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GLOBAL STABILITY FOR AN SEI EPIDEMIOLOGICAL MODEL WITH CONTINUOUS AGE-STRUCTURE IN THE EXPOSED AND INFECTIOUS CLASSES

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Abstract. We study a model of disease transmission with continuous age-structure for latently infected individuals and for infectious individuals. The model is very appropriate for tuberculosis. Key theorems, including asymptotic smoothness and uniform persistence, are proven by reformulating the system as a system of Volterra integral equations. The basic reproduction number $\mathcal{R}_0$ is calculated. For $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable. For $\mathcal{R}_0 > 1$, a Lyapunov functional is used to show that the endemic equilibrium is globally stable amongst solutions for which the disease is present. Finally, some special cases are considered.

1. **Introduction.**

Models of disease spread have been studied since Kermack and McKendrick [11] in 1927, particularly in the last thirty years. A review can be found in [7].

Many of these models are formulated as ordinary differential equations (ODE) with distinct variables to describe the size of groups such as susceptible, exposed and infectious, with possibly several compartments to further divide these groups [9, 10, 14]. The ODE formulation assumes that all individuals within a compartment behave identically, regardless of how much time they have spent in the compartment. For instance, it assumes that all individuals in an infectious compartment have the same level of infectiousness, and also that the waiting times in each compartment are exponentially distributed.

In this paper, we include the duration that an individual has spent in the exposed class and in the infectious class as variables. The state of the population at a particular time is given by the current number of susceptibles and two functions. One function describes the density of individuals who are exposed to the disease, and have been for a duration a. The other function describes the density of infectious individuals. This leads to a partial differential equation (PDE) formulation [27].

Models with continuous age-structure have been studied in many works including [4, 8, 11, 15, 24, 26, 27].

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Age-structured systems are well-suited to modelling tuberculosis (as well as other applications, such as antibiotic resistance [2]). Infectious tuberculosis is a deadly disease if not treated. However, an individual may have latent tuberculosis for months, years or even decades before the disease becomes infectious. The risk per unit time of activation appears to be higher in the early stages of latency than in later stages; see [1], where low dimensional ODE models have been used to study this phenomenon, with the global analysis provided in [17]. In [18], an staged progression ODE model with an arbitrary number of infectious stages in considered. As stated above, though, the ODE nature of the model puts limitations on the distribution of waiting times in the exposed population.

By including the duration a spent in the exposed class, we are able to model the risk of activation as a function of a, allowing more generality in the distribution of waiting times or latency periods. Similarly, the distribution of waiting times in the infectious class is made general by allowing the exit rate to be a function of the time spent in that class.

ODE models including [5, 9, 10, 14, 16] have included a version of infection-age dependent infectivity by using progression through multiple infectious stages. However, since the distribution of waiting times in each stage is exponential, there would be individuals in the first class for arbitrarily large times and others who have progressed to the final stage in arbitrarily small times. Thus, the ODE staged progression models give only a weak approximation of infection-age dependent infectivity.

Continuous age-structure in the infectious class allows the infectivity to truly be a function of the duration spent in the class. Furthermore, it allows the elevated death rate due to disease to depend on the duration for which one has been infectious.

Until recently [15, 19], full global stability results for continuous age-structure models were lacking. A key goal in this paper is to treat a continuous age-structure model from start to finish, including the global stability. The global stability approach used here is related to that used in [15, 19, 20, 21, 22].

Other aspects of the analysis follow the techniques laid out in the new book [25]. In that book (and in [15]), an SI model of disease transmission, with continuous age-structure for the infectives is studied; that is, a scalar age-structured variable is used. In [15], the SI model is reformulated as a non-densely defined Cauchy problem in order to study the asymptotic smoothness and persistence. The current approach is closer to that found in [25].

The SEI model considered here includes continuous age-structure for both the exposed and the infectious classes; that is, a two-dimensional age-structured variable is used. Thus, the application of the methods in [25] requires some care. On the other hand, we hope that the calculations here help to demonstrate the usefulness of the techniques given in [25].

In [24], an SEI model with continuous age-structure for the infectious class was studied. The model was reformulated as an infinite delay differential equation with most of the analysis, including asymptotic smoothness and persistence, performed in [24]. The global analysis appeared in [19]. That system is a special case of the one studied here.

2. Model equations.

Based on disease status, a population is divided into three classes: susceptible, exposed or infectious. The number of susceptibles at time t is given by $S(t)$. In order to model the time-course development of the disease within an individual, the exposed and infectious sub-populations include age-structure; that is, at time t, these classes are described by density functions $e(t,a)$ and $i(t,a)$, stratified by the duration a for which individuals have been in the class. Individuals who have been in the exposed class for duration a, progress to class i at rate $\gamma(a)$ and are removed from the population at rate $\mu(a)$. Individuals who have been in the infectious class for duration a are removed at rate $\nu(a)$ and infect susceptibles with mass-action coefficient $\beta(a)$. (Note that if $\beta(a)$ is zero for certain values of a, then the individuals are not truly infectious, even though they are in the infectious class.)

All recruitment into the population is into the susceptible class and occurs with constant flux Λ. Susceptibles are removed at rate μ_S. All new infections enter the exposed class. The model is described by the equations

$$\begin{aligned} \frac{dS(t)}{dt} &= \Lambda - \mu_S S(t) - S(t)\int_0^\infty \beta(a) i(t,a)da \\ \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} &= -\left(\gamma(a)+\mu(a)\right)e(t,a) \\ \frac{\partial i}{\partial t} + \frac{\partial i}{\partial a} &= -\nu(a) i(t,a), \end{aligned} \tag{1}$$

with boundary conditions

$$\begin{aligned} e(t,0) &= S(t)\int_0^\infty \beta(a) i(t,a)da \\ i(t,0) &= \int_0^\infty \gamma(a) e(t,a)da \end{aligned} \tag{2}$$

for $t \geq 0$. We make the following hypotheses about the parameters of the system.

(H1) $\Lambda, \mu_S > 0$.

(H2) $\beta, \gamma, \mu, \nu \in L^\infty_+$, with respective essential upper bounds $\bar\beta$, $\bar\gamma$, $\bar\mu$ and $\bar\nu$.

(H3) β and γ are Lipschitz continuous on $\mathbb{R}_{\geq 0}$, with Lipschitz coefficients M_β and M_γ, respectively.

(H4) For any $a > 0$, there exists $a_\beta, a_\gamma > a$ such that β is positive in a neighbourhood of a_β and γ is positive in a neighbourhood of a_γ.

(H5) There exists $\mu_0 \in (0, \mu_S]$ such that $\mu(a), \nu(a) \geq \mu_0$ for all $a > 0$.

Some special cases of Equation (1) are discussed in Sections 10 and 11.

Following [27], the phase space for the system is $\mathcal{Y} = \mathbb{R}_{\geq 0} \times L^1_+ \times L^1_+$, where L^1_+ is the space of functions on $(0,\infty)$ that are non-negative and Lebesgue integrable[1], and the norm on $\mathcal{Y}$ is taken to be

$$\|(x,\varphi,\phi)\| = |x| + \int_0^\infty |\varphi(a)|\,da + \int_0^\infty |\phi(a)|\,da.$$

The norm has the biological interpretation of giving the total population size.

[1]More precisely, L^1 is the space of equivalence classes of Lebesgue integrable functions, where two functions are equivalent if they are equal almost everywhere, and L^1_+ is the non-negative cone of L^1.

The initial condition for the system described by Equations (1) and (2) is

$$(S(0), e(0,\cdot), i(0,\cdot)) = (S_0, \varphi_e(\cdot), \varphi_i(\cdot)) \in \mathcal{Y}.$$

Standard existence, uniqueness and continuability results hold for Equations (1) and (2), and the system defines a continuous semi-flow $\Phi : \mathbb{R}_{\geq 0} \times \mathcal{Y} \to \mathcal{Y}$. Furthermore, solutions of this system have compact closure, and therefore have non-empty omega limit sets [27].

Notation. If $X(t)$ is the solution to Equations (1) and (2), which satisfies the initial condition $X(0) = X_0 \in \mathcal{Y}$, then for any $t \geq 0$, we use the following notations interchangeably:

$$X(t) = \Phi(t, X_0) = \Phi_t(X_0) = (S(t), e(t,\cdot), i(t,\cdot)).$$

Thus,

$$\|\Phi_t(X_0)\| = \|(S(t), e(t,\cdot), i(t,\cdot)\| = S(t) + \int_0^\infty e(t,a)da + \int_0^\infty i(t,a)da.$$

3. **Preliminaries and equilibria.** For $a \geq 0$, let

$$\Omega(a) = \mathrm{e}^{-\int_0^a (\gamma(\sigma)+\mu(\sigma))d\sigma} \qquad \text{and} \qquad \Gamma(a) = \mathrm{e}^{-\int_0^a \nu(\sigma)d\sigma}. \tag{3}$$

It follows from **(H2)** and **(H5)**, that

$$0 < \Omega(a), \Gamma(a) \leq \mathrm{e}^{-\mu_0 a} \tag{4}$$

for each $a \geq 0$. Additionally, the equations $\Omega'(a) = -(\gamma(a) + \mu(a))\,\Omega(a)$ and $\Gamma'(a) = -\nu(a)\Gamma(a)$ hold for almost all $a \geq 0$. Let

$$A = \int_0^\infty \gamma(a)\Omega(a)da \qquad \text{and} \qquad B = \int_0^\infty \beta(a)\Gamma(a)da. \tag{5}$$

It follows from **(H2)**, **(H4)** and Equation (4) that A and B are positive and finite.

For $t \geq 0$, let

$$J(t) = \int_0^\infty \beta(a)i(t,a)da \qquad \text{and} \qquad L(t) = \int_0^\infty \gamma(a)e(t,a)da.$$

Then the boundary conditions given in Equation (2) can be rewritten as $e(t,0) = S(t)J(t)$ and $i(t,0) = L(t)$. We follow [27] and solve the PDE part of Equation (1), obtaining

$$e(t,a) = \begin{cases} S(t-a)J(t-a)\Omega(a) & \text{for } 0 \leq a \leq t \\ \varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)} & \text{for } t < a \end{cases} \tag{6}$$

and

$$i(t,a) = \begin{cases} L(t-a)\Gamma(a) & \text{for } 0 \leq a \leq t \\ \varphi_i(a-t)\frac{\Gamma(a)}{\Gamma(a-t)} & \text{for } t < a. \end{cases} \tag{7}$$

It is useful to note that

$$e(t,a) = e(t-a,0)\Omega(a) \qquad \text{and} \qquad i(t,a) = i(t-a,0)\Gamma(a) \qquad \text{for } 0 \leq a \leq t. \tag{8}$$

Consider a general equilibrium $\left(\widetilde{S}, \widetilde{e}(\cdot), \widetilde{i}(\cdot)\right) \in \mathcal{Y}$. The PDE part of Equation (1) becomes an ODE in a, yielding $\widetilde{e}(a) = \Omega(a)\widetilde{e}(0)$ and $\widetilde{i}(a) = \Gamma(a)\widetilde{i}(0)$. The boundary conditions given in Equation (2) imply $\widetilde{e}(0) = B\widetilde{S}\widetilde{i}(0)$ and $\widetilde{i}(0) = A\widetilde{e}(0)$. Thus, if either of $\widetilde{e}(0)$ and $\widetilde{i}(0)$ is zero, then the other must be as well. That is, they are both zero or they are both non-zero. Also, by multiplying these two equations, we obtain

$$\widetilde{e}(0)\,\widetilde{i}(0) = AB\,\widetilde{S}\,\widetilde{e}(0)\,\widetilde{i}(0).$$

Suppose $\left(\widetilde{S}, \widetilde{e}(\cdot), \widetilde{i}(\cdot)\right)$ is a disease-free equilibrium. Then we take $\widetilde{e}(0) = \widetilde{i}(0) = 0$, and so $\widetilde{e} = \widetilde{i} = \mathbf{0}$, where $\mathbf{0} \in L^1_+$ is the zero function. Let the disease-free equilibrium be given by $E^0 = \left(S^0, \mathbf{0}, \mathbf{0}\right)$. Using $\frac{dS}{dt} = 0$, we find

$$E^0 = \left(S^0, \mathbf{0}, \mathbf{0}\right) = \left(\frac{\Lambda}{\mu_S}, \mathbf{0}, \mathbf{0}\right).$$

In order to find any endemic equilibria, we first determine the basic reproduction number $\mathcal{R}_0$ using the next generation operator approach [3]. We calculate

$$\mathcal{R}_0 \;=\; \frac{\Lambda}{\mu_S}\int_0^\infty \gamma(a)\Omega(a)da\int_0^\infty \beta(a)\Gamma(a)da \;=\; S^0 AB.$$

The quantity A is the probability that a newly infected individual survives the exposed class and proceeds to the infectious class. The product S^0B is the expected number of new infections that will be generated by a single newly infectious individual during the full period of infectiousness, in an otherwise disease-free population.

Now, taking $\widetilde{e}(0)$ and $\widetilde{i}(0)$ both to be non-zero gives $\widetilde{S} = \frac{1}{AB}$. Denote the endemic equilibrium by $E^* = (S^*, e^*(a), i^*(a))$. Then, $S^* = \frac{1}{AB}$ and using $0 = \frac{dS}{dt}$, we get $e^*(0) = \Lambda - \mu_S S^*$. Thus,

$$\begin{aligned} E^* = (S^*, e^*(a), i^*(a)) &= \left(\frac{1}{AB},\; e^*(0)\Omega(a),\; i^*(0)\Gamma(a)\right) \\ &= \left(\frac{S^0}{\mathcal{R}_0},\; \frac{\Lambda}{\mathcal{R}_0}(\mathcal{R}_0 - 1)\Omega(a),\; A\frac{\Lambda}{\mathcal{R}_0}(\mathcal{R}_0 - 1)\Gamma(a)\right). \end{aligned}$$

Theorem 3.1. *If $\mathcal{R}_0 \le 1$, then the only equilibrium in $\mathcal{Y}$ is E^0. If $\mathcal{R}_0 > 1$, then there are two equilibria, E^0 and E^*, which lie in $\mathcal{Y}$.*

4. **Boundedness.**

Proposition 1. *Let $X_0 \in \mathcal{Y}$. Then*

1. $\frac{d}{dt}\left\|\Phi_t(X_0)\right\| \le \Lambda - \mu_0\left\|\Phi_t(X_0)\right\|$ *for all* $t \ge 0$,
2. $\left\|\Phi_t(X_0)\right\| \le \max\left\{\frac{\Lambda}{\mu_0}, \frac{\Lambda}{\mu_0} + e^{-\mu_0 t}\left(\|X_0\| - \frac{\Lambda}{\mu_0}\right)\right\} \le \max\left\{\frac{\Lambda}{\mu_0}, \|X_0\|\right\}$ *for all* $t \ge 0$,
3. $\limsup_{t\to\infty}\left\|\Phi_t(X_0)\right\| \le \frac{\Lambda}{\mu_0}$,
4. Φ *is point dissipative; that is, there is a bounded set that attracts all points in* $\mathcal{Y}$.

Proof. We first note that

$$\frac{d}{dt}\left\|\Phi_t(X_0)\right\| = \frac{dS}{dt} + \frac{d}{dt}\int_0^\infty e(t,a)da + \frac{d}{dt}\int_0^\infty i(t,a)da. \tag{9}$$

By Equation (6), we have

$$\int_0^\infty e(t,a)da = \int_0^t S(t-a)J(t-a)\Omega(a)da + \int_t^\infty \varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)}da.$$

We make the substitution $a = t - \sigma$ in the first integral, and $a = t + \tau$ in the second integral, and differentiating by t, we get

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)da = &\frac{d}{dt}\int_0^t S(\sigma)J(\sigma)\Omega(t-\sigma)d\sigma + \frac{d}{dt}\int_0^\infty \varphi_e(\tau)\frac{\Omega(t+\tau)}{\Omega(\tau)}d\tau \\ =&S(t)J(t)\Omega(0) + \int_0^t S(\sigma)J(\sigma)\Omega'(t-\sigma)d\sigma + \int_0^\infty \varphi_e(\tau)\frac{\Omega'(t+\tau)}{\Omega(\tau)}d\tau.\end{aligned}$$

Converting the two integrals above back to integrals in terms of a, noting that $\Omega(0) = 1$ and $\Omega'(a) = -(\gamma(a) + \mu(a))\,\Omega(a)$ almost everywhere, and combining the two integrals into a single integral, we find

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)da &= S(t)J(t) - \int_0^\infty (\gamma(a)+\mu(a))\,e(t,a)da \\ &= S(t)\int_0^\infty \beta(a)i(t,a)da - \int_0^\infty (\gamma(a)+\mu(a))\,e(t,a)da.\end{aligned} \tag{10}$$

Similarly,

$$\frac{d}{dt}\int_0^\infty i(t,a)da = \int_0^\infty \gamma(a)e(t,a)da - \int_0^\infty \nu(a)i(t,a)da. \tag{11}$$

Combining Equation (10) and Equation (11) with the expression for $\frac{dS}{dt}$ given in Equation (1), we see that Equation (9) becomes

$$\frac{d}{dt}\|\Phi_t(X_0)\| = \Lambda - \mu_S S(t) - \int_0^\infty \mu(a)e(t,a)da - \int_0^\infty \nu(a)i(t,a)da.$$

Then, by **(H5)**, we have

$$\begin{aligned}\frac{d}{dt}\|\Phi_t(X_0)\| &\le \Lambda - \mu_0 S(t) - \mu_0\int_0^\infty e(t,a)da - \mu_0\int_0^\infty i(t,a)da \\ &= \Lambda - \mu_0\|\Phi_t(X_0)\|.\end{aligned}$$

This proves the first statement in the proposition. The second statement comes from solving the differential inequality and leads directly to the third statement, which implies the fourth. □

The following two propositions are direct consequences of the previous one.

Proposition 2. *If $X_0 \in \mathcal{Y}$ and $\|X_0\| \le K$ for some $K > \frac{\Lambda}{\mu_0}$, then the following hold for all $t \ge 0$:*

- $S(t), \int_0^\infty e(t,a)da, \int_0^\infty i(t,a)da \le K$,
- $J(t) \le \bar{\beta}K$ *and* $L(t) \le \bar{\gamma}K$,
- $e(t,0) \le \bar{\beta}K^2$ *and* $i(t,0) \le \bar{\gamma}K$.

Proposition 3. *Let $C \subseteq \mathcal{Y}$ be bounded. Then*

1. $\Phi(\mathbb{R}_{\ge 0}, C)$ *is bounded,*
2. Φ *is eventually bounded on C,*
3. *If C is bounded by $K > \frac{\Lambda}{\mu_0}$, then $\Phi(\mathbb{R}_{\ge 0}, C)$ is also bounded by K,*
4. *Given any $K > \frac{\Lambda}{\mu_0}$, there exists $T = T(C,K)$ such that $\|\Phi(t,C)\| \le K$ for all $t \ge T$.*

Similar to the proof of Proposition 1, the differential inequality $\frac{dS(t)}{dt} \le \Lambda - \mu_S S(t)$ yields the following result.

Proposition 4. *Let $X_0 \in \mathcal{Y}$. Then $\limsup_{t\to\infty} S(t) \le \frac{\Lambda}{\mu_S}$.*

5. **Asymptotic smoothness.** A semi-flow is called asymptotically smooth if each forward invariant bounded closed set is attracted by a nonempty compact set. In order to prove that the semi-flow Φ is asymptotically smooth, we use the following result, which is a special case of [25, Theorem 2.46] (which is based on [6, Lemma 3.2.3]).

Theorem 5.1. *The semi-flow $\Phi : \mathbb{R}_{\geq 0} \times \mathcal{Y} \to \mathcal{Y}$ is asymptotically smooth if there are maps $\Theta, \Psi : \mathbb{R}_{\geq 0} \times \mathcal{Y} \to \mathcal{Y}$ such that $\Phi(t, X) = \Theta(t, X) + \Psi(t, X)$, and the following hold for any bounded closed set C that is forward invariant under Φ:*

- $\lim_{t\to\infty} \operatorname{diam} \Theta(t, C) = 0$,
- *there exists $t_C \geq 0$ such that $\Psi(t, C)$ has compact closure for each $t \geq t_C$.*

We now give Theorem B.2. from [25], as it applies to $L^1_+(\mathbb{R}_{\geq 0})$.

Theorem 5.2. *A set $C \subseteq L^1_+(\mathbb{R}_{\geq 0})$ has compact closure if and only if the following conditions hold:*

1. $\sup_{f\in C} \int_0^\infty f(a)da < \infty$,
2. $\lim_{r\to\infty} \int_r^\infty f(a)da \to 0$ *uniformly in* $f \in C$,
3. $\lim_{h\to 0^+} \int_0^\infty |f(a + h) - f(a)|\, da \to 0$ *uniformly in* $f \in C$,
4. $\lim_{h\to 0^+} \int_0^h f(a)da \to 0$ *uniformly in* $f \in C$.

In order to apply Theorems 5.1 and 5.2 to the model, we first prove the following result.

Proposition 5. *The functions J and L are Lipschitz continuous on $\mathbb{R}_{\geq 0}$.*

Proof. Let $K \geq \max\left\{\frac{\Lambda}{\mu_0}, \|X_0\|\right\}$. Then, by Proposition 1, $\|X(t)\| \leq K$ for all $t \geq 0$.

Let $t \geq 0$ and let $h > 0$. Then

$$\begin{aligned}
J(t{+}h) - J(t) &= \int_0^\infty \beta(a)i(t + h, a)da - \int_0^\infty \beta(a)i(t, a)da \\
&= \int_0^h \beta(a)i(t + h, a)da + \int_h^\infty \beta(a)i(t + h, a)da - \int_0^\infty \beta(a)i(t, a)da \\
&= \int_0^h \beta(a)i(t + h - a, 0)\Gamma(a)da + \int_h^\infty \beta(a)i(t + h, a)da - \int_0^\infty \beta(a)i(t, a)da.
\end{aligned}$$

For the first integral, we use the bounds $\beta(a) \leq \bar{\beta}$, $i(t{+}h{-}a, 0) \leq \bar{\gamma}K$ and $\Gamma(a) \leq 1$; for the second integral, we make the substitution $\sigma = a - h$, obtaining

$$J(t + h) - J(t) \leq \bar{\beta}\bar{\gamma}Kh + \int_0^\infty \beta(\sigma + h)i(t + h, \sigma + h)d\sigma - \int_0^\infty \beta(a)i(t, a)da.$$

From Equation (8), we note that $i(t + h, \sigma + h) = i(t, \sigma)\frac{\Gamma(\sigma+h)}{\Gamma(\sigma)}$. Combining the integrals, we find that

$$\begin{aligned}
J(t + h) - J(t) &\leq \bar{\beta}\bar{\gamma}Kh + \int_0^\infty \left(\beta(a + h)\frac{\Gamma(a + h)}{\Gamma(a)} - \beta(a)\right) i(t, a)da \\
&= \bar{\beta}\bar{\gamma}Kh + \int_0^\infty \left(\beta(a + h)\mathrm{e}^{-\int_a^{a+h} \nu(\tau)d\tau} - \beta(a)\right) i(t, a)da \\
&= \bar{\beta}\bar{\gamma}Kh + \int_0^\infty \beta(a + h)\left(\mathrm{e}^{-\int_a^{a+h} \nu(\tau)d\tau} - 1\right) i(t, a)da \\
&\quad + \int_0^\infty (\beta(a + h) - \beta(a))\, i(t, a)da.
\end{aligned} \tag{12}$$

By **(H2)**, $0 \geq -\int_a^{a+h} \nu(\tau)d\tau \geq -\bar{\nu}h$. Thus, $1 \geq \mathrm{e}^{-\int_a^{a+h} \nu(\tau)d\tau} \geq \mathrm{e}^{-\bar{\nu}h} \geq 1 - \bar{\nu}h$, where the final inequality comes from the fact that e^x lies above its tangent at zero. Therefore, $0 \leq \beta(a+h)\left|\mathrm{e}^{-\int_a^{a+h} \nu(\tau)d\tau} - 1\right| \leq \bar{\beta}\bar{\nu}h$. Recalling, also, that $\int_0^\infty i(t,a)da \leq \|X(t)\| \leq K$, we see that Equation (12) implies

$$|J(t+h) - J(t)| \leq \bar{\beta}\bar{\gamma}Kh + \bar{\beta}\bar{\nu}Kh + \int_0^\infty |\beta(a+h) - \beta(a)|\, i(t,a)da. \tag{13}$$

Next, we show that the remaining integral in Equation (13) is of order h. Using **(H3)**, we find

$$\int_0^\infty |\beta(a+h) - \beta(a)|\, i(t,a)da \leq \int_0^\infty M_\beta h\, i(t,a)da \leq M_\beta hK.$$

Combining this with Equation (13), it follows that J is Lipschitz with coefficient $M_J = \left(\bar{\beta}\bar{\gamma} + \bar{\beta}\bar{\nu} + M_\beta\right)K$. Similarly, L is also Lipschitz. □

The following product rule will be used in the proof of Theorem 5.3. We omit the proof.

Proposition 6. *Let $D \subseteq \mathbb{R}$. For $j = 1,2$, suppose $f_j : D \to \mathbb{R}$ is a bounded Lipschitz continuous function with bound K_j and Lipschitz coefficient M_j. Then the product function $f_1 f_2$ is Lipschitz with coefficient $K_1M_2 + K_2M_1$.*

We are now prepared to prove the following, which is the main result of this section.

Theorem 5.3. *The flow Φ is asymptotically smooth.*

Proof. Let $C \subset \mathcal{Y}$ be bounded. Let $K > \frac{\Lambda}{\mu_0}$ be a bound for C. Let $X_0 \in C$. We consider the solution $\Phi(t, X_0) = (S(t), e(t,\cdot), i(t,\cdot))$, where e and i are given by Equation (6) and Equation (7).

For $t \geq 0$, let $\Psi(t, X_0) = \left(S(t), \widetilde{e}(t,\cdot), \widetilde{i}(t,\cdot)\right)$ and $\Theta(t, X_0) = (0, \widetilde{\varphi}_e(t,\cdot), \widetilde{\varphi}_i(t,\cdot))$, where

$$\widetilde{e}(t,a) = \left\{\begin{array}{ll} e(t,a) & \text{for } 0 \leq a \leq t \\ 0 & \text{for } t < a \end{array}\right\} = \left\{\begin{array}{ll} S(t-a)J(t-a)\Omega(a) & \text{for } 0 \leq a \leq t \\ 0 & \text{for } t < a, \end{array}\right.$$

$$\widetilde{i}(t,a) = \left\{\begin{array}{ll} i(t,a) & \text{for } 0 \leq a \leq t \\ 0 & \text{for } t < a \end{array}\right\} = \left\{\begin{array}{ll} L(t-a)\Gamma(a) & \text{for } 0 \leq a \leq t \\ 0 & \text{for } t < a, \end{array}\right.$$

$$\widetilde{\varphi}_e = e - \widetilde{e} \qquad \text{and} \qquad \widetilde{\varphi}_i = i - \widetilde{i}.$$

Then $\Phi = \Theta + \Psi$. For $t \geq 0$, we have

$$\widetilde{\varphi}_e(t,a) = \left\{\begin{array}{ll} 0 & \text{for } 0 \leq a \leq t \\ e(t,a) & \text{for } t < a \end{array}\right\} = \left\{\begin{array}{ll} 0 & \text{for } 0 \leq a \leq t \\ \varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)} & \text{for } t < a. \end{array}\right.$$

Let the standard norm on L^1 be denoted by $\|\cdot\|_1$. Then

$$\begin{aligned} \|\widetilde{\varphi}_e(t,\cdot)\|_1 &= \int_0^\infty |\widetilde{\varphi}_e(t,a)|\, da \\ &= \int_t^\infty \varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)}da \\ &= \int_0^\infty \varphi_e(\sigma)\frac{\Omega(\sigma+t)}{\Omega(\sigma)}d\sigma. \end{aligned}$$

Using Equation (3) to replace both instances of Ω, and then **(H5)**, we find

$$\begin{aligned}\|\widetilde{\varphi}_e(t,\cdot)\|_1 &= \int_0^\infty \varphi_e(\sigma)\mathrm{e}^{-\int_\sigma^{\sigma+t}(\gamma(\tau)+\mu(\tau))d\tau}d\sigma \\ &\le \mathrm{e}^{-\mu_0 t}\int_0^\infty \varphi_e(\sigma)d\sigma \\ &\le K\mathrm{e}^{-\mu_0 t},\end{aligned}$$

which tends to zero as t goes to ∞. Similarly, $\|\widetilde{\varphi}_i(t,\cdot)\|_1 \le K\mathrm{e}^{-\mu_0 t}$. This shows that $\Theta(t, X_0)$ approaches $\mathbf{0} \in \mathcal{Y}$ with uniform exponential speed, and therefore $\lim_{t\to\infty} \mathrm{diam}\Theta(t, C) = 0$, as required by Theorem 5.1.

It remains to be shown that there exists $t_C \ge 0$ such that $\Psi(t, C) \subseteq \mathcal{Y}$ has compact closure for each $t \ge t_C$. We do this with $t_C = 0$.

By part (3) of Proposition 3, we know that $S(t)$ remains in the compact set $[0, K]$. Next, we show that $\widetilde{e}$ remains in a pre-compact subset of L^1_+ that is independent of X_0. This is done by verifying conditions (1-4) of Theorem 5.2.

By Proposition 2 and Equation (4), we have

$$0 \le \widetilde{e}(t,a) = \left\{\begin{array}{ll} S(t-a)J(t-a)\Omega(a) & \text{for } 0 \le a \le t \\ 0 & \text{for } t < a \end{array}\right\} \le \bar{\beta}K^2\mathrm{e}^{-\mu_0 a},$$

from which conditions (1, 2, 4) of Theorem 5.2 follow directly. Now, we demonstrate that condition (3) holds. Because we are interested in the limit as h tends to 0^+, we consider $h \in (0, t)$. Then

$$\begin{aligned}&\int_0^\infty |\widetilde{e}(t,a+h) - \widetilde{e}(t,a)|da \\ &= \int_0^{t-h} |S(t-a-h)J(t-a-h)\Omega(a+h) - S(t-a)J(t-a)\Omega(a)|\,da \\ &\qquad + \int_{t-h}^t |0 - S(t-a)J(t-a)\Omega(a)|\,da \\ &\le \bar{\beta}K^2h + \int_0^{t-h} |S(t-a-h)J(t-a-h)\Omega(a+h) - S(t-a)J(t-a)\Omega(a)|\,da \\ &\le \bar{\beta}K^2h + \int_0^{t-h} S(t-a-h)J(t-a-h)\,|\Omega(a+h) - \Omega(a)|\,da \\ &\qquad + \int_0^{t-h} |S(t-a-h)J(t-a-h) - S(t-a)J(t-a)|\,\Omega(a)da \\ &\le \bar{\beta}K^2h + \bar{\beta}K^2\int_0^{t-h} |\Omega(a+h) - \Omega(a)|\,da \\ &\qquad + \int_0^{t-h} |S(t-a-h)J(t-a-h) - S(t-a)J(t-a)|\,\Omega(a)da.\end{aligned} \tag{14}$$

Recalling Equation (3) and Equation (4), we note that Ω is a decreasing function, which takes values in the unit interval. Thus,

$$\begin{aligned}\int_0^{t-h} |\Omega(a+h) - \Omega(a)|\,da &= \int_0^{t-h} (\Omega(a) - \Omega(a+h))\,da \\ &= \int_0^h \Omega(a)da - \int_{t-h}^t \Omega(a)da \le h.\end{aligned}$$

Combining this with Equation (14), we find

$$\begin{aligned}\int_0^\infty &|\widetilde{e}(t,a+h)-\widetilde{e}(t,a)|da\\ &\le 2\bar{\beta}K^2h+\int_0^{t-h}|S(t-a-h)J(t-a-h)-S(t-a)J(t-a)|\,\Omega(a)da.\end{aligned} \tag{15}$$

Finally, we determine a bound for the remaining integral on the right-hand side. Combining Proposition 2 with the expression for $\frac{dS}{dt}$ given in Equation (1), we find that $\left|\frac{dS}{dt}\right|$ is bounded by $M_S=\Lambda+\mu_S K+\bar{\beta}K^2$, and therefore $S(\cdot)$ is Lipschitz on $[0,\infty)$ with coefficient M_S. By Proposition 5, there exists a Lipschitz coefficient M_J for $J:[0,\infty)\to\mathbb{R}$. Thus, Proposition 6 implies that $S(\cdot)J(\cdot)$ is Lipschitz on $[0,\infty)$ with coefficient $M_{SJ}=KM_J+\bar{\beta}KM_S$. Therefore, for $a\in[0,t-h)$,

$$|S(t-a-h)J(t-a-h)-S(t-a)J(t-a)|\,\Omega(a)\le M_{SJ}h\Omega(a)\le M_{SJ}h\mathrm{e}^{-\mu_0 a}.$$

Thus, Equation (15) leads to

$$\begin{aligned}\int_0^\infty|\widetilde{e}(t,a+h)-\widetilde{e}(t,a)|da &\le 2\bar{\beta}K^2h+M_{SJ}h\int_0^{t-h}\mathrm{e}^{-\mu_0 a}da\\ &\le 2\bar{\beta}K^2h+\frac{M_{SJ}}{\mu_0}h\\ &=\left(2\bar{\beta}K^2+\frac{M_{SJ}}{\mu_0}\right)h.\end{aligned}$$

We note that M_{SJ} depends on K, which depends on the set C, but not on X_0. Therefore, this inequality holds for any $X_0\in C$, and so condition (3) of Theorem 5.2 is satisfied. Thus, $\widetilde{e}$ remains in a pre-compact subset C_K^e of L_+^1. Similarly, $\widetilde{i}$ remains in a pre-compact subset C_K^i of L_+^1. Thus, $\Psi(t,C)\subseteq[0,K]\times C_K^e\times C_K^i$, which has compact closure in $\mathcal{Y}$. It follows that $\Psi(t,C)$ has compact closure. Thus, the second condition of Theorem 5.1 is satisfied, and therefore Φ is asymptotically smooth. □

6. **Attractor.** A *total trajectory* of Φ is a function $X:\mathbb{R}\to\mathcal{Y}$ such that $\Phi_s(X(t))=X(t+s)$ for all $t\in\mathbb{R}$ and all $s\ge 0$. For a total trajectory,

$$e(t,a)=e(t-a,0)\Omega(a)\quad\text{and}\quad i(t,a)=i(t-a,0)\Gamma(a)\quad\text{for all } t\in\mathbb{R} \text{ and } a\in\mathbb{R}_{\ge 0}.$$

It is worth noting that total trajectories often have nice properties. For example:

Proposition 7. *If $Y:\mathbb{R}\to\mathcal{Y}$ is a total trajectory, then the corresponding functions J and L are Lipschitz on $[t,\infty)$ for any $t\in\mathbb{R}$.*

Proof. Let $X_0=Y(t)$. Then, the result follows from Proposition 5. □

A non-empty compact set $\widetilde{A}$ is a *compact attractor* of a class $\mathcal{C}$ of sets if $\widetilde{A}$ is invariant and $\mathrm{d}\left(\Phi_t(C),\widetilde{A}\right)\to 0$ for each $C\in\mathcal{C}$. Such a set consists of total trajectories; that is, for each $X_0\in\widetilde{A}$, there exists a total trajectory X such that $X(0)=X_0$ and $X(t)\in\widetilde{A}$ for all $t\in\mathbb{R}$.

Theorem 6.1. *There exists a set $\mathcal{A}$, which is a compact attractor of bounded sets.*

Proof. Propositions 1 and 3 and Theorem 5.3 show that Φ is point dissipative, eventually bounded on bounded sets, and asymptotically smooth. Thus, the result follows from Theorem 2.33 of [25]. □

The following corollary follows from Proposition 1 and Proposition 4.

Corollary 1. *If* $X_0 = (x, \varphi, \phi) \in \mathcal{A}$, *then* $\|X_0\| \leq \frac{\Lambda}{\mu_0}$ *and* $0 \leq x \leq \frac{\Lambda}{\mu_S}$.

7. **Behaviour for $\mathcal{R}_0 < 1$.** Suppose $\mathcal{R}_0 < 1$. Let $X_0 \in \mathcal{A}$ and let $X(t)$ be a total trajectory in $\mathcal{A}$, which passes through X_0 at time zero. Then

$$\begin{bmatrix} J(t) \\ \\ L(t) \end{bmatrix} = \begin{bmatrix} \int_0^\infty \beta(a) i(t-a,0)\Gamma(a)da \\ \\ \int_0^\infty \gamma(a) e(t-a,0)\Omega(a)da \end{bmatrix} = \begin{bmatrix} \int_0^\infty \beta(a) L(t-a)\Gamma(a)da \\ \\ \int_0^\infty \gamma(a) S(t-a)J(t-a)\Omega(a)da \end{bmatrix}. \tag{16}$$

Let $\bar{J} = \sup_{t\in\mathbb{R}} J(t)$ and $\bar{L} = \sup_{t\in\mathbb{R}} L(t)$. Then Equation (16) implies

$$J(t) \leq \bar{L} \int_0^\infty \beta(a)\Gamma(a)da = \bar{L}B$$

for all $t \in \mathbb{R}$. Taking the supremum on the left-hand side, we obtain

$$\bar{J} \leq \bar{L}B.$$

By also using Corollary 1, we similarly see that

$$\bar{L} \leq \frac{\Lambda}{\mu_S}\bar{J}\int_0^\infty \gamma(a)\Omega(a)da = \bar{J}\frac{\Lambda}{\mu_S}A.$$

Combining these inequalities, we find that

$$\bar{J} \leq \bar{J}\frac{\Lambda}{\mu_S}AB = \bar{J}\mathcal{R}_0.$$

Then, since $\bar{J}$ is non-negative and $\mathcal{R}_0 < 1$, it follows that $\bar{J} = 0$. Similarly, $\bar{L} = 0$. Thus, the attractor is a compact invariant subset of the disease-free space $\mathbb{R}\times\{\mathbf{0}\}\times\{\mathbf{0}\}$. The only such set is the singleton containing the disease-free equilibrium, and so we have the following result.

Theorem 7.1. *If* $\mathcal{R}_0 < 1$, *then the compact attractor of bounded sets is* $\mathcal{A} = \{E^0\}$.

Remark 1. Using the linearization method described in [27, Section 4.5], one can show that the disease-free equilibrium is locally asymptotically stable for $\mathcal{R}_0$ less than one.

8. **Uniform persistence for $\mathcal{R}_0 > 1$.** We first show that the system is uniformly weakly ρ-persistent by using a Laplace transform approach, with persistence function $\rho(X(t)) = J(t)$. We then show that the system is uniformly (strongly) ρ-persistent. We follow the approach used in [25, Chapter 9].

For any $X_0 \in \mathcal{Y}$, we have

$$\begin{aligned} J(t) &= \int_0^\infty \beta(a) i(t,a) \\ &= \int_0^t \beta(a)\Gamma(a) i(t-a,0)da + \widetilde{J}(t), \end{aligned}$$

where $\widetilde{J}(t) = \int_t^\infty \beta(a)\varphi_i(a-t)\frac{\Gamma(a)}{\Gamma(a-t)}da$. Using the boundary condition Equation (2) to rewrite $i(t-a,0)$, we find

$$\begin{aligned} J(t) &= \int_0^t \beta(a)\Gamma(a)\left[\int_0^\infty \gamma(\sigma)e(t-a,\sigma)d\sigma\right]da + \widetilde{J}(t) \\ &= \int_0^t \beta(a)\Gamma(a)\left[\int_0^{t-a} \gamma(\sigma)\Omega(\sigma)e(t-a-\sigma,0)d\sigma + \widetilde{L}(t-a)\right]da + \widetilde{J}(t), \end{aligned}$$

where $\widetilde{L}(t-a) = \int_{t-a}^{\infty} \gamma(\sigma)\varphi_e(\sigma+a-t)\frac{\Omega(\sigma)}{\Omega(\sigma+a-t)}d\sigma$. Next, we introduce the notation

$$K(t) = \int_0^t \beta(a)\Gamma(a)\widetilde{L}(t-a)da + \widetilde{J}(t), \tag{17}$$

and use the substitution $e(t-a-\sigma, 0) = S(t-a-\sigma)J(t-a-\sigma)$ to write

$$J(t) = \int_0^t \int_0^{t-a} \beta(a)\Gamma(a)\gamma(\sigma)\Omega(\sigma)S(t-a-\sigma)J(t-a-\sigma)d\sigma da + K(t). \tag{18}$$

It follows from Equation (18) that $J(t)$ can only be identically zero for $t \geq 0$ if $K(t)$ is identically zero, which in turn, due to **(H4)**, can only happen if φ_e and φ_i are identically zero. Thus, if the disease is initially present, then J takes on positive values. In fact, either $J(t)$ is zero for all $t \geq 0$ or $J(t)$ takes on positive values for arbitrarily large values of t. We have the following.

Proposition 8. *Either $\varphi_e = \varphi_i = \mathbf{0} \in L^1$ and therefore $e(t,\cdot) = i(t,\cdot) = \mathbf{0}$ for all $t > 0$, or $J(t)$ takes on positive values for arbitrarily large values of t.*

For the remainder of this section, we assume that the disease is initially present; that is, the support of at least one of φ_e and φ_i has positive measure, and therefore $J(t)$ takes on positive values for arbitrarily large values of t. Recalling that J is Lipschitz (see Proposition 5), it follows that J is positive on a set of positive measure.

Let

$$J^\infty = \limsup J(t)$$

and

$$S_\infty = \liminf S(t).$$

Let $\epsilon > 0$. Then there exists $T_1 \geq 0$ such that $J(t) \leq J^\infty + \frac{\epsilon}{2}$ for all $t \geq T_1$. Then, it follows from the expression for $\frac{dS}{dt}$ given in Equation (1) that $S_\infty \geq \frac{\Lambda}{\mu_S + J^\infty + \frac{\epsilon}{2}}$. Thus, there exists $T_2 \geq T_1$ such that

$$S(t) \geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \tag{19}$$

for all $t \geq T_2$. We now perform a time-shift of T_2 on the solution being studied; that is, we replace the initial condition X_0 with $X_1 = \Phi_{T_2}(X_0)$. The solution passing through X_1 at time 0 satisfies Equation (18), and also satisfies Equation (19) for all $t \geq 0$. Furthermore, the bound J^∞ is also valid for the new solution. Thus, Equation (18) leads to

$$J(t) \geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^t \int_0^{t-a} \beta(a)\Gamma(a)\gamma(\sigma)\Omega(\sigma)J(t-a-\sigma)d\sigma da + K(t).$$

Next, we make the substitution $\sigma = \tau - a$; then we change the order of integration:

$$\begin{aligned} J(t) &\geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^t \int_a^t \beta(a)\Gamma(a)\gamma(\tau-a)\Omega(\tau-a)J(t-\tau)d\tau da + K(t) \\ &\geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^t \int_0^\tau \beta(a)\Gamma(a)\gamma(\tau-a)\Omega(\tau-a)J(t-\tau)da d\tau + K(t). \end{aligned}$$

Let

$$l(\tau) = \int_0^\tau \beta(a)\Gamma(a)\gamma(\tau-a)\Omega(\tau-a)da.$$

Then we obtain the following inequality, which includes a convolution:

$$J(t) \geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^t l(\tau) J(t-\tau) d\tau + K(t). \tag{20}$$

Since $K \geq 0$, we can omit it and the inequality is preserved:

$$J(t) \geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^t l(\tau) J(t-\tau) d\tau.$$

Taking the Laplace transform of each side converts the convolution to a product:

$$\widehat{J}(\lambda) \geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \widehat{l}(\lambda) \widehat{J}(\lambda). \tag{21}$$

J is positive on a set of positive measure and therefore $\widehat{J}$ is strictly positive. Thus, Equation (21) yields

$$\begin{aligned} 1 &\geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \widehat{l}(\lambda) \\ &= \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^\infty \mathrm{e}^{-\lambda\tau} \int_0^\tau \beta(a)\Gamma(a)\gamma(\tau - a)\Omega(\tau - a) da d\tau. \end{aligned}$$

Change the order of integration, and then let $\tau = \sigma + a$, to obtain

$$\begin{aligned} 1 &\geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^\infty \int_a^\infty \mathrm{e}^{-\lambda\tau} \beta(a)\Gamma(a)\gamma(\tau - a)\Omega(\tau - a) d\tau da \\ &= \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda(\sigma + a)} \beta(a)\Gamma(a)\gamma(\sigma)\Omega(\sigma) d\sigma da \\ &= \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^\infty \mathrm{e}^{-\lambda\sigma} \gamma(\sigma)\Omega(\sigma) d\sigma \int_0^\infty \mathrm{e}^{-\lambda a} \beta(a)\Gamma(a) da. \end{aligned}$$

Taking limits as ϵ and λ tend to zero, we obtain

$$\begin{aligned} 1 &\geq \frac{\Lambda}{\mu_S + J^\infty} \int_0^\infty \gamma(\sigma)\Omega(\sigma) d\sigma \int_0^\infty \beta(a)\Gamma(a) da \\ &= \frac{\Lambda}{\mu_S + J^\infty} AB, \end{aligned}$$

where A and B are given in Equation (5). Rearranging, we find that

$$\begin{aligned} J^\infty &\geq \Lambda AB - \mu_S \\ &= \mu_S \left(\mathcal{R}_0 - 1\right), \end{aligned}$$

which is positive for $\mathcal{R}_0$ greater than one.

Define $\rho : \mathcal{Y} \to \mathbb{R}_{\geq 0}$ by

$$\rho\left(x, \varphi, \phi\right) = \int_0^\infty \beta(a)\phi(a) da. \tag{22}$$

Then for $t \geq 0$,

$$\rho\left(\Phi_t(X_0)\right) = \int_0^\infty \beta(a) i(t, a) da = J(t), \tag{23}$$

and so, if the disease is initially present, then

$$\limsup \rho\left(\Phi_t(X_0)\right) = J^\infty \geq \mu_S \left(\mathcal{R}_0 - 1\right).$$

We have proven the following.

Theorem 8.1. *If $\mathcal{R}_0 > 1$, then the semi-flow is uniformly weakly ρ-persistent.*

In order to move from uniform weak persistence to uniform persistence, we follow the approach found in [25, Lemma 9.12]. To this end, we prove the following.

Proposition 9. *For a total trajectory $X(\cdot)$ in $\mathcal{Y}$, $S(t)$ is strictly positive and either J is identically zero or J is strictly positive.*

Proof. Let $X(\cdot)$ be a total trajectory in $\mathcal{Y}$, with $X(t) = (S(t), e(t,\cdot), i(t,\cdot))$. For any $T \in \mathbb{R}$, the function $X_T : \mathbb{R}_{\geq 0} \to \mathcal{Y}$ defined by $X_T(t) = X(T+t)$ is a semi-trajectory of Equation (1) with initial condition $X_T(0) = X(T) \in \mathcal{Y}$.

If $S(T) = 0$ for some T, then Equation (1) dictates that $\frac{dS(T)}{dt} > 0$. Then, for sufficiently small $\epsilon > 0$, we would have $S(T-\epsilon) < 0$, contradicting the fact that the total trajectory X lies in $\mathcal{Y}$ for all $t \in \mathbb{R}$. Therefore, $S(\cdot)$ is strictly positive.

Suppose there exists $T \in \mathbb{R}$ such that $e(T,\cdot) = i(T,\cdot) = \mathbf{0}$. Then by Proposition 8, $J(t) = 0$ for all $t \geq T$. Additionally, for any $t < T$, we have $0 = e(T, T-t) = e(t,0)\Omega(T-t) = S(t)J(t)\Omega(T-t)$. By Equation (4), $\Omega(T-t)$ is positive, as is $S(t)$. Thus, $J(t) = 0$ for all $t < T$, and so J is identically zero.

We now assume that at least one of $e(T,\cdot)$ and $i(T,\cdot)$ is non-zero for each $T \in \mathbb{R}$. If there exists T_0 such that $e(T,\cdot) = \mathbf{0}$ for all $T \leq T_0$, then for any $a > 0$, we would have $i(T_0, a) = i(T_0 - a, 0)\Gamma(a) = \int_0^\infty \gamma(\sigma)e(T_0 - a, \sigma)d\sigma\Gamma(a) = 0$, implying that $i(T_0,\cdot)$ and $e(T_0,\cdot)$ would both be zero, giving a contradiction. Thus, there exists a sequence $\{T_n\}$ tending to $-\infty$ such that $e(T_n,\cdot)$ is non-zero for each n. That is, for each n, there exists $a_n > 0$ such that $0 \neq e(T_n, a_n) = e(T_n - a_n, 0)\Omega(a_n)$. Thus, we have $e(T_n^*, 0) \neq 0$ where $T_n^* := T_n - a_n$ tends to $-\infty$.

For each $n \in \mathbb{N}$, let $J_n(t) = J(T_n^* + t)$. Rewriting Equation (20) for J_n gives

$$J_n(t) \geq \frac{\Lambda}{\mu_S + J^\infty + \epsilon} \int_0^t l(\tau) J_n(t-\tau)d\tau + K_n(t),$$

where we refer to Equation (17) to define $K_n(t)$, noting that $K_n(t)$ is greater than $\widetilde{J}_n(t) = \int_t^\infty \beta(a) i(T_n^*, a-t)\frac{\Gamma(a)}{\Gamma(a-t)}da$. Note that $\widetilde{J}_n(0) = \int_0^\infty \beta(a) i(T_n^*, a)da = \frac{e(T_n^*,0)}{S(T_n^*)} > 0$. Similar to the proof of Proposition 5, we can show that $\widetilde{J}_n$ is Lipschitz on $\mathbb{R}_{\geq 0}$. Thus, it follows that $\widetilde{J}_n(t)$, and hence $K_n(t)$, are positive for sufficiently small t.

Note that the support of l has positive measure. Therefore, [25, Corollary B.6.] implies there exists $b > 0$ such that $J_n(t)$ is positive for all $t > b$. Furthermore, b depends only on l; thus the same b works for each J_n. Since each J_n is a shift of J by T_n^*, and T_n^* tends to $-\infty$, it follows that $J(t) > 0$ for all $t \in \mathbb{R}$, completing the proof. □

Let $\mathcal{Y}_0 = \{X_0 \in \mathcal{Y} : \rho\left(\Phi_t(X_0)\right) = 0 \text{ for all } t \in \mathbb{R}_{\geq 0}\}$, where ρ is given in Equation (22) and Equation (23). Then $\mathcal{Y}_0$ is the disease-free space and is non-empty. Let $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{Y}_0$. Let $\mathcal{A}_1 \subseteq \mathcal{A}$ be the compact attractor of compact sets in $\mathcal{Y} \setminus \mathcal{Y}_0$. Let $C \subseteq \mathcal{A}$ be the set consisting of points $X_0 \in \mathcal{A}$ such that there exists a total trajectory $X(\cdot)$ through X_0 with $X(t)$ approaching $\mathcal{A}_0$ as $t \to -\infty$ and approaching $\mathcal{A}_1$ as $t \to \infty$.

The next results are needed in Section 9 in order to use a particular Lyapunov functional.

Theorem 8.2. *If $\mathcal{R}_0 > 1$, then the semi-flow is uniformly ρ-persistent.*

Proof. The result follows from Proposition 5 (which implies J is continuous), Theorem 6.1, Theorem 8.1, Proposition 9 and [25, Theorem 5.2.]. □

The following result follows from Proposition 5, Theorem 8.1, Proposition 9 and [25, Theorem 5.7].

Theorem 8.3. *If $\mathcal{R}_0 > 1$, then the attractor $\mathcal{A}$ is the disjoint union*

$$\mathcal{A} = \mathcal{A}_0 \cup C \cup \mathcal{A}_1,$$

where

- *$\mathcal{A}_0$ and $\mathcal{A}_1$ are compact and invariant,*
- *$\mathcal{A}_0 = \mathcal{A} \cap \mathcal{Y}_0$ is the compact attractor of all bounded sets in $\mathcal{Y}_0$,*
- *ρ is bounded away from 0 on $\mathcal{A}_1$,*
- *$\mathcal{A}_1$ attracts all bounded sets in $\mathcal{Y} \setminus \mathcal{Y}_0$ on which $\rho \circ \Phi$ is eventually uniformly positive,*
- *$\mathcal{A}_1$ is stable,*
- *C is invariant and consists of total trajectories with alpha limit sets in $\mathcal{A}_0$ and omega limit sets in $\mathcal{A}_1$.*

The sets $\mathcal{A}_0$ and $\mathcal{A}_1$ are called the extinction attractor and the persistence attractor, respectively.

Corollary 2. *Suppose $\mathcal{R}_0 > 1$. Let $X(t) = (S(t), e(t,\cdot), i(t,\cdot))$ be a total trajectory in $\mathcal{A}_1$. Then there exists $\epsilon > 0$ such that $S(t), e(t,0), i(t,0) > \epsilon$ for all $t \in \mathbb{R}$.*

Proof. The right-hand side of Equation (19) provides a positive lower bound ϵ_1 for the S-coordinate for any point in $\mathcal{A} \supseteq \mathcal{A}_1$. By Theorem 8.3, there exists $\epsilon_2 > 0$ such that $\epsilon_2 < \rho(X(t)) = J(t)$ for all $t \in \mathbb{R}$. Thus, $e(t,0) = S(t)J(t) > \epsilon_1\epsilon_2$ for all t. Next, $i(t,0) = \int_0^\infty \gamma(a)e(t,a)da = \int_0^\infty \gamma(a)e(t-a,0)\Omega(a)da > \epsilon_1\epsilon_2 \int_0^\infty \gamma(a)\Omega(a)da = \epsilon_1\epsilon_2 A$, where A was shown to be positive when it was defined in Section 3. Letting $\epsilon = \min\{\epsilon_1, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2 A\}$ completes the proof. □

9. Behaviour for $\mathcal{R}_0 > 1$.

Theorem 9.1. *If the initial conditions satisfy $\varphi_e = \varphi_i = \mathbf{0}$, then $X(t)$ tends to the disease-free equilibrium E^0. Furthermore, $\mathcal{A}_0 = \{E^0\}$.*

Proof. One solution of Equation (1) that satisfies these initial conditions is given by $e(t,\cdot) = i(t,\cdot) = \mathbf{0}$ for all $t \geq 0$, with $S(t)$ satisfying $\frac{dS}{dt} = \Lambda - \mu_S S(t)$. This solution tends to E^0 with exponential speed. Since solutions to the initial value problem are unique, the first statement of the theorem follows.

Suppose $C \subseteq \mathcal{Y}_0$ is bounded by $K > 0$. Let $X_0 \in C$. Then $0 = \rho(X(t)) = \int_0^\infty \beta(a)i(t,a)da = J(t)$ for all $t \geq 0$. Thus, $e(t,0) = 0$ for all $t \geq 0$. Recall that $i(t,0) = L(t)$ and L is Lipschitz for $t \geq 0$. It follows from **(H4)** that in order to have $\int_0^\infty \beta(a)i(t,a)da$ identically zero for $t \geq 0$, that we must have $i(t,0)$ identically zero as well.

Thus, $e(t,\cdot) = i(t,\cdot) = \mathbf{0}$ and $\frac{dS}{dt} = \Lambda - \mu_S S$, with $|S(0)| \leq K$. Hence,

$$\left| S(t) - \frac{\Lambda}{\mu_S} \right| = \left| S(0) - \frac{\Lambda}{\mu_S} \right| \mathrm{e}^{-\mu_S t} \leq \left(K + \frac{\Lambda}{\mu_S} \right) \mathrm{e}^{-\mu_S t}.$$

It follows that $\mathrm{d}\left(\Phi_t(C), \{E^0\}\right) \leq \left(K + \frac{\Lambda}{\mu_S}\right) \mathrm{e}^{-\mu_S t}$, and so $\mathcal{A}_0 = \{E^0\}$. □

The following two lemmas will be used to cancel terms in the proof of Theorem 9.5.

Lemma 9.2. *Each solution of Equation (1) satisfies*

$$\int_0^\infty \beta(a)i^*(a)\left[\frac{S(t)}{S^*}\frac{i(t,a)}{i^*(a)} - \frac{e(t,0)}{e^*(0)}\right] da = 0. \tag{24}$$

Proof. Using the boundary condition given in Equation (2), we observe that

$$\begin{aligned} 0 &= \frac{1}{S^*}\left(e(t,0) - e^*(0)\frac{e(t,0)}{e^*(0)}\right) \\ &= \frac{1}{S^*}\left(\int_0^\infty \beta(a)S(t)i(t,a)da \; - \int_0^\infty \beta(a)S^*i^*(a)da\frac{e(t,0)}{e^*(0)}\right) \\ &= \int_0^\infty \beta(a)i^*(a)\left[\frac{S(t)}{S^*}\frac{i(t,a)}{i^*(a)} - \frac{e(t,0)}{e^*(0)}\right] da. \end{aligned}$$

□

Lemma 9.3. *Each solution of Equation (1) satisfies*

$$\int_0^\infty \gamma(a)e^*(a)\left[\frac{e(t,a)}{e^*(a)} - \frac{i(t,0)}{i^*(0)}\right] da = 0. \tag{25}$$

Proof. As in the previous proof, we use Equation (2), finding that

$$\begin{aligned} 0 &= i(t,0) - i^*(0)\frac{i(t,0)}{i^*(0)} \\ &= \int_0^\infty \gamma(a)e(t,a)da - \int_0^\infty \gamma(a)e^*(a)da\frac{i(t,0)}{i^*(0)} \\ &= \int_0^\infty \gamma(a)e^*(a)\left[\frac{e(t,a)}{e^*(a)} - \frac{i(t,0)}{i^*(0)}\right] da. \end{aligned}$$

□

The following lemma will be used in the proof of Theorem 9.5, and may prove useful in the global analysis of other models that include age-structure. We point out that when applied in the proof of Theorem 9.5, the lemma is used for total trajectories that exist for all t; thus, the following lemma is formulated for $t \in \mathbb{R}$, $a > 0$.

Lemma 9.4. *Let q be a non-negative, bounded Lebesgue measurable function. Let z_1 and z_2 be non-zero solutions of*

$$\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} = -q(a)z(t,a),$$

for $t \in \mathbb{R}$ and $a > 0$, with $z_j(t,0) = Z_j(t) > 0$ for all $t \in \mathbb{R}$, for $j = 1,2$. Let

$$U(t) = \int_0^\infty \alpha(a)G\left(\frac{z_1(t,a)}{z_2(t,a)}\right) da,$$

where G is continuous and $\alpha(a) = \int_a^\infty \xi(\sigma)d\sigma$, with $\xi, \alpha \in L^1_+$. Then

$$\frac{dU}{dt} = \int_0^\infty \xi(a)\left[G\left(\frac{z_1(t,0)}{z_2(t,0)}\right) - G\left(\frac{z_1(t,a)}{z_2(t,a)}\right)\right] da.$$

Proof. Let $Q(a) = \mathrm{e}^{-\int_0^a q(\sigma)d\sigma}$. Then $z_j(t,a) = Z_j(t-a)Q(a)$ for all $t \in \mathbb{R}$, $a > 0$. Thus, $z_j(t,a)$ is positive for all t and a, and $\frac{z_1(t,a)}{z_2(t,a)} = \frac{Z_1(t-a)}{Z_2(t-a)}$. Therefore,

$$\frac{dU}{dt} = \frac{d}{dt}\int_0^\infty \alpha(a)G\left(\frac{Z_1(t-a)}{Z_2(t-a)}\right) da.$$

We make the substitution $\sigma = t - a$, obtaining

$$\begin{aligned}\frac{dU}{dt} &= \frac{d}{dt}\int_{-\infty}^{t} \alpha(t-\sigma)G\left(\frac{Z_1(\sigma)}{Z_2(\sigma)}\right)d\sigma \\ &= \alpha(0)G\left(\frac{Z_1(t)}{Z_2(t)}\right) + \int_{-\infty}^{t} \alpha'(t-\sigma)G\left(\frac{Z_1(\sigma)}{Z_2(\sigma)}\right)d\sigma \\ &= \alpha(0)G\left(\frac{Z_1(t)}{Z_2(t)}\right) + \int_{0}^{\infty} \alpha'(a)G\left(\frac{Z_1(t-a)}{Z_2(t-a)}\right)da.\end{aligned}$$

Now, converting from Z_1/Z_2 to z_1/z_2, filling in for $\alpha(0)$, and noting that $\alpha'(a) = -\xi(a)$, we obtain

$$\begin{aligned}\frac{dU}{dt} &= \int_0^{\infty} \xi(\sigma)d\sigma\, G\left(\frac{z_1(t,0)}{z_2(t,0)}\right) - \int_0^{\infty} \xi(a)G\left(\frac{z_1(t,a)}{z_2(t,a)}\right)da \\ &= \int_0^{\infty} \xi(a)\left[G\left(\frac{z_1(t,0)}{z_2(t,0)}\right) - G\left(\frac{z_1(t,a)}{z_2(t,a)}\right)\right]da,\end{aligned}$$

completing the proof. □

The following theorem is the key result of this paper. Loosely, it states that the endemic equilibrium is globally attracting (amongst solutions for which disease is present) if the basic reproduction number is greater than one.

Theorem 9.5. *Suppose $\mathcal{R}_0 > 1$. Then $\mathcal{A}_1 = \{E^*\}$. Furthermore, each solution for which the disease is initially present tends to the endemic equilibrium E^*.*

Proof. Let

$$g(y) = y - 1 - \ln y.$$

Note that $g : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is continuous and concave up. Also, g has a unique minimum at 1, with $g(1) = 0$.

Let $X(t) = (S(t), e(t,\cdot), i(t,\cdot))$ be a total trajectory in $\mathcal{A}_1$. By Corollary 2, $S(t)$, $e(t,0)$ and $i(t,0)$ are bounded away from 0. Futhermore, by applying Corollary 1 and then Proposition 2, we can also find upper bounds for $S(t)$, $e(t,0)$ and $i(t,0)$.

Thus, there exists $\bar{g} > 0$ such that $0 \leq g(y) < \bar{g}$ for $y = \frac{S(t)}{S^*}, \frac{e(t,0)}{e^*(0)}, \frac{i(t,0)}{i^*(0)}$ for any $t \in \mathbb{R}$. Also, since $\frac{e(t,a)}{e^*(a)} = \frac{e(t-a,0)\Omega(a)}{e^*(0)\Omega(a)} = \frac{e(t-a,0)}{e^*(0)}$, and similarly, $\frac{i(t,a)}{i^*(a)} = \frac{i(t-a,0)}{i^*(0)}$, we see that $0 \leq g(y) < \bar{g}$ for $y = \frac{e(t,a)}{e^*(a)}, \frac{i(t,a)}{i^*(a)}$ for any $t \in \mathbb{R}$, $a \in \mathbb{R}_{\geq 0}$.

Let

$$\alpha_e(a) = \int_a^{\infty} \gamma(\sigma)e^*(\sigma)d\sigma \qquad \text{and} \qquad \alpha_i(a) = \int_a^{\infty} \beta(\sigma)i^*(\sigma)d\sigma.$$

Then by using the essential upper bounds for γ and β, and the expressions for the equilibrium coordinates e^* and i^*, it can easily be shown that $\alpha_e(a)$ and $\alpha_i(a)$ are each bounded above by a multiple of the decaying exponential $\mathrm{e}^{-\mu_0 a}$. Thus, the Lyapunov functional, which we define now, is bounded on the solution $X(\cdot)$. Let

$$V(t) = V_S(t) + V_e(t) + V_i(t), \qquad \text{with} \qquad \begin{aligned} V_S(t) &= g\left(\frac{S(t)}{S^*}\right) \\ V_e(t) &= B\int_0^{\infty} \alpha_e(a)g\left(\frac{e(t,a)}{e^*(a)}\right)da \\ V_i(t) &= \int_0^{\infty} \alpha_i(a)g\left(\frac{i(t,a)}{i^*(a)}\right)da, \end{aligned}$$

where B is given by Equation (5).

We now work to show that $\frac{dV}{dt}$ is non-positive. For clarity, we first find the derivatives of V_S, V_e and V_i individually, before combining. We begin with $\frac{dV_S}{dt}$:

$$\begin{aligned}\frac{dV_S}{dt} &= \frac{1}{S^*}\left(1-\frac{S^*}{S}\right)\left[\Lambda - \mu_S S - \int_0^\infty \beta(a) S i(t,a) da\right] \\ &= \frac{1}{S^*}\left(1-\frac{S^*}{S}\right)\left[\mu_S(S^*-S) + \int_0^\infty \beta(a) S^* i^*(a)\left(1-\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\right) da\right] \\ &= -\mu_S\frac{(S-S^*)^2}{SS^*} + \int_0^\infty \beta(a) i^*(a)\left[1-\frac{S^*}{S}-\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}+\frac{i(t,a)}{i^*(a)}\right] da.\end{aligned} \tag{26}$$

Next, we calculate $\frac{dV_e}{dt}$. Lemma 9.4 implies

$$\begin{aligned}\frac{dV_e}{dt} &= B\int_0^\infty \gamma(a) e^*(a)\left[g\left(\frac{e(t,0)}{e^*(0)}\right) - g\left(\frac{e(t,a)}{e^*(a)}\right)\right] da \\ &= B\int_0^\infty \gamma(a) e^*(a)\left[\frac{e(t,0)}{e^*(0)} - \frac{e(t,a)}{e^*(a)} + \ln\left(\frac{e(t,a)}{e^*(a)}\right) - \ln\left(\frac{e(t,0)}{e^*(0)}\right)\right] da.\end{aligned} \tag{27}$$

Similarly

$$\begin{aligned}\frac{dV_i}{dt} &= \int_0^\infty \beta(a) i^*(a)\left[g\left(\frac{i(t,0)}{i^*(0)}\right) - g\left(\frac{i(t,a)}{i^*(a)}\right)\right] da \\ &= \int_0^\infty \beta(a) i^*(a)\left[\frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)} + \ln\left(\frac{i(t,a)}{i^*(a)}\right) - \ln\left(\frac{i(t,0)}{i^*(0)}\right)\right] da.\end{aligned} \tag{28}$$

Combining Equations (26), (27) and (28), we get

$$\begin{aligned}\frac{dV}{dt} &= -\mu_S\frac{(S-S^*)^2}{SS^*} \\ &+ \int_0^\infty \beta(a) i^*(a)\left[1-\frac{S^*}{S}-\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}+\frac{i(t,0)}{i^*(0)}+\ln\left(\frac{i(t,a)}{i^*(a)}\right)-\ln\left(\frac{i(t,0)}{i^*(0)}\right)\right] da \\ &+ B\int_0^\infty \gamma(a) e^*(a)\left[\frac{e(t,0)}{e^*(0)} - \frac{e(t,a)}{e^*(a)} + \ln\left(\frac{e(t,a)}{e^*(a)}\right) - \ln\left(\frac{e(t,0)}{e^*(0)}\right)\right] da.\end{aligned} \tag{29}$$

We now use Lemma 9.2 and Lemma 9.3 to replace the appropriate term in each integral of Equation (29) with a different term:

$$\begin{aligned}\frac{dV}{dt} &= -\mu_S\frac{(S-S^*)^2}{SS^*} \\ &\quad + \int_0^\infty \beta(a) i^*(a)\left[1-\frac{S^*}{S}-\frac{e(t,0)}{e^*(0)}+\frac{i(t,0)}{i^*(0)}+\ln\left(\frac{i(t,a)}{i^*(a)}\right)-\ln\left(\frac{i(t,0)}{i^*(0)}\right)\right] da \\ &\quad + B\int_0^\infty \gamma(a) e^*(a)\left[\frac{e(t,0)}{e^*(0)} - \frac{i(t,0)}{i^*(0)} + \ln\left(\frac{e(t,a)}{e^*(a)}\right) - \ln\left(\frac{e(t,0)}{e^*(0)}\right)\right] da.\end{aligned} \tag{30}$$

Next, we note that for an H that does not depend on a, we have

$$\begin{aligned}\int_0^\infty \beta(a) i^*(a) H da &= H i^*(0)\int_0^\infty \beta(a)\Gamma(a) da \\ &= H i^*(0) B \\ &= B\int_0^\infty \gamma(a) e^*(a) H da.\end{aligned} \tag{31}$$

This allows terms that are independent of a to be moved from one integral in Equation (30) to the other. We now use Equation (31) with $H = \frac{e(t,0)}{e^*(0)} - \frac{i(t,0)}{i^*(0)}$ to cancel terms, obtaining

$$\frac{dV}{dt} = -\mu_S \frac{(S-S^*)^2}{SS^*} + \int_0^\infty \beta(a) i^*(a) \left[1 - \frac{S^*}{S} + \ln\left(\frac{i(t,a)}{i^*(a)}\right) - \ln\left(\frac{i(t,0)}{i^*(0)}\right)\right] da$$
$$+ B\int_0^\infty \gamma(a) e^*(a) \left[\ln\left(\frac{e(t,a)}{e^*(a)}\right) - \ln\left(\frac{e(t,0)}{e^*(0)}\right)\right] da.$$

Next use Equation (31) to move each of the terms $\ln\left(\frac{i(t,0)}{i^*(0)}\right)$ and $\ln\left(\frac{e(t,0)}{e^*(0)}\right)$ to the other integral, obtaining

$$\frac{dV}{dt} = -\mu_S \frac{(S-S^*)^2}{SS^*} + \int_0^\infty \beta(a) i^*(a) \left[1 - \frac{S^*}{S} + \ln\left(\frac{i(t,a)}{i^*(a)}\right) - \ln\left(\frac{e(t,0)}{e^*(0)}\right)\right] da$$
$$+ B\int_0^\infty \gamma(a) e^*(a) \left[\ln\left(\frac{e(t,a)}{e^*(a)}\right) - \ln\left(\frac{i(t,0)}{i^*(0)}\right)\right] da.$$

Multiplying Equation (24) by $\frac{e^*(0)}{e(t,0)}$, we see that $\left(1 - \frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)}\right)$ can be added inside the first integral without changing the value. Similarly, multiplying Equation (25) by $\frac{i^*(0)}{i(t,0)}$, we see that $\left(1 - \frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)}\right)$ can be added inside the second integral. Also, in the first integral we add and subtract $\ln \frac{S^*}{S}$, and use properties of logarithms to combine terms, finding

$$\begin{aligned}
\frac{dV}{dt} =& -\mu_S \frac{(S-S^*)^2}{SS^*} \\
&+ \int_0^\infty \beta(a) i^*(a) \left[\left(1 - \frac{S^*}{S} + \ln\frac{S^*}{S}\right)\right. \\
&\qquad \left. + \left(1 - \frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)} + \ln\left(\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)}\right)\right)\right] da \\
&+ B\int_0^\infty \gamma(a) e^*(a) \left[1 - \frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)} + \ln\left(\frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)}\right)\right] da \\
=& -\mu_S \frac{(S-S^*)^2}{SS^*} - \int_0^\infty \beta(a) i^*(a) \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)}\right)\right] da \\
&\qquad - B\int_0^\infty \gamma(a) e^*(a)\, g\left(\frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)}\right) da.
\end{aligned}$$

Since g is non-negative, it follows that $\frac{dV}{dt} \le 0$ and therefore V is non-increasing. Thus, since V was bounded on $X(\cdot)$, the alpha limit set of $X(\cdot)$ must be contained in $\mathcal{M}$, the largest invariant subset of $\left\{\frac{dV}{dt} = 0\right\}$.

We now determine $\mathcal{M}$. In order to have $\frac{dV}{dt}$ equal to zero it is necessary to have $S = S^*$. Thus, at each point in $\mathcal{M}$, we have $S = S^*$ and therefore $\frac{dS}{dt} = 0$ in $\mathcal{M}$. This implies

$$0 = \Lambda - \mu_S S^* - S^* \int_0^\infty \beta(a) i(t,a) da$$

for all t, which can only happen if $\int_0^\infty \beta(a) i(t,a) da = \int_0^\infty \beta(a) i^*(a) da$ for all t. Combining this with the boundary condition given in Equation (2), we see that $e(t,0) \equiv e^*(0)$ and so $e(t,a) = e^*(a)$ for all t and a. This, using Equation (2), implies $i(t,0) \equiv i^*(0)$ and so $i(t,a) = i^*(a)$ for all t and a. Thus, we may conclude that $\mathcal{M} = \{E^*\}$.

Thus, the alpha limit set of $X(\cdot)$ consists of just the endemic equilibrium E^*, and therefore $V(X(t)) \leq V(E^*)$ for all $t \in \mathbb{R}$. Noting that E^* is the point in $\mathcal{Y}$ that minimizes V, it follows that $X(t) \equiv E^*$. That is, $\mathcal{A}_1 = \{E^*\}$.

The remaining statement of the theorem follows from the definition of $\mathcal{A}_1$ as the persistence attractor (see Theorem 8.3). □

Corollary 3. *If $\mathcal{R}_0 > 1$, then the attractor $\mathcal{A}$ consists of the disease-free equilibrium, which is unstable, the endemic equilibrium, which is stable, and heteroclinic connectors between the two equilibria.*

10. **Special cases. Example 1: ODE.** Suppose $\beta(a) \equiv \beta$, $\gamma(a) \equiv \gamma$, $\mu(a) \equiv \mu$ and $\nu(a) \equiv \nu$ for some $\beta, \gamma, \mu, \nu > 0$. Let $E(t) = \int_0^\infty e(t,a)da$ and $I(t) = \int_0^\infty i(t,a)da$. Then Equation (1) becomes

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \mu_0 S - \beta SI \\
\frac{dE}{dt} &= \beta SI - (\gamma + \mu) E \\
\frac{dI}{dt} &= \gamma E - \nu I.
\end{aligned}$$

The global behaviour of this system was resolved in [13] using compound matrix techniques, and again in [12] using a Lyapunov function.

Example 2: Age-structure for infecteds. Suppose $\gamma(a) \equiv \gamma$ and $\mu(a) \equiv \mu$ for some $\gamma, \mu > 0$. Let $E(t) = \int_0^\infty e(t,a)da$. Then Equation (1) becomes

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \mu_0 S(t) - S(t) \int_0^\infty \beta(a) i(t,a) da \\
\frac{dE(t)}{dt} &= S(t) \int_0^\infty \beta(a) i(t,a) da - (\gamma + \mu) E(t) \\
\frac{\partial i}{\partial t} + \frac{\partial i}{\partial a} &= -\nu(a) i(t,a),
\end{aligned}$$

with boundary condition

$$i(t,0) = \gamma E(t)$$

for $t \geq 0$. A special case of this model with ν constant was presented and studied in [24], with the global analysis being completed in [19]. A similar model with a finite upper bound on the integrals was studied in [23]. It follows from Theorem 7.1 and Theorem 9.5 that the same behaviour holds for non-constant ν as well.

Example 3: Non-exponential distribution of waiting times in the latent class. Suppose $\beta(a) \equiv \beta$ and $\nu(a) \equiv \nu$ for some $\beta, \nu > 0$. Let $I(t) = \int_0^\infty i(t,a)da$.

Then Equation (1) becomes

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda - \mu_S S(t) - \beta S(t)I(t)\\ \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} &= -\left(\gamma(a) + \mu(a)\right)e(t,a)\\ \frac{dI(t)}{dt} &= \int_0^\infty \gamma(a)e(t,a)da - \nu I(t),\end{aligned}$$

with boundary condition

$$e(t,0) = \beta S(t)I(t)$$

for $t \geq 0$. The dynamics of this system are determined by the value of $\mathcal{R}_0$, as described by Theorem 7.1 and Theorem 9.5.

A model of this form would seem appropriate for tuberculosis where the disease often remains latent for an extended period, but the activation rate appears to decline over time [1]. The function $\Omega(a) = \mathrm{e}^{-\int_0^a (\gamma(\sigma)+\mu(\sigma))d\sigma}$ gives the fraction of infected individuals that are still latently infected a time units later. Thus, detailed knowledge of the distribution of latency durations can be explicitly included in the model.

11. **Discussion.** We stress that the model here is distinct from the SI model studied in [15]. One may wonder if an SI model with age-structure for the infected population with appropriately chosen parameters is equivalent to the general model studied here. This is not the case. This can be seen most readily by considering a population in which all individuals that are actively infectious have been detected and removed or quarantined, but latently infected individuals remain. This corresponds to an initial condition with $\varphi_e \neq \mathbf{0} = \varphi_i$, and according to Theorem 9.5 the semi-trajectory tends to the endemic equilibrium. In this situation, for any $\varphi_e \neq \mathbf{0}$ we would have $e(0,0) = 0$. Thus, quite reasonably, a quarantine would appear to at least slow down an outbreak. A model that combines the latently and actively infected into a single age-structured population, will not capture this, as it would predict that latently infected individuals left out of the quarantine could cause new infections immediately. In this latter case, a quarantine would not seem as effective in slowing an outbreak. For a disease such as tuberculosis, where the detection of infectious individuals is easier than the detection of latently infected individuals, this distinction is important.

The model studied in this paper is particularly good for diseases such as tuberculosis where there is a clear biological difference between individuals that are exposed or latent, and those that are actively infectious.

As presented here, it is necessary that the coefficient functions β and γ be Lipschitz continuous. This allows the initial conditions for e and i to be taken in L^1_+. Then, the functions J and L, related to the boundary conditions $e(t,0)$ and $i(t,0)$, can be shown to be Lipschitz continuous (see Proposition 5). This is necessary to show that the semi-flow is asymptotically smooth (see Theorem 5.3).

Alternatively, one may assume less regularity in β and γ, and more regularity in the initial conditions. For example, in order to obtain discrete delay equations, one chooses β or γ or both to be step functions, which are not Lipschtz or even continuous. For these particular cases, the phase space must be chosen differently. Let $\widetilde{C} = L^1_+ \cap L^\infty$. By taking $e(0,\cdot), i(0,\cdot) \in \widetilde{C}$, the phase space $\widetilde{\mathcal{Y}} = \mathbb{R}_{\geq 0} \times \widetilde{C} \times \widetilde{C}$ is positively invariant and the functions J and L are Lipschitz continuous; asymptotic smoothness follows.

The next two examples are cases where the age-structure model reduces to delay models.

Example 4: Two delays for the infecteds. Suppose $\gamma(a) \equiv \gamma$ and $\mu(a) \equiv \mu$ for some $\gamma, \mu > 0$. Also, suppose

$$\beta(a) = \begin{cases} 0 & \text{if } 0 \leq a < \tau_\beta \\ \beta & \text{if } \tau_\beta < a \end{cases}$$

and

$$\nu(a) = \begin{cases} \eta & \text{if } 0 \leq a < \tau_\nu \\ \nu & \text{if } \tau_\nu < a. \end{cases}$$

for some $\beta, \eta, \nu > 0$ and $0 < \tau_\beta < \tau_\nu$. This is a further specialization of Example 2.

Let $E(t) = \int_0^\infty e(t,a)da$, $I_1 = \int_{\tau_\beta}^{\tau_\nu} i(t,a)da$ and $I_2 = \int_{\tau_\nu}^\infty i(t,a)da$. Then Equation (1) becomes

$$\begin{aligned} \frac{dS(t)}{dt} &= \Lambda - \mu_0 S(t) - \beta S\left(I_1(t) + I_2(t)\right) \\ \frac{dE(t)}{dt} &= \beta S(t)\left(I_1(t) + I_2(t)\right) - \left(\gamma + \mu\right) E(t) \\ \frac{dI_1(t)}{dt} &= \gamma \mathrm{e}^{-\eta\tau_\beta} E(t - \tau_\beta) - \gamma \mathrm{e}^{-\eta\tau_\nu} E(t - \tau_\nu) - \eta I_1(t) \\ \frac{dI_2(t)}{dt} &= \gamma \mathrm{e}^{-\eta\tau_\nu} E(t - \tau_\nu) - \nu I_2(t). \end{aligned}$$

The dynamics of this system are determined by the value of $\mathcal{R}_0$, as described by Theorem 7.1 and Theorem 9.5.

Example 5: Letting τ_β be greater than τ_ν. Suppose the functions γ, μ, β, ν and E are as given in Example 4. Now, suppose $0 < \tau_\nu \leq \tau_\beta$. Let $I = \int_{\tau_\beta}^\infty i(t,a)da$. Then Equation (1) becomes

$$\begin{aligned} \frac{dS(t)}{dt} &= \Lambda - \mu_0 S(t) - \beta S I(t) \\ \frac{dE(t)}{dt} &= \beta S(t) I(t) - \left(\gamma + \mu\right) E(t) \\ \frac{dI(t)}{dt} &= \gamma_0 E(t - \tau_\beta) - \nu I(t), \end{aligned}$$

where $\gamma_0 = \gamma \mathrm{e}^{-(\eta\tau_\nu + \nu(\tau_\beta - \tau_\nu))}$. The dynamics of this system are determined by the value of $\mathcal{R}_0$, as described by Theorem 7.1 and Theorem 9.5.

It is interesting to note that the difference in the differential equations for Examples 4 and 5 is brought about through a simple change in the sign of $\tau_\beta - \tau_\nu$. Biologically, this change relates to whether the onset of disease symptoms is before or after the time when individuals become infectious.

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