambda_2 \psi_t(\Delta t)}}{\lambda_1 - \lambda_2} \right] \right. \\
\left. - \alpha \left(\ln y(t) - \frac{\beta \gamma - \varrho \eta}{\alpha \beta} \right) \frac{e^{\lambda_1 \psi_t(\Delta t)} - e^{\lambda_2 \psi_t(\Delta t)}}{\lambda_1 - \lambda_2} \right\},
\end{aligned} \tag{52}$$

$$\begin{aligned}
y(t + \Delta t) = y(t) \exp \left\{ \beta \left(\ln x(t) - \frac{\eta}{\beta} \right) \frac{e^{\lambda_1 \psi_t(\Delta t)} - e^{\lambda_2 \psi_t(\Delta t)}}{\lambda_1 - \lambda_2} \right. \\
\left. - \left(\ln y(t) - \frac{\beta \gamma - \varrho \eta}{\alpha \beta} \right) \left[1 + \frac{\lambda_2 e^{\lambda_1 \psi_t(\Delta t)} - \lambda_1 e^{\lambda_2 \psi_t(\Delta t)}}{\lambda_1 - \lambda_2} \right] \right\},
\end{aligned}$$

where

$$\psi_t(\Delta t) = \psi(t + \Delta t) - \psi(t) = \int_t^{t+\Delta t} k(\tau) d\tau.$$

Under the assumption of random environment, starting from (52), we obtain the system of stochastic equations related to the prey and predator populations

$$\begin{aligned}
X(t + \Delta t) - X(t) = X(t) \left[\exp \left\{ \left(\Delta t \ln X(t) - \frac{W_2(\Delta t)}{\beta} \right) \frac{\lambda_1 R_{1,t}(\Delta t) - \lambda_2 R_{2,t}(\Delta t)}{\lambda_1 - \lambda_2} \right. \right. \\
\left. \left. - \alpha \left(\Delta t \ln Y(t) - \frac{\beta W_1(\Delta t) - \varrho W_2(\Delta t)}{\alpha \beta} \right) \frac{R_{1,t}(\Delta t) - R_{2,t}(\Delta t)}{\lambda_1 - \lambda_2} \right\} - 1 \right],
\end{aligned} \tag{53}$$

$$\begin{aligned}
Y(t + \Delta t) - Y(t) = Y(t) \left[\exp \left\{ \beta \left(\Delta t \ln X(t) - \frac{W_2(\Delta t)}{\beta} \right) \frac{R_{1,t}(\Delta t) - R_{2,t}(\Delta t)}{\lambda_1 - \lambda_2} \right. \right. \\
\left. \left. - \left(\Delta t \ln Y(t) - \frac{\beta W_1(\Delta t) - \varrho W_2(\Delta t)}{\alpha \beta} \right) \frac{\lambda_2 R_{1,t}(\Delta t) - \lambda_1 R_{2,t}(\Delta t)}{\lambda_1 - \lambda_2} \right\} - 1 \right],
\end{aligned}$$

where

$$R_{j,t}(\Delta t) = \frac{e^{\lambda_j \psi_t(\Delta t)} - 1}{\Delta t} \quad (j = 1, 2).$$

Recalling that

$$\lim_{\Delta t \rightarrow 0} R_{j,t}(\Delta t) = \lambda_j k(t) \quad (j = 1, 2),$$

from (53) we easily obtain (12).

Case (ii). For $\varrho^2 - 4\alpha\beta < 0$, by virtue of (9), the state of the system at time $t + \Delta t$ is:

$$\begin{aligned}
x(t + \Delta t) = x(t) \exp \left\{ \left(\ln x(t) - \frac{\eta}{\beta} \right) \left[-1 + \cos[z_t(\Delta t)] e^{-\varrho \psi_t(\Delta t)/2} \right. \right. \\
\left. \left. - \frac{\varrho e^{-\varrho \psi_t(\Delta t)/2}}{\sqrt{4\alpha\beta - \varrho^2}} \sin[z_t(\Delta t)] \right] - \frac{2\alpha e^{-\varrho \psi_t(\Delta t)/2}}{\sqrt{4\alpha\beta - \varrho^2}} \left(\ln y(t) - \frac{\beta \gamma - \varrho \eta}{\alpha \beta} \right) \sin[z_t(\Delta t)] \right\},
\end{aligned} \tag{54}$$

$$\begin{aligned}
y(t + \Delta t) = y(t) \exp \left\{ \frac{2\beta e^{-\varrho \psi_t(\Delta t)/2}}{\sqrt{4\alpha\beta - \varrho^2}} \left(\ln x(t) - \frac{\eta}{\beta} \right) \sin[z_t(\Delta t)] \right. \\
\left. + \left(\ln y(t) - \frac{\beta \gamma - \varrho \eta}{\alpha \beta} \right) \left[-1 + \cos[z_t(\Delta t)] e^{-\varrho \psi_t(\Delta t)/2} + \frac{\varrho e^{-\varrho \psi_t(\Delta t)/2}}{\sqrt{4\alpha\beta - \varrho^2}} \sin[z_t(\Delta t)] \right] \right\},
\end{aligned}$$

where

$$z_t(\Delta t) = z(t + \Delta t) - z(t) = \frac{1}{2}\psi_t(\Delta t)\sqrt{4\alpha\beta - \varrho^2}.$$

Under the assumption of random environment, from (54) we derive the system of stochastic equations:

$$\begin{aligned} X(t + \Delta t) - X(t) = X(t) \left[\exp \left\{ - \left(\Delta t \ln X(t) - \frac{W_2(\Delta t)}{\beta} \right) \left[C_t(\Delta t) + \frac{\varrho S_t(\Delta t)}{\sqrt{4\alpha\beta - \varrho^2}} \right] \right. \right. \\ \left. \left. - \frac{2\alpha}{\sqrt{4\alpha\beta - \varrho^2}} \left(\Delta t \ln Y(t) - \frac{\beta W_1(\Delta t) - \varrho W_2(\Delta t)}{\alpha\beta} \right) S_t(\Delta t) \right\} - 1 \right], \end{aligned} \quad (55)$$

$$\begin{aligned} Y(t + \Delta t) - Y(t) = Y(t) \left[\exp \left\{ \frac{2\beta}{\sqrt{4\alpha\beta - \varrho^2}} \left(\Delta t \ln X(t) - \frac{W_2(\Delta t)}{\beta} \right) S_t(\Delta t) \right. \right. \\ \left. \left. - \left(\Delta t \ln Y(t) - \frac{\beta W_1(\Delta t) - \varrho W_2(\Delta t)}{\alpha\beta} \right) \left[C_t(\Delta t) - \frac{\varrho S_t(\Delta t)}{\sqrt{4\alpha\beta - \varrho^2}} \right] \right\} - 1 \right], \end{aligned}$$

where

$$\begin{aligned} C_t(\Delta t) &= \frac{1 - e^{-\varrho\psi_t(\Delta t)/2}}{\Delta t} + \frac{1 - \cos[z_t(\Delta t)]}{\Delta t} e^{-\varrho\psi_t(\Delta t)/2}, \\ S_t(\Delta t) &= \frac{\sin[z_t(\Delta t)]}{\Delta t} e^{-\varrho\psi_t(\Delta t)/2}. \end{aligned}$$

Since

$$\lim_{\Delta t \rightarrow 0} C_t(\Delta t) = \frac{\varrho}{2}k(t), \quad \lim_{\Delta t \rightarrow 0} S_t(\Delta t) = \frac{k(t)}{2} \sqrt{4\alpha\beta - \varrho^2},$$

from (55) we obtain the infinitesimal moments (12).

Case (iii). For $\varrho^2 - 4\alpha\beta = 0$, by using (11), the state of the system at time $t + \Delta t$ is:

$$\begin{aligned} x(t + \Delta t) = x(t) \exp \left\{ \left(\ln x(t) - \frac{\eta}{\beta} \right) \left[-1 + \left(1 - \sqrt{\alpha\beta}\psi_t(\Delta t) \right) e^{-\sqrt{\alpha\beta}\psi_t(\Delta t)} \right] \right. \\ \left. - \alpha\psi_t(\Delta t) \left(\ln y(t) - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) e^{-\sqrt{\alpha\beta}\psi_t(\Delta t)} \right\}, \end{aligned} \quad (56)$$

$$\begin{aligned} y(t + \Delta t) = y(t) \exp \left\{ \beta\psi_t(\Delta t) \left(\ln x(t) - \frac{\eta}{\beta} \right) e^{-\sqrt{\alpha\beta}\psi_t(\Delta t)} \right. \\ \left. + \left(\ln y(t) - \frac{\beta\gamma - \varrho\eta}{\alpha\beta} \right) \left[-1 + \left(1 + \sqrt{\alpha\beta}\psi_t(\Delta t) \right) e^{-\sqrt{\alpha\beta}\psi_t(\Delta t)} \right] \right\}. \end{aligned}$$

Under the assumption of random environment, by virtue of (56) we obtain

$$\begin{aligned} X(t + \Delta t) - X(t) &= X(t) \left[\exp \left\{ - \left(\Delta t \ln X(t) - \frac{W_2(\Delta t)}{\beta} \right) \left[U_t(\Delta t) + \sqrt{\alpha\beta} V_t(\Delta t) \right] \right. \right. \\ &\quad \left. \left. - \alpha \left(\Delta t \ln Y(t) - \frac{\beta W_1(\Delta t) - \varrho W_2(\Delta t)}{\alpha\beta} \right) V_t(\Delta t) \right\} - 1 \right], \end{aligned} \quad (57)$$

$$\begin{aligned} Y(t + \Delta t) - Y(t) &= Y(t) \left[\exp \left\{ \beta \left(\Delta t \ln X(t) - \frac{W_2(\Delta t)}{\beta} \right) V_t(\Delta t) \right. \right. \\ &\quad \left. \left. - \left(\Delta t \ln Y(t) - \frac{\beta W_1(\Delta t) - \varrho W_2(\Delta t)}{\alpha\beta} \right) \left[U_t(\Delta t) - \sqrt{\alpha\beta} V_t(\Delta t) \right] \right\} - 1 \right], \end{aligned}$$

where

$$U_t(\Delta t) = \frac{1 - e^{-\sqrt{\alpha\beta} \psi_t(\Delta t)}}{\Delta t}, \quad V_t(\Delta t) = \frac{\psi_t(\Delta t)}{\Delta t} e^{-\sqrt{\alpha\beta} \psi_t(\Delta t)}.$$

Being

$$\lim_{\Delta t \rightarrow 0} U_t(\Delta t) = \sqrt{\alpha\beta} k(t), \quad \lim_{\Delta t \rightarrow 0} V_t(\Delta t) = k(t),$$

from (57) we derive the infinitesimal moments (12), with $\varrho = 2\sqrt{\alpha\beta}$.

In conclusion, $\{X(t), Y(t), t \geq 0\}$ is a two-dimensional diffusion process characterized by infinitesimal moments (12).

Appendix B. Proof of Proposition 1. When $\varrho^2 - 4\alpha\beta > 0$, by setting $c_1 = 1$ and $c_2 = -1$ in (17) and in (18), we obtain:

$$\begin{aligned} \widehat{f}(z, w, t | z_0, w_0) &= \frac{\beta^2}{2\pi\sqrt{\varphi(t)}} \exp \left\{ - \frac{\beta^2}{2\varphi(t)} \left[D_2 E_2(t) \left(z - z_0 e^{\lambda_1 \psi(t)} \right)^2 \right. \right. \\ &\quad \left. \left. + D_1 E_1(t) \left(w - w_0 e^{\lambda_2 \psi(t)} \right)^2 + 2D_3 E_3(t) \left(z - z_0 e^{\lambda_1 \psi(t)} \right) \left(w - w_0 e^{\lambda_2 \psi(t)} \right) \right] \right\}, \end{aligned} \quad (58)$$

where $E_i(t)$ ($i = 1, 2, 3$) and $\varphi(t)$ are given in (21) and (23), respectively. Note that $\varphi(t) > 0$ for $t > 0$. Indeed, since

$$D_1 D_2 - D_3^2 = \beta^2 (\varrho^2 - 4\alpha\beta) (\sigma_1^2 \sigma_2^2 - \sigma_{12}^2) > 0,$$

making use of the Cauchy-Schwarz inequality for integrals, from (23) for $t > 0$ one has:

$$\varphi(t) \geq E_1(t) E_2(t) (D_1 D_2 - D_3^2) > 0.$$

By virtue of (17), with $c_1 = 1$ and $c_2 = -1$, and of (58) one obtains the joint probability density function of the diffusion process $\{X(t), Y(t), t \geq 0\}$ as in (22). For $t > 0$, one has $P_2(t) > 0$ and $Q_2(t) > 0$. Indeed, they can be re-written as

$$\begin{aligned} P_2(t) &= \int_0^{\psi(t)} \left[D_2 e^{2\lambda_1 x} + D_1 e^{2\lambda_2 x} - 2D_3 e^{-\varrho x} \right] dx, \\ Q_2(t) &= \int_0^{\psi(t)} \left[D_2 \left(\frac{\lambda_2}{\beta} \right)^2 e^{2\lambda_2 x} + D_1 \left(\frac{\lambda_1}{\beta} \right)^2 e^{2\lambda_1 x} - 2D_3 \frac{\alpha}{\beta} e^{-\varrho x} \right] dx, \end{aligned}$$

whose integrated functions satisfy the inequalities:

$$\begin{aligned} D_2 e^{2\lambda_1 x} + D_1 e^{2\lambda_2 x} - 2D_3 e^{-\varrho x} &> [\beta\sigma_1(e^{\lambda_1 x} - e^{\lambda_2 x}) - \sigma_2(\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x})]^2, \\ D_2 \left(\frac{\lambda_2}{\beta}\right)^2 e^{2\lambda_2 x} + D_1 \left(\frac{\lambda_1}{\beta}\right)^2 e^{2\lambda_1 x} - 2D_3 \frac{\alpha}{\beta} e^{-\varrho x} \\ &> [\sigma_1(\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}) - \alpha\sigma_2(e^{\lambda_1 x} - e^{\lambda_2 x})]^2. \end{aligned}$$

This completes the proof of Proposition 1.

Appendix C. Proof of Proposition 2. When $\varrho^2 - 4\alpha\beta < 0$, by setting $c_1 = 1$ and $c_2 = i$ in (17) and (18), one obtains

$$\begin{aligned} \hat{f}(z, w, t|z_0, w_0) &= \frac{\beta^2}{2\pi\sqrt{\tilde{\varphi}(t)}} \exp\left\{-\frac{\beta^2}{2\tilde{\varphi}(t)} \left[-D_2 E_2(t) \left(z - z_0 e^{\lambda_1 \psi(t)}\right)^2 \right. \right. \\ &\quad \left. \left. + D_1 E_1(t) \left(w - w_0 e^{\lambda_2 \psi(t)}\right)^2 - 2iD_3 E_3(t) \left(z - z_0 e^{\lambda_1 \psi(t)}\right) \left(w - w_0 e^{\lambda_2 \psi(t)}\right)\right]\right\}, \end{aligned} \quad (59)$$

with $z(t)$ and $\tilde{\varphi}(t)$ defined in (8) and (27), respectively. For $t > 0$, we note that $\tilde{\varphi}(t) > 0$. Indeed, since

$$\frac{D_3^2}{\varrho^2} - \frac{D_1 D_2}{4\alpha\beta} = \frac{(4\alpha\beta - \varrho^2)(\beta\sigma_1^2 + \alpha\sigma_2^2 - \sigma_1\sigma_2\varrho)^2}{4\alpha\beta\varrho^2} > 0,$$

from (27) it follows

$$\begin{aligned} \tilde{\varphi}(t) &= \frac{D_3^2}{\varrho^2} - \frac{D_1 D_2}{4\alpha\beta} + e^{-2\varrho\psi(t)} \left(\frac{D_3^2}{\varrho^2} - \frac{D_1 D_2}{4\alpha\beta}\right) - 2e^{-\varrho\psi(t)} \\ &\quad \times \left\{\frac{D_3^2}{\varrho^2} - \frac{D_1 D_2}{4\alpha\beta} \cos[2z(t)]\right\} > \left(\frac{D_3^2}{\varrho^2} - \frac{D_1 D_2}{4\alpha\beta}\right) \left(1 - e^{-\varrho\psi(t)}\right)^2 > 0. \end{aligned}$$

Making use of (17), with $c_1 = 1$ and $c_2 = i$, by virtue of (59) the joint probability density function of the diffusion process $\{X(t), Y(t), t \geq 0\}$ is the one in (26).

Recalling (24), we derive alternative expressions for $\tilde{P}_2(t)$ and $\tilde{Q}_2(t)$:

$$\begin{aligned} \tilde{P}_2(t) &= (\alpha\beta\sigma_2^2 - \beta^2\sigma_1^2)[E_1(t) + E_2(t)] + (\varrho\sigma_2^2 + 2\beta\sigma_{12})[\lambda_1 E_2(t) + \lambda_2 E_1(t)] \\ &\quad + 2D_3 E_3(t), \end{aligned} \quad (60)$$

$$\begin{aligned} \tilde{Q}_2(t) &= (\varrho\sigma_1^2 + 2\alpha\sigma_{12})[\lambda_1 E_1(t) + \lambda_2 E_2(t)] + (\alpha\beta\sigma_1^2 - \alpha^2\sigma_2^2)[E_1(t) + E_2(t)] \\ &\quad + \frac{2\alpha}{\beta} D_3 E_3(t), \end{aligned}$$

where

$$\begin{aligned} E_1(t) + E_2(t) &= \frac{1}{2\alpha\beta} \left\{ \varrho - \varrho e^{-\varrho\psi(t)} \cos[2z(t)] + \sqrt{4\alpha\beta - \varrho^2} e^{-\varrho\psi(t)} \sin[2z(t)] \right\}, \\ \lambda_1 E_2(t) + \lambda_2 E_1(t) &= \frac{1}{2\alpha\beta} \left\{ 2\alpha\beta - \varrho^2 - (2\alpha\beta - \varrho^2) e^{-\varrho\psi(t)} \cos[2z(t)] \right. \\ &\quad \left. - \varrho\sqrt{4\alpha\beta - \varrho^2} e^{-\varrho\psi(t)} \sin[2z(t)] \right\}, \\ \lambda_1 E_1(t) + \lambda_2 E_2(t) &= -1 + e^{-\varrho\psi(t)} \cos[2z(t)]. \end{aligned}$$

Relations (60) show that $\tilde{P}_2(t) > 0$ and $\tilde{Q}_2(t) > 0$ for $t > 0$. This completes the proof of Proposition 2.

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