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EQUILIBRIUM SOLUTIONS FOR MICROSCOPIC STOCHASTIC SYSTEMS IN POPULATION DYNAMICS

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Abstract. The present paper deals with the problem of existence of equilibrium solutions of equations describing the general population dynamics at the microscopic level of modified Liouville equation (individually–based model) corresponding to a Markov jump process. We show the existence of factorized equilibrium solutions and discuss uniqueness. The conditions guaranteeing uniqueness or non-uniqueness are proposed under the assumption of periodic structures.

1. **Introduction.** In many animal societies individuals placed together in a group enter into self–organizing process leading to the formation of hierarchies in domination (see [14] and references therein). Such process occurs through dominance–subordination interactions (usually pairwise) between individuals. The interested reader is addressed to mathematical and computer models that base on empirical data in a primitive wasp society (*Polistes dominulus*) — see [14] and references therein. The self-organizing processes involve a double reinforcement mechanism: winners reinforce their probability of winning and losers reinforce their probability of losing, see [5, 14, 15] and references therein.

This leads to the conclusion that the adequate mathematical model should take into account the individual state and its time evolution. In other words the microscopic description becomes more satisfactory than the standard macroscopic description in terms of densities.

Similar situation may be observed in the case of cancer — immune system competition when every entity (cell) may be characterized by a level of activity. The microscopic model takes into account the individual states (activities) of interacting entities of the system.

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In the simple situations of anonymous organisms a mathematical model at the mesoscopic level — a nonlinear Boltzmann–type integro-differential equation — describing the time evolution of the distribution of dominance parameter $u \in [0, 1]$ was proposed by Jäger and Segel [6] and related to a large population of bumble bees. A Boltzmann–type equation for periodic functions was proposed in paper [7] for the one–dimensional angular distribution in the context of angular self–organization of the actin cytoskeleton in the process of instant changing of filament orientation in the course of specific actin–actin interactions.

The mesoscopic population dynamics models were considered and extended to various situations in biology, medicine and other applied sciences in [1, 2, 4, 9, 10, 11, 12] (see references therein). An interesting approach of modelling the competition between cancer and immune system was proposed by Bellomo and Forni [3] (see also [4]). The general bilinear equations of the Boltzmann–type were studied in [12] (see also [9]).

In Refs. [9, 10, 11] a general framework for the program of finding possible transitions between the different levels of description — microscopic (individually–based) **(Mi)**, mesoscopic **(Me)**, and macroscopic **(Ma)** was discussed. The levels are

(Mi) – the level of interacting entities ("*micro–scale*"), in mathematical terms of jump Markov processes, that lead to continuous (linear) stochastic semigroups that is related to the modified Liouville equation;

(Me) – the level of the statistical description of a test–entity ("*meso–scale*"), in terms of continuous nonlinear semigroups related to the solutions of nonlinear Boltzmann–type;

(Ma) – the level of densities of subpopulations ("*macro–scale*"), in terms of dynamical systems related to nonlinear systems of ODEs or reaction–diffusion (–chemotaxis) equations.

Refs. [9, 10, 11] consider some important examples for various situations of biological interest, in particular a large class of models that correspond to ODEs of Lotka–Volterra–type and reaction–diffusion systems (with small diffusion), reaction–diffusion–chemotaxis systems (i.e. reaction–diffusion equations with a chemotaxis–type term) in the context of tumour invasion at the macroscopic level. Ref. [10] studies the microscopic and mesoscopic models that correspond to very well known models in biomathematics: the Verhulst logistic equation and the Lotka–Volterra system of equations. The asymptotic time behaviour for the mesoscopic model corresponding to the Verhulst logistic equation is defined. The mesoscopic model corresponding to the Verhulst equation is modified to a mesoscopic model of DNA denaturation. Ref. [11] generalizes the previous approach resulting in bilinear equations of the Boltzmann–type at the mesoscopic level and then related to the bilinear models at the macroscopic level to the general nonlinear case. These methods may lead to new and more accurate modelling of complex processes.

The present paper is devoted to identification of the equilibrium solutions of equations describing the general population dynamics at the microscopic level of modified Liouville equation. It is an individually-based model of finite number of entities (individuals) in terms of a Markov jump process. We show the existence of a factorized equilibrium solution and discuss its uniqueness. The conditions guaranteeing uniqueness or non-uniqueness are proposed under the assumption of periodic structures. In particular the conditions guaranteeing the existence of non–factorized equilibrium solution are stated.

The paper is organized as follows. In Section 2 we introduce the modified Liouville equations representing a population dynamics model with binary interactions of individuals at the microscopic level. In Section 3 the corresponding bilinear Boltzmann–type equation is introduced. In Section 4 the existence of factorized equilibrium solutions for the microscopic model is proved. Section 5 deals with possible uniqueness of the equilibrium solution. Under the assumption on periodic structures we propose the conditions guaranteeing uniqueness or non-uniqueness. In Section 6 we discuss a possibility of generalizations to the case of multiple interactions.

2. **Microscopic population dynamics models.** Following [9, 10, 11] we consider a stochastic system of (large) number N of entities (individuals) of various sub–populations. Every n–th entity, $n = 1, \dots, N$, is characterized by the pair of parameters

$$(j_n, u_n) \equiv \mathbf{u}_n \in \mathbf{U} \equiv \mathcal{J} \times \mathcal{U}\,,$$

where the variable $j_n \in \mathcal{J}$, $\mathcal{J} \subset \mathbb{N}$, represents the sub–population of n–th entity and $u_n \in \mathcal{U} \subset \mathbb{R}^d$, $d \geq 1$, is its biological (or physical) inner state (a dimensionless variable). We assume that each entity changes its population and/or state at a random time.

We consider the probability density

$$f^N = f^N(t, \mathbf{u}_1, \dots, \mathbf{u}_N)\,,$$
$$f^N : [0,\infty) \times \mathbf{U}^N \to [0,\infty)\,.$$

We assume that such a stochastic system is defined by the Markow jump processes of N entities through the following generator (see [9, 10]) — a modified Liouville operator —

$$\Lambda_N^* f^N(t, \mathbf{u}_1, \dots, \mathbf{u}_N) = \frac{1}{N} \sum_{\substack{1\leq n,m\leq N \\ m\neq n}} \Big(\int_{\mathbf{U}} A(\mathbf{u}_n; \mathbf{v}, \mathbf{u}_m)\, a(\mathbf{v}, \mathbf{u}_m) \times f^N(t, \mathbf{u}_1, \dots, \mathbf{u}_{n-1}, \mathbf{v}, \mathbf{u}_{n+1}, \dots, \mathbf{u}_N)\,\mathrm{d}\mathbf{v} - a(\mathbf{u}_n, \mathbf{u}_m) f^N(t, \mathbf{u}_1, \dots, \mathbf{u}_N) \Big)\,. \tag{1}$$

We adhere here to the following convection

$$\sum_{j\in\mathcal{J}} \int_{\mathcal{U}} \dots \,\mathrm{d}u \;=: \int_{\mathbf{U}} \dots \,\mathrm{d}\mathbf{u}\,, \tag{2}$$

and "for a.a. $\mathbf{u} \in \mathbf{U}$" means "for all $j \in \mathcal{J}$ and a.a. (with respect to the Lebesgue measure) $u \in \mathcal{U}$".

The microscopic model is defined by functions $a = a(\mathbf{u}, \mathbf{v})$ and $A = A(\mathbf{u}; \mathbf{v}, \mathbf{w})$, where

- $a = a(\mathbf{u}, \mathbf{v})$ is the *rate of interaction* of an entity of $j \in \mathcal{J}$ sub–population and with state $u \in \mathcal{U}$, where $\mathbf{u} = (j, u)$, and an entity of $k \in \mathcal{J}$ sub–population and with state $v \in \mathcal{U}$, where $\mathbf{v} = (k, v)$; a is a measurable, bounded function
$$a \,:\, \mathbf{U}^2 \to [0,\infty)\,; \tag{3}$$
- $A = A(\mathbf{u}; \mathbf{v}, \mathbf{w})$ is the *transition probability function* into j–th sub–population and with state u, $\mathbf{u} = (j, u)$, of an entity of k–th sub–population and with state v, $\mathbf{v} = (k, v)$, due to the interaction with an entity of l–th population and with state w, $\mathbf{w} = (l, w)$; A is a measurable function
$$A \,:\, \mathbf{U}^3 \to [0,\infty)\,, \tag{4}$$

such that for a.a. $\mathbf{v}, \mathbf{w} \in \mathbf{U}$

$$\int\limits_{\mathbf{U}} A(\mathbf{u}; \mathbf{v}, \mathbf{w}) \, \mathrm{d}\mathbf{u} \; = \; 1 \, . \tag{5}$$

The stochastic model (at the microscopic level) will be completely determined by the choice of functions a and A.

Let $L^{1,N} = L^1\big(\mathbf{U}^N\big)$ be the Banach space of integrable functions and equipped with the norm

$$\|f^N\|_{L^{1,N}} = \int\limits_{\mathbf{U}} \cdots \int\limits_{\mathbf{U}} \big|f^N(\mathbf{u}_1, \ldots, \mathbf{u}_N)\big| \, \mathrm{d}\mathbf{u}_1 \, \cdots \, \mathrm{d}\mathbf{u}_N \, .$$

The probability density satisfies the Cauchy problem for the *modified Liouville equation*

$$\begin{array}{lcll} \partial_t f^N & = & \Lambda_N^* f^N \, , & \text{in} \quad (0, \infty) \times \mathbf{U}^N \, , \\ f^N\Big|_{t=0} & = & F^N \, , & \text{in} \quad \mathbf{U}^N \, . \end{array} \tag{6}$$

The operator Λ_N^* is a bounded linear operator in $L^{1,N}$. Therefore there exists a continuous semigroup that defines the solutions to Eq. (6) in $L^{1,N}$ for $t > 0$

$$f^N\big(t, \mathbf{u}_1, \ldots, \mathbf{u}_N\big) = \exp\Big(t \, \Lambda_N^*\Big) F^N\big(\mathbf{u}_1, \ldots, \mathbf{u}_N\big) \, .$$

It is easy to see that for any $t > 0$ the function $\exp\big(t \, \Lambda_N^*\big) F^N$ is a probability density provided that F^N is a probability density. Therefore the semigroup is a continuous (linear) semigroup of Markov operators — a continuous stochastic semigroup.

3. **Mesoscopic population dynamics models.** In the limit $N \to \infty$ the linear problem (6) results ([9, 11]) in a bilinear Boltzmann–like integro–differential equation (*Generalized Kinetic Models*) – that can be related to the mesoscopic description. In fact assuming that the process starts with a chaotic (i.e. factorized) probability density

$$F^N = \underbrace{F \otimes \ldots \otimes F}_{N \times} , \tag{7}$$

$$\underbrace{F \otimes \ldots \otimes F}_{N \times} \big(\mathbf{u}_1, \ldots, \mathbf{u}_N\big) = \prod_{n=1}^{N} F(\mathbf{u}_n) \, ,$$

i.e. N–fold outer product of a probability density F, one may rigorously relate (6) with the solution of

$$\begin{array}{lcll} \partial_t f & = & \Gamma[f] \, , & \text{for} \quad t > 0 \, , \quad \mathbf{u} \in \mathbf{U} \, , \\ f\Big|_{t=0} & = & F \, , & \text{for} \quad \mathbf{u} \in \mathbf{U} \, , \end{array} \tag{8}$$

where

$$\begin{array}{rcl} \Gamma[f](t, \mathbf{u}) & = & \int\limits_{\mathbf{U}} \int\limits_{\mathbf{U}} A(\mathbf{u}; \mathbf{v}, \mathbf{w}) a(\mathbf{v}, \mathbf{w}) f(t, \mathbf{v}) f(t, \mathbf{w}) \, \mathrm{d}\mathbf{v} \, \mathrm{d}\mathbf{w} \\ & - & f(t, \mathbf{u}) \int\limits_{\mathbf{U}} a(\mathbf{u}, \mathbf{v}) f(t, \mathbf{v}) \, \mathrm{d}\mathbf{v} \, . \end{array}$$

In the context of biological or medical processes various Boltzmann–like equations were considered by various authors – see e.g. [2, 4, 6, 9, 10, 11] and references therein.

The existence and uniqueness theory (in $L^{1,1}$) for Eq. (8) is standard [9, 12].

4. **Equilibrium solutions.** We are interested in the solution of the following equations

$$\Lambda_N^* f^N(\mathbf{u}_1 \ldots \mathbf{u}_N) = 0\,, \tag{9}$$

and

$$\Gamma[f](\mathbf{u}) = 0\,, \tag{10}$$

in the sets of all probability densities $\mathbb{D}^{(N)}$ in $L^{1,N}$, and $\mathbb{D}^{(1)}$ in $L^{1,1}$, respectively, i.e. the equilibrium solutions of Equations (6) and (8), respectively, where

Definition 4.1.

$$\mathbb{D}^{(N)} = \left\{ f^N \in L^{1,N} \;:\; f^N \geq 0 \text{ and } \int\limits_{\mathbf{U}^N} f^N(\mathbf{u}_1, \ldots, \mathbf{u}_N)\,\mathrm{d}\mathbf{u}_1 \ldots \mathrm{d}\mathbf{u}_N = 1 \right\}.$$

Arlotti and Bellomo [1] studied problem (10) and, by the Schauder fixed point theorem, they proved that for $|\mathcal{J}| = 1, \mathcal{U} = [0,1]$, a – constant, and A – continuous on $[0,1]^3$ there exists a solution of (10) in $\mathbb{D}^{(1)}$. It is easy to see that their proof holds true in the case of $|\mathcal{J}| < \infty$ and any compact $\mathcal{U}$. The theorem does not deliver the uniqueness of equilibrium solution.

Lachowicz and Wrzosek [12] studied problem (10) for the periodic functions and the interactions in terms of convolution operators. Under some conditions they proved the existence of unstable equilibrium solutions which are inhomogeneous with respect to $\mathbf{u}$–variable additional to the homogeneous (i.e. constant) solution. The set of equilibrium solutions is finite but as large as we want. On the other hand the condition that guarantees the existence of the only homogeneous (i.e. constant) equilibrium solution was stated.

We use the similar idea of [1] in the case of the microscopic equation (6). Instead of the C^0–setting we are using here the L^2–setting. We are looking for the solution of Eq. (10) in the factorized form

$$f^N = \underbrace{f \otimes \ldots \otimes f}_{N\times}\,, \tag{11}$$

with $f \in \mathbb{D}^{(1)}$.

We need a stronger assumption than (3)

Assumption 1.

$$a(\mathbf{u},\mathbf{v}) > 0\,, \qquad \text{for a.a.} \quad \mathbf{u},\mathbf{v} \in \mathbf{U}\,,$$

as well as

Assumption 2.

$$|\mathcal{J}| < \infty\,, \qquad \mathcal{U} \text{ is a compact set in } \mathbb{R}^d\,.$$

Under Assumption 1 **if**

$$\int\limits_{\mathbf{U}} \frac{A(\mathbf{u};\mathbf{v},\mathbf{w})a(\mathbf{v},\mathbf{w})}{a(\mathbf{u},\mathbf{w})} f(\mathbf{v})\,\mathrm{d}\mathbf{v} = f(\mathbf{u})\,, \qquad \text{for a.a.} \quad \mathbf{u},\mathbf{w} \in \mathbf{U}\,, \tag{12}$$

for some $f \in \mathbb{D}^{(1)}$, **then** $f^N \in \mathbb{D}^{(N)}$ given by (11) is a solution of Eq. (9) and therefore an equilibrium solution of Eq. (6).

The problem reduces to showing that function $f \in \mathbb{D}^{(1)}$ satisfying Eq. (12) exists.

Assumption 3. *We assume that* $\frac{A(\mathbf{u};\mathbf{v},\mathbf{w})a(\mathbf{v},\mathbf{w})}{a(\mathbf{u},\mathbf{w})}$ *is independent of variable* w, *for a.a.* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{U}$, *i.e.*

$$\frac{A(\mathbf{u};\mathbf{v},\mathbf{w})a(\mathbf{v},\mathbf{w})}{a(\mathbf{u},\mathbf{w})} =: \alpha(\mathbf{u},\mathbf{v})\,, \qquad \text{for a.a.} \quad \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{U}\,. \tag{13}$$

Moreover α *satisfies*

$$\int\limits_{\mathbf{U}} \alpha(\mathbf{u},\mathbf{v})\,\mathrm{d}\mathbf{u} = 1\,, \qquad \text{for a.a.} \quad \mathbf{v} \in \mathbf{U}\,, \tag{14}$$

and

$$\int\limits_{\mathbf{U}}\int\limits_{\mathbf{U}} \Big(\alpha(\mathbf{u},\mathbf{v})\Big)^2 \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v} < \infty\,. \tag{15}$$

Under Assumptions 1 and 3 the linear operator defined by the LHS of Eq. (12) is an integral operator acting in $L^2(\mathbf{U}) \cap \mathbb{D}^{(1)}$ and it is compact as an operator in $L\big(L^2(\mathbf{U}), L^2(\mathbf{U})\big)$. Thus the conditions of the Schauder fixed point theorem are verified and there exists a fixed point of the operator defined by the LHS of Eq. (12). Therefore we have

Theorem 4.2. *Let Assumptions 1, 2, 3 be satisfied. Then there exists a solution* $f^N \in \mathbb{D}^{(N)}$ *of Eq. (9).*

We may note that various applications are consistent with Assumption 3 — e.g. the microscopic equation corresponding to the Verhulst logistic growth — see [9, 10].

5. **Uniqueness.**

Theorem 4.2 does not deliver uniqueness of the equilibrium solution even in the class of factorized functions. In general uniqueness actually can be a quite difficult problem as Example 1 below shows.

Following [12] we may discuss the number of equilibrium solutions directly to Eq. (6), i.e. solutions of Eq. (9). We assume that both A and a are periodic functions with respect to each variable on $\mathbb{Z} \times \mathbb{R}^d$ with period $(p, 1, \ldots, 1) \in \mathbb{Z} \times \mathbb{R}^d$, where $p > 0$ is an integer. This leads to the assumption that $\mathbf{U} = \mathbb{Z}_p \times \mathbb{T}^d$, where $\mathbb{Z}_p$ is the group of integers $\mathbb{Z}$ modulo p and $\mathbb{T}^d$ is a d–dimensional (normalized) torus.

The periodic structures in the mesoscopic description (Eq. (8)) were considered e.g. in [7, 8, 13], see also examples in [12]. In paper [7] the equation of type (8) for the one–dimensional angular distribution was proposed in the context of angular self–organization of the actin cytoskeleton in the process of instant changing of filament orientation in the course of specific actin–actin interactions. The mathematical properties of the model were studied in [8, 13].

Assumption 4.

$$\begin{array}{rclcl} 0 & \le & a(\mathbf{u},\mathbf{v}) & = & \tilde{a}(\mathbf{u}-\mathbf{v})\,, \\ 0 & \le & A(\mathbf{u};\mathbf{v},\mathbf{w}) & = & \tilde{A}(\mathbf{u}-\mathbf{v})\,, \end{array} \tag{16}$$

for a.a. $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{Z}_p \times \mathbb{T}^d$, *where* $\tilde{a}$, $\tilde{A}$ *are given measurable functions defined on* $\mathbb{Z}_p \times \mathbb{T}^d$; *and*

$$\int\limits_{\mathbb{Z}_p \times \mathbb{T}^d} \tilde{A}(\mathbf{u})\,\mathrm{d}\mathbf{u} = 1\,. \tag{17}$$

As in [12] we use some elements of Fourier analysis. For any function $f \in L^2\big((\mathbb{Z}_p \times \mathbb{T}^d)^N\big)$ we define the Fourier transform $\mathfrak{F}_N f$

$$\big(\mathfrak{F}_N f\big)_{(\mathbf{y}_1,\dots,\mathbf{y}_N)} = \int\limits_{(\mathbb{Z}_p\times\mathbb{T}^d)^N} \exp\Big(-2\pi\, i \sum_{k=1}^{N} \mathbf{y}_{k,p}\cdot\mathbf{u}_k\Big) f(\mathbf{u}_1,\dots,\mathbf{u}_N)\,\mathrm{d}\mathbf{u}_1\dots\mathrm{d}\mathbf{u}_N\,,$$

where $i=\sqrt{-1}$, and $(\mathbf{y}_1,\dots,\mathbf{y}_N) = (y_{1,0}, y_1,\dots,y_{N,0},y_N) \in (\mathbb{Z}_p\times\mathbb{Z}^d)^N$ is the Fourier variable, $\mathbf{y}_{k,p} = (\frac{y_{k,0}}{p}, y_k)$, $\mathbf{u}_k = (j_k, u_k)$, $\mathbf{y}_{k,p}\cdot\mathbf{u}_k = \frac{y_{k,0}}{p} j_k + y_k\cdot u_k$.

Example 1. Let $a = \mathrm{const} > 0$ and α be given by (13) such that

$$\alpha(\mathbf{u},\mathbf{v}) = \tilde{\alpha}(\mathbf{u}-\mathbf{v})\,,$$

where $\tilde{\alpha}$ is a periodic function with respect to each variable on $\mathbb{Z}\times\mathbb{R}^d$ with period $(p,1,\dots,1)\in\mathbb{Z}\times\mathbb{R}^d$. We assume that $\tilde{\alpha}$ is non–negative and Eq. (14) is satisfied. The operator defined by the LHS of (12) is expressed by means of convolution and Eq. (12) takes the form

$$\tilde{\alpha}\star f = f\,. \tag{18}$$

It is easy to see that the (positive normalized) constant function is a solution of Eq. (18). Assume now that

$$\tilde{\alpha}\in L^2(\mathbb{Z}_p\times\mathbb{T}^d)\,.$$

We consider the Fourier transform $\mathfrak{F}_1 f$ of any function $f\in L^2(\mathbb{Z}_p\times\mathbb{T}^d)$ with the Fourier variable $\mathbf{y}\in\mathbb{Z}_p\times\mathbb{Z}^d$. The function $f\in L^2(\mathbb{Z}_p\times\mathbb{T}^d)$ satisfies Eq. (18) if and only if

$$\big(\mathfrak{F}_1\tilde{\alpha}\big)_{\mathbf{y}} = 1 \qquad \text{or} \qquad \big(\mathfrak{F}_1 f\big)_{\mathbf{y}} = 0\,, \tag{19}$$

for any $\mathbf{y}\in\mathbb{Z}_p\times\mathbb{Z}^d$. Therefore introducing

$$\mathcal{S}(f) = \Big\{\mathbf{y}\in\mathbb{Z}_p\times\mathbb{Z}^d \,:\, \big(\mathfrak{F}_1 f\big)_{\mathbf{y}} \neq 0\Big\}\,,$$

and

$$\mathcal{K}(\tilde{\alpha}) = \Big\{\mathbf{y}\in\mathbb{Z}_p\times\mathbb{Z}^d \,:\, \big(\mathfrak{F}_1 \tilde{\alpha}\big)_{\mathbf{y}} = 1\Big\}\,,$$

we see that the condition for f to be a solution of Eq. (18) is

$$\mathcal{S}(f)\subset\mathcal{K}(\tilde{\alpha})\,. \tag{20}$$

Condition (20) defines the number of possible solutions of Eq. (18). In particular, if $\mathcal{K} = \{\mathbf{0}\}$ then only the (positive normalized) constant function is a solution of Eq. (18) in $\mathbb{D}^{(1)}$.

Under Assumption 4 problem (9) reads

$$\sum_{\substack{1\le n,m\le N\\ m\neq n}} \Big((\mathfrak{F}_1\tilde{A})_{\mathbf{y}_n} - 1\Big)(\mathfrak{F}_N H_{n,m} f)_{(\mathbf{y}_1,\dots,\mathbf{y}_N)} = 0\,, \tag{21}$$

where

$$H_{n,m} f(\mathbf{u}_1,\dots,\mathbf{u}_N) = \tilde{a}(\mathbf{u}_n-\mathbf{u}_m) f(\mathbf{u}_1,\dots,\mathbf{u}_N)\,.$$

Let

$$\mathcal{S}_N(f) = \big\{(\mathbf{y}_1,\dots,\mathbf{y}_N)\in(\mathbb{Z}_p\times\mathbb{Z}^d)^N \,:\, \big(\mathfrak{F}_N f\big)_{(\mathbf{y}_1,\dots,\mathbf{y}_N)} \neq 0\big\}\,,$$

$$\mathcal{K}_N(\tilde{A}) = \big\{(\mathbf{y}_1,\dots,\mathbf{y}_N)\in(\mathbb{Z}_p\times\mathbb{Z}^d)^N \,:\, (\mathfrak{F}_1\tilde{A})_{\mathbf{y}_n} = 1 \quad \forall\, n=1,\dots,N\big\}\,,$$

$$\tilde{\mathcal{K}}_N(\tilde{A}) = \big\{(\mathbf{y}_1,\dots,\mathbf{y}_N)\in(\mathbb{Z}_p\times\mathbb{Z}^d)^N \,:\, \sum_{n=1}^{N}(\mathfrak{F}_1\tilde{A})_{\mathbf{y}_n} = N\big\}\,,$$

and

$$\mathcal{H}_{n,m}(\tilde{a},f) = \Big\{(\mathbf{z}_1,\ldots,\mathbf{z}_N) \in (\mathbb{Z}_p \times \mathbb{Z}^d)^N \ : \\ \mathbf{z}_k = \mathbf{y}_k + \bar{\mathbf{w}}_k\,,\ \forall\, k=1,\ldots,N\,,\ (\mathbf{y}_1,\ldots,\mathbf{y}_N) \in \mathcal{S}_N(f)\,, \\ \bar{\mathbf{w}}_n = \mathbf{w}_n\,,\ \bar{\mathbf{w}}_m = -\mathbf{w}_n\,,\ \bar{\mathbf{w}}_r = 0\,,\ \forall\, r \notin \{n,m\}\,,\ \mathbf{w}_n \in \mathcal{S}_1(\tilde{a})\Big\}\,.$$

By Assumption (17) it follows that $\mathcal{K}_N(\tilde{A}) \neq \emptyset$, in fact

$$(\mathbf{0},\ldots,\mathbf{0}) \in \mathcal{K}_N(\tilde{A})\,.$$

Moreover, if $\tilde{A} \in L^2(\mathbb{Z}_p \times \mathbb{T}^d)$ then $|\mathcal{K}_N(\tilde{A})| < \infty$.

Then we have

Theorem 5.1. *Let Assumption 4 be satisfied and*

$$\tilde{A}, \tilde{a} \in L^2(\mathbb{Z}_p \times \mathbb{T}^d)\,.$$

If *$f \in L^2\big((\mathbb{Z}_p \times \mathbb{T}^d)^N\big)$ is such that*

$$\mathcal{H}_{n,m}(\tilde{a},f) \subset \mathcal{K}_N(\tilde{A})\,, \qquad \forall\, n \neq m\,, \tag{22}$$

then *f is a solutions to Eq. (9).*

Corollary 1. *If $\tilde{a} = \mathrm{const}$ and $\tilde{A}$ satisfies Assumption 4 then $f \in L^2\big((\mathbb{Z}_p \times \mathbb{T}^d)^N\big)$ satisfies Eq. (9) if and only if*

$$\mathcal{S}_N(f) \subset \tilde{\mathcal{K}}_N(\tilde{A})\,. \tag{23}$$

Remark 1. In paper [12] the existence of non–negative continuous function $\tilde{A} : \mathbb{T}^1 \to \mathbb{R}^1_+$ that satisfies

$$\big(\mathfrak{F}_1\tilde{A}\big)_0 = 1 \tag{24}$$

together with

$$\big(\mathfrak{F}_1\tilde{A}\big)_\xi = 1 \qquad \text{for}\ \ \xi \in \mathbb{Z}\setminus\{0\} \tag{25}$$

was proved. Therefore from Corollary 1 it follows that the corresponding Fourier coefficient $\big(\mathfrak{F}_N f\big)_{(\xi,\ldots,\xi)}$ of a function $f \in L^2\big((\mathbb{T}^1)^N\big)$ satisfying Eq. (9), may be arbitrary. Thus one may construct nonconstant and nonfactorized equilibrium solutions in $\mathbb{D}^{(N)}$. For example the function

$$f(u_1,\ldots,u_N) = 1 + c_1\cos\Big(2\pi\xi\sum_{j=1}^N u_j\Big) + c_2\sin\Big(2\pi\xi\sum_{j=1}^N u_j\Big)\,,$$

where the constants c_1, c_2 are such that $\max\{|c_1|,|c_2|\} < \frac{1}{\sqrt{2}}$, belongs to $\mathbb{D}^{(N)}$ and satisfies Eq. (9).

Corollary 2. *Let Assumption 4 be satisfied, $\tilde{a} = \mathrm{const}$ and*

$$\tilde{A} \in L^2\big(\mathbb{Z}_p \times \mathbb{T}^d\big)\,.$$

Then

1. *the normalized constant function is a solution of Eq. (9)*
2. *$f \in L^2\big((\mathbb{Z}_p \times \mathbb{T}^d)^N\big)$ is a solution of Eq. (9) if and only if*

$$\mathcal{S}_N(f) \subset \mathcal{K}_N(\tilde{A})\,.$$

Therefore, if $|\tilde{\mathcal{K}}_N(\tilde{A})| = 1$, i.e. $\tilde{\mathcal{K}}_N(\tilde{A}) = \{(\mathbf{0},\ldots,\mathbf{0})\}$, then only the constant normalized function is a solution of Eq. (9) in $\mathbb{D}^{(N)}$.

Similarly as in the case of mesoscopic equation (6) — see [12] — we may formulate the negative result concerning the asymptotic stability of equilibrium solutions to Eq. (6), given by condition (22). In fact under the assumption that $|\mathcal{K}_N(\tilde{A})| \geq 2$ and under suitable assumption on $\tilde{a}$, there exist equilibrium solutions arbitrary close to the given one (the corresponding Fourier coefficients may be arbitrary). Therefore, for any of the norms L^1, L^2, or C^0, none of the equilibrium solutions given by (22) can be asymptotically stable.

6. **Generalizations.** Ref. [11] generalizes the previous approach ([9, 10]) resulting in bilinear equations of the Boltzmann–type at the mesoscopic level to the general nonlinear case. The general framework is applied to propose the microscopic and mesoscopic models that correspond to well known systems of nonlinear equations in biomathematics.

We may note that the result of the previous section may be repeated in this general case.

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