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STABILITY AND HOPF BIFURCATION IN A DIFFUSIVE PREDATOR-PREY SYSTEM INCORPORATING A PREY REFUGE

Xiaoyuan Chang[a,b]

[a]Department of Mathematics, Harbin Institute of Technology
Harbin, Heilongjiang 150001, China
[b]School of Applied Sciences, Harbin University of Science and Technology
52 Xuefu Street, Harbin, Heilongjiang 150080, China

Junjie Wei

Department of Mathematics, Harbin Institute of Technology
Harbin, Heilongjiang 150001, China

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Abstract. A diffusive predator-prey model with Holling type II functional response and the no-flux boundary condition incorporating a constant prey refuge is considered. Globally asymptotically stability of the positive equilibrium is obtained. Regarding the constant number of prey refuge m as a bifurcation parameter, by analyzing the distribution of the eigenvalues, the existence of Hopf bifurcation is given. Employing the center manifold theory and normal form method, an algorithm for determining the properties of the Hopf bifurcation is derived. Some numerical simulations for illustrating the analysis results are carried out.

1. **Introduction.** One of the most important and popular interactions between species in ecology environment is the predation interaction (see [17]), which has been modeled by the predator-prey system and considered extensively in many aspects with many different functional responses without diffusion (see [5, 15, 22, 23, 25, 39]) and with diffusion (see [3, 4, 6, 7, 19, 24, 29, 30, 31, 37, 38, 40]). After Crombic, a biologist, did the beetles experiment and discussed the effect of prey refuge in 1946, who points out that the term of the prey using refuge impacts the density of the equilibrium which is so called the stabilizing effect (see [11, 13, 35]), scientists turn to consider the dynamics of the system with all kinds of response functions incorporating a prey refuge without diffusion (see [1, 2, 8, 12, 16, 18, 21, 26, 27, 28, 32, 33, 34] and with diffusion (see [9, 20]). Especially, Gonzalez-Olivares and Ramos-Jiliberto in [8] incorporate a new ingredient in an original Lotka-Volterra predator-prey model by adding a refuge term for the prey and consider the dynamics of the Rosenzweig-MacArthur predator-prey model. They show that adding two different types of prey refuge, including adding a constant proportion of prey using refuge

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and a constant number of prey using refuge (see [36]), enhance the stability of the equilibrium when the prey refuge is small. Chen et. al. in [1] show the instability and global stability properties of the equilibria and the existence and uniqueness of limit cycle for a predator-prey model with Holling type II functional response incorporating a constant prey refuge. Ko and Ryu in [20] and Guan et. al. in [9] consider the dynamics of a diffusive predator-prey model incorporating a constant proportion of prey refuge, including the existence and the stability of steady-state solutions and the Turing space. However, to our knowledge, there are surprisingly few conclusions in the diffusive predator-prey model with Holling type II functional response incorporating a constant number of prey using refuge, which is motivated for our paper.

Chen et. al. in [1] and Gonzalez-Olivares and Ramos-Jiliberto in [8] have considered the following Holling-II predator-prey model incorporating a prey refuge:

$$\begin{aligned} \frac{du}{dt} &= \alpha u\left(1-\frac{u}{K}\right)-\frac{\beta(u-m)v}{1+a(u-m)}, \\ \frac{dv}{\partial t} &= v\left(-r+\frac{c\beta(u-m)}{1+a(u-m)}\right), \end{aligned} \tag{1}$$

where u represents the prey density and v represents the predator density; α is the intrinsic per capita growth rate of prey; K is the prey environmental carrying capacity; β is the maximal per capita consumption rate of predators; a is the amount of prey needed to achieve one-half of β; r is the per capita death rate of predators; c is the efficiency with which predators convert consumed prey into new predators. m is a positive constant representing taking m $(u \geq m)$ of the prey from predation and leaving $u-m$ of the prey available to the predator. The parameter ecological meaning can also be found in [1] and [8].

For simplicity, we take the following scaling:

$$\overline{u}=u-m,\ \overline{v}=\frac{\beta}{a}v,\ \overline{m}=am,\ \overline{K}=aK,\ \overline{r}=\frac{a}{\beta}r$$

and still denote $\overline{u}, \overline{v}, \overline{m}, \overline{K}, \overline{r}$ as u, v, m, K, r, respectively. Then system (1) with the diffusion, the Neumann boundary condition and the initial value takes the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\triangle u+\alpha(u+m)\left(1-\dfrac{u+m}{K}\right)-\dfrac{uv}{1+u}, & x\in\Omega,\ t>0, \\ \dfrac{\partial v}{\partial t} = d_2\triangle v+v\left(-r+\dfrac{cu}{1+u}\right), & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu}=0, & t\geq 0,\ x\in\partial\Omega, \\ u(0,x)=u_0(x)\geq 0,\ v(0,x)=v_0(x)\geq 0, & x\in\overline{\Omega}, \end{cases} \tag{2}$$

where d_j $(j=1,2)$ is the diffusive coefficient and the region Ω represents the living environment of the prey and the predator. In this paper, we always consider $\Omega=(0,l\pi)$ and $l\in\mathbb{R}^+$. In biological meaning, the Neumann boundary condition represents no-flux in the boundary region, that is there is no species or individuals in or out the living environment and the living environment is closed. In this paper, we focus on the stability and Hopf bifurcation of system (2).

The rest of our paper is organized as follows. In Section 2, by analyzing the distribution of the roots of the characteristic equations, the stability of the positive equilibrium for system (2) is obtained. And by applying Poincaré-Andronov-Hopf bifurcation theory, we get the existence of periodic solution bifurcating from the positive constant solution for some ranges of the parameter m. In Section 3, an algorithm for determining the direction of the Hopf bifurcation, the stability and the period of bifurcating periodic solutions are derived by applying the normal form theory and the center manifold method of partial differential equations (see [10]). Finally, some numerical simulations are presented to illustrate the analytic results in Section 4.

2. Stability of the positive equilibrium and Hopf bifurcation.

In this section, we consider the stability of the positive equilibrium for system (2) by analyzing the distribution of eigenvalues in corresponding linear system of system (2).

By the biological significance of system (2), we are interested in the stability of the positive constant equilibrium. It is not difficult to calculate that under the following hypothesis

$$(\mathbf{H1}) \quad c > r \text{ and } 0 < m \le \min\{K-\theta, \theta\},$$

system (2) has a unique positive constant solution $E_* = (\theta, v_\theta)$, where

$$\theta = \frac{r}{c-r} \quad \text{and} \quad v_\theta = \frac{\alpha c}{r}(\theta+m)\left(1-\frac{\theta+m}{K}\right). \tag{3}$$

Applying the similar method mentioned in [14, 20, 40], we have the following conclusion for the globally asymptotically stability of $E_*(\theta, v_\theta)$.

Theorem 2.1. *Suppose that either* $0 < K \le 1$ *or* $K > 1$ *and* $(K-1)/2 \le m \le K$. *Then the positive equilibrium* $E_*(\theta, v_\theta)$ *is globally asymptotically stable.*

Proof. Define

$$E(u(t,x), v(t,x)) = \int_0^{l\pi}\int_\theta^u \frac{cp(\xi)-r}{p(\xi)} d\xi dx + \int_0^{l\pi}\int_{v_\theta}^v \frac{\eta - v_\theta}{\eta} d\eta dx,$$

where $p(u) = u/(1+u)$. Then

$$E_t(u,v) = \int_0^{l\pi} \frac{cp(u)-r}{p(u)} u_t dx + \int_0^{l\pi} \frac{v-v_\theta}{v} v_t dx = I_1(t) + I_2(t),$$

where

$$I_1(t) = -\int_0^{l\pi} \frac{rd_1}{u^2}u_x^2 + \frac{d_2}{v^2}v_x^2 dx,\ \ I_2(t) = \int_0^{l\pi} c[p(u)-p(\theta)][g(u)-g(\theta)]dx$$

and

$$g(u) = \frac{\alpha(u+m)(K-u-m)(1+u)}{Ku},$$
$$g'(u) = \frac{\alpha u^2(-2u+K-2m-1) - m(K-m)}{Ku^2}.$$

Notice that, for any $u > 0$, $g'(u) > 0$ when $0 < K \le 1$ or $K > 1$ but $(K-1)/2 \le m \le K$. That is, $[p(u)-p(\theta)][g(u)-g(\theta)] < 0$, which leads to $I_2(t) < 0$ for any $u > 0$. Thus, $E_t < 0$ along an orbit $(u(t,x), v(t,x))$ of system (2) with any non-negative initial value $(u_0, v_0) \not\equiv (0,0)$ and $E_t = 0$ only if $(u(t,x), v(t,x)) = (\theta, v_\theta) = E_*$. □

Due to the conclusion of Theorem 2.1, we next always assume that the parameters d_1, d_2, α, r and K are all fixed positive numbers satisfying $K > 1$ and $0 < m < (K-1)/2$, the parameters m and c are two arbitrary positive quantities. In view of the definition of θ in (3), we know θ is an arbitrary $c-$dependent quantity. Now, we consider the stability of the positive equilibrium $E_*(\theta, v_\theta)$ in system (2) as the positive numbers m and c (also θ) vary.

Applying the conclusions of Yi and Wei in [40], we make some notations as follows. Define the real-valued Sobolev space

$$X := \left\{(u,v)^T : \ u,v \in H^2((0,l\pi)), \ (u_x,v_x)|_{x=0,l\pi} = (0,0)\right\}$$

and the complexification of X to be

$$X_{\mathbb{C}} := X \oplus iX = \left\{(x_1+y_1)+i(x_2+y_2) : \ (x_1,y_1)^T, (x_2,y_2)^T \in X\right\}.$$

For the sake of convenience, we denote $u_1(t) = u(t,\cdot)$, $u_2(t) = v(t,\cdot)$ and $U = (u_1,u_2)^T \in X$. Then system (2) can be rewritten as an abstract differential equation as follows:

$$\dot{U}(t) = d\triangle U(t) + L(m)U(t) + F(U(t)), \tag{4}$$

where

$$\begin{aligned} L(m) &= \begin{pmatrix} A(m) & B(m) \\ C(m) & 0 \end{pmatrix}, \ d = \operatorname{diag}(d_1,d_2), \ \operatorname{dom}(d\triangle) = X, \\ F(U) &= \begin{pmatrix} \alpha(u+m)\left(1-\dfrac{u+m}{K}\right) - \dfrac{uv}{1+u} - A(m)u - B(m)v \\ \left(-r + \dfrac{cu}{1+u}\right)v - C(m)u \end{pmatrix} \end{aligned} \tag{5}$$

and

$$\begin{aligned} A(m) &= \frac{\alpha}{Kc\theta^2}(rm^2 + (2\theta r - 2c\theta^2 - Kr)m + \theta(-2c\theta^2 + (cK+r)\theta - Kr)), \\ B(m) &= -\frac{r}{c} < 0, \ C(m) = \frac{\alpha r}{K\theta^2}(-m^2 + (K-2\theta)m + \theta(K-\theta)) > 0. \end{aligned} \tag{6}$$

The linearized equation of system (4) at $E_*(\theta, v_\theta)$ has the form:

$$\dot{U}(t) = d\triangle U(t) + L(m)U(t) \tag{7}$$

and its characteristic equation is

$$\lambda y - d\triangle y - L(m)y = 0, \quad \text{for some } y \in \operatorname{dom}(d\triangle)\backslash\{0\}. \tag{8}$$

It is well known that the operator $u \mapsto \triangle u$ with $\partial_\nu u = 0$ at 0 and $l\pi$ has eigenvalues $-n^2/l^2$ ($n \in \mathbb{N}_0$) with the corresponding eigenfunctions $\cos(nx/l)$. Let

$$\phi = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos\left(\frac{n}{l}x\right)$$

be an eigenfunction for $\triangle + L(m)$ with the eigenvalue λ, see also [10]. Hence, from the nth equation of Eq. (8), we have

$$\lambda^2 + T_n(m)\lambda + D_n(m) = 0, \tag*{$(9)_n$}$$

where

$$T_n(m) = (d_1+d_2)\frac{n^2}{l^2} - A(m), \tag*{$(10)_n$}$$

$$D_n(m) = d_1 d_2 \frac{n^4}{l^4} - d_2 A(m)\frac{n^2}{l^2} - B(m)C(m). \tag*{$(11)_n$}$$

We know from [40] that system (2) undergoes a Hopf bifurcation at the bifurcating point m^* which satisfies that

$$T_n(m^*) = 0, \ D_n(m^*) > 0 \text{ and } T_j(m^*) \neq 0, \ D_j(m^*) \neq 0 \text{ for any } j \neq n$$

and for the simple pair of complex eigenvalues near the imaginary axis $\omega_1(m) \pm i\omega_2(m)$, $\omega_1(m^*) = 0$, $\omega_2(m^*) \neq 0$ and $\omega_1'(m^*) \neq 0$.

Next, we consider the sign of $T_n(m)$ and $D_n(m)$ in detail.

When $n = 0$, $D_0(0) > 0$ and $T_0(m) = -A(m)$. E_* is locally asymptotically stable if $A(m) < 0$, simultaneously, E_* is unstable if $A(m) > 0$ and a potential spatially homogenous periodic solutions occur if $A(m) = 0$.

A straight computation gives that

$$m_0(\theta) = \frac{1}{2r}\left(Kr + 2c\theta^2 - 2\theta r - \sqrt{4\theta^4 c^2 + r^2 K^2}\right) \tag{12}$$

is a root of $A(m) = 0$ (also is a root of $T_0(m) = 0$), which also is a potential Hopf bifurcating point as the hypothesis

(**H2**) $K > 1$ and $0 < \theta < (K-1)/2$ (or equivalently $c > (K+1)r/(K-1)$)

holds via Theorem 2.1, and it is not difficult to see that m_0 is a function of θ which we write as $m_0(\theta)$.

Now, we assume **(H2)** holds and consider the case of $n \geq 1$ in $(10)_n$ and $(11)_n$.

From $(10)_n$, $T_n(m) = 0$ has a solution leads to

$$A(m) = (d_1 + d_2)\frac{n^2}{l^2} \tag{13}$$

has a solution. From the formula of $A(m)$ in (6) and visually explanation in the following Figure 1,

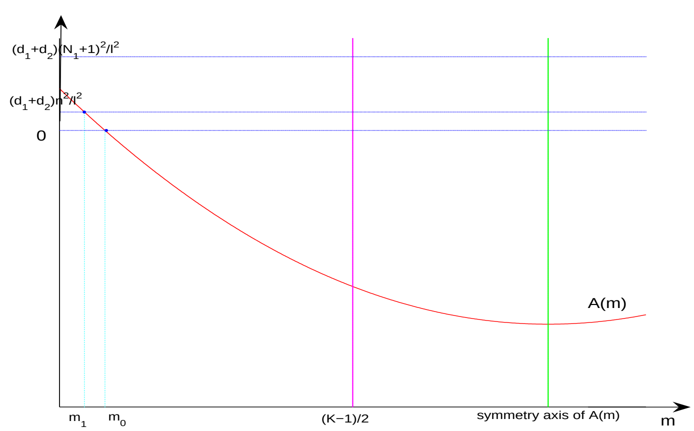

FIGURE 1. The relationship among $A(m)$, $(d_1 + d_2)n^2/l^2$ and m. The green solid line represents the symmetry axis of $A(m)$; the pink solid line represents the line of $m = (K-1)/2$; the red solid curve represents the curve trend of $A(m)$ as m varies and the blue ones represent the value of $(d_1 + d_2)n^2/l^2$ as n varies.

we have

$$A(m)\begin{cases} > 0, & \text{as } m \in (0, m_0), \\ = 0, & \text{as } m = m_0, \\ < 0, & \text{as } m \in (m_0, (K-1)/2) \end{cases}$$

and

$$A(0) = \max_{m\in[0,(K-1)/2]} A(m) = \frac{\alpha}{c\theta K}\left(-2c\theta^2 + (Kc+r)\theta - Kr\right),$$

which implies that the potential bifurcating points of system (2) must belong to $(0, m_0]$.

Solving Eq. (13) gives

$$m_n = m_n(\theta) = \frac{2c\theta^2 + Kr - 2\theta r - \sqrt{r^2K^2 + 4c^2\theta^4 + 4cK\theta^3(d_1+d_2)\dfrac{n^2}{l^2}}}{2r} \tag{14}$$

satisfying $0 < m_n < (K-1)/2$ and the following properties.

Lemma 2.2. *The bifurcating points $m_n(\theta)$ are finite, that is, there is a non-negative integer $N_1 \in \mathbb{N}_0$, such that $m_n(\theta)$ are bifurcating points as $0 \le n \le N_1$, for otherwise, $m_n(\theta)$ are not bifurcating points for any positive θ and $n > N_1$.*

Proof. We note that $A(0)$ is a locally maximum value of $A(m)$ as $m \in [0,(K-1)/2]$. That is, when n is big enough to $A(0) > (d_1+d_2)n^2/l^2$, we have $T_n(m) > 0$, which means system (2) does not exist any purely imaginary roots. Solving the equation $A(0) = (d_1+d_2)n^2/l^2$, we have $T_n(m) = 0$ has a unique solution $m_n(\theta)$ as $0 \le n \le N_1$ and $T_n(m) = 0$ has no solutions as $n > N_1$, where

$$N_1 = \left[\frac{\alpha l^2}{c\theta K(d_1+d_2)}(-2c\theta^2 + (cK+r)\theta - Kr)\right].$$

□

Lemma 2.3. *For any $\theta > 0$ and $0 \le n \le N_1$, $m_n(\theta)$ have the following relationships*

$$0 < m_{N_1}(\theta) < \cdots < m_n(\theta) < m_{n-1}(\theta) < \cdots < m_1(\theta) < m_0(\theta) < (K-1)/2.$$

Proof. From the formula of $m_n(\theta)$ in (12) and (14), we have the conclusion directly. □

Figure 2 shows the phenomena of Lemma 2.2 and Lemma 2.3:

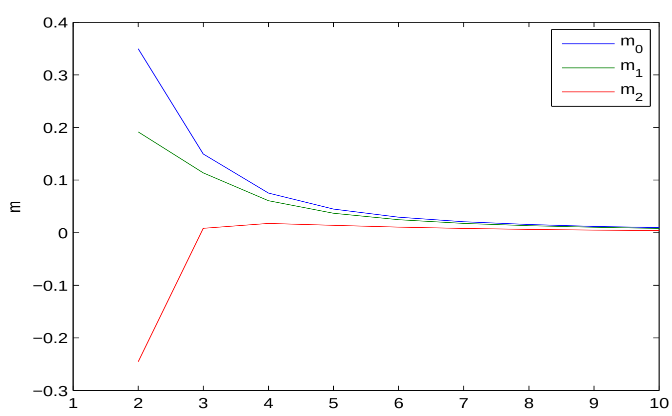

FIGURE 2. The phenomena of Lemma 2.2 and Lemma 2.3. Here, $r = 1, K = 5.5, \alpha = 1, d_1 = 2, d_2 = 0.5$.

Apparently, we have $T_n(m_n)=0,\ D_n(m_n)>0$ as $0\le n\le N_1$ and $T_j(m_n)\ne 0$ for any $j\ne n$. For convenience, we make the hypothesis

$$(\mathbf{H3})\ d_1/d_2 \ge \alpha(K-1-2\theta)^2/(4cK).$$

Clearly, $(\mathbf{H_3})$ holds when $D_j(m)>0$ is satisfied for any $j\ne n$ and any $m\in(0,(K-1)/2)$.

A straight calculation shows that the transversality condition is satisfied.

Lemma 2.4. *Assume* **(H1)** *and* **(H2)** *hold. Then for any* $m\in(0,(K-1)/2)$*, we have*

$$\mathrm{Re}(\lambda'(m))<0.$$

Proof. Suppose that the root of Eq. $(9)_n$ has the form $\lambda(m)=\omega_1(m)+i\omega_2(m)$, where $\omega_j(m)\in\mathbb{R}$ for any $j=1,2$ and

$$\mathrm{Re}(\lambda'(m))=\omega_1'(m)=\frac{A'(m)}{2}=\frac{\alpha}{2Kc\theta^2}(2rm+2\theta r-2c\theta^2-Kr).$$

By the admitted range of m, we have $\mathrm{Re}(\lambda'(m))=\omega_1'(m)<0$ for any $m\in(0,(K-1)/2)$. □

Summing up the above arguments, we obtain the following significant conclusions:

Theorem 2.5. *Suppose the hypotheses* **(H1)**−**(H3)** *hold. Then there are* N_1+1 *bifurcating points* m_n *satisfying*

$$0<m_{N_1}<m_{N_1-1}<\cdots<m_1<m_0<(K-1)/2,$$

such that system (2) *undergoes Hopf bifurcation at* $m=m_n$ *for any* $0\le n\le N_1$*. Moreover, we have*

(i) *if* $0<m<m_0$*, then* E_* *is unstable;*
(ii) *if* $m_0<m<(K-1)/2$*, then* E_* *is locally asymptotically stable;*
(iii) *the periodic solutions bifurcating from* $m=m_0$ *are spatially homogeneous and the periodic solutions bifurcating from* $m=m_n$ $(1\le m\le N_1)$ *are spatially non-homogeneous.*

Remark 1. Comparing with the system (1) or (2) with $m=0$ considered in [40], we obtain that adding a constant number of prey using refuge m does enhance the stability of the positive equilibrium which coincide with the results in [8] and other corresponding papers concerned the system incorporating a constant number prey refuge.

We give a visual illustration to the stability of E_* by Figure 3 and Figure 4.

3. Direction of Hopf bifurcation.

In this section, we shall study the direction of Hopf bifurcation near the positive equilibrium, stability and the period of bifurcating periodic solutions by applying the normal formal theory and the center manifold theorem of differential equations presented in [10]. For some fixed $0\le n\le N_1$, we denote $\widetilde{m}=m_n$, $\widetilde{\omega}=\omega_2(m_n)$ and compute the bifurcation direction near E_* at $m=\widetilde{m}$ and the purely imaginary root denoted as $\lambda=\omega_2(\widetilde{m})i=\widetilde{\omega}i$.

We make a variable change in system (2) by denoting $\widetilde{u}$ and $\widetilde{v}$ as $u-\theta$ and $v-v_\theta$, respectively. Drop the tilde as a matter of convenience, then the first two equations

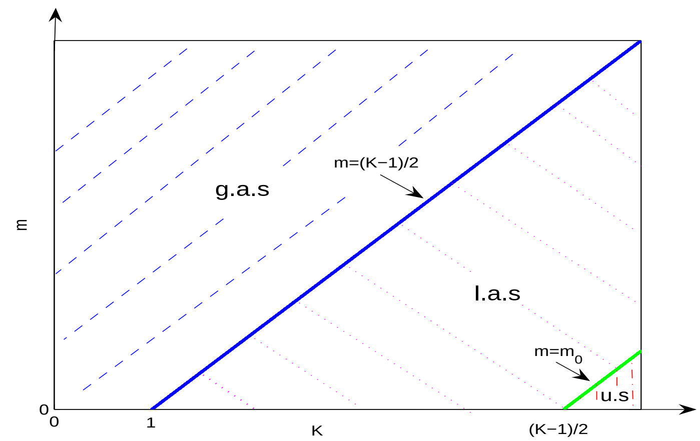

FIGURE 3. The stability diagram of E^* with m and K. Here, the green line $m = m_0$ represents the first Hopf bifurcating curve.

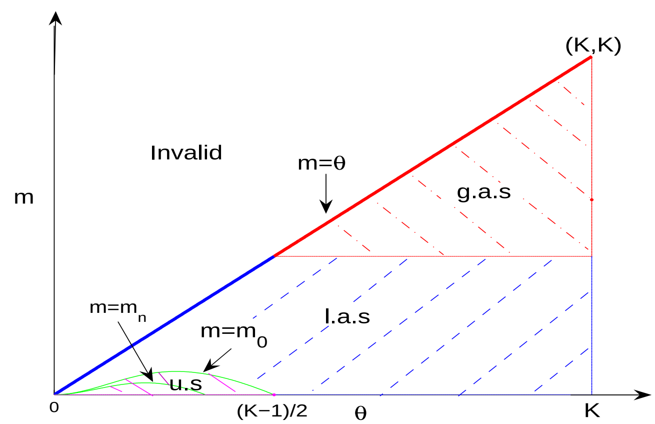

FIGURE 4. The stability diagram of E^* with m and θ as $K > 1$. Here, the green line $m = m_0$ and $m = m_n$ represent the Hopf bifurcating curve.

of system (2) can be transformed into

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \triangle u + \alpha(u + m + \theta)\left(1 - \dfrac{u + m + \theta}{K}\right) - \dfrac{(u+\theta)(v + v_\theta)}{1 + u + \theta}, \\ \dfrac{\partial v}{\partial t} = d_2 \triangle v + (v + v_\theta)\left(-r + \dfrac{c(u+\theta)}{1 + u + \theta}\right), \end{cases} \tag{15}$$

for $x \in \Omega = (0, l\pi)$ and $t \in (0, +\infty)$. Using the similar notations as in (4) and (5), we rewrite system (15) as the following abstract differential equation:

$$\dot{U}(t) = \widetilde{L}U + F(U, \widetilde{m}), \text{ for any } U = (u, v)^T \in X, \tag{16}$$

where

$$\widetilde{L} = d\triangle + L(\widetilde{m}) = \begin{pmatrix} \widetilde{A} + d_1 \frac{\partial^2}{\partial x^2} & \widetilde{B} \\ \widetilde{C} & d_2 \frac{\partial^2}{\partial x^2} \end{pmatrix}, \tag{17}$$

$$F(U,\widetilde{m}) = \begin{pmatrix} F_1(U,\widetilde{m}) \\ F_2(U,\widetilde{m}) \end{pmatrix} = \begin{pmatrix} \alpha(u+m+\theta)\left(1 - \dfrac{u+m+\theta}{K}\right) - \dfrac{(u+\theta)(v+v_\theta)}{1+u+\theta} - \widetilde{A}u - \widetilde{B}v \\ (v+v_\theta)\left(-r + \dfrac{c(u+\theta)}{1+u+\theta}\right) - \widetilde{C}u \end{pmatrix}, \tag{18}$$

and

$$\begin{aligned} \widetilde{A} = A(\widetilde{m}) &= \alpha\left(1 - \frac{2(\theta+\widetilde{m})}{K} - \frac{r(\theta+\widetilde{m})}{c\theta^2}\left(1 - \frac{\theta+\widetilde{m}}{K}\right)\right), \\ \widetilde{B} = B(\widetilde{m}) &= -\frac{r}{c} < 0, \ \widetilde{C} = C(\widetilde{m}) = \frac{cv_\theta}{(1+\theta)^2} > 0. \end{aligned} \tag{19}$$

Let $< \cdot, \cdot >$ be the complex-valued L^2 inner product on Hilbert space $X_{\mathbb{C}}$, defined by

$$< U_1, U_2 >= \int_0^{l\pi} \bar{u}_1 u_2 + \bar{v}_1 v_2 dx, \text{ for any } U_j = (u_j, v_j)^T \in X_{\mathbb{C}} \text{ and } j = 1, 2. \tag{20}$$

With the help of the definition of inner product in (20), we define the adjoint operator of the operator $\widetilde{L}$ as $\widetilde{L}^*$ on $\mathcal{D}_{\widetilde{L}^*} = X_{\mathbb{C}}$ satisfying

$$\widetilde{L}^* = \begin{pmatrix} d_1 \dfrac{\partial^2}{\partial x^2} + \widetilde{A} & \widetilde{C} \\ \widetilde{B} & d_2 \dfrac{\partial^2}{\partial x^2} \end{pmatrix}.$$

By a direct computation, we obtain respectively the eigenfunctions of $\widetilde{L}$ and $\widetilde{L}^*$ corresponding to the eigenvalue $i\widetilde{\omega}$ and $-i\widetilde{\omega}$ on $X_{\mathbb{C}}$ denoted by

$$q = \begin{pmatrix} 1, & b_n \end{pmatrix}^T \cos\left(\frac{n}{l}x\right) \quad \text{and} \quad q^* = \begin{pmatrix} a_n^*, & b_n^* \end{pmatrix}^T \cos\left(\frac{n}{l}x\right)$$

satisfying $< q^*, q >= 1$ and $< q^*, \bar{q} >= 0$, where

$$b_n = -\frac{d_2 n^2}{\widetilde{B} l^2} + \frac{\widetilde{\omega}}{\widetilde{B}} i, \ a_n^* = \frac{1}{l\pi} + \frac{d_2 n^2}{\widetilde{\omega} l^3 \pi} i \ \text{ and } \ b_n^* = \frac{\widetilde{B}}{l\pi\widetilde{\omega}} i.$$

We decompose $X = X^C \oplus X^S$ with the center subspace $X^C := \{zq + \bar{z}\bar{q} : \ z \in \mathbb{C}\}$ and the stable subspace $X^S := \{u \in X : < q^*, u >= 0\}$. For any $(u, v)^T \in X$, there exists $z \in \mathbb{C}$ and $w = (w_1, \ w_2)^T \in X^S$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = zq + \bar{z}\bar{q} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \ \text{ or } \ \begin{cases} u = z\cos\left(\frac{n}{l}x\right) + \bar{z}\cos\left(\frac{n}{l}x\right) + w_1, \\ v = b_n z \cos\left(\frac{n}{l}x\right) + \overline{b_n}\bar{z}\cos\left(\frac{n}{l}x\right) + w_2. \end{cases} \tag{21}$$

From (21), system (16) is equivalent to the following system:

$$\begin{cases} \dfrac{dz}{dt} = i\widetilde{\omega} z + < q^*, F_0 >, \\ \dfrac{dw}{dt} = \widetilde{L}w + H(z, \bar{z}, w), \end{cases} \tag{22}$$

where

$$H(z, \bar{z}, w) = F_0 - < q^*, F_0 > q - < \overline{q^*}, F_0 > \bar{q} \tag{23}$$

and $F_0 := F_0(zq + \bar{z}\bar{q} + w)$.

On the other hand, F_0 has the form from [10]:

$$F_0(U) := \frac{1}{2}Q(U,U) + \frac{1}{6}C(U,U,U) + \mathcal{O}(|U|^4),$$

where $U = (u,v)^T$, Q and C are symmetric multilinear forms. Here, using the notations mentioned in [10] and [40], we denote $Q_{XY} = Q(X,Y)$ and $C_{XYZ} = C(X,Y,Z)$. A straight calculation shows

$$\begin{aligned} Q_{qq} &= \begin{pmatrix} c_n \\ d_n \end{pmatrix} \cos^2\left(\frac{n}{l}x\right),\ Q_{\bar{q}\bar{q}} = \overline{Q_{qq}}, \\ Q_{q\bar{q}} &= \begin{pmatrix} e_n \\ f_n \end{pmatrix} \cos^2\left(\frac{n}{l}x\right),\ C_{qq\bar{q}} = \begin{pmatrix} g_n \\ h_n \end{pmatrix} \cos^3\left(\frac{n}{l}x\right), \end{aligned} \tag{24}$$

where

$$\begin{aligned} c_n &= F_{1uu} + 2F_{1uv}b_n,\ d_n = F_{2uu} + 2F_{2uv}b_n, \\ e_n &= F_{1uu} + F_{1uv}(\overline{b_n} + b_n),\ f_n = F_{2uu} + F_{2uv}(\overline{b_n} + b_n), \\ g_n &= F_{1uuu} + F_{1uuv}(2b_n + \overline{b_n}),\ h_n = F_{2uuu} + F_{2uuv}(2b_n + \overline{b_n}) \end{aligned} \tag{25}$$

and

$$\begin{aligned} F_{1uu} &= \frac{2(Kv_\theta - \alpha(1+\theta)^3)}{K(1+\theta)^3},\ F_{1uv} = -\frac{1}{(1+\theta)^2},\ F_{2uu} = -\frac{2cv_\theta}{(1+\theta)^3}, \\ F_{2uv} &= \frac{c}{(1+\theta)^2},\ F_{1uuu} = -\frac{6v_\theta}{(1+\theta)^4},\ F_{1uuv} = \frac{2}{(1+\theta)^3}, \\ F_{2uuu} &= \frac{6cv_\theta}{(1+\theta)^4},\ F_{2uuv} = -\frac{2c}{(1+\theta)^3}, \\ F_{1vv} &= F_{2vv} = F_{1uvv} = F_{1vvv} = F_{2uvv} = F_{2vvv} = 0. \end{aligned} \tag{26}$$

From the formula of H in (23), we denote

$$H(z,\bar{z},w) = \frac{H_{20}}{2}z^2 + H_{11}z\bar{z} + \frac{H_{02}}{2}\bar{z}^2 + \mathcal{O}(|z|\cdot w), \tag{27}$$

then we have

$$\begin{aligned} H_{20} &= Q_{qq} - < q^*, Q_{qq} > q - < \overline{q^*}, Q_{qq} > \bar{q} \\ &= \begin{cases} \cos^2\left(\frac{n}{l}x\right)\begin{pmatrix} c_n \\ d_n \end{pmatrix}, & n \in \mathbb{N}, \\ \frac{1}{\widetilde{B}^2 l\pi}\begin{pmatrix} \widetilde{B}(l\pi - 2)(G_1 + 2\widetilde{\omega}F_{1uv}i) \\ G_2 + 2\widetilde{\omega}G_3 i \end{pmatrix}, & n = 0, \end{cases} \\ H_{11} &= Q_{q\bar{q}} - < q^*, Q_{q\bar{q}} > q - < \overline{q^*}, Q_{q\bar{q}} > \bar{q} \\ &= \begin{cases} \cos^2\left(\frac{n}{l}x\right)\begin{pmatrix} e_n \\ f_n \end{pmatrix}, & n \in \mathbb{N}, \\ \frac{1}{\widetilde{B}^2 l\pi}\begin{pmatrix} \widetilde{B}(l\pi - 2)G_1 \\ G_2 \end{pmatrix}, & n = 0, \end{cases} \\ H_{20} &= H_{02}, \end{aligned} \tag{28}$$

where

$$
\begin{aligned}
< q^*, Q_{qq} > &= \begin{cases} 0, & n \in \mathbb{N}, \\ \dfrac{\widetilde{\omega} G_1 + 2\widetilde{B}\widetilde{\omega} F_{2uv} + i(G_4 + 2\widetilde{\omega}^2 F_{1uv})}{\widetilde{B}\widetilde{\omega}}, & n = 0, \end{cases} \\
< \overline{q^*}, Q_{qq} > &= \begin{cases} 0, & n \in \mathbb{N}, \\ \dfrac{\widetilde{\omega} G_1 - 2\widetilde{B}\widetilde{\omega} F_{2uv} + i(2\widetilde{\omega}^2 F_{1uv} - G_4)}{\widetilde{B}\widetilde{\omega}}, & n = 0, \end{cases} \\
< q^*, Q_{q\bar{q}} > &= \begin{cases} 0, & n \in \mathbb{N}, \\ \dfrac{\widetilde{\omega} G_1 + iG_4}{\widetilde{B}\widetilde{\omega}}, & n = 0, \end{cases} \\
< \overline{q^*}, Q_{q\bar{q}} > &= \begin{cases} 0, & n \in \mathbb{N}, \\ \dfrac{\widetilde{\omega} G_1 - iG_4}{\widetilde{B}\widetilde{\omega}}, & n = 0, \end{cases}
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
&G_1 = \widetilde{B} F_{1uu} - 2\widetilde{A} F_{1uv}, \\
&G_2 = \widetilde{B}^2 (l\pi - 2) F_{2uu} - 4\widetilde{A}^2 F_{1uv} + 2\widetilde{A}\widetilde{B}(F_{1uu} + (2 - l\pi) F_{2uv}), \\
&G_3 = 2\widetilde{A} F_{1uv} + \widetilde{B}(l\pi - 2) F_{2uv}, \\
&G_4 = 2\widetilde{B}(\widetilde{A} F_{2uv} - \widetilde{B} F_{2uu}), \\
&\int_0^{l\pi} \cos^2\left(\frac{n}{l}x\right) dx = \frac{l\pi}{2}, \qquad \int_0^{l\pi} \cos^3\left(\frac{n}{l}x\right) dx = 0, \\
&\int_0^{l\pi} \cos^4\left(\frac{n}{l}x\right) dx = \frac{3l\pi}{8}, \qquad \int_0^{l\pi} \cos\left(\frac{2n}{l}x\right) \cos^2\left(\frac{n}{l}x\right) dx = \frac{l\pi}{4}.
\end{aligned}
$$

Denote

$$
w = \frac{w_{20}}{2} z^2 + w_{11} z\bar{z} + \frac{w_{02}}{2} \bar{z}^2 + \mathcal{O}(|z|^3),
$$

then we have

$$
w_{20} = (2i\widetilde{\omega} I - \widetilde{L})^{-1} H_{20}, \; w_{11} = -\widetilde{L}^{-1} H_{11}. \tag{30}
$$

We compute the values of w_{20} and w_{11} in the case of $n \neq 0$ and $n = 0$, respectively. At first, we consider the case of $n \neq 0$. A straightforward calculation shows

$$
[2i\widetilde{\omega} I - \widetilde{L}_{2n}]^{-1} = (\beta_1 + i\beta_2) \begin{pmatrix} 2i\widetilde{\omega} + \dfrac{4d_2 n^2}{l^2} & \widetilde{B} \\ \widetilde{C} & 2i\widetilde{\omega} - \dfrac{(d_2 - 3d_1)n^2}{l^2} \end{pmatrix},
$$

$$
[2i\widetilde{\omega} I - \widetilde{L}_0]^{-1} = (\beta_3 + i\beta_3) \begin{pmatrix} 2i\widetilde{\omega} & \widetilde{B} \\ \widetilde{C} & 2i\widetilde{\omega} - \dfrac{(d_1 + d_2)n^2}{l^2} \end{pmatrix},
$$

$$
\widetilde{L}_{2n} = \begin{pmatrix} \dfrac{(3d_1 - d_2)n^2}{l^2} & \widetilde{B} \\ \widetilde{C} & 4d_2 \dfrac{n^2}{l^2} \end{pmatrix}, \qquad \widetilde{L}_0 = \begin{pmatrix} -\dfrac{(d_1 + d_2)n^2}{l^2} & B(\widetilde{m}) \\ C(\widetilde{m}) & 0 \end{pmatrix}
$$

and

$$\widetilde{L}_{2n}^{-1} = \beta_5 \begin{pmatrix} 4d_2\dfrac{n^2}{l^2} & \widetilde{B} \\ \widetilde{C} & \dfrac{(3d_1-d_2)n^2}{l^2} \end{pmatrix}, \qquad \widetilde{L}_0^{-1} = \beta_6 \begin{pmatrix} 0 & B(\widetilde{m}) \\ C(\widetilde{m}) & -\dfrac{(d_1+d_2)n^2}{l^2} \end{pmatrix},$$

with

$$\begin{aligned}
\beta_1 &= \frac{3d_2(4d_1-d_2)n^4 - 3\widetilde{\omega}^2 l^4}{9(d_2^2(d_2^2+4d_1^2)n^8 - 8d_1d_2^3n^6 + 2\widetilde{\omega}^2 l^4(3d_2^2+2d_2^2)n^4 + l^8\widetilde{\omega}^4)},\\
\beta_2 &= \frac{-6\widetilde{\omega}(d_1+d_2)l^2n^2 i}{9(d_2^2(d_2^2+4d_1^2)n^8 - 8d_1d_2^3n^6 + 2\widetilde{\omega}^2 l^4(3d_2^2+2d_2^2)n^4 + l^8\widetilde{\omega}^4)},\\
\beta_3 &= \frac{d_2^2 l^4 - 3\widetilde{\omega}^2 l^4}{d_2^2 n^8 + 9l^8\widetilde{\omega}^4 + n^4(4d_1^2 l^4\widetilde{\omega}^2 + 2(4d_1-3)d_2 l^4\widetilde{\omega}^2 + 4d^2 l^4\widetilde{\omega}^2)},\\
\beta_4 &= \frac{2\widetilde{\omega}(d_1+d_2)l^2n^2}{d_2^2 n^8 + 9l^8\widetilde{\omega}^4 + n^4(4d_1^2 l^4\widetilde{\omega}^2 + 2(4d_1-3)d_2 l^4\widetilde{\omega}^2 + 4d^2 l^4\widetilde{\omega}^2)},\\
\beta_5 &= \frac{l^4}{\widetilde{\omega}^2 l^4 - 3d_2(d_2-4d_1)n^4},\\
\beta_6 &= \frac{l^4}{d_2^2 n^4 - \widetilde{\omega}^2 l^4}.
\end{aligned}$$

From (30), we have

$$w_{20} = \begin{pmatrix} \xi_{20n1} \\ \xi_{20n2} \end{pmatrix} \cos\left(\frac{2n}{l}x\right) + \begin{pmatrix} \xi_{2001} \\ \xi_{2002} \end{pmatrix}, \ w_{11} = \begin{pmatrix} \xi_{11n1} \\ \xi_{11n2} \end{pmatrix} \cos\left(\frac{2n}{l}x\right) + \begin{pmatrix} \xi_{1101} \\ \xi_{1102} \end{pmatrix},$$

where

$$\begin{aligned}
\xi_{20n1} &= \frac{\beta_1 + i\beta_2}{2}\left(\left(2i\widetilde{\omega} + \frac{4d_2n^2}{l^2}\right)c_n + \widetilde{B}d_n\right),\\
\xi_{20n2} &= \frac{\beta_1 + i\beta_2}{2}\left(\widetilde{C}c_n + \left(2i\widetilde{\omega} + \frac{(3d_1-d_2)n^2}{l^2}\right)d_n\right),\\
\xi_{2001} &= \frac{\beta_3 + i\beta_4}{2}(2i\widetilde{\omega}c_n + \widetilde{B}d_n),\\
\xi_{2002} &= \frac{\beta_3 + i\beta_4}{2}\left(\widetilde{C}c_n + \left(2i\widetilde{\omega} - \frac{(d_1+d_2)n^2}{l^2}\right)d_n\right),\\
\xi_{11n1} &= \frac{\beta_5}{2}\left(\frac{4d_2n^2}{l^2}e_n + \widetilde{B}f_n\right),\\
\xi_{11n2} &= \frac{\beta_5}{2}\left(\widetilde{C}e_n + \frac{(3d_1-d_2)n^2}{l^2}f_n\right),\\
\xi_{1101} &= \frac{\beta_6}{2}\widetilde{B}f_n,\\
\xi_{1102} &= \frac{\beta_6}{2}\left(\widetilde{C}e_n - \frac{(d_1+d_2)n^2}{l^2}f_n\right).
\end{aligned}$$

When $n = 0$, we have

$$w_{20} = (w_{20}^1, w_{20}^2)^T, \ w_{11} = (w_{11}^1, w_{11}^2)^T,$$

where

$$
\begin{aligned}
w_{20}^1 =& \frac{1}{\widetilde{B}^2\widetilde{C}l\pi(9\widetilde{B}\widetilde{C}-4\widetilde{A}^2)}\{\widetilde{B}\widetilde{C}(4\widetilde{A}^2(3-2l\pi)F_{1uv}+3\widetilde{B}(4\widetilde{C}F_{1uv}+\widetilde{B}(l\pi-2)F_{2uu})\\
&-2\widetilde{A}\widetilde{B}(F_{1uu}(1-2l\pi)+F_{2uv}(l\pi-2)))+2i\widetilde{\omega}(3(l\pi-2)\widetilde{B}^2\widetilde{C}(F_{1uu}+F_{2uv})\\
&-4\widetilde{A}^3F_{1uv}+2\widetilde{A}^2\widetilde{B}(F_{1uu}+(2-l\pi)F_{2uv})+\widetilde{A}\widetilde{B}(2\widetilde{C}F_{1uv}(5-l\pi)\\
&+\widetilde{B}(l\pi-2)F_{2uu}))\},\\
w_{20}^2 =& \frac{1}{\widetilde{B}^3\widetilde{C}l\pi(9\widetilde{B}\widetilde{C}-4\widetilde{A}^2)}\{\widetilde{B}\widetilde{C}(3(l\pi-2)\widetilde{B}^2\widetilde{C}(F_{1uu}+4F_{2uv})-12\widetilde{A}^3F_{1uv}+2\widetilde{A}^2\widetilde{B}\\
&(F_{1uu}+6F_{2uv}-3l\pi F_{2uv})+\widetilde{A}\widetilde{B}(2\widetilde{C}F_{1uv}(14-l\pi))+\widetilde{B}(l\pi-2)F_{2uu})+2\widetilde{\omega}i\\
&(4\widetilde{A}^4F_{1uv}+3\widetilde{B}^2\widetilde{C}(\widetilde{C}F_{1uv}+\widetilde{B}F_{2uv})(l\pi-2)-2\widetilde{A}^3\widetilde{B}(F_{1uu}+(2-l\pi)F_{2uv})\\
&-\widetilde{A}^2\widetilde{B}(\widetilde{B}F_{2uu}(l\pi-2)+2(3+l\pi)\widetilde{C}F_{1uv})+\widetilde{A}\widetilde{B}^2\widetilde{C}(5(2-l\pi)F_{2uv}\\
&+F_{1uu}(4+l\pi)))\},\\
w_{11}^1 =& \frac{1}{\widetilde{B}^2\widetilde{C}l\pi}\{4\widetilde{A}^2F_{1uv}+\widetilde{B}^2F_{2uu}(2-l\pi)-2\widetilde{A}\widetilde{B}(F_{1uu}+(2-l\pi)F_{2uv})\},\\
w_{11}^2 =& \frac{1}{\widetilde{B}^3\widetilde{C}l\pi}\{\widetilde{B}^2\widetilde{C}F_{1uu}(2-l\pi)-4\widetilde{A}^3F_{1uv}+\widetilde{A}\widetilde{B}(2\widetilde{C}F_{1uv}+\widetilde{B}F_{2uu})(l\pi-2)\\
&+2\widetilde{A}^2\widetilde{B}(F_{1uu}+F_{2uv}(2-l\pi))\}.
\end{aligned}
$$

From [10], we know that system (15) restricted to the center manifold is given by

$$
\frac{dz}{dt}=i\widetilde{\omega}z+<q^*,F_0>=i\widetilde{\omega}z+\sum_{2\le i+j\le 3}\frac{g_{ij}}{i!j!}z^i\bar{z}^j+\mathcal{O}(|z|^4),
$$

where

$$
g_{20}=<q^*,Q_{qq}>,\ \ g_{11}=<q^*,Q_{q\bar{q}}>,\ \ g_{02}=<q^*,Q_{\bar{q}\bar{q}}> \tag{31}
$$

and

$$
g_{21}=2<q^*,Q_{w_{11}q}>+<q^*,Q_{w_{20}\bar{q}}>+<q^*,C_{qq\bar{q}}>. \tag{32}
$$

Denote

$$
\begin{aligned}
c_1(0)&=\frac{i}{2\widetilde{\omega}}(g_{20}g_{11}-2|g_{11}|^2-\frac{|g_{02}|^2}{3})+\frac{g_{21}}{2},\quad \mu_2=-\frac{\operatorname{Re}(c_1(0))}{\operatorname{Re}(\lambda'(\widetilde{m}))},\\
T_2&=-\frac{1}{\widetilde{\omega}}(\operatorname{Im}(c_1(0))+\mu_2\operatorname{Im}(\lambda'(\widetilde{m}))),\quad \beta_2^*=2\operatorname{Re}(c_1(0)).
\end{aligned}
$$

We know the following conclusions from [10]:

Theorem 3.1. *For any critical value $\widetilde{m}$, we have*

(i) *μ_2 determines the directions of the Hopf bifurcation: if $\mu_2>0$ (<0), then the direction of the Hopf bifurcation is forward (backward), that is, the bifurcating periodic solutions exists for $m>\widetilde{m}$ $(m<\widetilde{m})$;*
(ii) *β_2^* determines the stability of the bifurcating periodic solutions on the center manifold: if $\beta_2^*<0$ (>0), then the bifurcating periodic solutions are orbitally asymptotically stable(unstable);*
(iii) *T_2 determines the period of the bifurcating periodic solutions: if $T_2>0$ (<0), then the period increases (decreases).*

Now, we compute the value of g_{21} defining in (32) as $n \neq 0$ and $n = 0$, since the value of g_{20} and g_{11} is determined by (29) and (31).As $n \neq 0$, $g_{20} = g_{11} = g_{02} = 0$,

$$
\begin{aligned}
< q^*, Q_{w_{20}\bar{q}} >=& \frac{l\pi}{4}\{\overline{a_n^*}(F_{1uu}\xi_{20n1} + F_{1uv}(\xi_{20n1}\overline{b_n} + \xi_{20n2})) + \overline{b_n^*}(F_{2uu}\xi_{20n1} + F_{2uv} \\
& (\xi_{20n1}\overline{b_n} + \xi_{20n2}))\} + \frac{l\pi}{2}\{\overline{a_n^*}(F_{1uu}\xi_{2001} + F_{1uv}(\xi_{2001}\overline{b_n} + \xi_{2002})) \\
& + \overline{b_n^*}(F_{2uu}\xi_{2001} + F_{2uv}(\xi_{2001}\overline{b_n} + \xi_{2002}))\}, \\
< q^*, Q_{w_{11}q} >=& \frac{l\pi}{4}\{\overline{a_n^*}(F_{1uu}\xi_{11n1} + F_{1uv}(\xi_{11n1}\overline{b_n} + \xi_{11n2})) + \overline{b_n^*}(F_{2uu}\xi_{11n1} + F_{2uv} \\
& (\xi_{11n1}\overline{b_n} + \xi_{11n2}))\} + \frac{l\pi}{2}\{\overline{a_n^*}(F_{1uu}\xi_{1101} + F_{1uv}(\xi_{1101}\overline{b_n} + \xi_{1102})) \\
& + \overline{b_n^*}(F_{2uu}\xi_{1101} + F_{2uv}(\xi_{1101}\overline{b_n} + \xi_{1102}))\}, \\
< q^*, C_{qq\bar{q}} >=& \frac{3l\pi}{8}(\overline{a_n^*}g_n + \overline{b_n^*}h_j)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Re}(g_{21}) =& \frac{1}{8\widetilde{B}\widetilde{\omega}}\{2\widetilde{B}\widetilde{\omega}(F_{1uu} + b_n^1F_{1uv})((\xi_{20n1}^1 + 2\xi_{2001}^1) + 2(\xi_{11n1}^1 + 2\xi_{1101}^1)) - 2\widetilde{B} \\
& (b_n^2d_2F_{1uv} + \widetilde{B}b_n^2F_{2uv})((\xi_{20n1}^1 + 2\xi_{2001}^1) - 2(\xi_{11n1}^1 + 2\xi_{1101}^1)) + 2\widetilde{\omega}\widetilde{B}b_n^2 \\
& F_{1uv}((\xi_{20n1}^2 + 2\xi_{2001}^2) - 2(\xi_{11n1}^2 + 2\xi_{1101}^2)) + 2\widetilde{B}(d_2F_{1uu} + b_n^1d_2F_{1uv} + \\
& \widetilde{B}F_{2uu} + \widetilde{B}b_n^1F_{2uv})((\xi_{20n1}^2 + 2\xi_{2001}^2) + 2(\xi_{11n1}^2 + 2\xi_{1101}^2)) + 2\widetilde{B}(d_2F_{1uv} \\
& + \widetilde{B}F_{2uv})((\xi_{20n2}^2 + 2\xi_{2002}^2) + 2(\xi_{11n2}^2 + 2\xi_{1102}^2)) + 2\widetilde{\omega}\widetilde{B}F_{1uv}((2\xi_{2002}^1 \\
& + \xi_{20n2}^1) + 2(\xi_{11n2}^1 + 2\xi_{1102}^1)) + 3\widetilde{\omega}(-2d_2F_{1uuv} + \widetilde{B}(F_{1uuu} + F_{2uuv}))\}, \\
\beta_2^* =& 2\,\mathrm{Re}(c_1(0)) = \mathrm{Re}(g_{21}),\ \ \mu_2 = \mathrm{Re}(c_1(0)) = \mathrm{Re}(g_{21})/2,
\end{aligned}
$$

where for $j = 0$ or n and $k = 1, 2$,

$$
b_n = b_n^1 + ib_n^2, \quad \xi_{20jk} = \xi_{20jk}^1 + i\xi_{20jk}^2, \quad \xi_{11jk} = \xi_{11jk}^1 + i\xi_{11jk}^2.
$$

As $n = 0$,

$$
\begin{aligned}
c_0 =& \frac{2(-\alpha(1+\theta)(k - 2m + (-1+\theta)\theta) + k(v_\theta + i\sqrt{rv_\theta}))}{k\theta(1+\theta)^2}, \\
d_0 =& \frac{2c\left(-ik(-iv_\theta + \sqrt{rv_\theta}) + \alpha(1+\theta)(k - 2(m+\theta))\right)}{k\theta(1+\theta)^2}, \\
e_0 =& \frac{2kv_\theta - 2al(1+\theta)(k - 2m + (-1+\theta)\theta)}{k\theta(1+\theta)^2}, \\
f_0 =& \frac{2c(-kv_\theta + al(1+\theta)(k - 2(m+\theta)))}{k\theta(1+\theta)^2}, \\
g_0 =& \frac{2(-ik\left(-3iv_\theta + \sqrt{rv_\theta}\right) + 3\alpha(1+\theta)(k - 2(m+\theta)))}{k\theta(1+\theta)^3}, \\
h_0 =& \frac{2c\left(k\left(3v_\theta + i\sqrt{rv_\theta}\right) - 3\mathrm{al}(1+\theta)(k - 2(m+\theta))\right)}{k\theta(1+\theta)^3}
\end{aligned}
$$

and

$$
\begin{aligned}
g_{21} =& \frac{1}{\widetilde{B}^8\widetilde{C}^2 l^4\pi^4(9\widetilde{B}\widetilde{C}-4(d_1+d_2)^2)}((-2(9\widetilde{B}\widetilde{C}-4(d_1+d_2)^2)(-4(d_1+d_2)^4 {F_{1uv}}^2 \\
&+\widetilde{B}^4(F_{1uu}+F_{1uv})F_{2uu}(l\pi-2)+2\widetilde{B}(d_1+d_2)^2F_{1uv}(\widetilde{C}F_{1uv}(l\pi-2)+d_2 \\
&(F_{1uu}+4F_{2uv}-F_{2uv}l\pi)+d_1(F_{1uu}+F_{2uv}(4-l\pi)))-\widetilde{B}^2(d_1+d_2)(\widetilde{C}F_{1uv} \\
&(F_{1uu}+2F_{2uv})(l\pi-2)+(d_1+d_2)(4{F_{1uv}}^2+2F_{1uu}(2F_{1uv}+F_{2uv})+F_{1uv} \\
&F_{2uu}(2-l\pi)-2{F_{2uv}}^2(l\pi-2)))+\widetilde{B}^3(\widetilde{C}F_{1uu}F_{2uv}(l\pi-2)+d_1(2{F_{1uu}}^2-(2 \\
&F_{1uv}+F_{2uu})F_{2uv}(l\pi-2)+2F_{1uu}(F_{1uv}+2F_{2uv}-F_{2uv}l\pi))+d_2(2{F_{1uu}}^2 \\
&-(2F_{1uv}+F_{2uu})F_{2uv}(l\pi-2)+2F_{1uu}(F_{1uv}+2F_{2uv}-F_{2uv}l\pi))))+\widetilde{B}^5\widetilde{C}l^2 \\
&\pi^2(4(d_1+d_2)^4F_{1uv}(5\widetilde{C}F_{1uv}-2(d_1+d_2)F_{2uv})+\widetilde{B}^4(2(d_1+d_2)F_{2uu}(F_{2uu} \\
&+F_{2uv})+3\widetilde{C}(F_{1uv}F_{2uu}+2F_{2uv}(F_{2uu}+F_{2uv})+F_{1uu}(3F_{2uu}+2F_{2uv})))(l\pi \\
&-2)-2\widetilde{B}(d_1+d_2)^2(\widetilde{C}^2{F_{1uv}}^2(20+l\pi)+\widetilde{C}(d_1+d_2)F_{1uv}(3F_{1uu}+F_{2uv}(10 \\
&-7l\pi))-2(d_1+d_2)^2F_{2uv}(F_{1uu}+F_{2uv}(2-l\pi)))+\widetilde{B}^3(3\widetilde{C}^2(4{F_{1uv}}^2+2F_{1uv} \\
&F_{2uu}+4{F_{2uv}}^2+F_{1uu}(4F_{1uv}+F_{2uv}))(l\pi-2)+4(d_1+d_2)^2(F_{2uu}+F_{2uv}) \\
&(F_{1uu}+F_{2uv}(2-l\pi))+\widetilde{C}(d_1+d_2)(-5F_{2uu}F_{2uv}(l\pi-2)+{F_{1uu}}^2(4l\pi-2) \\
&+F_{1uv}(20F_{2uu}+24F_{2uv}-4F_{2uu}l\pi-6F_{2uv}l\pi)+F_{1uu}(-2F_{1uv}+4F_{2uv} \\
&+4F_{1uv}l\pi-2F_{2uv}l\pi)))+\widetilde{B}^2(6\widetilde{C}^3{F_{1uv}}^2(-2+l\pi)-2(d_1+d_2)^3(4F_{1uv}(F_{2uu} \\
&+F_{2uv})+F_{2uu}F2_{uv}(2-l\pi))-\widetilde{C}^2(d_1+d_2)F_{1uv}(F_{1uu}(l\pi-14)+6F_{2uv}(5l\pi \\
&-14))-\widetilde{C}(d_1+d_2)^2(3F_{1uv}F_{2uu}(l\pi-2)-4F_{2uv}^2(l\pi-2)+4F_{1uv}^2(2l\pi-3) \\
&+2F_{1uu}(F_{2uv}(3+l\pi)+F_{1uv}(4l\pi-6))))))).
\end{aligned}
$$

4. **Numerical simulations.** In this section, we present some numerical simulations to illustrate the theoretical analysis and symbolic mathematical software Matlab is used to plot numerical graphs.

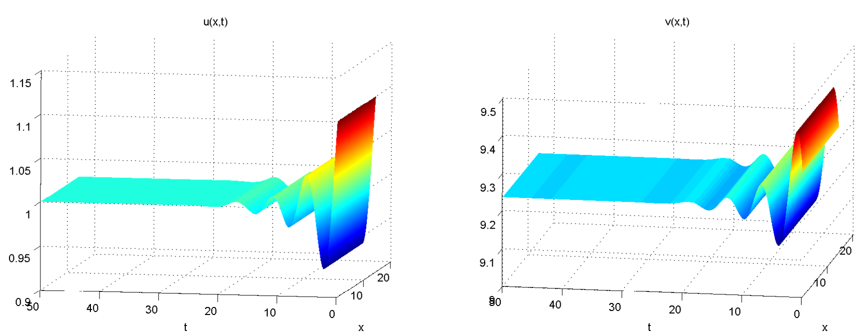

FIGURE 5. The numerical simulations of system (2.1) with $m = 0.9 > m_0 = 0.6148$. Left: component u (stable). Right: component v (stable).

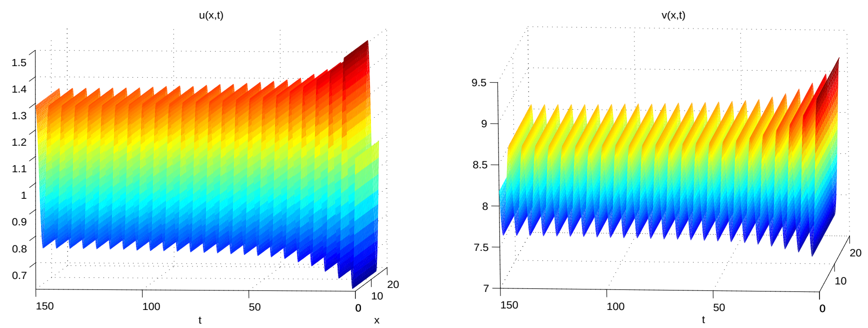

FIGURE 6. The numerical simulations of a stable homogeneous equilibrium solution of system (2.1) with $m = 0.61 < m_0 = 0.6148$. Left: component u (stable). Right: component v (stable).

We consider the system (2) with $d_1 = 1$, $d_2 = 0.2$, $r = 0.5$, $K = 10$, $\alpha = 3$, $c = 1$, $l = 5$ and the initial value $(u_0, v_0) = (1.1, 9.334)$. By a direct computation, we have $m_0 \approx 0.6148$, $m_1 \approx 0.4394$, $m_2 \approx -0.0564$. When $m = 0.9$, $E_*(u_*, v_*) \approx (1, 9.234)$ and when $m = 0.61$, $E_*(u_*, v_*) \approx (1, 8.1047)$. Moreover, the numerical simulation system has two Hopf bifurcation points and the hypotheses **(H1)** -**(H3)** hold. Theorem 2.5 yields the following results: if $0 < m = 0.61 < m_0$, then $E_* \approx (1, 8.1047)$ is unstable; if $0.6148 < m = 0.9 < 1$, then $E_* \approx (1, 9.234)$ is locally asymptotically stable; and when $m = m_0$, spatially homogeneous bifurcating periodic solutions occur; when $m = m_1$, spatially non-homogeneous bifurcating periodic solutions occur. From Theorem 3.1, if $m = m_0$, $c_1(0) = -0.0456 - 0.017i$, $\lambda'(m_0) = -0.9568 + 0.2654i$, $\omega_0 = 1.00399$, $\mu_2 = -0.056 < 0$, $\beta_2^* = -0.0912 < 0$, $T_2 = 0.0021 > 0$, which implies that the homogeneous bifurcating periodic solution is locally asymptotically stable, the bifurcating direction is backward and the period of the bifurcating periodic solution increases. If $m = m_1$, $c_1(0) = -0.1425 - 0.4709i$, $\lambda'(m_1) = -0.8595 + 0.2616i$, $\omega_1 = 1.00383$, $\mu_2 = -0.1484 < 0$, $\beta_2^* = -0.285 < 0$, $T_2 = 0.4303 > 0$, which implies that the non-homogeneous bifurcating periodic solutions is locally asymptotically stable, the bifurcating direction is backward and the period of the bifurcating periodic solution increases.

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E-mail address: smilingchang@126.com
E-mail address: weijj@hit.edu.cn