MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **10**, Number **4**, August **2013**

doi:10.3934/mbe.2013.10.1067

pp. **1067–1094**

PARAMETRIZATION OF THE ATTAINABLE SET FOR A NONLINEAR CONTROL MODEL OF A BIOCHEMICAL PROCESS

Ellina V. Grigorieva

Department of Mathematics and Computer Sciences
Texas Woman's University
Denton, TX 76204, USA

Evgenii N. Khailov

Department of Computational Mathematics and Cybernetics
Moscow State Lomonosov University
Moscow, 119992, Russia

Andrei Korobeinikov

Centre de Recerca Matemàtica
Campus de Bellaterra, Edifici C
08193 Bellaterra, Barcelona, Spain

(Communicated by Patrick De Leenheer)

Abstract. In this paper, we study a three-dimensional nonlinear model of a controllable reaction $[X]+[Y]+[Z]\to[Z]$, where the reaction rate is given by a unspecified nonlinear function. A model of this type describes a variety of real-life processes in chemical kinetics and biology; in this paper our particular interests is in its application to waste water biotreatment. For this control model, we analytically study the corresponding attainable set and parameterize it by the moments of switching of piecewise constant control functions. This allows us to visualize the attainable sets using a numerical procedure.

These analytical results generalize the earlier findings, which were obtained for a trilinear reaction rate (which corresponds to the law of mass action) and reported in [18, 19], to the case of a general rate of reaction. These results allow to reduce the problem of constructing the optimal control to a straightforward constrained finite dimensional optimization problem.

1. Introduction.

A need to optimize frequently arises in processes in chemical kinetics, bio-engineering and medicine, including applications such as production of biological materials with pre-determined properties, and treatments of diseases (including cancer and HIV) [31, 32, 33]. However, fundamentally nonlinear nature and complexity of the problems originated in biology make the construction of an optimal control a challenging task. The usual approach to this kind of problems is numerical methods and dynamical programming, whereas analytical results are rare. An apparent drawback of the numerical methods is that a numerical procedure is dealing with a model based on a number of specific assumptions regarding

2010 *Mathematics Subject Classification.* Primary: 49J15, 49N90; Secondary: 93C10, 93C95.
Key words and phrases. Environmental problem, waste water biotreatment, nonlinear reaction rate, attainable set.

parametrization and the forms of functional feedbacks. Correctness of these assumptions for a biological or a bio-engineering problem is usually very difficult to justify. This drawback is generally not regarded as significant, because of intuitive expectation that a specific choice of parametrization is not of principal importance while the basic properties of the corresponding functions, such as monotonicity, convexity/concavity, etc., remain the same. However, while this common believe is probably justified when principal qualitative properties of a system have to be studied, its correctness for the task of constructing the optimal control for a real-life problem causes considerable doubts. Thus, Gross et al. [20] show an example of a biosystem, where a small perturbation of functional responses can lead to principal changes in the system stability; for a control system, the impact of such changes on the corresponding optimal control could be dramatic.

An advance in the application of the classic optimal control theory and the Pontryagin Maximum Principle to higher dimension nonlinear problems in biology and biotechnology was made in recent papers of Grigorieva, Khailov and their collaborators [18, 19], where the process known as the autothermal thermophilic aerobic digestion was considered as a case study. Autothermal thermophilic aerobic digestion (ATAD) is a bacterial process occurring in the presence of oxygen, which is used to treat accumulated sewage sludge with the aim of reducing the organic contents and the concentration of pathogenic microorganisms in the sludge. Aeration promotes growth of bacteria, which feed on and thus reduce the organic substrates in the sludge (converting it into carbon dioxide) and kill the pathogens with realizing metabolic heat. While the aeration significantly speeds up the process, it also greatly increases the operating costs, because of the energy which is spent to pump air into the sludge. Aeration can be controlled, and optimizing of the energy use is, therefore, essential for reduction of these costs. A review of the ATAD origin, design and operation can be found in [2, 13, 6].

Specifically, in [18, 19] a nonlinear control model of the ATAD was considered. For this model, properties of the corresponding attainable set were studied in details, and the parametrization of the set by the moments of switching of piecewise constant controls was constructed. The exact or approximate knowledge of attainable sets of a control system allows to estimate the limit possibilities for the control system and to choose the optimal or a suboptimal control. This consideration makes results in [18, 19] of immediate practically relevance, as they allow to considerably narrow the class of functions, which should be considered as candidates for the optimal controls, and thus serve as a sound basis for a numerical procedure. Substantial discussion of the properties of attainable sets can be founded in monographs [9, 1]. Techniques for describing of attainable sets are provided in [25, 37, 39, 40, 38, 14, 16, 17], and a number of methods of approximating these sets are given in [30, 54, 35, 22, 48, 45, 46, 21, 8, 52, 53].

A certain shortcoming of the above mentioned results is that they are obtained for a rather simple model based on a number of specific assumptions. The mathematical model, considered in [18, 19], is due to Brune [5], and is composed of three variables. It was postulated that the reaction occurs according to the law of mass action, while the other functional responses were assumed linear. The law of mass action is a mathematically convenient assumption, and is generally assumed to describe the process with a reasonable degree of accuracy (qualitatively, at least). This makes its use very common in mathematical modeling. However, the law of mass action does not necessary describes the reaction with the accuracy sufficient for the real-life practice. The deficiency of the mass action law was recognized long time ago, and a number of alternative nonlinear reaction rates, such as the Michaelis-Menten, or Monod kinetics, were proposed.

It should be taken in consideration, that neither actual functional responses, nor parametrization for ATAD is known in sufficient details. Furthermore, either of these can differ for different plants, and even vary in a single plant in response to varying environmental conditions (temperature, humidity) or the composition of the threaded sludge. Bacterial composition is also different for different plants, and varies within a single reactor with time. This makes sensible considering a system with a non-specified nonlinear functional responses and looking for the generic properties of these, in order to verify the robustness of the earlier obtained results.

Due to complexity of biological systems, the situations where functional responses are unknown or undefinable are rather common. A sound mathematical approach of dealing with this situation is to study generic properties of the systems assuming that the functional responses are given by non-specified functions. This concept goes back to the seminal work of A.N. Kolmogorov [24], and is currently considered as one of the major directions in mathematical biology [4, 7, 49, 50, 12, 26, 27, 28]. In line with this idea, in this paper we assume that the reaction rate is given by an unspecified non-linear separable function and establish the properties of this function, which determine the control characteristics for the attainable set of the considered control system.

This paper deals with the construction of a parametric description by the moments of switchings of a piecewise constant control of the attainable set (its interior and boundary) for the three-dimensional nonlinear control model. The paper is organized as follows. In Section 2, we formulate the mathematical model and describe its properties. In Section 3, we introduce the corresponding attainable set and study its properties. The main result of this Section is Theorem 3.3, which describes the structure of controls corresponding to points on the boundary of the attainable set. Constructing of the parametrization for this set is conducted in Section 4. Examples for specific models and pictures of the corresponding attainable sets are presented in Section 5. Finally, appropriate conclusions are provided in Section 6.

Background on ATAD [6]**.** Sewage water contains inorganic and organic chemical species and potentially pathogenic microorganisms. The objectives of the treatment are, accordingly, (i) a reduction of the organic content to an acceptable predetermined level, and (ii) elimination, or at least a reduction to a safe level, of pathogenic microorganisms. At the initial stage of the treatment, filtered concentrated sludge is produced in a process known as an activated sludge process (ASP); the reduction of the organic content and elimination of pathogens are the objectives of the next stage, that is ATAD. In this paper we assume that the above mentioned objectives of ATAD can be separated; that is, following [6], we assume that the reduction of the organic content to an acceptable level automatically implies the elimination of the pathogenic microorganisms.

ATAD can be operated as a batch or semi-batch process. An additional volume of untreated concentrated sludge is added into a reactor containing sludge at the start of a batch. Air is pumped continuously into the reactor providing the oxygenation required for aerobic bacterial digestion and the mechanical mixing of the sludge. Bacterial growth and an increase of temperature follow the digestion of organic substance. At the end of the batch period, which is typically set at 24 hours for staffing reasons, a part of the treated volume is removed and is immediately replaced by the next batch of untreated sludge.

The process is efficient, but it is also costly, as the aeration is energy-consuming. Optimizing the aeration can significantly reduce the cost of operation.

2. **Mathematical model and its properties.** The majority of existing ATAD models are overloaded with details and, as a result, are very large and complex. This complexity prevents the application of both the analysis and the usual optimization techniques. In this paper, our intention is to study a model, which we are able to investigate analytically and optimize numerically and which includes the essential mechanisms of the ATAD process. Accordingly, we formulate and investigate a simple model of the ATAD reaction, which is based on Brune's model [1].

In order to describe the process of aerobic biotreatment, we consider a simple mathematical model, which represent the process as a chemical reaction with three reagents,

$$[X] + [Y] + [Z] \to [Z].$$

For the particular case of ATAD, $x(t)$ is the concentration of oxygen, $y(t)$ is the concentration of organic matter, and $z(t)$ is that of the thermophilic aerobic bacteria. We assume that the mass in the reactor is well stirred, and hence the reactant concentrations are homogeneous in the volume. We also assume that all the biological activity takes place only in the reactor, and that anaerobic metabolic activity is negligible. Moreover, we assume that the aeration rate u is the only control, and that the control function is bounded. These assumption are common in the literature on ATAD [18, 19, 1].

In order to study the impact of a non-linearity of the reaction rate on the control and the attainable set, in this paper we assume that the reaction rate is a product of three unspecified functions $f(x)$, $g(y)$ and $h(z)$, that is $f(x)g(y)h(z)$. Under this assumptions, the changes of concentrations of the reagents are described by a three-dimensional nonlinear control system of differential equations

$$\begin{cases} \dot{x} = -d_x f(x)g(y)h(z) + u(m - x), \\ \dot{y} = -d_y f(x)g(y)h(z), \\ \dot{z} = d_z f(x)g(y)h(z) - bz. \end{cases} \tag{1}$$

Here, the first equation represents the evolution of oxygen concentration: the first term, $-f(x)g(y)h(z)$, describes the process of its absorption in the reaction, whereas the second term describes inflow of oxygen by pumping into the reactor. The second equation describes a decrease of the organic matter in the reaction. The third equation of system (1) shows an evolution of the active biomass concentration; the bacteria mass grows at the rate $f(x)g(y)h(z)$ and decays (due to natural mortality) at a rate bz. In the equations of this system d_x, d_y, d_z are positive reaction constants. System (1) also includes positive initial conditions

$$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0;\ x_0 \in (0, m);\ y_0,\ z_0 > 0, \tag{2}$$

and a restriction on the rate of pumping air.

Introduce the values:

$$\alpha_s = \sqrt{\frac{d_y d_z}{d_x}},\ \ \beta_s = \sqrt{\frac{d_x d_z}{d_y}},\ \ \gamma_s = \sqrt{\frac{d_x d_y}{d_z}}.$$

In order to reduce the number of parameters in system (1), let us make the following substitutions:

$$\widetilde{x} = \alpha_s x, \;\; \widetilde{y} = \beta_s y, \;\; \widetilde{z} = \gamma_s z, \;\; \widetilde{m} = \alpha_s m,$$

where $\widetilde{x}$, $\widetilde{y}$ and $\widetilde{z}$ are new variables, and $\widetilde{m}$ is the new parameter. The new initial conditions and unspecified functions then should be defined as

$$\widetilde{x}_0 = \alpha_s x_0, \;\; \widetilde{y}_0 = \beta_s y_0, \;\; \widetilde{z}_0 = \gamma_s z_0,$$

$$\widetilde{f}(\widetilde{x}) = \alpha_s f\left(\frac{\widetilde{x}}{\alpha_s}\right), \;\; \widetilde{g}(\widetilde{y}) = \beta_s g\left(\frac{\widetilde{y}}{\beta_s}\right), \;\; \widetilde{h}(\widetilde{z}) = \gamma_s h\left(\frac{\widetilde{z}}{\gamma_s}\right).$$

As a result, after dropping the tildes above solutions $\widetilde{x}(t)$, $\widetilde{y}(t)$, $\widetilde{z}(t)$, functions $\widetilde{f}(\widetilde{x})$, $\widetilde{g}(\widetilde{y})$, $\widetilde{h}(\widetilde{z})$ and values $\widetilde{m}$, $\widetilde{x}_0$, $\widetilde{y}_0$, $\widetilde{z}_0$, we obtain the system of equations (1) with initial conditions (2) and $d_x = d_y = d_z = 1$. Therefore, in the following arguments we will assume these values of the constants d_x, d_y, d_z in system (1).

We assume that functions $f(x)$, $g(y)$ and $h(z)$ are twice continuously differentiable for all $x, \; y, \; z > -\delta$, and that

$$f(0) = g(0) = h(0) = 0 \tag{3}$$

hold. Here δ is a small positive number. Moreover, we assume that functions $f(x)$, $g(y)$ and $h(z)$ are monotonically increasing and concave for all $x, \; y, \; z \geq 0$. That is, the inequalities

$$f'(x) > 0, \;\; g'(y) > 0, \;\; h'(z) > 0 \tag{4}$$

and

$$f''(x) \leq 0, \;\; g''(y) \leq 0, \;\; h''(z) \leq 0 \tag{5}$$

hold. By (3) and (4), functions $f(x)$, $g(y)$ and $h(z)$ are positive for all $x, \; y, \; z > 0$. Furthermore, by (3)–(5), the inequalities

$$f(x) \leq f'(0)x, \;\; g(y) \leq g'(0)y, \;\; h(z) \leq h'(0)z, \tag{6}$$

and

$$f'(x) \leq f'(0), \;\; g'(y) \leq g'(0), \;\; h'(z) \leq h'(0), \tag{7}$$

hold for all $x, \; y, \; z > 0$.

The model is nonlinear and assumes a bounded control; these features make the model very interesting from the mathematical point of view. Numerical experiments [6] confirm that this model is capable to describe the process with a sufficient degree of accuracy while providing a suitable basis for further optimization.

In system (1), the value $u \in [0, u_{\max}]$ is a control. We consider the control u from the set of all Lebesgue measurable functions $u(t)$, $t \geq 0$ satisfying the inequalities $0 \leq u(t) \leq u_{\max}$. Further, we will consider such controls as admissible.

Let us fix an admissible control $u(t)$ and consider for $t \geq 0$ corresponding solution $w_u(t) = (x_u(t), y_u(t), z_u(t))^\top$ of system (1) with initial conditions (2). Here symbol $^\top$ means transpose. Please note, that the solution $w_u(t)$ exists and unique on the maximum interval $[0, \gamma_u)$, where γ_u is either a finite positive number, or $+\infty$ [34, 23]. The following statement holds for $w_u(t)$.

Lemma 2.1. *For admissible control $u(t)$, $t \in [0, \gamma_u)$, for the corresponding solution $w_u(t) = (x_u(t), y_u(t), z_u(t))^\top$ of the problem (1),(2),*

$$0 < x_u(t) < m, \;\; 0 < y_u(t) y_0, \;\; 0 < z_u(t) < z_{\max} \tag{8}$$

hold for all $t \in [0, \gamma_u)$, where $z_{\max} = y_0 + z_0$.

Proof. The positiveness of components $x_u(t)$, $y_u(t)$, $z_u(t)$ of the solution $w_u(t)$, that is

$$x_u(t) > 0, \ \ y_u(t) > 0, \ \ z_u(t) > 0, \ \ t \in [0, \gamma_u), \tag{9}$$

follows from relationships (2),(3) and Theorem 4.6 (§4, Chapter 1) in [29]. The similar arguments lead to a validity of inequality

$$x_u(t) < m, \ \ t \in [0, \gamma_u). \tag{10}$$

From inequalities (9), the positiveness of functions $f(x)$, $g(y)$ and $h(z)$ for all $x, \ y, \ z > 0$ and the second equation of system (1), we obtain inequality

$$y_u(t) < y_0, \ \ t \in (0, \gamma_u). \tag{11}$$

Finally, combining the second and the third equations of system (1), and then integrating the result, we come to the equality

$$z_u(t) = y_0 + z_0 - y_u(t) - b \int_0^t z_u(s) ds, \ \ t \in [0, \gamma_u).$$

By (9), this relationship implies the inequality

$$z_u(t) < y_0 + z_0 = z_{\max}, \ \ t \in [0, \gamma_u). \tag{12}$$

Combining inequalities (9)–(12), we find the desired relationships (8). This completes the proof. □

The next statements immediately follow from Lemma 2.1.

Corollary 1. *The positive octant $\mathbb{R}^3_+$ is a positively invariant set of system* (1) *with respect to the given set of admissible controls.*

Corollary 2. *For admissible control $u(t)$, $t \in [0, \gamma_u)$ the corresponding solution $w_u(t) = (x_u(t), y_u(t), z_u(t))^\top$ of the problem* (1),(2) *is bounded.*

Moreover, it immediately follows (cf. [23]) that for this problem the value γ_u is equal to $+\infty$, and that for any $T > 0$ the solution $w(t)$ of system (1),(2) corresponding to an arbitrary admissible control $u(t)$, can be continued to the segment $[0, T]$ under simultaneous satisfaction of the inequalities

$$0 < x(t) < m, \ 0 < y(t) < y_0, \ 0 < z(t) < z_{\max}, \ \ t \in (0, T]. \tag{13}$$

Finally, let us define the set of admissible controls $D(T)$ as the set of all Lebesgue measurable functions $u(t)$, such that for almost all $t \in [0, T]$ the inequalities

$$0 \le u(t) \le u_{\max} \tag{14}$$

hold.

3. **Attainable set and its properties.** For problem (1),(2), the attainable set $X(T) \subset \mathbb{R}^3$ from the initial point w_0 at the moment of time T is the set of values $w(T) = (x(T), y(T), z(T))^\top$ of solutions $w(t) = (x(t), y(t), z(t))^\top$ of system (1) with initial conditions (2) corresponding to all possible controls $u(\cdot) \in D(T)$. From inequalities (13) and Theorem 2 (Chapter 4 in [34]), it follows that the set $X(T)$ is a compact set in $\mathbb{R}^3$ located in the region

$$\left\{ w = (x, y, z)^\top \in \mathbb{R}^3 : 0 < x < m, \ 0 < y < y_0, \ 0 < z < z_{\max} \right\}.$$

In order to study the boundary of the attainable set $X(T)$, we use the Pontryagin Maximum Principle (Theorem 3, Chapter 4 in [34]). Define the Hamiltonian

$$H(w,\psi,u) = u(m-x)\psi_1 - f(x)g(y)h(z)(\psi_1+\psi_2-\psi_3) - bz\psi_3,$$

where $\psi = (\psi_1,\psi_2,\psi_3)^\top$ are adjoint variables. Let the point $w = (x,y,z)^\top$ be a boundary point of set $X(T)$. Then there are a corresponding control $u(\cdot) \in D(T)$ and a trajectory $w(t) = (x(t),y(t),z(t))^\top$ of problem (1),(2), such that the equalities

$$x(T) = x,\ y(T) = y,\ z(T) = z.$$

hold. Moreover, there exists a nontrivial solution $\psi(t) = (\psi_1(t),\psi_2(t),\psi_3(t))^\top$ of the adjoint system

$$\begin{cases} \dot{\psi}_1(t) = u(t)\psi_1(t) + f'(x(t))g(y(t))h(z(t))(\psi_1(t)+\psi_2(t)-\psi_3(t)), \\ \dot{\psi}_2(t) = f(x(t))g'(y(t))h(z(t))(\psi_1(t)+\psi_2(t)-\psi_3(t)), \\ \dot{\psi}_3(t) = f(x(t))g(y(t))h'(z(t))(\psi_1(t)+\psi_2(t)-\psi_3(t)) + b\psi_3(t), \end{cases} \tag{15}$$

for which the control $u(t)$ is defined from the condition of maximum

$$H(w(t),\psi(t),u(t)) = \max_{v\in[0,u_{\max}]} H(w(t),\psi(t),v), \tag{16}$$

which is valid for almost all $t \in [0,T]$. By the first inequality of (13), relationship (16) can be rewritten as

$$u(t) = \begin{cases} 0, & \text{if} \quad L(t) < 0, \\ [0,u_{\max}], & \text{if} \quad L(t) = 0, \\ u_{\max}, & \text{if} \quad L(t) > 0. \end{cases} \tag{17}$$

Here function $L(t) = \psi_1(t)$ is the switching function, which behavior completely determines the control $u(t)$.

For convenience at further analysis, we introduce, together with switching function $L(t)$, the following axillary functions

$$G(t) = \psi_1(t)+\psi_2(t)-\psi_3(t),\ P(t) = -\psi_3(t),$$

$$\alpha(t) = f'(x(t))g(y(t))h(z(t)),\ \beta(t) = f(x(t))g(y(t))h'(z(t)),$$

$$\sigma(t) = f'(x(t))g(y(t))h(z(t)) + f(x(t))g'(y(t))h(z(t)) - f(x(t))g(y(t))h'(z(t)).$$

By (4) and the positiveness of functions $f(x)$, $g(y)$ and $h(z)$,

$$\alpha(t) > 0,\ \beta(t) > 0,\ t \in [0,T]. \tag{18}$$

Using adjoint system (15), we can now write a system of differential equations

$$\begin{cases} \dot{L}(t) = u(t)L(t) + \alpha(t)G(t), \\ \dot{G}(t) = u(t)L(t) + \sigma(t)G(t) + bP(t), \\ \dot{P}(t) = -\beta(t)G(t) + bP(t). \end{cases} \tag{19}$$

for the functions $L(t)$, $G(t)$ and $P(t)$.

We are now ready to proceed to properties of the switching function $L(t)$. The following statement is valid.

Lemma 3.1. *The switching function $L(t)$ is not equal to zero on any finite subinterval of the interval $[0,T]$.*

Proof. Assume the contradiction. Let $L(t) = 0$ hold on some subinterval $\Delta \subset [0, T]$. Then $\dot{L}(t) = 0$ for all $t \in \Delta$. Then, by the first equation of system (19) and inequalities (18), $G(t) = 0$ on the subinterval Δ, and hence $\dot{G}(t) = 0$ for all $t \in \Delta$ as well. Furthermore, from the second equation of this system we see that $P(t) = 0$ holds on the interval Δ. Moreover, the third equation of system (19) is also satisfied on this interval. System (19) is a linear and homogeneous system, and hence we must conclude that the relationship

$$L(t) = G(t) = P(t) = 0$$

holds on the entire interval $[0, T]$. Then the definitions of these functions immediately lead to the equalities

$$\psi_1(t) = \psi_2(t) = \psi_3(t) = 0, \; t \in [0, T].$$

This contradicts to the nontriviality of the solution $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))^\top$ of adjoint system (15), and hence the hypothesis is incorrect, and the function $L(t)$ cannot be equal to zero on the subinterval $\Delta \subset [0, T]$. The proof is completed. $\square$

Remark 1. The same result can be obtained by verifying the standard Lie bracket condition for existence of a singular arc [3, 43].

Remark 2. From (17) and Lemma 3.1 it follows that the control $u(t)$, corresponding to a boundary point $w = (x, y, z)^\top$ of the attainable set $X(T)$, is a piecewise constant function taking values $\{0; u_{\max}\}$.

The following statement estimates the maximum number of zeroes of function $L(t)$.

Lemma 3.2. *The switching function $L(t)$ has at most two zeroes on the interval $[0, T]$.*

Proof. Firstly, we outline the idea of the proof. System (19) is a linear nonautonomous system of differential equations, defined on the given finite time interval. We will transform the matrix of this system to the upper triangular form. Functions, which are responsible for this transformation, are given by a system of quadratic differential equations and, therefore, are locally defined in a small neighborhood of the value $t = 0$. Arguing by contradiction and using differential inequalities and the comparison theorem, we will show the existence of such solutions to the system of quadratic differential equations, which are defined on the entire interval. Therefore, the triangular system also is defined on this interval, and from its analysis we will make a conclusion about the number of zeros of function $L(t)$.

Now we proceed to the proof. Let us make the nonlinear substitution

$$\rho(t) = L(t), \;\; \mu(t) = G(t), \;\; \lambda(t) = P(t) + q_1(t)L(t) + q_2(t)G(t),$$

to system (19), where functions $q_1(t)$, $q_2(t)$ are to be defined. In new variables $\rho(t)$, $\mu(t)$ and $\lambda(t)$, system (19) is

$$\begin{cases} \dot{\rho}(t) = u(t)\rho(t) + \alpha(t)\mu(t), \\ \dot{\mu}(t) = (u(t) - bq_1(t))\rho(t) + (\sigma(t) - bq_2(t))\mu(t) + b\lambda(t), \\ \dot{\lambda}(t) = \Big[\dot{q}_1(t) + (u(t) - b)q_1(t) + u(t)q_2(t) - bq_1(t)q_2(t)\Big]\rho(t) + \\ \qquad + \Big[\dot{q}_2(t) + \alpha(t)q_1(t) + (\sigma(t) - b)q_2(t) - bq_2^2(t) - \beta(t)\Big]\mu(t) + \\ \qquad + b(1 + q_2(t))\lambda(t). \end{cases} \tag{20}$$

Here, we choose the functions $q_1(t)$, $q_2(t)$ to makes the expressions inside the square brackets equal to zero. Then the functions $q_1(t)$, $q_2(t)$ satisfy the differential equations

$$\begin{cases} \dot q_1(t) + (u(t) - b)q_1(t) + u(t)q_2(t) - bq_1(t)q_2(t) = 0, \\ \dot q_2(t) + \alpha(t)q_1(t) + (\sigma(t) - b)q_2(t) - bq_2^2(t) - \beta(t) = 0, \end{cases} \tag{21}$$

and system (20) is

$$\begin{cases} \dot\rho(t) = u(t)\rho(t) + \alpha(t)\mu(t), \\ \dot\mu(t) = (u(t) - bq_1(t))\rho(t) + (\sigma(t) - bq_2(t))\mu(t) + b\lambda(t), \\ \dot\lambda(t) = b(1 + q_2(t))\lambda(t). \end{cases} \tag{22}$$

Next, in system (22) we make again the nonlinear substitution

$$\widetilde\rho(t) = \rho(t), \quad \widetilde\mu(t) = \mu(t) + q_3(t)\rho(t), \quad \widetilde\lambda(t) = \lambda(t),$$

where function $q_3(t)$ is to be defined. In the new variables, system (22) takes the form

$$\begin{cases} \dot{\widetilde\rho}(t) = (u(t) - \alpha(t)q_3(t))\widetilde\rho(t) + \alpha(t)\widetilde\mu(t), \\ \dot{\widetilde\mu}(t) = \Big[\dot q_3(t) - bq_1(t) + (u(t) - \sigma(t))q_3(t) + \\ \qquad + bq_2(t)q_3(t) - \alpha(t)q_3^2(t) + u(t)\Big]\widetilde\rho(t) + \\ \qquad + (\sigma(t) - bq_2(t) + \alpha(t)q_3(t))\widetilde\mu(t) + b\widetilde\lambda(t), \\ \dot{\widetilde\lambda}(t) = b(1 + q_2(t))\widetilde\lambda(t). \end{cases} \tag{23}$$

As above, we choose function $q_3(t)$ such to make the expression inside the square brackets equal to zero. Then we have the differential equation

$$\dot q_3(t) - bq_1(t) + (u(t) - \sigma(t))q_3(t) + bq_2(t)q_3(t) - \alpha(t)q_3^2(t) + u(t) = 0. \tag{24}$$

for function $q_3(t)$, and system (23) is now

$$\begin{cases} \dot{\widetilde\rho}(t) = (u(t) - \alpha(t)q_3(t))\widetilde\rho(t) + \alpha(t)\widetilde\mu(t), \\ \dot{\widetilde\mu}(t) = (\sigma(t) - bq_2(t) + \alpha(t)q_3(t))\widetilde\mu(t) + b\widetilde\lambda(t), \\ \dot{\widetilde\lambda}(t) = b(1 + q_2(t))\widetilde\lambda(t). \end{cases} \tag{25}$$

Combining the differential equations (21),(24), for functions $q_1(t)$, $q_2(t)$ and $q_3(t)$, we obtain the following system

$$\begin{cases} \dot q_1(t) = -(u(t) - b)q_1(t) - u(t)q_2(t) + bq_1(t)q_2(t), \\ \dot q_2(t) = -\alpha(t)q_1(t) - (\sigma(t) - b)q_2(t) + bq_2^2(t) + \beta(t), \\ \dot q_3(t) = bq_1(t) - (u(t) - \sigma(t))q_3(t) - bq_2(t)q_3(t) + \alpha(t)q_3^2(t) - u(t). \end{cases} \tag{26}$$

Let us now rewrite this system in a matrix form. In order to do this, we define symmetric matrices $A_1(t)$, $A_2(t)$ and $A_3(t)$ as follows

$$A_1(t) = \begin{pmatrix} 0 & \frac{b}{2} & 0 \\ \frac{b}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A_2(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A_3(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{b}{2} \\ 0 & -\frac{b}{2} & \alpha(t) \end{pmatrix}.$$

We introduce vectors $b_1(t)$, $b_2(t)$ and $b_3(t)$ as

$$b_1(t) = \begin{pmatrix} b - u(t) \\ -u(t) \\ 0 \end{pmatrix}, \ b_2(t) = \begin{pmatrix} -\alpha(t) \\ b - \sigma(t) \\ 0 \end{pmatrix}, \ b_3(t) = \begin{pmatrix} b \\ 0 \\ \sigma(t) - u(t) \end{pmatrix},$$

and functions $c_1(t)$, $c_2(t)$ and $c_3(t)$ as

$$c_1(t) = 0, \quad c_2(t) = \beta(t), \quad c_3(t) = -u(t).$$

Then the matrix form for system (26) is

$$\begin{cases} \dot{q}_1(t) = \big(A_1(t)q(t), q(t)\big) + (b_1(t), q(t)) + c_1(t), \\ \dot{q}_2(t) = \big(A_2(t)q(t), q(t)\big) + (b_2(t), q(t)) + c_2(t), \\ \dot{q}_3(t) = \big(A_3(t)q(t), q(t)\big) + (b_3(t), q(t)) + c_3(t), \end{cases} \tag{27}$$

where $q(t) = (q_1(t), q_2(t), q_3(t))^\top$ and (p, e) is the scalar product of vectors $p, e \in \mathbb{R}^3$.

Our task now is to show that system (27) has a solution, defined on the entire interval $[0, T]$. Assume the contradiction, that is let an arbitrary solution $q(t)$ of system (27) be defined on the subinterval $[0, t_1)$, $t_1 \in (0, T]$, which is the maximum possible subinterval, where this solution exists. Then, by Lemma (§14, Chapter 4) in [10], it follows that

$$\lim_{t \to t_1 - 0} \|q(t)\| = +\infty, \tag{28}$$

and hence there necessary exists a number $\nu > 0$, a value $t_0 \in [0, t_1)$, and a set $\Pi = \big\{q \in \mathbb{R}^3 : \|q\| \geq \nu\big\}$ such, that the inclusion $q(t) \in \Pi$ holds for all $t \in [t_0, t_1)$. Here, the values ν and t_0 will be defined below.

Let us evaluate the derivative of the function $\|q(t)\|$ on the interval $[t_0, t_1)$. From (27) we have

$$\frac{d}{dt}\Big(\|q(t)\|\Big) = \|q(t)\|^{-1} \cdot \Big(\xi_1(t) + \xi_2(t) + \xi_3(t)\Big), \tag{29}$$

where

$$\begin{aligned} \xi_1(t) &= q_1(t)\big(A_1(t)q(t), q(t)\big) + q_2(t)\big(A_2(t)q(t), q(t)\big) + q_3(t)\big(A_3(t)q(t), q(t)\big), \\ \xi_2(t) &= q_1(t)(b_1(t), q(t)) + q_2(t)(b_2(t), q(t)) + q_3(t)(b_3(t), q(t)), \\ \xi_3(t) &= c_1(t)q_1(t) + c_2(t)q_2(t) + c_3(t)q_3(t). \end{aligned}$$

Using inequities (6),(7),(13), and (14) we can estimate the upper boundary for $\xi_1(t)$, $\xi_2(t)$ and $\xi_3(t)$ on the interval $[t_0, t_1)$. For $\xi_3(t)$ we have the following chain of relationships

$$\begin{aligned} \xi_3(t) \;&\leq\; \sqrt{\beta^2(t) + u^2(t)} \cdot \|q(t)\| = \sqrt{(f(x(t))g(y(t))h'(z(t)))^2 + u^2(t)} \cdot \|q(t)\| \\ &\leq\; \sqrt{(f'(0)g'(0)h'(0))^2 x^2(t) y^2(t) + u^2(t)} \cdot \|q(t)\| < C \cdot \|q(t)\|, \end{aligned}$$

where

$$C = \sqrt{Q^2 m^2 y_0^2 + u_{\max}^2}.$$

Here and further we consider $Q = f'(0)g'(0)h'(0)$.

For $\xi_2(t)$ we have the following chain of relationships

$$\begin{aligned} \xi_2 \;&\leq\; \sqrt{\|b_1(t)\|^2 + \|b_2(t)\|^2 + \|b_3(t)\|^2} \cdot \|q(t)\|^2 \\ &=\; \sqrt{(b - u(t))^2 + u^2(t) + \alpha^2(t) + (b - \sigma(t))^2 + b^2 + (\sigma(t) - u(t))^2} \cdot \|q(t)\|^2. \end{aligned} \tag{30}$$

Furthermore, we have the following inequalities

$$(b - u(t))^2 \leq 2b^2 + 2u^2(t) \leq 2b^2 + 2u_{\max}^2,$$

$$\alpha^2(t) = (f'(x(t))g(y(t))h(z(t)))^2 \leq Q^2 y^2(t) z^2(t) < Q^2 y_0^2 z_{\max}^2,$$

$$
\begin{aligned}
(b-\sigma(t))^2 &= \Big(\big(f'(x(t))g(y(t))h(z(t)) + f(x(t))g'(y(t))h(z(t))\big) \\
&\quad -\big(f(x(t))g(y(t))h'(z(t)) + b\big)\Big)^2 \\
&\le 2\big(f'(x(t))g(y(t))h(z(t)) + f(x(t))g'(y(t))h(z(t))\big)^2 \\
&\quad + 2\big(f(x(t))g(y(t))h'(z(t)) + b\big)^2 \\
&\le 4\big(f'(x(t))g(y(t))h(z(t))\big)^2 + 4\big(f(x(t))g'(y(t))h(z(t))\big)^2 \\
&\quad + 4\big(f(x(t))g(y(t))h'(z(t))\big)^2 + 4b^2 \\
&\le 4Q^2\big(x^2(t)y^2(t) + y^2(t)z^2(t) + z^2(t)x^2(t)\big) + 4b^2 \\
&< 4Q^2\big(m^2y_0^2 + y_0^2z_{\max}^2 + z_{\max}^2m^2\big) + 4b^2,
\end{aligned}
$$

$$
\begin{aligned}
(\sigma(t)-u(t))^2 &= \Big(\big(f'(x(t))g(y(t))h(z(t)) + f(x(t))g'(y(t))h(z(t))\big) \\
&\quad -\big(f(x(t))g(y(t))h'(z(t)) + u(t)\big)\Big)^2 \\
&\le 2\big(f'(x(t))g(y(t))h(z(t)) + f(x(t))g'(y(t))h(z(t))\big)^2 \\
&\quad + 2\big(f(x(t))g(y(t))h'(z(t)) + u(t)\big)^2 \\
&\le 4\big(f'(x(t))g(y(t))h(z(t))\big)^2 + 4\big(f(x(t))g'(y(t))h(z(t))\big)^2 \\
&\quad + 4\big(f(x(t))g(y(t))h'(z(t))\big)^2 + 4u^2(t) \\
&\le 4Q^2\big(x^2(t)y^2(t) + y^2(t)z^2(t) + z^2(t)x^2(t)\big) + 4u^2(t) \\
&< 4Q^2\big(m^2y_0^2 + y_0^2z_{\max}^2 + z_{\max}^2m^2\big) + 4u_{\max}^2.
\end{aligned}
$$

Substituting these inequalities into (30), we find that

$$
\sqrt{\|b_1(t)\|^2 + \|b_2(t)\|^2 + \|b_3(t)\|^2}\cdot\|q(t)\|^2 < B\cdot\|q(t)\|^2,
$$

where

$$
B = \sqrt{7b^2 + 7u_{\max}^2 + Q^2\big(8m^2y_0^2 + 9y_0^2z_{\max}^2 + 8z_{\max}^2m^2\big)}.
$$

Finally, for $\xi_1(t)$ we have the inequality

$$
\begin{aligned}
\xi_1(t) &\le |q_1(t)|\cdot\|A_1(t)q(t)\|\cdot\|q(t)\| + |q_2(t)|\cdot\|A_2(t)q(t)\|\cdot\|q(t)\| \\
&\quad + |q_3(t)|\cdot\big|\big(A_3(t)q(t),q(t)\big)\big|.
\end{aligned}
\tag{31}
$$

Here,

$$
\|A_1(t)q(t)\| \le \frac{b}{2}\|q(t)\|, \quad \|A_2(t)q(t)\| \le b\|q(t)\|.
$$

The eigenvalues of matrix $A_3(t)$ are

$$
\eta_1(t) = 0,\ \eta_2(t) = \frac{\alpha(t) - \sqrt{\alpha^2(t)+b^2}}{2},\ \eta_3(t) = \frac{\alpha(t) + \sqrt{\alpha^2(t)+b^2}}{2}.
$$

Hence, for the last term in (31), the inequality

$$
\big|\big(A_3(t)q(t),q(t)\big)\big| \le \eta_3(t)\|q(t)\|^2
$$

holds. Combining the estimations, obtained for (31), we have the following chain of inequalities

$$
|q_1(t)|\cdot\|A_1(t)q(t)\|\cdot\|q(t)\| + |q_2(t)|\cdot\|A_2(t)q(t)\|\cdot\|q(t)\| + |q_3(t)|\cdot\big|\big(A_3(t)q(t),q(t)\big)\big|
$$

$$\leq \left(\frac{b}{2}|q_1(t)| + b|q_2(t)| + \frac{\alpha(t) + \sqrt{\alpha^2(t) + b^2}}{2}|q_3(t)|\right) \cdot \|q(t)\|^2$$

$$\leq \sqrt{\frac{5b^2}{4} + \frac{\left(\alpha(t) + \sqrt{\alpha^2(t) + b^2}\right)^2}{4}} \cdot \|q(t)\|^3$$

$$\leq \sqrt{\frac{7b^2}{4} + \alpha^2(t)} \cdot \|q(t)\|^3 < A \cdot \|q(t)\|^3,$$

where

$$A = \sqrt{\frac{7b^2}{4} + Q^2 y_0^2 z_{\max}^2}.$$

Substituting the inequalities for $\xi_1(t)$, $\xi_2(t)$ and $\xi_3(t)$ into (29), we finally obtain the differential inequality

$$\frac{d}{dt}\left(\|q(t)\|\right) \leq A\|q(t)\|^2 + B\|q(t)\| + C, \; t \in [t_0, t_1). \tag{32}$$

Now, we consider a quadratic equation

$$AK^2 - BK + C = 0. \tag{33}$$

For its discriminant D, we have the following chain of equalities

$$\begin{aligned} D =& B^2 - 4AC \\ =& 7b^2 + 7u_{\max}^2 + Q^2\left(8m^2y_0^2 + 9y_0^2z_{\max}^2 + 8z_{\max}^2m^2\right) \\ & - 4\sqrt{\frac{7b^2}{4} + Q^2y_0^2z_{\max}^2} \cdot \sqrt{Q^2m^2y_0^2 + u_{\max}^2} \\ =& 7b^2 + 7u_{\max}^2 + Q^2\left(8m^2y_0^2 + 9y_0^2z_{\max}^2 + 8z_{\max}^2m^2\right) \\ & - \sqrt{7b^2 + 4Q^2y_0^2z_{\max}^2} \cdot \sqrt{4Q^2m^2y_0^2 + 4u_{\max}^2} \\ =& \left(\sqrt{7b^2 + 4Q^2y_0^2z_{\max}^2}\right)^2 + \left(\sqrt{4Q^2m^2y_0^2 + 4u_{\max}^2}\right)^2 \\ & - \sqrt{7b^2 + 4Q^2y_0^2z_{\max}^2} \cdot \sqrt{4Q^2m^2y_0^2 + 4u_{\max}^2} \\ & + \left(3u_{\max}^2 + Q^2\left(4m^2y_0^2 + 5y_0^2z_{\max}^2 + 8z_{\max}^2m^2\right)\right). \end{aligned}$$

It is easy to see that the discriminant D is positive.

Next, we introduce a Lyapunov function $V(q) = \|q\| + K_0$, where $q \in \Pi$ and K_0 is the biggest root of equation (33), that is

$$K_0 = \frac{B + \sqrt{B^2 - 4AC}}{2A}.$$

By (32), the function $V(q)$ satisfies

$$\frac{d}{dt}\left(V(q(t))\right) \leq A\left(V(q(t)) - K_0\right)^2 + B\left(V(q(t)) - K_0\right) + C.$$

Here, by definition, $AK_0^2 - BK_0 + C = 0$, and hence

$$\frac{d}{dt}\Big(V(q(t))\Big) \leq AV^2(q(t)) - \Big(2AK_0 - B\Big)V(q(t)), \; t \in [t_0, t_1). \tag{34}$$

Now, let consider the auxiliary Cauchy problem

$$\begin{cases} \dot{h}(t) = Ah^2(t) - \Big(2AK_0 - B\Big)h(t), \ t \in [t_0, t_1], \\ h(t_0) = h_0, \ h_0 \geq K_0 + \nu. \end{cases} \tag{35}$$

Here the value h_0 satisfies

$$h_0 > 2K_0 - \frac{B}{A}. \tag{36}$$

Solving the corresponding Bernoulli equation and satisfying the initial condition, we find the solution to (35):

$$h(t) = \Big(\frac{A}{2AK_0 - B} + \Big[\frac{1}{h_0} - \frac{A}{2AK_0 - B}\Big]e^{(2AK_0 - B)(t - t_0)}\Big)^{-1}, \ t \in [t_0, t_1]. \tag{37}$$

In this equality, we assume that the values of ν and t_0 are such that the expression in brackets is defined for all $t \in [t_0, t_1]$. We can do this, for example, by choosing, for a given ν, a value t_0 such that the difference $(t_1 - t_0)$ was sufficiently small. By (36), the sum inside of the square brackets in (37) is negative, and hence $h(t)$ is a finite positive monotonically increasing on the segment $[t_0, t_1]$ function. Therefore, $h(t) < h(t_1)$ for all $t \in [t_0, t_1)$. Hence, by differential inequality (34), Cauchy problem (35) and Chaplygin's Theorem (Theorem 1.1 in [51]), and under the condition

$$h_0 = V(q(t_0)) = K_0 + \|q(t_0)\|,$$

we have the inequalities

$$\|q(t)\| < h(t) - K_0 < h(t_1) - K_0, \ t \in (t_0, t_1).$$

This contradicts to (28), and hence the hypothesis is incorrect, and system (26) has a solution $q(t) = (q_1(t), q_2(t), q_3(t))^\top$ defined on the entire interval $[0, T]$. Therefore, system (25) is defined on this interval as well. Applying to this system the generalized Rolle's Theorem [11], we conclude that the switching function $L(t) = \widetilde{\rho}(t)$ has at most two zeros on the interval $[0, T]$. This completes the proof. □

From Lemma 3.2, Remark 2 to Lemma 3.1 and relationship (17) lead to the following theorem.

Theorem 3.3. *Let point $w = (x, y, z)^\top$ be on the boundary of the attainable set $X(T)$. Then, its corresponding control $u(t)$ is a piecewise constant function taking values $\{0; u_{\max}\}$ and having at most two switchings on the interval $(0, T)$.*

4. **Parametric description of the attainable set.** The results in the previous Section enable us to parameterize the attainable set $X(T)$ by the moments of switching of piecewise constant controls.

Let us consider the set

$$\Lambda(T) = \left\{\theta = (\theta_1, \theta_2, \theta_3)^\top \in \mathbb{R}^3 : 0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq T\right\}.$$

For each point $\theta \in \Lambda(T)$, we define the corresponding control $u_\theta(\cdot) \in D(T)$

$$u_\theta(t) = \begin{cases} u_{\max}, & \text{if} \quad 0 \leq t \leq \theta_1, \\ 0, & \text{if} \quad \theta_1 < t \leq \theta_2, \\ u_{\max}, & \text{if} \quad \theta_2 < t \leq \theta_3, \\ 0, & \text{if} \quad \theta_3 < t \leq T. \end{cases} \tag{38}$$

Let $w_\theta(t)$ be the solution of problem (1),(2) corresponding to control $u_\theta(t)$. We define a mapping $F(\cdot,T):\Lambda(T)\to\mathbb{R}^3$ as

$$F(\theta,T)=w_\theta(T),\quad \theta\in\Lambda(T).$$

For this mapping we have the statement, which follows from well-known results in theory of ODEs about a continuous dependence of solutions to ODEs with respect to parameters [23].

Lemma 4.1. *The mapping $F(\cdot,T)$ is continuous on the set $\Lambda(T)$.*

Using the mapping $F(\cdot,T)$, we introduce the auxiliary set $Z(T)=F(\Lambda(T),T)$, which consists of all ends $w_\theta(T)$ of trajectories $w_\theta(t)$ of problem (1),(2) under all possible controls $u_\theta(t)$, $t\in[0,T]$, defined by formula (38). Every element of the set $Z(T)$ is a result of a bang-bang control $u_\theta(t)$, $t\in[0,T]$, with at most three switchings on the interval $(0,T)$.

Now, we have to discuss some properties of the auxiliary set $Z(T)$. Considering a point $\theta\in\mathrm{int}\Lambda(T)$, its corresponding control $u_\theta(t)$ defined by (38) and a trajectory $w_\theta(t)$, $t\in[0,T]$, we can reformulate the Cauchy problem (1),(2) in the form

$$\begin{cases} \dot w_\theta(t)=Aw_\theta(t)+\varphi(w_\theta(t))c+u_\theta(t)\rho(w_\theta(t)),\ t\in[0,T],\\ w_\theta(0)=w_0=(x_0,y_0,z_0)^\top, \end{cases} \tag{39}$$

where A is a 3×3 matrix, $c\in\mathrm{R}^3$, and functions $\rho(w)$ and $\varphi(w)$ are a vector and a scalar functions, respectively, such that

$$A=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-b\end{pmatrix},\ c=\begin{pmatrix}-1\\-1\\1\end{pmatrix},\ \rho(w)=\begin{pmatrix}m-x\\0\\0\end{pmatrix},\ \varphi(w)=f(x)g(y)h(z). \tag{40}$$

For the system (39), we define a function $\Phi_\theta(t)$, $t\in[0,T]$, as a solution of the matrix Cauchy problem

$$\begin{cases} \dot\Phi_\theta(t)=\left(A+c\left(\dfrac{\partial\varphi}{\partial w}(w_\theta(t))\right)^\top+u_\theta(t)\dfrac{\partial\rho}{\partial w}(w_\theta(t))\right)\Phi_\theta(t),\ t\in[0,T],\\ \Phi_\theta(T)=E, \end{cases} \tag{41}$$

where E is the identity matrix.

Let us evaluate the derivatives $\dfrac{\partial w_\theta}{\partial\theta_i}(T)$, $i=\overline{1,3}$. Using well-known results in theory of ODEs about differentiation of solutions to ODEs with respect to parameters [23] one can find that the derivatives satisfy the following equalities

$$\frac{\partial w_\theta}{\partial\theta_i}(T)=(-1)^{i-1}u_{\max}\Phi_\theta^{-1}(\theta_i)\rho(w_\theta(\theta_i)),\ i=\overline{1,3}. \tag{42}$$

Remark 3. More detailed calculations for systems, in which the control appears linearly, are presented in [36].

Now we are in a position to state the following statement.

Lemma 4.2. *The following equalities hold*

$$F(\mathrm{int}\Lambda(T),T)=\mathrm{int}Z(T),\ F(\partial\Lambda(T),T)=\partial Z(T), \tag{43}$$

and the restriction of mapping $F(\cdot,T)$ onto the interior of set $\Lambda(T)$ is one-to-one.

Here, $\partial\Theta$ and $\mathrm{int}\Theta$ denote the boundary and the interior of a compact set $\Theta\subset\mathbb{R}^3$.

Proof. Firstly, we consider the set $\mathrm{int}\Lambda(T)$. The mapping $F(\cdot,T)$ is continuously differentiable on the set $\mathrm{int}\Lambda(T)$, and for every point $\theta \in \mathrm{int}\Lambda(T)$, by (42), the following equalities hold

$$\frac{\partial F}{\partial \theta_i}(\theta,T) = (-1)^{i-1} u_{\max} \Phi_\theta^{-1}(\theta_i)\rho(w_\theta(\theta_i)),\ i=\overline{1,3}. \tag{44}$$

The continuity of these derivatives on the set $\mathrm{int}\Lambda(T)$ is determined by a continuous dependence of the trajectory $w_\theta(t)$ and solution $\Phi_\theta(t)$ of the matrix Cauchy problem (41) in variables θ_i, $i=\overline{1,3}$. It is established by arguments, which are similar to the arguments presented in Lemma 4.1.

We have to show that the Jacobi matrix of the restriction of mapping $F(\cdot,T)$ onto $\mathrm{int}\Lambda(T)$ is nonsingular. Suppose the opposite. Then there is a point $\bar\theta \in \mathrm{int}\Lambda(T)$ for which vectors $\frac{\partial F}{\partial \theta_i}(\bar\theta,T)$, $i=\overline{1,3}$, are linearly dependent. With respect to (44), it means the existence of a nonzero vector $q \in \mathrm{R}^3$ such that the equalities

$$(\rho(w_\theta(\bar\theta_i)),\eta(\bar\theta_i)) = 0,\ i=\overline{1,3}, \tag{45}$$

hold. Here $\eta(t) = (\Phi_\theta^{-1}(t))^\top q$. By (41), we can see that function $\eta(t)$, $t\in[0,T]$, satisfies the adjoint system (15), which is written as

$$\dot\psi(t) = -\left(A + c\left(\frac{\partial\varphi}{\partial w}(w_\theta(t))\right)^\top + u_\theta(t)\frac{\partial\rho}{\partial w}(w_\theta(t))\right)^\top \psi(t),$$

where $\psi(t) = (\psi_1(t),\psi_2(t),\psi_3(t))^\top$. Then, applying inequalities (13) and Lemma 3.2 to the function $r(t) = (\rho(w_\theta(t)),\eta(t))$, we find that function $r(t)$ has two zeros on interval $(0,T)$ at most. This fact contradicts the equalities

$$r(\bar\theta_i) = 0,\ i=\overline{1,3},$$

resulting from (45). Therefore, the assumption is wrong, and hence the proposition is true. By this and by the Theorem on the invariance of interior points [41], the first equality of (43) follows.

Furthermore, set $\mathrm{int}\Lambda(T)$ is a convex set, and the set $\mathrm{int}Z(T)$ is path connected. Indeed, the mapping $F(\cdot,T)$ transforms any segment of $\mathrm{int}\Lambda(T)$ into a curve located completely inside $\mathrm{int}Z(T)$. For every point of $\mathrm{int}\Lambda(T)$ the Local Theorem on an implicit function [41] holds. Then the last fact of the statement follows from the Global Theorem 3 on an implicit function [47]. Hence the validity of the second equality of (43) follows. The proof is completed. □

Remark 4. We extend by continuity the derivatives $\frac{\partial F}{\partial \theta_i}(\theta,T)$, $i=\overline{1,3}$ of the mapping $F(\cdot,T)$ onto the boundary of the set $\Lambda(T)$. As a result, we have continuous partial derivatives of the mapping $F(\cdot,T)$ on the entire set $\Lambda(T)$.

From the definitions of the attainable set $X(T)$ and the auxiliary set $Z(T)$, Theorem 3.3, and Lemma 4.2 the following inclusions hold

$$Z(T) \subseteq X(T),\ \partial X(T) \subseteq \partial Z(T). \tag{46}$$

These explain why the set $Z(T)$ plays such an important role in the study of the attainable set $X(T)$.

Let us establish another important property of the auxiliary set $Z(T)$. It shows the uniqueness of controls corresponding to points on the boundary of this set and is a direct consequence of Lemma 4.2. This property was not before in [18, 19], though it strengthens results presented there.

Lemma 4.3. *Let θ_1, θ_2 be different points of the set $\partial\Lambda(T)$, and $u_{\theta_1}(t)$, $u_{\theta_2}(t)$ corresponding controls, defined on the interval $[0,T]$ by (38), for which*

$$\text{meas}\Big\{t \in [0,T] : u_{\theta_1}(t) \neq u_{\theta_2}(t)\Big\} > 0. \tag{47}$$

Suppose that the controls $u_{\theta_1}(t)$ and $u_{\theta_2}(t)$ correspond to the same point w on the boundary of the set $Z(T)$. Then, these controls coincide; that is $u_{\theta_1}(t) = u_{\theta_2}(t)$ for all $t \in [0,T]$.

Here meas Θ is the Lebesgue measure of the set $\Theta \subset \mathbb{R}^1$.

Remark 5. Condition (47) is essential for this Lemma. For example, the points $\theta_1 = (\frac{T}{4}, \frac{T}{2}, \frac{T}{2})^\top$ and $\theta_2 = (\frac{T}{4}, \frac{3T}{4}, \frac{3T}{4})^\top$ are different and belong to set $\partial\Lambda(T)$. However, they give by (38) the same piecewise constant control, defined on the interval $[0,T]$, which takes the value $u_{\max}$ on the interval $[0, \frac{T}{4}]$ and the value 0 on the interval $(\frac{T}{4}, T]$. Therefore, points θ_1 and θ_2 correspond to the same point on the boundary of the set $Z(T)$.

Proof. First, we have to introduce additional concepts and notations. Let us fix a small number $\mu > 0$ and integrate on the interval $[-\mu, 0]$ from right to left the problem (1),(2) with control $u(t) = u_{\max}$. The integration yields the point $\widehat{w}_0^\mu$. Let us now consider the system

$$\dot{w}(t) = Aw(t) + \varphi(w(t))c + u(t)\rho(w(t)), \quad t \in [-\mu, T+\mu], \tag{48}$$

with initial condition

$$w(-\mu) = \widehat{w}_0^\mu. \tag{49}$$

Here matrix A, vector c and functions $\varphi(w)$, $\rho(w)$ are defined by (40).

By analogy with the set $D(T)$, for problem (48),(49) we define the set of admissible controls $D_\mu(T)$ as the set of all Lebesgue measurable functions, which for almost all $t \in [0,T]$ satisfy inequalities (14).

We introduce the set

$$\Lambda_\mu(T) = \left\{\theta = (\theta_1, \theta_2, \theta_3)^\top \in \mathbb{R}^3 : -\mu \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq T + \mu\right\}.$$

For each point $\theta \in \Lambda_\mu(T)$ we construct the control $\widehat{u}_\theta^\mu(\cdot) \in D_\mu(T)$ as

$$\widehat{u}_\theta^\mu(t) = \begin{cases} u_{\max}, & \text{if} \quad -\mu \leq t \leq \theta_1, \\ 0, & \text{if} \quad \theta_1 < t \leq \theta_2, \\ u_{\max}, & \text{if} \quad \theta_2 < t \leq \theta_3, \\ 0, & \text{if} \quad \theta_3 < t \leq T + \mu. \end{cases} \tag{50}$$

Let $\widehat{w}_\theta^\mu(t)$, $t \in [-\mu, T+\mu]$ be the solution to the problem (48),(49) corresponding to the control $\widehat{u}_\theta^\mu(t)$. Finally, define a mapping $\widehat{F}_\mu(\cdot, T) : \Lambda_\mu(T) \to \mathbb{R}^3$ by the rule

$$\widehat{F}_\mu(\theta, T) = \widehat{w}_\theta^\mu(T+\mu), \quad \theta \in \Lambda_\mu(T).$$

Using arguments similar to those in Lemma 4.2 applied to this mapping lead to the following statement.

Proposition 1. *The restriction of mapping $\widehat{F}_\mu(\cdot, T)$ onto the interior of the set $\Lambda_\mu(T)$ is one-to-one.*

Next, we integrate on the interval $[-\mu, 0]$ from right to left the problem (1),(2) with control $u(t) = 0$ and, as a result, obtain the point $\widetilde{w}_0^\mu$. Let us consider on the interval $[-\mu, T+\mu]$ system (48) with the initial condition

$$w(-\mu) = \widetilde{w}_0^\mu. \tag{51}$$

For problem (48),(51) we take $D_\mu(T)$ as the set of admissible controls.

For each point $\theta \in \Lambda_\mu(T)$ we construct the control $\widetilde{u}_\theta^\mu(\cdot) \in D_\mu(T)$ by formula

$$\widetilde{u}_\theta^\mu(t) = \begin{cases} 0, & \text{if} \quad -\mu \le t \le \theta_1, \\ u_{\max}, & \text{if} \quad \theta_1 < t \le \theta_2, \\ 0, & \text{if} \quad \theta_2 < t \le \theta_3, \\ u_{\max}, & \text{if} \quad \theta_3 < t \le T+\mu. \end{cases} \tag{52}$$

Let $\widetilde{w}_\theta^\mu(t)$, $t \in [-\mu, T+\mu]$ be the solution to the problem (48),(51) corresponding to the control $\widetilde{u}_\theta^\mu(t)$. Finally, we define a mapping $\widetilde{F}_\mu(\cdot, T) : \Lambda_\mu(T) \to \mathbb{R}^3$ by the rule

$$\widetilde{F}_\mu(\theta, T) = \widetilde{w}_\theta^\mu(T+\mu), \quad \theta \in \Lambda_\mu(T).$$

Again, the arguments similar to those presented in Lemma 4.2 applied to this mapping lead to the following statement.

Proposition 2. *The restriction of mapping $\widetilde{F}_\mu(\cdot, T)$ onto the interior of the set $\Lambda_\mu(T)$ is one-to-one.*

Now, we are in a position to finish the proof of Lemma 4.3. Let θ_1 and θ_2 be the points of set $\partial\Lambda(T)$, for which the corresponding controls $u_{\theta_1}(t)$, $u_{\theta_2}(t)$, $t \in [0, T]$, defined by (38) and satisfying (47), correspond to the same point $w \in \partial\Lambda(T)$. There are three cases.

Case 1. Let the control $u_{\theta_1}(t)$ be one of three types, namely either

$$u_{\theta_1}(t) = \begin{cases} u_{\max}, & \text{if} \quad 0 \le t \le \tau_1, \\ 0, & \text{if} \quad \tau_1 < t \le \tau_2, \\ u_{\max}, & \text{if} \quad \tau_2 < t \le T, \end{cases} \tag{53}$$

or

$$u_{\theta_1}(t) = \begin{cases} 0, & \text{if} \quad 0 \le t \le \tau_1, \\ u_{\max}, & \text{if} \quad \tau_1 < t \le \tau_2, \\ 0, & \text{if} \quad \tau_2 < t \le T, \end{cases} \tag{54}$$

or

$$u_{\theta_1}(t) = \begin{cases} 0, & \text{if} \quad 0 \le t \le \tau, \\ u_{\max}, & \text{if} \quad \tau < t \le T. \end{cases} \tag{55}$$

Here $\tau \in (0, T)$, and $\tau_1, \tau_2 \in (0, T)$, $\tau_1 < \tau_2$ are the moments of switching. We continue controls $u_{\theta_1}(t)$, $u_{\theta_2}(t)$, $t \in [0, T]$ to the interval $[-\mu, T+\mu]$ by the value $u_{\max}$ on the interval $[-\mu, 0)$ and the value 0 on the interval $(T, T+\mu]$. This yields new controls $\widehat{u}_{\theta_1}^\mu(t)$, $\widehat{u}_{\theta_2}^\mu(t)$ of type (50) corresponding to points $\widehat{\theta}_1^\mu, \widehat{\theta}_2^\mu \in \Lambda_\mu(T)$, which steer system (48) on the interval $[-\mu, T+\mu]$ from initial point (49) to the same point $\widehat{w}_\mu$. In this case, the point $\widehat{\theta}_1^\mu$ is uniquely associated with switchings of control $\widehat{u}_{\theta_1}^\mu(t)$. Namely, $\widehat{\theta}_1^\mu = (\tau_1, \tau_2, T)^\top$ for (53), $\widehat{\theta}_1^\mu = (0, \tau_1, \tau_2)^\top$ for (54), and $\widehat{\theta}_1^\mu = (0, \tau, T)^\top$ for (55). For all these formulas, the inclusion $\widehat{\theta}_1^\mu \in \mathrm{int}\Lambda_\mu(T)$ holds. Therefore, by Proposition 1 the coincidence of points $\widehat{\theta}_1^\mu$ and $\widehat{\theta}_2^\mu$ immediately follows. Hence, the corresponding controls $\widehat{u}_{\theta_1}^\mu(t)$ and $\widehat{u}_{\theta_2}^\mu(t)$ also coincide on the interval $[-\mu, T+\mu]$. Therefore, the same conclusion holds for controls $u_{\theta_1}(t)$ and $u_{\theta_2}(t)$ on the interval $[0, T]$.

Case 2. Let the control $u_{\theta_1}(t)$ be of the type

$$u_{\theta_1}(t) = \begin{cases} u_{\max}, & \text{if} \quad 0 \le t \le \tau, \\ 0, & \text{if} \quad \tau < t \le T, \end{cases} \tag{56}$$

where $\tau \in (0, T)$ is the moment of switching. Without loss of generality, we assume that control $u_{\theta_2}(t)$ is not of types (53)–(55).

Let us continue controls $u_{\theta_1}(t)$, $u_{\theta_2}(t)$, $t \in [0, T]$ to the interval $[-\mu, T+\mu]$ by the value 0 on the interval $[-\mu, 0)$ and the value $u_{\max}$ on the interval $(T, T+\mu]$. This yields new controls $\widetilde{u}^{\mu}_{\theta_1}(t)$, $\widetilde{u}^{\mu}_{\theta_2}(t)$ of type (52) corresponding to points $\widetilde{\theta}^{\mu}_1, \widetilde{\theta}^{\mu}_2 \in \Lambda_\mu(T)$, which steer system (48) on the interval $[-\mu, T+\mu]$ from initial point (51) to the same point $\widetilde{w}_\mu$. In this case, the point $\widetilde{\theta}^{\mu}_1$ is uniquely associated with switchings of control $\widetilde{u}^{\mu}_{\theta_1}(t)$ by formula $\widetilde{\theta}^{\mu}_1 = (0, \tau, T)^\top$. Then, the inclusion $\widetilde{\theta}^{\mu}_1 \in \mathrm{int}\Lambda_\mu(T)$ holds. Therefore, by Proposition 2 the coincidence of points $\widetilde{\theta}^{\mu}_1$ and $\widetilde{\theta}^{\mu}_2$ immediately follows. Hence, the corresponding controls $\widetilde{u}^{\mu}_{\theta_1}(t)$, $\widetilde{u}^{\mu}_{\theta_2}(t)$ also coincide on the interval $[-\mu, T+\mu]$, and the same conclusion is correct for controls $u_{\theta_1}(t)$, $u_{\theta_2}(t)$ on the interval $[0, T]$.

Case 3. Suppose for definiteness that controls $u_{\theta_1}(t)$, $u_{\theta_2}(t)$ are of the types

$$u_{\theta_1}(t) = u_{\max}, \;\; u_{\theta_2}(t) = 0, \;\; t \in [0, T].$$

We continue these controls to the interval $[-\mu, T+\mu]$ in the same manner as in Case 2, obtaining controls $\widetilde{u}^{\mu}_{\theta_1}(t)$ and $\widetilde{u}^{\mu}_{\theta_2}(t)$ of the types

$$\widetilde{u}^{\mu}_{\theta_1}(t) = \begin{cases} 0, & \text{if} \quad -\mu \le t \le 0, \\ u_{\max}, & \text{if} \quad \;\; 0 < t \le T+\mu, \end{cases}$$

and

$$\widetilde{u}^{\mu}_{\theta_2}(t) = \begin{cases} 0, & \text{if} \quad -\mu \le t \le T, \\ u_{\max}, & \text{if} \quad \;\; T < t \le T+\mu. \end{cases}$$

These controls are of the type (55) and therefore, by the arguments from Case 1 applied on the interval $[-2\mu, T+2\mu]$, we come to the conclusion that controls $u_{\theta_1}(t)$ and $u_{\theta_2}(t)$ also coincide on the interval $[0, T]$.

All cases are considered, and the proof is completed. □

The following Corollary immediately follows from Lemma 4.3.

Corollary 3. *Each point w on the boundary of the auxiliary set $Z(T)$ can be reached by unique piecewise constant control $u(t)$, $t \in [0, T]$, which takes values $\{0; u_{\max}\}$ and has at most two switchings on the interval $(0, T)$.*

Further investigation of the auxiliary set $Z(T)$ involves studies of its supplements $\mathrm{R}^3 \setminus \mathrm{int}Z(T)$ and $\mathrm{R}^3 \setminus Z(T)$. First, we prove the following statement about the set $\mathrm{R}^3 \setminus \mathrm{int}Z(T)$.

Lemma 4.4. *The set $\mathrm{R}^3 \setminus \mathrm{int}Z(T)$ is path connected.*

Proof. Inequalities (13) imply the existence of the ball $P_K(0) = \Big\{ w \in \mathbb{R}^3 : \|w\| \le K \Big\}$ such that $Z(T) \subset \mathrm{int}P_K(0)$. Let us consider the closed sets $A_1 = Z(T)$ and $A_2 = P_K(0) \setminus \mathrm{int}Z(T)$. It is easy to see that the sets $A_1 \cup A_2 = P_K(0)$ and

$A_1 \cap A_2 = \partial Z(T)$ are path connected. Then, by Proposition 14.11 in [55], the set A_2 is path connected as well. Since the set $\mathrm{R}^3 \setminus \mathrm{int} P_K(0)$ is path connected and has a non-empty intersection with the set A_2, then, by Proposition 14.F in [55], we conclude that the set $\mathrm{R}^3 \setminus \mathrm{int} Z(T)$ is path connected as well. This completes the proof. □

Second, we prove the statement below associated with the set $\mathrm{R}^3 \setminus Z(T)$. The proof employs an extension method of the mapping $F(\cdot, T)$, which is different from the method presented in [18, 19] and which is more natural for problem (1),(2).

Lemma 4.5. *The set* $\mathrm{R}^3 \setminus Z(T)$ *is path connected.*

Proof. The justification of this statement consists of two steps.
Step 1. Let us fix a small number $\mu > 0$. Using arguments from Lemma 4.3 we construct an extension $F_\mu(\cdot, T)$ of the mapping $F(\cdot, T)$ from the set $\Lambda(T)$ to the set $\Lambda_\mu(T)$ as follows. For each point $\theta \in \Lambda_\mu(T)$ we define by (50) the control $u^\mu_\theta(\cdot) \in D_\mu(T)$. Then, let $w^\mu_\theta(t)$, $t \in [-\mu, T+\mu]$ be the solution of system (48) with initial condition $w^\mu_0 = \widehat{w}^\mu_0$ corresponding to control $u^\mu_\theta(t)$. Next, we integrate this system with initial condition $w^\mu_\theta(T+\mu)$ from right to left on the interval $[T, T+\mu]$ with control $u^\mu_*(t) = 0$, and denote $w^\mu_*(t)$ the corresponding solution. Finally, we define a mapping $F_\mu(\cdot, T) : \Lambda_\mu(T) \to \mathbb{R}^3$ by the rule

$$F_\mu(\theta, T) = w^\mu_\theta(T+\mu) - \int\limits_T^{T+\mu} \Big(Aw^\mu_*(t) + \varphi(w^\mu_*(t))c\Big)dt, \quad \theta \in \Lambda_\mu(T).$$

From this definition it follows that

$$F_\mu(\theta, T) = F(\theta, T), \quad \theta \in \Lambda(T).$$

Hence, the mapping $F_\mu(\cdot, T)$ is the desired extension of the mapping $F(\cdot, T)$.

The mapping $F_\mu(\cdot, T)$ is continuously differentiable on the set $\mathrm{int} Z_\mu(T)$, and for each point $\theta \in \mathrm{int}\Lambda_\mu(T)$ the equalities

$$\frac{\partial F_\mu}{\partial \theta_i}(\theta, T) = (-1)^{i-1} u_{\max} \Phi^\mu_*(T) \Big[\Phi^\mu_\theta(\theta_i)\Big]^{-1} \rho(w^\mu_\theta(\theta_i)),\ i = \overline{1,3} \tag{57}$$

hold. These equalities are corollaries of the well-known results in theory of ODEs about differentiation of solutions to ODEs with respect to parameters and initial conditions [23]. Here, $\Phi^\mu_\theta(t)$ and $\Phi^\mu_*(t)$ are solutions of Cauchy problems similar to (41). These solutions are respectively defined on intervals $[-\mu, T+\mu]$ and $[T, T+\mu]$. The first function corresponds to control $u^\mu_\theta(t)$ and solution $w^\mu_\theta(t)$, and the second function corresponds to control $u^\mu_*(t)$ and solution $w^\mu_*(t)$. These functions satisfy initial conditions

$$\Phi^\mu_\theta(T+\mu) = E = \Phi^\mu_*(T+\mu).$$

Now, applying to mapping $F_\mu(\cdot, T)$ the arguments similar to those in Lemma 4.2 we obtain the following statement.

Proposition 3. *The restriction of mapping* $F_\mu(\cdot, T)$ *onto the interior of the set* $\Lambda_\mu(T)$ *is one-to-one.*

Let us consider the following subsets on the boundary of the set $\Lambda(T)$:

$$\Lambda_1(T) = \Big\{\theta \in \Lambda(T) : 0 = \theta_1 < \theta_2 < \theta_3 < T\Big\},$$

$$\Lambda_2(T) = \Big\{\theta \in \Lambda(T) : 0 < \theta_1 < \theta_2 < \theta_3 = T\Big\},$$

$$\Lambda_3(T) = \Big\{\theta \in \Lambda(T) : 0 = \theta_1 < \theta_2 < \theta_3 = T\Big\}.$$

Lemma 4.3 implies that the mapping $F(\cdot, T)$ transfers one-to-one these sets onto corresponding surfaces and a curve on the boundary of the set $Z(T)$. Since sets $\Lambda_i(T)$, $i = \overline{1,3}$ are located in the set $\mathrm{int}\Lambda_\mu(T)$, then by Proposition 3 the mapping $F_\mu(\cdot, T)$, as an extension of the mapping $F(\cdot, T)$, transfers one-to-one path connected neighborhoods $B_\mu(\theta) \cap \Big(\mathrm{R}^3 \setminus \Lambda(T)\Big)$ of points $\theta \in \Lambda_i(T)$, $i = \overline{1,3}$ onto path connected neighborhoods of points

$$w = F(\theta, T) = F_\mu(\theta, T) \in \partial Z(T).$$

Here $B_\mu(\theta) = \Big\{\eta \in \mathbb{R}^3 : \|\eta - \theta\| < \mu\Big\}$.

We note that at each point of sets $\Lambda_i(T)$, $i = \overline{1,3}$ mappings $F(\cdot, T)$, $F_\mu(\cdot, T)$ coincide, and the corresponding Jacobi matrix is nonsingular. Therefore, points of these sets are regular points of these mappings (cf. [42]). It is easy to see, from Remark 4 to Lemma 4.2 and formulas (44),(57), that at all other points on the boundary of the set $\Lambda(T)$ the Jacobi matrix is singular. Therefore, all these points form a set of critical points of mappings $F(\cdot, T)$, $F_\mu(\cdot, T)$. From the Sard's Theorem [42], it follows that the image of this set has zero Lebesgue measure on the set $\partial Z(T)$. Indeed, on the boundary of the set $Z(T)$ such points form a curve. The proof of Step 1 is completed.

Step 2. Let us consider arbitrary points $w_1, w_2 \in \mathrm{R}^3 \setminus Z(T)$. At the same time, we have $w_1, w_2 \in \mathrm{R}^3 \setminus \mathrm{int}Z(T)$ and by Lemma 4.4 the set $\mathrm{R}^3 \setminus \mathrm{int}Z(T)$ is path connected. Therefore, there exists the path $h(t)$, $t \in [0, 1]$ connecting these points; that is

$$h(0) = w_1,\ h(1) = w_2; \quad h(t) \in \mathrm{R}^3 \setminus \mathrm{int}Z(T),\ t \in (0, 1).$$

Next, we consider two cases.

Case 1. Let the inclusion $h(t) \in \mathrm{R}^3 \setminus Z(T)$ hold for all $t \in [0, 1]$. Then, points w_1 and w_2 are connected by path $h(t)$, $t \in [0, 1]$ in the set $\mathrm{R}^3 \setminus Z(T)$. From the arbitrariness of these points the desired fact follows.

Case 2. Let there be a value $t_0 \in (0, 1)$ such that $h(t_0) \notin \mathrm{R}^3 \setminus Z(T)$. The set $\mathrm{R}^3 \setminus Z(T)$ is open and therefore, there are intervals adjoining to $t = 0$ and $t = 1$, such that for each value t in these intervals the inclusion $h(t) \in \mathrm{R}^3 \setminus Z(T)$ holds. Then, it determined the values

$$\bar{t} = \sup\Big\{t \in (0, t_0) : \forall s \in [0, t]\ \ h(s) \in \mathrm{R}^3 \setminus Z(T)\Big\},$$

$$\widetilde{t} = \inf\Big\{t \in (t_0, 1) : \forall s \in [t, 1]\ \ h(s) \in \mathrm{R}^3 \setminus Z(T)\Big\}.$$

Obviously that $\bar{t} \le \widetilde{t}$.

Let us consider the case $\bar{t} < \widetilde{t}$. The case $\bar{t} = \widetilde{t}$ will follow from arguments below. By definitions of values $\bar{t}$, $\widetilde{t}$ we have $h(\bar{t}), h(\widetilde{t}\,) \in \partial Z(T)$. The set $\partial Z(T)$ is path connected and therefore, there exists the path $g(t)$, $t \in [\bar{t}, \widetilde{t}\,]$ connecting points $h(\bar{t})$ and $h(\widetilde{t}\,)$; that is

$$g(\bar{t}) = h(\bar{t}),\ g(\widetilde{t}\,) = h(\widetilde{t}\,); \quad g(t) \in \partial Z(T),\ t \in (\bar{t}, \widetilde{t}\,).$$

The continuous curve $g(t)$, $t \in [\bar{t}, \widetilde{t}\,]$ is a compact set in $\mathbb{R}^3$. Therefore, by Proposition 17.H in [55], it is covered by a finite number of balls $B_\epsilon(g(t_i))$, $i = \overline{0, (m+1)}$. Here $\{t_i\}_{i=0}^{m+1}$ is the partition of the interval $[\bar{t}, \widetilde{t}\,]$: $\bar{t} = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = \widetilde{t}$, and $\{g(t_i)\}_{i=0}^{m+1}$ are centers of these balls. On the basis of results of

Step 1, without loss of generality, we consider, first, that all points $\{g(t_i)\}_{i=0}^{m+1}$ are regular points of the mapping $F(\cdot, T)$. Second, the value $\epsilon > 0$ is chosen so that sets $B_\epsilon(g(t_i)) \cap \Big(\mathrm{R}^3 \setminus Z(T)\Big)$, $i = \overline{0, (m+1)}$ are path connected.

Now, for each $i = \overline{0, m}$, we consider two neighboring balls $B_\epsilon(g(t_i))$, $B_\epsilon(g(t_{i+1}))$. There exists a point η of the curve $g(t)$, $[\bar{t}, \widetilde{t}\ ]$ such that $\eta \in B_\epsilon(g(t_i))$, $\eta \in B_\epsilon(g(t_{i+1}))$. Then, there is an open neighborhood V of point η such that $V \subset B_\epsilon(g(t_i))$, $V \subset B_\epsilon(g(t_{i+1}))$. The point η is on the boundary of the set $Z(T)$ and therefore, in neighborhood V there is the point $\widehat{\eta} \in \mathrm{R}^3 \setminus Z(T)$. Hence, the point $\widehat{\eta}$ belongs to both these balls and therefore, the path connected sets $B_\epsilon(g(t_i)) \cap \Big(\mathrm{R}^3 \setminus Z(T)\Big)$, $B_\epsilon(g(t_{i+1})) \cap \Big(\mathrm{R}^3 \setminus Z(T)\Big)$ have a non-empty intersection. Then, by Proposition 14.F in [55], the set

$$\Omega = \bigcup_{i=0}^{m+1} \Big(B_\epsilon(g(t_i)) \cap \Big(\mathrm{R}^3 \setminus Z(T)\Big)\Big) \subset \mathrm{R}^3 \setminus Z(T),$$

and is path connected.

Let us define the small number $\delta > 0$ such that inclusions

$$h(\bar{t} - \delta) \in B_\epsilon(g(t_0)) \cap \Big(\mathrm{R}^3 \setminus Z(T)\Big), \quad h(\widetilde{t} + \delta) \in B_\epsilon(g(t_{m+1})) \cap \Big(\mathrm{R}^3 \setminus Z(T)\Big)$$

simultaneously hold. Hence, we can conclude that $h(\bar{t} - \delta), h(\widetilde{t} + \delta) \in \Omega$. Therefore, from the path connection of the set Ω, it follows the existence of the path $q(t)$, $t \in [\bar{t} - \delta, \widetilde{t} + \delta]$ connecting points $h(\bar{t} - \delta)$ and $h(\widetilde{t} + \delta)$; that is

$$q(\bar{t} - \delta) = h(\bar{t} - \delta),\ q(\widetilde{t} + \delta) = h(\widetilde{t} + \delta); \quad q(t) \in \mathrm{R}^3 \setminus Z(T),\ t \in (\bar{t} - \delta, \widetilde{t} + \delta).$$

Now, we define the path $\chi(t)$, $t \in [0, 1]$ by the formula

$$\chi(t) = \begin{cases} h(t), & \text{if} \quad t \in [0, \bar{t} - \delta], \\ q(t), & \text{if} \quad t \in [\bar{t} - \delta, \widetilde{t} + \delta], \\ h(t), & \text{if} \quad t \in [\widetilde{t} + \delta, 1]. \end{cases}$$

Obviously, the inclusion $\chi(t) \in \mathrm{R}^3 \setminus Z(T)$ holds for all $t \in [0, 1]$, and hence, the points w_1 and w_2 are connected by path $\chi(t)$, $t \in [0, 1]$ in the set $\mathrm{R}^3 \setminus Z(T)$. From the arbitrariness of these points the desired fact follows. The proof of Step 2 and hence, that of the entire statement is now completed. □

Finally, we are now able to establish the validity of the main result of this paper.

Theorem 4.6. *For the attainable set $X(T)$ and the auxiliary set $Z(T)$, the equality $X(T) = Z(T)$ holds.*

Proof. It follows from the first inclusion in (46) that in order to prove the hypothesis it is sufficient to show the validity of the inclusion $X(T) \subseteq Z(T)$. Let us assume the opposite, i.e. assume that there exists a point $\widetilde{w}$ such that

$$\widetilde{w} \notin Z(T),\ \widetilde{w} \in X(T)$$

holds. Consider a point $\widehat{w} \notin X(T)$.

The arguments presented in Lemmas 4.2 and 4.5 show that the boundary of the set $Z(T)$ divides R^3 into two path connected subsets $\mathrm{int}Z(T)$ and $\mathrm{R}^3 \setminus Z(T)$. The path connectedness of the second set ensures the existence of a continuous curve $\sigma(s)$, $s \in [0, 1]$, as well as $\widetilde{w} = \sigma(0)$, $\widehat{w} = \sigma(1)$, and $\sigma(s) \notin Z(T)$ for all $s \in (0, 1)$. By Theorem 36 on "transition through customs" in [44], there is a value $s_\star \in (0, 1)$

such that $\sigma(s_\star) \in \partial X(T)$. Therefore, there is a defined point $\bar{w} = \sigma(s_\star)$, such that the relationships

$$\bar{w} \in \partial X(T), \ \bar{w} \notin \partial Z(T),$$

simultaneously hold. This contradicts to the second inclusion in (46). Hence the assumption is incorrect, and the required inclusion holds. The proof is completed. □

Theorem 4.6, Lemma 4.2, and Corollary 3 of Lemma 4.3 imply, that the set $\Lambda(T)$ and the mapping $F(\cdot, T)$ form a parametric description of the attainable set $X(T)$ (its boundary and interior) by the moments of switching of control $u_\theta(t)$. Moreover, each interior point of the attainable set $X(T)$ can be obtained under unique control $u_\theta(t)$ with precisely three switchings, and each boundary point of this set can be reached by a unique control with at most two switchings.

Remark 6. It may be noteworthy that a similar parametrization of the attainable set was obtained for a nonlinear control model for the process of production and sales of a consumer good [15].

5. Numerical simulations.

We consider numerical examples for two specific types of the reaction rate in system (1).

Law of mass action. Let $f(x) = x$, $g(y) = y$, $h(z) = z$. It is easy to see that constrains (3)–(5) are satisfied. With this reaction rate, the system (1) takes the form

$$\begin{cases} \dot{x} = -xyz + u(m - x), \ t \in [0, T], \\ \dot{y} = -xyz, \\ \dot{z} = xyz - bz. \end{cases} \tag{58}$$

This reaction rate is the law of mass action, and this model was studied in detail in [18, 19].

Figures 1 to 3 were constructed with MATLAB using Theorem 4.6, and previously reported in [18, 19]. These Figures show attainable sets $X(T)$ for system (58) for three sets of initial conditions and system parameters, respectively.

Michaelis–Menten kinetics. Let $f(x) = \dfrac{x}{x + K_x}$, $g(y) = \dfrac{y}{y + K_y}$, $h(z) = z$. Constrains (3)–(5) are also satisfied, and system (1) takes the form

$$\begin{cases} \dot{x} = -\dfrac{x}{x + K_x}\dfrac{y}{y + K_y}z + u(m - x), \ t \in [0, T], \\ \dot{y} = -\dfrac{x}{x + K_x}\dfrac{y}{y + K_y}z, \\ \dot{z} = \dfrac{x}{x + K_x}\dfrac{y}{y + K_y}z - bz. \end{cases} \tag{59}$$

Figures 4 to 6, which are also constructed with MATLAB using Theorem 4.6, show Examples 4 to 6 of attainable sets $X(T)$ for system (59) for three different sets of initial conditions and system parameters, which are taken from [6].

6. Conclusions.

Many processes in biotechnology and medicine (treatments) can be controlled. High costs of reagents and medications, as well as possible severe side effects, imply that optimizing controls for these processes may be beneficial. However, complexity of biological processes and insufficient data usually prevent straightforward application of standard optimization techniques to these problems.

In this paper we consider a control model of a process of the autothermal thermophilic aerobic digestion. This is a biological process, where bacteria are employed

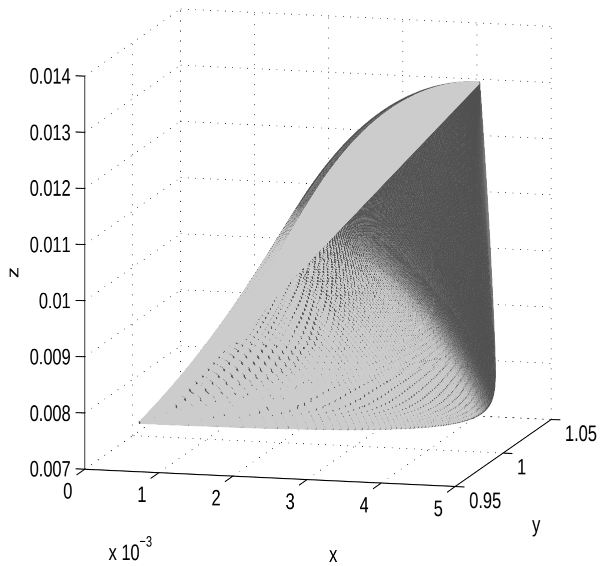

FIGURE 1. Attainable set $X(T)$ of system (58),(2). Here, $x_0 = 0.0019$, $y_0 = 2.498$, $z_0 = 0.0874$, $m = 0.048$, $b = 0.24$, $u_{\max} = 4.0$ and $T = 6.0$ (Example 1).

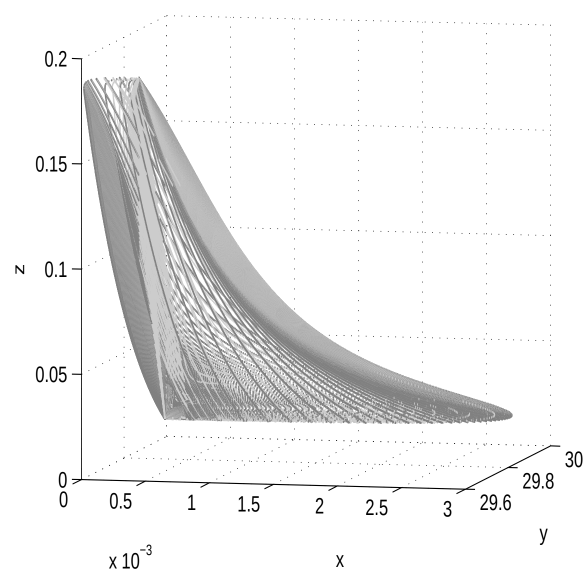

FIGURE 2. Attainable set $X(T)$ of system (58),(2). Here, $x_0 = 0.0192$, $y_0 = 74.94$, $z_0 = 0.0874$, $m = 0.048$, $b = 0.24$, $u_{\max} = 4.0$ and $T = 6.0$ (Example 2).

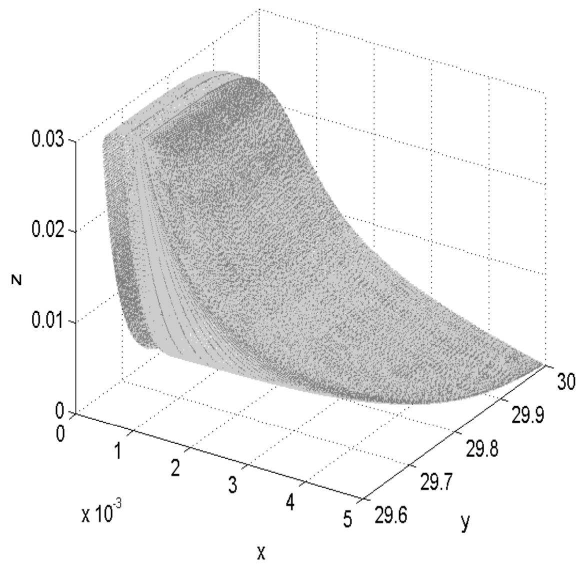

FIGURE 3. Attainable set $X(T)$ of system (58),(2). Here, $x_0 = 0.001$, $y_0 = 146.9694$, $z_0 = 0.1715$, $m = 0.0245$, $b = 0.5$, $u_{\max} = 4.0$ and $T = 20.0$ (Example 3).

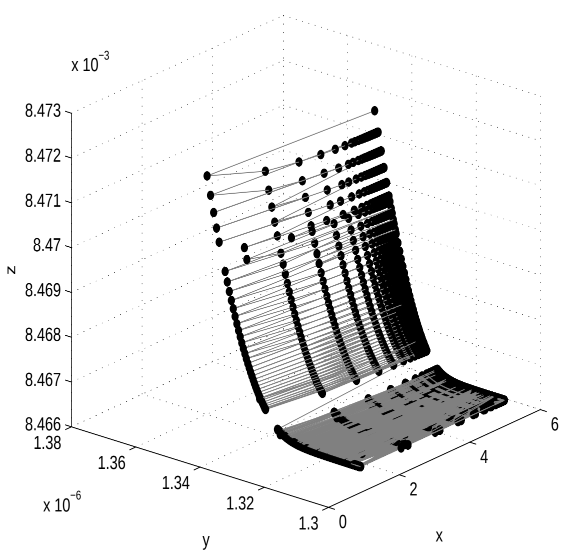

FIGURE 4. Attainable set $X(T)$ of system (59),(2). Here, $x_0 = 0.9541 \cdot 10^{-5}$, $y_0 = 0.4989$, $z_0 = 0.5612$, $K_x = 0.0006361$, $K_y = 0.1247$, $m = 5.6125$, $b = 0.5$, $u_{\max} = 20.0$ and $T = 10.0$ (Example 4).

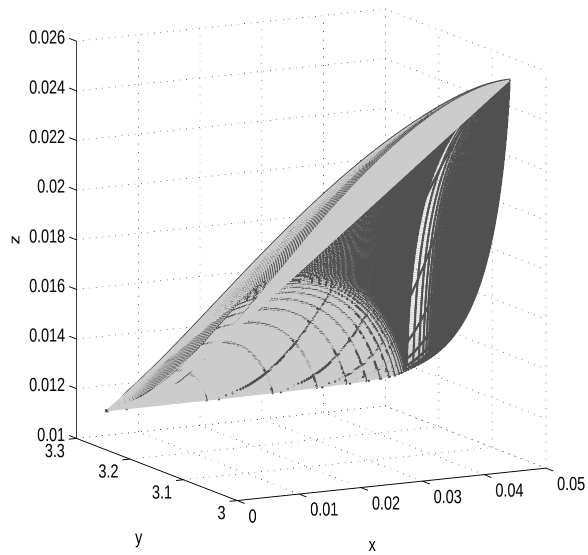

FIGURE 5. Attainable set $X(T)$ of system (59),(2). Here, $x_0 = 0.004874$, $y_0 = 3.2496$, $z_0 = 1.0155$, $K_x = 0.32496$, $K_y = 0.8124$, $m = 0.04874$, $b = 0.5625$, $u_{\max} = 12.5$ and $T = 8.0$ (Example 5).

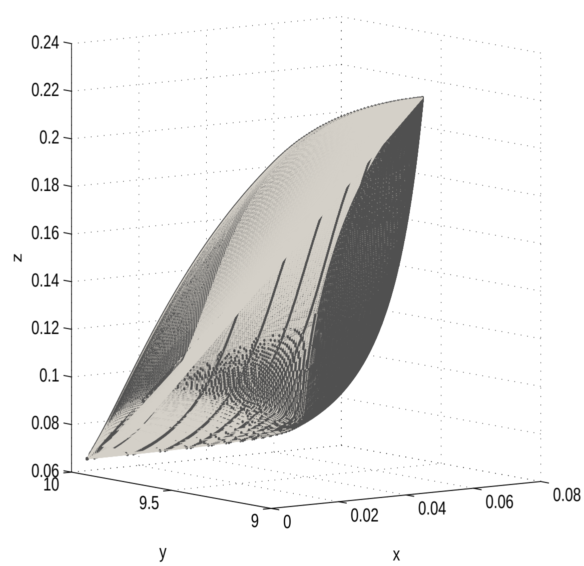

FIGURE 6. Attainable set $X(T)$ of system (59),(2). Here, $x_0 = 0.002647$, $y_0 = 9.9247$, $z_0 = 1.3233$, $K_x = 0.26466$, $K_y = 5.9548$, $m = 0.06616$, $b = 0.25$, $u_{\max} = 4.1666$ and $T = 12.0$ (Example 6).

for waste water treatment. Our objective was to investigate the impact of the nonlinearity of the reaction rate on the attainable set and the control function. For this reason, we considered a very general model of the process, where the reaction rate is given by a product of three unspecified nonlinear functions, $f(x)g(y)h(z)$, constrained by a few biologically motivated generic conditions (such as monotonicity and concavity of the functions). For this model, we analytically obtained the detailed structure of an attainable set $X(T)$ and found the form of controls corresponding to points of this set. In particular, for this model the moments of switching of the controls $u_\theta(t)$, which form the set $\Lambda(T)$, together with the mapping $F(\cdot, T)$, play the role of parametrization for the set $X(T)$ (for both its interior and boundary). It was proved that each point on the boundary of set $X(T)$ can be reached by a control from the above mentioned class (a bang-bang control with at most two switchings), and every point of the interior of $X(T)$ is the result of a bang-bang control with precisely three switchings. An original computer program (written in MATLAB) allows us to numerically construct attainable sets for a variety of reaction rates, initial conditions and the system parameters.

It is noteworthy, that this results are in agreement with the earlier reported results for the model, where the reaction rate is given by the law of mass action. This finding rigorously confirm the intuitive expectation, that this type of control is the same for any model of the process where a reaction rate possesses certain generic properties, such as the monotonicity and the concaveness with respect to all arguments.

These results are also of apparent practical significance and can be straightforwardly applied to practice, as they allow significantly narrow the set of possible controls and thus considerably reduce the amount of computations required to numerically find optimal controls for real-life optimal control problems.

Acknowledgments. A. Korobeinikov is supported by the Ministry of Science and Innovation of Spain via Ramón y Cajal Fellowship RYC-2011-08061.

REFERENCES

[1] J.-P. Aubin and A. Cellina, "Differential Inclusions: Set-Valued Maps and Viability Theory," Springer, Berlin-New York, 1984.

[2] A. D. Bojarski, J. Rojas and T. Zhelev, *Modelling and sensitivity analysis of ATAD*, Computers & Chemical Engineering, **34** (2010), 802–811.

[3] B. Bonnard and M. Chyba, "Singular Trajectories and Their Role in Control Theory," Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[4] G. Bromström and H. Drange, *On the mathematical formulation and parameter estimation of the norwegian sea plankton system*, Sarsia, **85** (2000), 211–225.

[5] D. Brune, *Optimal control of the complete-mix activated sludge process*, Environmental Technology Letters, **6** (1985), 467–476.

[6] M. Burke, M. Chapwanya, K. Doherty, I. Hewitt, A. Korobeinikov, M. Meere, S. McCarthy, M. O'Brien, V. T. N. Tuoi, H. Winstanley and T. Zhelev, *Modeling of autothermal thermophylic aerobic digestion*, Mathematics-in-Industry Case Studies Journal, **2** (2010), 34–63.

[7] S. Busenberg, S. Kumar, P. Austin and G. Wake, *The Dynamics of a model of a plankton-nutrient interaction*, Bulletin of Mathematical Biology, **52** (1990), 677–696.

[8] F. L. Chernous'ko and V. B. Kolmanovskii, *Computational and approximate methods of optimal control*, Journal of Mathematical Sciences, **12** (1979), 310–353.

[9] F. L. Chernous'ko, "Ellipsoidal state Estimation for Dynamical Systems," CRS Press, Boca Raton, Florida, 1994.

[10] B. P. Demidovich, "Lectures on Stability Theory," Nauka, Moscow, 1967.

[11] A. V. Dmitruk, *A generalized estimate on the number of zeroes for solutions of a class of linear differential equations*, SIAM Journal of Control and Optimization, **30** (1992), 1087–1091.

[12] P. Georgescu and Y.-H. Hsieh, *Global stability for a virus dynamics model with nonlinear incidence of infection and removal*, SIAM Journal on Applied Mathematics, **67** (2007), 337–353.

[13] M. Graells, J. Rojas and T. Zhelev, *Energy efficiency optimization of wastewater treatment. Study of ATAD*, Computer Aided Chemical Engineering, **28** (2010), 967–972.

[14] E. N. Khailov and E. V. Grigorieva, *On the attainability set for a nonlinear system in the plane*, Moscow University. Computational Mathematics and Cybernetics, (2001), 27–32.

[15] E. V. Grigorieva and E. N. Khailov, *A nonlinear controlled system of differential equations describing the process of production and sales of a consumer good*, Dynamical Systems and Differential Equations, (2003), 359–364.

[16] E. N. Khailov and E. V. Grigorieva, *Description of the attainability set of a nonlinear controlled system in the plane*, Moscow University. Computational Mathematics and Cybernetics, (2005), 23–28.

[17] E. V. Grigorieva and E. N. Khailov, *Attainable set of a nonlinear controlled microeconomic model*, Journal of Dynamical and Control Systems, **11** (2005), 157–176.

[18] E. V. Grigorieva, N. V. Bondarenko, E. N. Khailov and A. Korobeinikov, *Three-dimensional nonlinear control model of wastewater biotreatment*, Neural, Parallel, and Scientific Computations, **20** (2012), 23–35.

[19] E. V. Grigorieva, N. V. Bondarenko, E. N. Khailov and A. Korobeinikov, *Finite-dimensional methods for optimal control of autothermal thermophilic aerobic digestion*, in "Industrial Waste" (eds. K.-Y. Show and X. Guo), InTech, Croatia, (2012), 91–120.

[20] T. Gross, W. Ebenhöh and U. Feudel, *Enrichment and foodchain stability: The impact of different forms of predator-prey interaction*, Journal of Theoretical Biology, **227** (2004), 349–358.

[21] V. I. Gurman and E. A. Trushkova, *Estimates for attainability sets of control systems*, Differential Equations, **45** (2009), 1636–1644.

[22] Kh. G. Guseinov, A. N. Moiseev and V. N. Ushakov, *On the approximation of reachable domains of control systems*, Journal of Applied Mathematics and Mechanics, **62** (1998), 169–175.

[23] P. Hartman, "Ordinary Differential Equations," Jorn Wiley & Sons, New York-London-Sydney, 1964.

[24] A. N. Kolmogorov, *Sulla teoria di Volterra della lotta per l'esistenza*, Giorn. Ist. Ital. Attuari, **7** (1936), 74–80.

[25] V. A. Komarov, *Estimates of the attainable set for differential inclusions*, Mathematical Notes, **37** (1985), 916–925.

[26] A. Korobeinikov, *Stability of ecosystem: Global properties of a general prey-predator model*, Mathematical Medicine and Biology, **26** (2009), 309–321.

[27] A. Korobeinikov, *Global asymptotic properties of virus dynamics models with dose dependent parasite reproduction and virulence, and nonlinear incidence rate*, Mathematical Medicine and Biology, **26** (2009), 225–239.

[28] A. Korobeinikov, *Global properties of a general predator-prey model with non-symmetric attack and consumption rate*, Discrete and Continuous Dynamical System. Ser. B, **14** (2010), 1095–1103.

[29] M. A. Krasnosel'skii, "The Operator of Translation along the Trajectories of Differential Equations," American Mathematical Society, Providence, RI, 1968.

[30] A. B. Kurzhanski and I. Valyi, "Ellipsoidal Calculus for Estimation and Control," Birkhäuser, Boston, 1997.

[31] U. Ledzewicz, J. Marriott, H. Maurer and H. Schättler, *Realizable protocols for optimal administration of drugs in mathematical models for anti-angiogenic treatment*, Mathematical Medicine and Biology, **27** (2010), 157–179.

[32] U. Ledzewicz, E. Kashdan and H. Schättler, *Optimal and suboptimal protocols for a mathematical model for tumor anti-angiogenesis in combination with chemotherapy*, Mathematical Biosciences and Engineering, **8** (2011), 307–323.

[33] U. Ledzewicz, M. Naghnaeian and H. Schättler, *Optimal response to chemotherapy for a mathematical model of tumor-immune dynamics*, Journal of Mathematical Biology, **64** (2012), 557–577.

[34] E. B. Lee and L. Markus, "Foundations of Optimal Control Theory," Jorn Wiley & Sons, New York, 1967.

[35] M. S. Nikol'skii, *Approximation of the attainability set for a controlled process*, Mat. Zametki, **41** (1987), 71–76, 121.

[36] N. P. Osmolovskii and H. Maurer, *Equivalence of second order optimality conditions for bang-bang control problems. Part 2: proofs, variational derivatives and representations*, Control and Cybernetics, **36** (2007), 5–45.

[37] A. I. Ovseevich and F. L. Chernous'ko, *Two-sided estimates on the attainability domains of controlled systems*, Journal of Applied Mathematics and Mechanics, **46** (1982), 590–595.

[38] A. I. Panasyuk and V. I. Panasyuk, *An equation generated by a differential inclusion*, Mathematical Notes, **27** (1980), 429–437, 494.

[39] A. I. Panasyuk, *Equations of attainable set dynamics. I. Integral funnel equation*, Journal of Optimization Theory and Applications, **64** (1990), 349–366.

[40] A. I. Panasyuk, *Equations of attainable set dynamics. II. Partial differential equations*, Journal of Optimization Theory and Applications, **64** (1990), 367–377.

[41] T. Partasarathy, "On Global Univalence Theorems," Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[42] A. Sard, *The measure of the critical values of differentiable maps*, Bulletin of the American Mathematical Society, **48** (1942), 883–890.

[43] H. Schättler and U. Ledzewicz, "Geometric Optimal Control. Theory, Methods and Examples," Springer, New York-Heidelberg-Dordrecht-London, 2012.

[44] L. Schwartz, "Analyse Mathematique 1," Hermann, Paris, 1967.

[45] G. V. Shevchenko, *Numerical method for solving a nonlinear time-optimal control problem with additive control*, Computational Mathematics and Mathematical Physics, **47** (2007), 1768–1778.

[46] G. V. Shevchenko, *Numerical solution of a nonlinear time optimal control problem*, Computational Mathematics and Mathematical Physics, **51** (2011), 537–549.

[47] S. Ichiraku, *A note on global implicit function theorems*, IEEE Transactions on Curcuits and Systems, **32** (1985), 503–505.

[48] D. Szolnoki, *Set-oriented methods for computing reachable sets and control sets*, Discrete and Continuous Dynamical Systems. Ser. B, **3** (2003), 361–382.

[49] Z. Feng and H. R. Thieme, *Endemic models with arbitrarily distributed periods of infection I: Fundamental properties of the model*, SIAM Journal on Applied Mathematics, **61** (2000), 803–833.

[50] H. R. Thieme and Z. Feng, *Endemic models with arbitrarily distributed periods of infection II: Fast disease dynamics and permanent recovery*, SIAM Journal on Applied Mathematics, **61** (2000), 983–1012.

[51] A. N. Tikhonov, A. B. Vasil'eva and A. G. Sveshnikov, "Differential Equations," Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[52] A. I. Tyatyushkin and O. V. Morzhin, *Constructive methods of control optimization in nonlinear systems*, Automation and Remote Control, **70** (2009), 772–786.

[53] A. I. Tyatyushkin and O. V. Morzhin, *Numerical investigation of attainability sets of nonlinear controlled differential systems*, Automation and Remote Control, **72** (2011), 1291–1300.

[54] P. Varaiya and A. B. Kurzhanski, *Ellipsoidal methods for dynamics and control. Part I*, Journal of Mathematical Sciences, **139** (2006), 6863–6901.

[55] O. Ya. Viro, O. A. Ivanov, N. Yu. Netsvetaev and V. M. Kharlamov, "Elementary Topology: Problem Textbook," AMS, 2008.

Received October 16, 2012; Accepted April 03, 2013.

E-mail address: egrigorieva@twu.edu
E-mail address: khailov@cs.msu.su
E-mail address: AKorobeinikov@crm.cat