MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **9**, Number **3**, July **2012**

doi:10.3934/mbe.2012.9.577

pp. **577–599**

MULTIPLE ENDEMIC STATES IN AGE-STRUCTURED *SIR* EPIDEMIC MODELS

Andrea Franceschetti and Andrea Pugliese

Dept. Mathematics, Università di Trento
Via Sommarive 14, 38123 Povo (TN), Italy

Dimitri Breda

Dept. Mathematics and Computer Sciences, Università di Udine
Via delle Scienze 206, 33100 Udine, Italy

(Communicated by Azmy S. Ackleh)

Abstract. *SIR* age-structured models are very often used as a basic model of epidemic spread. Yet, their behaviour, under generic assumptions on contact rates between different age classes, is not completely known, and, in the most detailed analysis so far, Inaba (1990) was able to prove uniqueness of the endemic equilibrium only under a rather restrictive condition.

Here, we show an example in the form of a 3×3 contact matrix in which multiple non-trivial steady states exist. This instance of non-uniqueness of positive equilibria differs from most existing ones for epidemic models, since it arises not from a backward transcritical bifurcation at the disease free equilibrium, but through two saddle-node bifurcations of the positive equilibrium. The dynamical behaviour of the model is analysed numerically around the range where multiple endemic equilibria exist; many other features are shown to occur, from coexistence of multiple attractive periodic solutions, some with extremely long period, to quasi-periodic and chaotic attractors.

It is also shown that, if the contact rates are in the form of a 2×2 WAIFW matrix, uniqueness of non-trivial steady states always holds, so that 3 is the minimum dimension of the contact matrix to allow for multiple endemic equilibria.

1. Introduction.

SIR epidemic models are at the basis of research aiming at predicting the epidemic dynamics and assessing control measures for many kinds of infectious diseases, from measles, chickenpox and other childhood diseases, to influenza, and to emerging diseases. Either the models are exactly age–sructured (generally with discrete age) *SIR* models or some variants of that, allowing for incubation periods or other complications.

A crucial point in the application of these models is the estimation of an adequate kernel that describes infection transmission between age classes. An approach going back to Anderson and May [2] is to divide ages into few (n) groups, so as to use a $n \times n$ matrix (often named WAIFW, "Who Acquires Infection From Whom") to model the transmission between (and within) groups. This is the approach currently

2000 *Mathematics Subject Classification.* Primary: 92D30; Secondary: 45G15, 65P30.

Key words and phrases. Age-structured epidemic model, multiple endemic equilibria, periodic and chaotic solutions, numerical bifurcation analysis, fixed point index.

AP acknowledges the partial support of PRIN-MIUR Grant 200777BWEP_001.

used in many publications, and several methods have been used for the estimation of the WAIFW matrix, one of which [17] has required an extensive survey of daily individual contacts in several European countries.

In contrast with the steady application of this type of models, lies the incomplete theoretical analysis of their dynamic behaviour. To date, the most general results have been obtained by Inaba [15] who, defining R_0 as the spectral radius of an appropriate operator, proved that for $R_0 < 1$ the disease–free equilibrium is asymptotically stable and there are no positive stationary solutions, while for $R_0 > 1$, the disease–free equilibrium is unstable and there exists at least one positive stationary solution.

If transmission follows 'separable mixing' (precisely defined below), the conclusion is sharper [7]: for $R_0 > 1$ there exists a unique positive stationary solution. The same conclusion has been reached by Inaba [15] under hypothesis (H4) (see Section 3), which is however unlikely to hold for empirical transmission kernels. Thus, the conjecture raised by Greenhalgh [11] that for $R_0 > 1$ there exists a unique positive stationary solution, is still open in general. In this paper, we show that the conjecture is false by exhibiting a WAIFW matrix such that the age-structured SIR model with that matrix as transmission kernel may have up to 3 positive equilibria for appropriate parameter values.

More generally, we show a method to build WAIFW matrices with this property; in the way, we prove that a 2×2 WAIFW matrix always leads to uniqueness of the endemic equilibrium when $R_0 > 1$, while one obtain non-uniqueness with a different 2×2 matrix-like structure with infectivity increasing with age.

A related issue concerns stability of the equilibria; while it is well known that the disease-free equilibrium is stable for $R_0 < 1$ and unstable for $R_0 > 1$, little is known about the stability of the endemic equilibria, except that general principles about transcritical bifurcations guarantee that it will be asymptotically stable for $R_0 > 1$ sufficiently close to 1. Thieme [18] showed that an endemic steady state could be unstable, even with proportionate mixing. Andreasen [3] studied this problem through perturbation methods under the assumption that the infectious period is small compared to the individuals life span: he proved [3] that if the disease transmission is independent of age then an endemic equilibrium is always locally stable, while the introduction of age-dependent susceptibility may lead to loss of local stability [4].

The numerical bifurcation study (based on the method in [5]) proposed in the final section shows that much more complex bifurcation patterns are possible for this type of model. This contrasts with the invariable convergence to the endemic equilibrium occurring in all simulations (we are aware of) aiming at reproducing empirical patterns. It is possible that the structure of realistic WAIFW matrices is such that, for $R_0 > 1$, the endemic equilibrium is always unique and globally stable. We hope that the current study helps in elucidating this feature.

2. The model.

The population is divided into three compartments, the susceptible, infective and immune individuals; as usual [1, 11, 15], we assume that the total population is in a demographical stationary state. $n(a)$ is the stationary age density of the total population, i.e. a stationary positive solution of the system

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) n(a,t) &= -\mu(a) n(a,t)\ ,\ 0 < a < \omega\ ,\ t > 0 \\ n(0,t) &= \int_0^\omega b(a) n(a,t) da \end{cases} \tag{1}$$

where $b(a)$ is the fertility rate, $\mu(a)$ is the death rate and a maximal age ω is assumed; the assumption of demographical stationarity is equivalent ([14], chap.1) to the condition

$$\int_0^\omega b(a)\pi(a)\, da = 1\,,$$

where

$$\pi(a) = \exp\{-\int_0^a \mu(s)\, ds\}$$

represents the probability of surviving up to age a.

We do not delve into the technical conditions [14] on the functions b and μ, because they are not actually needed for the analysis of the epidemic models. Note only that $n(a) = K\pi(a)$ for some constant $K > 0$.

Let now $S(a,t)$, $I(a,t)$, $R(a,t)$ be the age densities at time t of the susceptible, infective and immune individuals, respectively. Susceptible may become infected at rate $\lambda(a,t)$ ("force of infection"); infectives become removed at rate γ. Considering also deaths (at rate $\mu(a)$), the equations satisfied by S, I and R are

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) S &= -(\lambda(a,t) + \mu(a))S \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) I &= \lambda(a,t)S - (\mu(a)+\gamma)I \qquad 0 < a < \omega\,,\, t > 0 \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) R &= \gamma I - \mu(a)R\,. \end{cases} \tag{2}$$

Initial and boundary conditions have to be added to the equations. Neglecting transient maternal immunity, it is assumed that all newborns are susceptible, namely

$$S(0,t) = \int_0^\omega b(a)n(a)da\ ,\ I(0,t) = 0\ ,\ R(0,t) = 0 \quad t \geq 0. \tag{3}$$

In (3) it has been implicitly assumed $S(a,t) + I(a,t) + R(a,t) \equiv n(a)$. This is possible, since the sum $S(a,t) + I(a,t) + R(a,t)$ satisfies equation (1), as long as the initial conditions satisfy

$$S(a,0) + I(a,o) + R(a,0) = K\pi(a) \qquad \text{for some } K > 0$$

and then $n(a) = K\pi(a)$.

Under these assumptions, it is convenient to change variables to the fractions $x(a,t) = S(a,t)/n(a)$ and $y(a,t) = I(a,t)/n(a)$ (it is not necessary to consider also $z(a,t) = R(a,t)/n(a)$, since it can be obtained as $z(a,t) = 1 - x(a,t) - y(a,t)$). The system of equations satisfied by x and y is [15, p. 414]

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) x(a,t) = -\lambda(a,t)x(a,t) & \\ & 0 < a < \omega\,,\, t > 0 \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y(a,t) = \lambda(a,t)x(a,t) - \gamma y(a,t) & \\ x(0,t) = 1\ , \qquad y(0,t) = 0\ , & t > 0 \end{cases} \tag{4}$$

with initial conditions

$$x(a,0) = x_0(a), \quad y(a,0) = y_0(a)\,, \quad 0 < a < \omega\,,$$

with $x_0(a),\ y_0(a) \geq 0$ and $x_0(a) + y_0(a) \leq 1$.

To complete model (4), one still has to assign a constitutive rule for the force of infection $\lambda(a,t)$. The standard assumption ([2], chap.9.2) is of a general linear functional on infective density

$$\lambda(a,t) = \int_0^\omega \beta(a,s)I(s,t)\, ds = \int_0^\omega \beta(a,s)n(s)y(s,t)\, ds \tag{5}$$

with $\beta(a, a')$ the contact coefficient between susceptible individuals aged a and the infectives aged a'.

3. **Stationary solutions.** A first problem concerning system (4)–(5) is the existence of stationary solutions (equilibria). For any values of parameters and functions involved, there exists the disease-free equilibrium $(x(a), y(a)) \equiv (1, 0)$. Concerning the existence of other equilibria, Greenhalgh [11] showed the equivalence of the problem of existence and uniqueness of the steady states of system (2) with the problem of existence and uniqueness of the solutions of the following nonlinear integral equation

$$\lambda(a) = \int_0^\omega \lambda(\sigma) e^{-\int_0^\sigma \lambda(\tau)d\tau} \phi(a,\sigma) d\sigma \,, \tag{6}$$

where the kernel $\phi(\cdot,\cdot)$ is given as

$$\phi(a,\sigma) = \int_\sigma^\omega \beta(a,\xi) n(\xi) e^{-\gamma(\xi-\sigma)} d\xi \,. \tag{7}$$

The most general result to date concerning this problem has been obtained by Inaba [15] under the following assumptions about the contact coefficient $\beta(\cdot,\cdot)$.

(H1) $\beta(a,\xi) \in L_+^\infty\big((0,\omega)\times(0,\omega)\big)$.

(H2) $\lim\limits_{h\to 0} \int_0^\omega |\beta(a+h,\xi) - \beta(a,\xi)|\, da = 0$ uniformly for $\xi \in \mathbb{R}$, where β is extended by $\beta(a,\xi) = 0$ for $a, \xi \in (-\infty, 0) \cup (\omega, +\infty)$.

(H3) There exist numbers α with $\omega > \alpha > 0$ and $\epsilon > 0$ such that $\beta(a,\xi) \geq \epsilon$ for a.e. $(a,\xi) \in (0,\omega) \times (\omega - \alpha, \omega)$.

He considered the positive nonlinear operator $\Phi : L_+^1(0,\omega) \longrightarrow L_+^1(0,\omega)$ defined by

$$(\Phi\psi)(a) = \int_0^\omega \psi(\sigma) e^{-\int_0^\sigma \psi(s)ds} \phi(a,\sigma) d\sigma \,,\ a \in (0,\omega) \tag{8}$$

with $\phi(\cdot,\cdot)$ given by (7). Solutions of (6) can be written as solutions ψ of the nonlinear equation $\Phi\psi = \psi$, i.e. fixed points of the operator Φ.

Let finally $T = \Phi'(0)$ be the positive linear operator given by the Frechet derivative of Φ in zero, $r(T)$ the spectral radius of T. Then

Theorem 1. *(Inaba,1990) Assume (H1)–(H3); then*

1. *if $r(T) \leq 1$ then the only non-negative fixed point of Φ is $\psi = 0$;*
2. *if $r(T) > 1$, Φ has at least one non-zero fixed point (a positive solution of the equation $\Phi\psi = \psi$).*

To prove uniqueness, Inaba added the assumption

(H4) For all $(a,\sigma) \in [0,\omega]\times[0,\omega]$ the inequality $\beta(a,s)n(s) - \gamma\phi(a,s) \geq 0$ holds.

He proved, using monotonicity methods, that under (H4) there exists at most one non-trivial positive solution of equation (6) [15, Prop. 4.10]; hence in case $r(T) > 1$ there exists one and only one non-trivial positive solution under Assumptions (H1)–(H4).

Another assumption under which is easy to prove uniqueness of positive solutions is the so-called separable mixing $\beta(a,s) = \varphi(a)q(s)$. In that case equation (6) can be reduced to a one-dimensional problem, and $r(T)$ can be explicitly computed.

The problem arises of understanding whether or not uniqueness of positive solutions of (6) holds in general. We will consider especially contacts that take the form of WAIFW matrices [2]; age is divided into n age groups $I_j = (\alpha_{j-1}, \alpha_j)$, $j = 1 \dots n$ where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \omega$. Then

$$\beta(a, a') = \sum_{i,j=1}^{n} \beta_{ij} \mathbf{1}_{I_i}(a) \mathbf{1}_{I_j}(a') \tag{9}$$

where $\beta_{ij} \geq 0 \; \forall \, i, j = 1, \dots, n$.

In this case it is not difficult to see that, if $\psi \in L^1_+(0, \omega)$ is a fixed point of Φ, it must be $\psi = \sum_{i=1}^n x_i \mathbf{1}_{I_i}$ with $(x_1, \dots, x_n) \in \mathbb{R}^n_+$. Formally, one can define a map $J : \mathbb{R}^n \to L^1$ by

$$J(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \mathbf{1}_{I_i}$$

so that $\mathrm{Im}(\Phi) \subset \mathrm{Im}(J)$ and a map $\tilde{\Phi} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that $J \circ \tilde{\Phi} = \Phi \circ J$.

If $\bar{x}$ is a fixed point of $\tilde{\Phi}$, it is immediate that $J(\bar{x})$ is a fixed point of Φ. A similar reduction can be reached in the more general case of finite-dimensional mixing

$$\beta(a, s) = \sum_{j=1}^{n} \varphi_j(a) q_j(s)$$

for arbitrary functions φ_j and q_j; this case will not be dealt with in this paper, although similar methods would apply.

Through simple computations, the map $\tilde{\Phi}$ can be written explicitly as

$$\tilde{\Phi}_i(x) = \sum_{j=1}^{n} \beta_{ij} \, g_j(x) \; , \; i = 1, \dots, n \tag{10}$$

where

$$g_1(x) = x_1 \int_0^{\alpha_1} n(\xi) \int_0^{\xi} e^{-\gamma(\xi - \tau)} e^{-\tau x_1} d\tau \, d\xi$$

$$g_j(x) = \sum_{k=1}^{j-1} x_k \int_{\alpha_{j-1}}^{\alpha_j} n(\xi) \exp\{-\sum_{h=1}^{k-1} x_h(\alpha_h - \alpha_{h-1})\} \int_{\alpha_{k-1}}^{\alpha_k} e^{-\gamma(\xi-\tau) - x_k(\tau - \alpha_{k-1})} d\tau \, d\xi$$

$$+ x_j \int_{\alpha_{j-1}}^{\alpha_j} n(\xi) \exp\{-\sum_{h=1}^{j-1} x_h(\alpha_h - \alpha_{h-1})\} \int_{\alpha_{j-1}}^{\xi} e^{-\gamma(\xi-\tau) - x_j(\tau - \alpha_{j-1})} d\tau \, d\xi \, . \tag{11}$$

It is possible to prove the existence of non-zero fixed points of $\tilde{\Phi}$ under weaker conditions than those of Theorem 1. Indeed, conditions (H1) and (H2) are automatic when (9) holds, and (H3) can be weakened, or dropped altogether allowing for nonnegative fixed points. Precisely, consider the assumption

(H3') $\sum_{j=1}^{n} \beta_{ij} > 0 \quad \forall \; i = 1, \dots, n$ and there exist $m \in \{1 \dots, n-1\}$ and indices $i_1, \dots, i_m, \, j_1, \dots, j_m$ with $i_1 = 1$, $j_m = n$, $i_l \leq j_{l-1}$, $l = 2, \dots m$, such that

$$\beta_{i_1, j_1} \cdot \beta_{i_2, j_2} \cdots \beta_{i_m, j_m} > 0.$$

Theorem 2. *Assume* (9). *Then*

a) *If (H3') and* $r(T) > 1$, *there exists a positive fixed point of* $\tilde{\Phi}$, *hence of* Φ.

b) *If* $r(T) > 1$*, there exists a nonnegative fixed point of* $\tilde{\Phi}$*, hence of* Φ*.*

The proof is in Appendix A where it is shown that (H3') is sufficient to exclude existence of non-trivial fixed points of $\tilde{\Phi}$ on the boundary of $\mathbb{R}^n_+$. The first part of (H3') ensures that $\tilde{\Phi}_i(x)$, i.e. the force of infection in each age class I_i , is positive for some $x \in \mathbb{R}^n_+$. The second condition can be described as asking that, if there are some infective individuals in age-class n, eventually the force of infection will be positive in age class 1. Only this condition is needed, because the passage of time ensures that if there are some infective individuals in age class i, there will be infectives in age class j for each $j > i$ at some later time.

It can be seen that (H3') is essentially necessary to exclude existence of fixed points of $\tilde{\Phi}$ on the boundary of $\mathbb{R}^n_+$ in the following sense. Without lack of generality, one can assume that there exists j such that $\beta_{jn} > 0$; otherwise, the first $n-1$ components of $\tilde{\Phi}$ do not depend on x_n and one can restrict $\tilde{\Phi}$ to $\mathbb{R}^{n-1}$ (in biological terms, the n-th age class would be an epidemiological dead-end). Under this assumption, it is not difficult to see that, if (H3') is violated, the system $\tilde{\Phi}(x) = x$ can be split into subsystems, so there exist matrices with the same sign structure that have non-trivial fixed points on the boundary of $\mathbb{R}^n_+$.

In the course of the proof, we prove the following Proposition concerning the fixed point index $i(\tilde{\Phi}, \cdot)$; it will also be used in later Sections, and may be of independent interest.

Proposition 1. *Assume* $r(T) > 1$ *and (H3'). Then there exists a bounded open set* $G \subset \mathbb{R}^n_+$ *such that*

$$i(\tilde{\Phi}, G) = 1 \tag{12}$$

and all fixed points $x \in \mathbb{R}^n_+$*,* $x \neq 0$*, belong to* G*.*

Actually, the proof of Proposition 1 works even in the infinite dimensional case [15], using a generalization of the fixed point index [9, Ch. 20] needed because the positive cone of $L^1(0, \omega)$ has empty interior.

4. **An example of non-uniqueness.**

The example of non-uniqueness of stationary solutions has been obtained choosing coefficients β_{ij} that allowed for drastic simplifications. Specifically, in Appendix B we show the following

Example 1. *Let* $\beta_{ij} = 0$ *for* $(i,j) \notin \big\{(1,3),(2,2),(3,1)\big\}$, $i,j = 1,2,3$, $n(\xi) \equiv 1$ *on* $(0,\omega)$*, and let* $\tilde{\Phi}(\vartheta, x)$ *be the map defined in (10), making it explicit its dependence on all parameters* $\vartheta = (\beta_{13}, \beta_{22}, \beta_{31}, \alpha_1, \alpha_2, \omega, \gamma)$ *beyond* $x = (x_1, x_2, x_3)$*.*

Then there exist values $\vartheta^* \in \mathbb{R}^7_+$ *and* $x^* \in \mathbb{R}^3_+$ *such that*

$$\tilde{\Phi}(\vartheta^*, x^*) = x^* \tag{13}$$

$$\det\Big(I - \tilde{\Phi}_x(\vartheta^*, x^*)\Big) = 0. \tag{14}$$

More precisely, the construction of the example yields values of γ (=1), β_{13} ($\approx 1.8702 \cdot 10^4$), β_{22} (≈ 2.4633), β_{31} (≈ 1.6835), α_1 ($= 2$) such that for each ω large enough there exists α_2 with $\alpha_1 < \alpha_2 < \omega$ such that (13) and (14) are satisfied at $x^* = (1,1,1)$. It is a constructive recipe that, for instance, for $\omega = 100$, yields $\alpha_2 \approx 12.4342$.

Condition (13) says that x^* is a fixed point of $\tilde{\Phi}$ when $\vartheta = \vartheta^*$, while (14) is the necessary condition for saddle-node bifurcations of fixed points of maps. Given a saddle-node bifurcation, one can choose a one-dimensional parameter from ϑ (for instance α_2, which has been used in the numerical computations) and obtain that for α_2 close to α_2^* there exist two fixed points of $\tilde{\Phi}$ close to x^* for $\alpha_2 > \alpha_2^*$ [or $\alpha_2 < \alpha_2^*$] and none for $\alpha_2 < \alpha_2^*$ [or $\alpha_2 > \alpha_2^*$]. This would suffice to prove that, for certain parameter values, equation $x = \tilde{\Phi}(x)$ has multiple positive solutions, hence (4) has multiple positive equilibria.

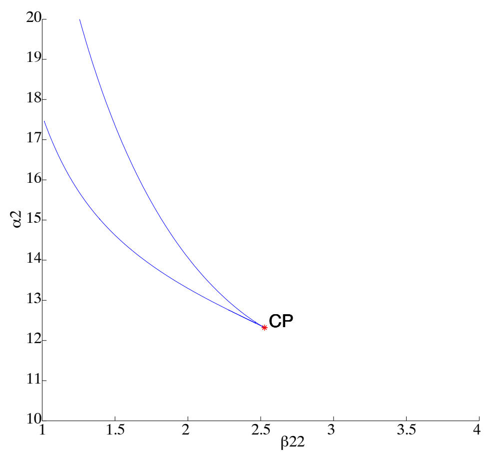

FIGURE 1. The curve in parameter space (β_{22}, α_2) at which the two conditions (13)–(14) occur at some point $x^* \in \mathbb{R}_+^3$. Other parameter values are $\beta_{13} = 1.8702 \cdot 10^4$, $\beta_{31} = 1.6835$, $\alpha_1 = 2$, $\omega = 100$, $\gamma = 1$.

In order to prove the occurrence of a saddle-node bifurcation in (α_2^*, x^*), one would need to check the transversality conditions. This can be performed through very long computations. We are satisfied here with checking them numerically, and showing the bifurcation diagram obtained numerically through the use of MATLAB and MATCONT software (see Figures 1 and 2).

One may note that the choice of β_{ij}'s in Proposition 1 does not satisfy condition (H3) but only (H3'); thus it cannot be technically considered a counter-example to the conjecture of uniqueness of equilibria under (H1)-(H3); from the computations it is clear, however, that by continuity the results would be the same if a small $\varepsilon > 0$ were added to all values of β_{ij}.

5. Numerical results.

First of all, using MATCONT, we computed, starting from the value found analytically $(2.4633, 12.4342)$ the curve of points in the parameter space (β_{22}, α_2) at which the two conditions (13)–(14) occur at some point x^* (Fig. 1). It can be seen that the curve has a cusp around $(2.51, 12.3)$ from which two branches depart for $\beta_{22} < 2.51$: for such values two saddle-node bifurcations occur at (α_2^*, x^*) and (α_2^{**}, x^{**}), which get further apart as β_{22} decreases away from 2.51.

We then looked at a bifurcation diagram in the single parameter α_2, setting $\beta_{22} \approx 1.83$, so that the two saddle-node bifurcation points are visibly away from each other. The overall bifurcation structure (Fig. 2) confirms that for $\alpha_2 \in (\alpha_2^*, \alpha_2^{**})$ equation (6) has actually 3 positive solutions, while only 1 (unless other solution branches exist) for $\alpha_2 < \alpha_2^*$ and $\alpha_2 > \alpha_2^{**}$.

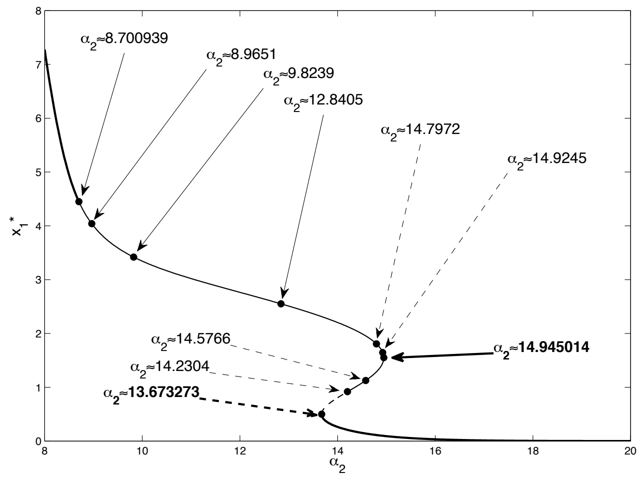

FIGURE 2. Coordinate x_1^* of the stationary positive solutions of system (4) against the value of α_2. Other parameter values are $\beta_{13} \approx 1.8702 \cdot 10^4$, $\beta_{22} \approx 1.83$, $\beta_{31} \approx 1.6835$, $\alpha_1 = 2$, $\omega = 100$, $\gamma = 1$. The points at which roots of the characteristic equation cross the imaginary axis are also shown: moving on the curve from right to left dashed labels denote crossings left-to-right, solid labels denote crossing right-to-left, thick labels are for real crossings, normal labels are for complex crossings. Traits between bifurcation points are shown in different style: thick solid denotes stable equilibrium, dashed unstable equilibrium, solid periodic orbits. More details are in the text.

The stability of the corresponding stationary solutions of (4) has been studied numerically through the approximation of the eigenvalues of the linearisation at equilibria, as depicted in Fig. 2. The technique, proposed in [5] and applied similarly in [6], is based on the reduction to finite dimension of the infinitesimal generator of the semigroup associated to the model. Starting the description from the right, along the lower thick solid curve all eigenvalues have negative real part (and thus the equilibrium is asymptotically stable) for all $\alpha_2 > \alpha_2^* \approx 13.673273$, the point of saddle-node bifurcation (thick dashed label). Proceeding along the middle branch of the curve, in the dashed part there is one real positive eigenvalue, while both at $\alpha_2 \approx 14.2304$ and $\alpha_2 \approx 14.5766$ (dashed labels) a pair of complex conjugate roots crosses the imaginary axis from left to right, thus giving rise through Hopf bifurcations to periodic solutions that inherit the instability of the equilibrium they are bifurcating from (thin solid part). At $\alpha_2 = \alpha_2^{**} \approx 14.945014$ a second saddle-node bifurcation occurs with the real positive eigenvalue becoming negative (thick solid label); the upper solution branch starts however unstable (still thin solid), because two pairs of complex conjugate eigenvalues have positive real part. At $\alpha_2 \approx 14.9245$ and $\alpha_2 \approx 14.7972$ (dashed labels) two further pairs of complex conjugate eigenvalues cross the imaginary axis from left to right; thus for α_2 to the left of 14.7972, the linearisation at the equilibrium on the upper solution branch has four pairs of complex conjugate eigenvalues in the positive half-plane. These move back in the negative half-plane at the bifurcation points at $\alpha_2 \approx 12.8405$, 9.8239, 8.9651 and 8.700939 (solid labels). For $\alpha_2 < 8.70093$ all eigenvalues have negative real part (thick solid curve), so that the equilibrium is asymptotically stable. Looking at the bifurcation diagram in the opposite direction, one sees that at $\alpha_2 = 8.70093$ a Hopf bifurcation occurs with the emergence of a periodic solution that, in the supercrititcal case, would inherit the asymptotical stability. Because of the complexity of the system, a numerical continuation of the branches of periodic solutions and the corresponding analysis of their stability was unfeasible. Instead we approximated the solutions of the PDE system (6) at selected values of α_2, chosen in relation to the diagram of Fig. 2.

Some results have been selected out of many simulations and are shown in Figures 3 and 4 where α_2 increases going from left to right and from top to bottom. As initial conditions, we choose, as described in the legends, either the final conditions of other simulations, or fractions of susceptible and infectives that would be at equilibrium if the force of infection were fixed at (x_1, x_2, x_3). Precisely

$$x(a) = \begin{cases} e^{-x_1 a} & \text{if } 0 < a < \alpha_1 \\ e^{-x_1\alpha_1 - x_2(a-\alpha_1)} & \text{if } \alpha_1 < a < \alpha_2 \\ e^{-x_1\alpha_1 - x_2(\alpha_2-\alpha_1) - x_3(a-\alpha_2)} & \text{if } \alpha_2 < a < \omega \end{cases} \tag{15}$$

$$y(a) = \int_0^a (x_1 \mathbf{1}_{(0,\alpha_1)}(\tau) + x_2 \mathbf{1}_{(\alpha_1,\alpha_2)}(\tau) + x_3 \mathbf{1}_{(\alpha_2,\omega)}(\tau)) e^{-\gamma(a-\tau)} x(\tau)\, d\tau\,.$$

From Fig. 3 it can be seen that, as expected, solutions converge to the (unique) equilibrium for $\alpha_2 = 8.5$ (first panel) and, passing the Hopf bifurcation, to a periodic solution (of period approximately 20 time-units) for $\alpha_2 = 9$ (second panel). Increasing furthermore α_2, the attractive periodic solution acquires a more complex structure (at $\alpha_2 = 11$), then a parameter region (α_2 between 11.59 and 11.67) emerges in which solutions appear chaotic, followed by a region (we show $\alpha_2 = 12$) where solutions converge to a periodic solutions of much shorter period (around 8 time units).

Looking at Fig. 4, obtained with larger values of α_2, one sees that at $\alpha_2 = 13$, two different periodic solutions exist, one of long period (around 50 time units), the other one much shorter (around 9 time units). They appear to coexist for α_2 up to around 13.5, where the period of the long-period solution approaches 80 time-units, while the solution of short period undergoes a Neimarck-Sacker bifurcation, giving rise to a quasi-periodic solution (fourth panel). This solution disappears into a chaotic region (not shown) for α_2 around 13.6, while the long-period solution persists and, at $\alpha_2 = 13.66$, has period around 200 years. Increasing α_2 further, the period lengthens but, as α_2 passes the saddle-node value around 13.675, all solutions appear to converge to the stable equilibrium solution (the lower branch in Fig. 2). The suspicion arises that the stable periodic solution disappears through a homoclinic bifurcation at the same time as the saddle-node bifurcation occurs. The complexity of the system prevents us from reaching definite conclusions in this respect as well as in the apparent transitions to chaos.

6. **A uniqueness result.** The example of non-uniqueness of equilibria of (4) has been obtained assuming that the contact rates have the structure (9) with $n = 3$. Since uniqueness is obvious with $n = 1$, one may wonder whether it is possible to find an example of non-uniqueness with the structure (9) with $n = 2$. The answer is actually negative, as shown below.

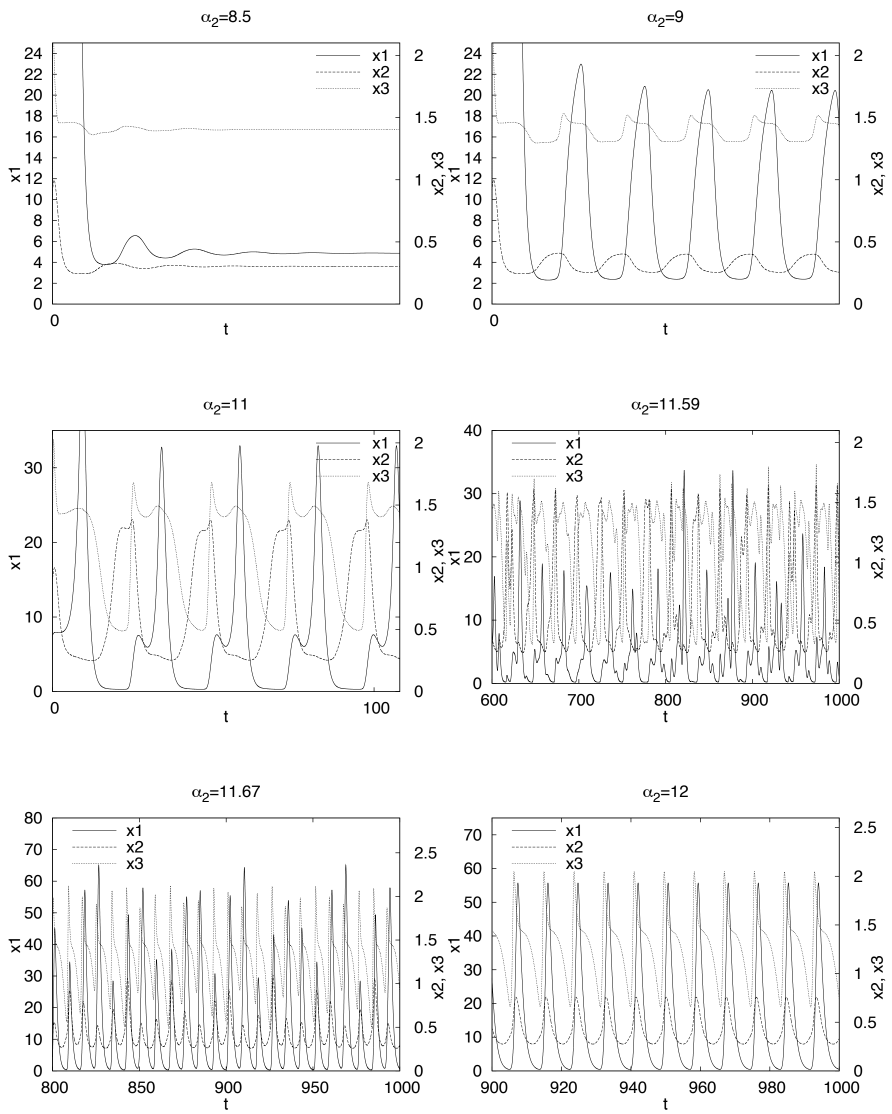

FIGURE 3. The values of $x_1(t) = \beta_{13}\int_{\alpha_2}^{\omega} n(s)y(s,t)\,ds$, $x_2(t) = \beta_{22}\int_{\alpha_1}^{\alpha_2} n(s)y(s,t)\,ds$, $x_3(t) = \beta_{31}\int_{0}^{\alpha_1} n(s)y(s,t)\,ds$ obtained by approximating (4) with a finite-difference scheme. The value of α_2 (from 8.5 to 12) is reported in each panel. All other parameter values as in Fig. 2. The initial conditions were, for the first three panels, the final conditions of the panel in Fig. 4a; in the last three panels, the final conditions of the third panel.

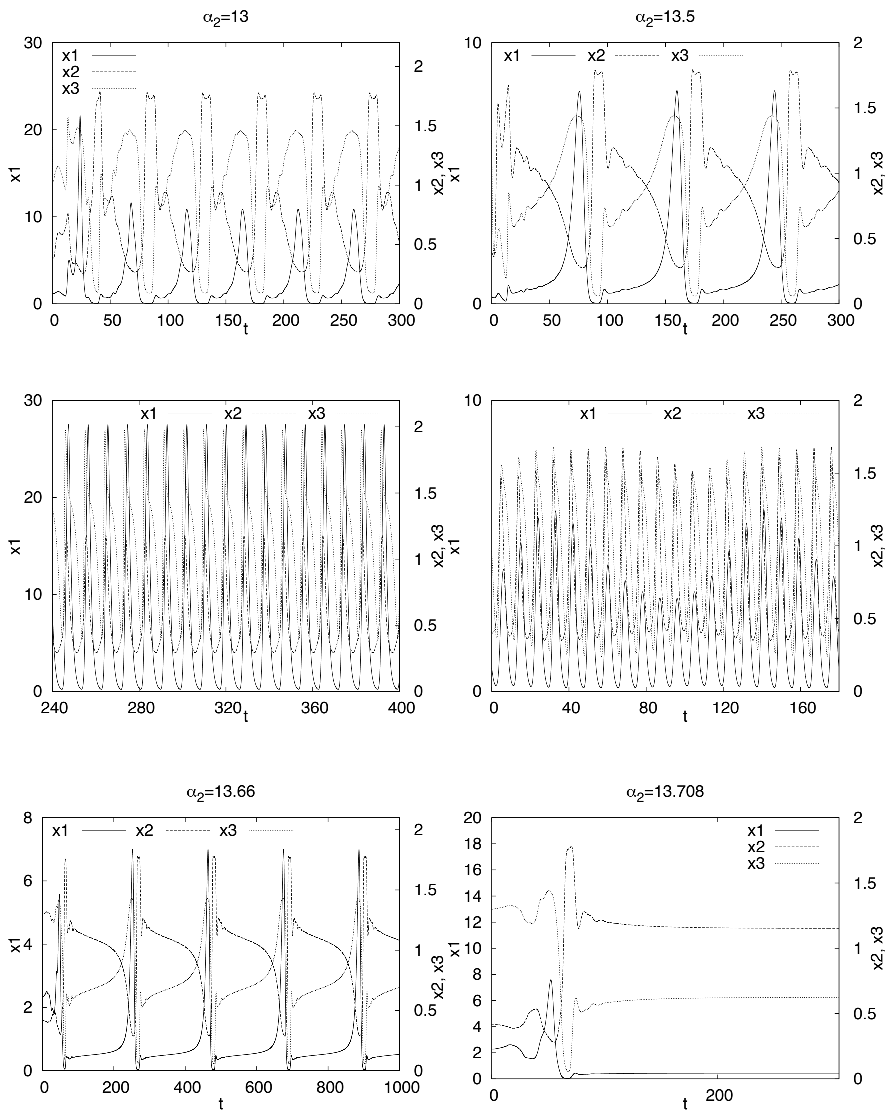

FIGURE 4. Same as Fig. 3 with α_2 between 13 and 13.7. Initial conditions were of the type (15): with, from top to bottom, $(x_1, x_2, x_3) = (2.56, 0.47, 1.44)$, $(2.60, 0.61, 1.55)$, $(2.58, 0.49, 1.46)$, $(2.48, 0.49, 1.43)$ and $(2.302, 0.423, 1.293)$ in the last row.

First note that, assuming (9) with $n = 2$ as contact rates, then the operator equation (6) takes the form $x_i = \tilde{\Phi}_i(x)$, $i = 1, 2$, where $\tilde{\Phi} : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ is the operator defined in (10) with $n = 2$ and that we now rewrite as

$$\tilde{\Phi}_i(x) = \beta_{i1} \int_0^{\alpha} n(\xi) e^{-\gamma\xi} x_1 \varphi_\xi(x_1) d\xi + \beta_{i2} \int_{\alpha}^{\omega} n(\xi) e^{-\gamma\xi} \times \Big(x_1 \varphi_\alpha(x_1) + x_2 e^{\alpha(x_2 - x_1)} \big(\varphi_\xi(x_2) - \varphi_\alpha(x_2) \big) \Big) d\xi \,, \quad i = 1, 2 \tag{16}$$

where

$$\varphi_s(t) = \int_0^s e^{\tau(\gamma - t)} d\tau = \begin{cases} \frac{e^{s(\gamma-t)}-1}{\gamma - t} & \gamma \neq t \\ s & \gamma = t \end{cases} \qquad s \geq 0 \,, \, t \geq 0 \tag{17}$$

and now $I_1 = (0, \alpha)$, $I_2 = (\alpha, \omega)$.

We assume in what follows that the entries of the matrix $\{\beta_{ij}\}_{i,j=1,2}$ satisfy the condition (H3'), that now is equivalent to the condition

$$\beta_{12} > 0 \;,\; \beta_{21} + \beta_{22} > 0 \,.$$

This assumption ensures that non-zero fixed points are in the interior of the positive cone $\mathbb{R}^2_+$.

The main result we obtain in this Section is

Theorem 3. *Assume (H3') and let $\tilde{\Phi}$ be defined as in (16) and $T = \tilde{\Phi}'(0)$. If $r(T) > 1$, then there exists a unique $x^* \in \mathrm{Int}\big(\mathbb{R}^2_+\big)$ s.t. $x^* = \tilde{\Phi}(x^*)$.*

The existence part was firstly proved on Theorem 1 by Inaba under assumption (H3) [15] and it is proved in Theorem 2, for the finite-dimensional case, under the more general assumption (H3'). Uniqueness is proved through a series of intermediate results. The crucial one is summarised as

Proposition 2. *Assume $x \in \mathrm{Int}\big(\mathbb{R}^2_+\big)$ is a fixed point of (16). Then*

$$\det \big(I - \tilde{\Phi}_x(x) \big) > 0.$$

Proposition 2 shows that the necessary condition (14) for a saddle-node bifurcation cannot occur for $n = 2$.

Proposition 2 is proved in Appendix C through a series of Lemmas.

In order to prove Theorem 3, first note that all nontrivial fixed points (that are in the interior of $\mathbb{R}^2_+$ because of (H3')) are isolated since, by Proposition 2, $\tilde{\Phi}$ is a local diffeomorphism in each fixed point x. Then, the local fixed point index in x is defined [19, Ch. 12.3] and given by

$$i(\tilde{\Phi}, x) = \mathrm{sign} \Big(\det \big(I - \tilde{\Phi}_x(x) \big) \Big) = 1. \tag{18}$$

We can finally prove the main result of this Section.

Proof of Theorem 3. Consider the open set G defined in Proposition 1. The set of positive fixed points of $\tilde{\Phi}$ is contained in G. We already know that such fixed points are isolated. Hence their number is finite for G is bounded. Let $x_1, \dots, x_m$ be such points and $B_\rho(x_i)$, $i = 1, \dots, m$, be disjoint neighborhoods of the x_i, $B_\rho(x_i) \subseteq G$. Then for the index sum Theorem [19, Prop. 12.6] and Propositions 2 and 1 we have

$$1 = i(\tilde{\Phi}, G) = \sum_{i=1}^{m} i(\tilde{\Phi}, B_\rho(x_i)) = \sum_{i=1}^{m} \mathrm{sign}\big(\det\big(I - \tilde{\Phi}_x(x_i)\big)\big) = m \,.$$

Then we have $m = 1$. ∎

We have seen above that, if the contact rates have the form (9) with $n = 2$, the positive equilibrium is unique above the threshold. It is instead possible to choose $\beta(a, a')$ in a way such that the operator Φ is essentially 2-dimensional, but the condition (14) for saddle-node bifurcation of equilibrium (thus giving rise to multiple positive equilibria) is satisfied.

Choose in fact

$$\beta(a,\xi) = \frac{\sum_{i,j=1}^{2} \beta_{ij} \mathbf{1}_{I_i}(a) \mathbf{1}_{I_j}(\xi)}{n(\xi) e^{-\gamma\xi}}. \tag{19}$$

With this choice of $\beta(\cdot,\cdot)$, a positive fixed point ψ for Φ must be $\psi = \sum_{i=1}^{2} x_i \mathbf{1}_{I_i}$ where (x_1, x_2) is a solution of the equation on $\mathbb{R}^2_+$:

$$x_i = \beta_{i1} x_1 \int_0^\alpha \varphi_\xi(x_1) d\xi + \beta_{i2} x_1 \varphi_\alpha(x_1)(\omega - \alpha) + \beta_{i2} x_2 \, e^{\alpha(x_2 - x_1)} \int_\alpha^\omega \big(\varphi_\xi(x_2) - \varphi_\alpha(x_2)\big) d\xi \,, \qquad i = 1, 2 \tag{20}$$

with φ given as in (17). Considering now $\tilde{\Phi} : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ defined as the right hand side of (20), one can find, with computations similar to those performed in Section 4, that there exist values of the parameters and of x^* such that $\tilde{\Phi}(x^*) = x^*$ and $\det\left(I - \tilde{\Phi}_x(\vartheta^*, x^*)\right) = 0$. One such example is obtained at $x^* = (1, 1)$ with $\beta_{11} = \beta_{22} = 0$, $\gamma = 1$, $\alpha = 4$, $\omega = 28 + \sqrt{720}$, $\beta_{12} = \frac{2}{\omega^2 - \alpha^2}$, $\beta_{21} = \frac{2}{\alpha^2}$.

It must be remarked that the choice (19) entails that $\beta(a, \xi)$ is unbounded as ξ goes to ω if, as often but not always assumed, $\lim_{a \to \omega} n(a) = 0$. This problem can be adjusted by modifying $\beta(a, \xi)$ when ξ is in a suitably small neighbourhood of ω while, by continuity, multiple solutions will continue to exist. However, the choice (19) appears anyway unrealistic, because with that infectiousness increases exponentially with age ξ.

7. Conclusions.

The analysis presented in this paper shows that the age-structured *SIR* epidemic model may have rather unexpected dynamical behaviour, with multiple endemic equilibria, coexistence of multiple attractive periodic solutions, some with extremely long period, occurrence of quasi-periodic and chaotic attractors and, more generally, a rather complicated bifurcation structure, of which we believe we have captured only some partial information.

Although the contact rates have been chosen according to the widely used pattern of WAIFW matrices, it must be admitted that the parameter values employed do not aim at being realistic, but have been chosen purely out of mathematical simplicity. Indeed, it is difficult to conceive that individuals in the intermediate age class have contact only within themselves, while those in the youngest age class can be infected only by those in the oldest class and vice versa. Furthermore in the example shown in Sections 4 and 5 the strength of infection old-to-young is orders of magnitude higher than that of infection young-to-old. Thus, the astonishing pattern of periodic epidemics occurring every around 200 years (Fig. 4e), assuming an average infectious period of 1 year and expected life of 100 years, may look just a mathematical curiosity.

An intuitive explanation of the multiple equilibria appears appropriate. If there were no population fluxes among age classes, with the WAIFW matrix chosen in the example, there would be two separate epidemics: one involving youngest and oldest age class, and another one involving the intermediate age class. Since there is instead population flux, an endemic infection in young and old age classes is such that (when parameter values are suitable) too few individuals arrive susceptible at the intermediate age class, so that the infection in that age class is not self-sustaining; this corresponds to the upper equilibrium branch in Fig. 2 that exists only if α_2 is not too large (otherwise, the size of the intermediate class becomes large enough that the infection in that age class becomes self-sustaining); in the alternative equilibrium (lower equilibrium branch in Fig. 2), the infection is endemic in the intermediate age class and too few susceptibles are left for the older age class to be able to spread sufficiently the infection to the young age class and back.

A similar pattern can be seen in the periodic solutions, where infections in the intermediate age class on one side, and in the young and old age class, on the other side, are desynchronized. One can possibly conceive less extreme examples in which partial desynchronizations of epidemics in different age classes occur.

The examples shown, though admittedly unrealistic, give insight on the mechanisms through which, when a dynamical state (such as age) affects epidemic transmission, complicated patterns may arise that are impossible when states are static; as well known, in multigroup SIR models, global convergence to a unique equilibrium occurs above threshold [13, 12]. The complex behaviour of this system may appear reminiscent of what found for the LPA model of *Tribolium* [8], where again there are 3 stages with simple chain transitions and nonlinear interactions.

One may finally wonder whether, if the contact matrices are 'realistic' in some sense, uniqueness of the endemic equilibrium occurs also in age-structured epidemic model. The question becomes which are the features of 'realistic' contact matrices: for instance, one may require that they are not too far from symmetry and that their diagonal elements are larger than the non-diagonal (as in the empirically identified matrices in POLYMOD [17]). Looking at the same question from a different perspective, one could aim at identifying which properties of contact matrices are required to obtain a saddle-node bifurcation, although computations appear prohibitive without simplifying assumptions of the type we used in Section 4.

Appendix A. **Proof of Theorem 2.** We now use the structure of $\tilde{\Phi}$ (10) to prove the following

Lemma 1. *If (H3') holds, then* $\tilde{\Phi}$ *has no fixed points on the boundary of* $\mathbb{R}^n_+$ *but for* 0.

Proof. One immediately see that, if $x \in \mathbb{R}^n_+$ with $x_i > 0$ then $g_j(x) > 0$ (11) for all $j \geq i$; in particular, if $x \in \mathbb{R}^n_+ \setminus \{0\}$, then $g_n(x) > 0$.

Assume $x \in \mathbb{R}^n_+$, $x \neq 0$, is a fixed point of $\tilde{\Phi}$. Since $g_n(x) > 0$, from (H3') $\tilde{\Phi}_{i_m}(x) > 0$, so that $x_{i_m} > 0$; similarly from $x_{i_l} > 0$, one obtains $g_{j_{l-1}}(x) > 0$, implying $x_{i_{l-1}} = \tilde{\Phi}_{i_{l-1}}(x) > 0$. Applying this recursively backwards with l from m to 2, one arrives at $x_{i_1} = x_1 > 0$. Then from (H3'), one gets $g_j(x) > 0$ for all j, so that $x_i = \tilde{\Phi}_i(x) > 0$ for all $i = 1, \ldots, n$. □

Proof of Proposition 1. We observe that

$$M_i = \sup_{x \in \mathbb{R}^n_+} \Phi_i(x) < +\infty\,,\ i = 1, \ldots, n\,,$$

so we can choose $R > \sum_{i=1}^n M_i(\alpha_i - \alpha_{i-1})$ and define

$$G = \left\{x \in \mathbb{R}^n_+ : x_i > 0\ \forall\ i,\ \rho < \|J(x)\|_1 < R\right\},$$

with $\rho > 0$ to be selected below, $\|\cdot\|_1$ the L^1-norm on $(0,\omega)$ and recalling the definition $J(x) = \sum_{i=1}^n x_i \mathbf{1}_{I_i}$.

Choose x_0 in the interior of $\mathbb{R}^n_+$ with $\|J(x_0)\|_1 < R$, and let

$$q(t) = \max\left\{\ln\left(r\big((1-t)T\big)\right), t\|J(x_0)\|_1\right\}$$

for $t \in [0,1]$. q is continuous since $\ln\left(r((1-t)T)\right) = \ln\left((1-t)r(T)\right)$ because of the spectral mapping theorem ([16, Chapt. 1.6]). We notice that $q(t) > 0$ for all $t \in [0,1]$ because of the assumption on $r(T)$ and the choice of x_0. Hence $\min_{t\in[0,1]} q(t) > 0$ and we can choose ρ with $0 < \rho \le \min_{t\in[0,1]} q(t)$.

Now we define the homotopy

$$H : \overline{G} \times [0,1] \longrightarrow \mathbb{R}^n \quad \text{by} \quad H(x,t) = (1-t)\tilde{\Phi}(x) + tx_0.$$

We need to show that $H(x,t) \neq x$ for all x on the boundary of G and $t \in [0,1]$. By construction

$$\|J(H(x,t))\|_1 = (1-t)\|J(\tilde{\Phi}(x))\|_1 + t\|J(x_0)\|_1 < (1-t)\sum_{j=1}^n M_j(\alpha_j - \alpha_{j-1}) + tR < R.$$

Furthermore, in Remark 6 of [10], it is shown that, if $\bar{\psi}$ is a fixed point in $L^1_+(0,\omega)$ of a map $\Phi(\psi) + u_0$ with Φ given by (8) for some kernel $\beta(a,a')$, $u_0 \in L^1_+(0,\omega)$ and $r(\Phi'(0)) > 1$, then $\|\bar{\psi}\|_1 > \ln(r(\Phi'(0))$. Note that, when β has the structure (9), we do not need to distinguish between $r(\Phi'(0))$ and $r(\tilde{\Phi}'(0))$, because the two operators have the same eigenvalues.

Assume then that $\bar{x} = (1-t)\tilde{\Phi}(\bar{x}) + tx_0$ for some $t \in [0,1]$ such that $r((1-t)T) > 1$ (this occurs for $0 \le t < 1 - 1/r(T)$). Applying the previous observation to $(1-t)\Phi$ with $u_0 = J(tx_0)$, one obtains $\|J(\bar{x})\|_1 > \ln(r((1-t)T))$. Instead, if $r((1-t)T) \le 1$, then $\|J(\bar{x})\|_1 > \ln(r((1-t)T))$ is obvious.

By the monotonicity of J and the L^1-norm, one has $J(\bar{x}) \ge J(tx_0)$, hence $\|J(\bar{x})\|_1 \ge \|J(tx_0)\|_1 = t\|J(x_0)\|_1$. Thus $\|J(\bar{x})\|_1 \ge q(t)$ for each $t \in [0,1]$, hence it must be $\|J(\bar{x})\|_1 > \rho$.

To conclude, we must exclude that there exists fixed points of $H(\cdot,t)$ on the boundaries where $x_i = 0$ for some $i = 1,\dots n$. If $t = 0$ this is impossible by the previous Lemma; if $t > 0$, this is prevented from $\bar{x} \ge tx_0 \in \mathrm{Int}(\mathbb{R}^n_+)$.

Hence H is an admissible homotopy; it follows for homotopy invariance [19, Chapt. 12.3]

$$1 = i(x_0, G) = i(\tilde{\Phi}, G)\,. \tag{21}$$

□

Proof of Theorem 2. a) If (H3') holds, then $i(\tilde{\Phi}, G) = 1$, which implies that $\tilde{\Phi}$ has at least one fixed point in G ([19, Theor. 12.4(2)]), as claimed.

b) Returning to the Proof of Proposition 1, without condition (H3') we cannot exclude that $\tilde{\Phi}$ has nonzero fixed points on the boundary of $\mathbb{R}^n_+$. However since $x_0 \in \mathrm{Int}(\mathbb{R}^n_+)$, $H(\cdot,t)$ cannot have fixed points on the boundary of G for $t > 0$, so we can use the homotopy $H(x,t)$ for $t \in [\varepsilon, 1]$ for any $\varepsilon > 0$ and conclude that $H(x,\varepsilon)$ will have a fixed point, say $\bar{x}_\varepsilon$ in G for each $\varepsilon > 0$.

By the boundedness of G, we can find a sequence ε_k with $\lim_{k\to\infty} \varepsilon_k = 0$ such that $\bar{x}_{\varepsilon_k}$ converges to some $\bar{x} \in \bar{G}$. Passing to the limit in

$$\bar{x}_{\varepsilon_k} = (1-\varepsilon_k)\tilde{\Phi}(\bar{x}_{\varepsilon_k}) + \varepsilon_k x_0$$

and using the continuity of $\tilde{\Phi}$, we can conclude that $\bar{x} = \tilde{\Phi}(\bar{x})$. □

Appendix B. **Construction of Example 1.** Let us start by rewriting (10) with $n = 3$, as

$$\begin{aligned}
\tilde{\Phi}_i(x) =&\beta_{i1}\int_0^{\alpha_1} n(\xi)e^{-\gamma\xi}x_1\varphi_\xi(x_1)d\xi \\
&+\beta_{i2}\int_{\alpha_1}^{\alpha_2} n(\xi)e^{-\gamma\xi}\Big(x_1\varphi_{\alpha_1}(x_1)+x_2e^{\alpha_1(x_2-x_1)}\big(\varphi_\xi(x_2)-\varphi_{\alpha_1}(x_2)\big)\Big)d\xi \\
&+\beta_{i3}\int_{\alpha_2}^{\omega} n(\xi)e^{-\gamma\xi}\Bigg(x_1\varphi_{\alpha_1}(x_1)+x_2e^{\alpha_1(x_2-x_1)}\big(\varphi_{\alpha_2}(x_2)-\varphi_{\alpha_1}(x_2)\big) \\
&\quad + x_3e^{\alpha_1(x_2-x_1)}e^{\alpha_2(x_3-x_2)}\cdot\big(\varphi_\xi(x_3)-\varphi_{\alpha_2}(x_3)\big)\Bigg)d\xi\,, \qquad i=1,2,3\,,
\end{aligned} \tag{22}$$

where $\varphi_s(t)$, $s \geq 0$ and $t \geq 0$, is defined in (17).

If β_{ij} are chosen as in the Example 1 and $x \in \mathrm{Int}\big(\mathbb{R}^3_+\big)$ is a fixed point of (22), then

$$\det\Big(I-\tilde{\Phi}_x(x)\Big) = \beta_{13}\beta_{22}\beta_{31}(A_1-A_2-A_3)$$

where

$$\begin{aligned}
A_1 =& \int_{\alpha_2}^{\omega} n(\xi)e^{-\gamma\xi}\Big[\left(\frac{1}{x_1}+\alpha_1\right)x_3e^{\alpha_1(x_2-x_1)+\alpha_2(x_3-x_2)}\left(\varphi_\xi(x_3)-\varphi_{\alpha_2}(x_3)\right) \\
&+\left(\frac{1}{x_1}+\alpha_1\right)x_2e^{\alpha_1(x_2-x_1)}\left(\varphi_{\alpha_2}(x_2)-\varphi_{\alpha_1}(x_2)\right)-x_1\varphi'_{\alpha_1}(x_1)\Big]d\xi \\
&\cdot\int_{\alpha_1}^{\alpha_2} n(\xi)e^{-\gamma\xi}\Big[\frac{x_1}{x_2}\varphi_{\alpha_1}(x_1)+x_2e^{\alpha_1(x_2-x_1)}\big(\alpha_1\left(\varphi_{\alpha_1}(x_2)-\varphi_\xi(x_2)\right)\big) \\
&+\big(\varphi'_{\alpha_1}(x_2)-\varphi'_\xi(x_2)\big)\Big]d\xi\cdot\int_0^{\alpha_1} n(\xi)e^{-\gamma\xi}\frac{x_1}{x_3}\varphi_\xi(x_1)d\xi
\end{aligned} \tag{23}$$

$$\begin{aligned}
A_2 =& \int_{\alpha_2}^{\omega} n(\xi)e^{-\gamma\xi}e^{\alpha_1(x_2-x_1)}e^{\alpha_2(x_3-x_2)}\Big[(1+\alpha_2x_3)\left(\varphi_\xi(x_3)-\varphi_{\alpha_2}(x_3)\right) \\
&+x_3\left(\varphi'_\xi(x_3)-\varphi'_{\alpha_2}(x_3)\right)\Big]d\xi \\
&\cdot\int_{\alpha_1}^{\alpha_2} n(\xi)e^{-\gamma\xi}\Big[\frac{x_1}{x_2}\varphi_{\alpha_1}(x_1)+x_2e^{\alpha_1(x_2-x_1)}\big(\alpha_1\left(\varphi_{\alpha_1}(x_2)-\varphi_\xi(x_2)\right)\big) \\
&+\big(\varphi'_{\alpha_1}(x_2)-\varphi'_\xi(x_2)\big)\Big]d\xi\;\cdot\int_0^{\alpha_1} n(\xi)e^{-\gamma\xi}\left(\varphi_\xi(x_1)+x_1\varphi'_\xi(x_1)\right)d\xi
\end{aligned} \tag{24}$$

$$A_3 = \int_{\alpha_2}^{\omega} n(\xi)e^{-\gamma\xi}\Big[(\alpha_1-\alpha_2)x_3 e^{\alpha_1(x_2-x_1)}e^{\alpha_2(x_3-x_2)}\left(\varphi_\xi(x_3)-\varphi_{\alpha_2}(x_3)\right)$$
$$+ e^{\alpha_1(x_2-x_1)}\Big((1+\alpha_1 x_2)\left(\varphi_{\alpha_2}(x_2)-\varphi_{\alpha_1}(x_2)\right)+x_2\left(\varphi'_{\alpha_2}(x_2)-\varphi'_{\alpha_1}(x_2)\right)\Big)\Big]d\xi$$
$$\cdot \int_{\alpha_1}^{\alpha_2} n(\xi)e^{-\gamma\xi}\Big[\varphi_{\alpha_1}(x_1)+x_1\varphi'_{\alpha_1}(x_1)-\alpha_1 x_2 e^{\alpha_1(x_2-x_1)}\left(\varphi_\xi(x_2)-\varphi_{\alpha_1}(x_2)\right)\Big]d\xi$$
$$\cdot \int_0^{\alpha_1} n(\xi)e^{-\gamma\xi}\frac{x_1}{x_3}\varphi_\xi(x_1)d\xi\Bigg\}. \tag{25}$$

To make the computations as simple as possible, we choose

$$\gamma = 1 \quad , \; n(\xi) = 1 \;\; \forall\, \xi \in (0,\omega)\,, \tag{26}$$

and set the other parameters in such a way that $x^* = (1,1,1)$ is a solution of (22).

From (22) we have

$$1 = \beta_{13}\int_{\alpha_2}^{\omega} e^{-\xi}\xi\, d\xi = \beta_{13}\big(g_1(\omega)-g_1(\alpha_2)\big)$$
$$1 = \beta_{22}\int_{\alpha_1}^{\alpha_2} e^{-\xi}\xi\, d\xi = \beta_{22}\big(g_1(\alpha_2)-g_1(\alpha_1)\big)$$
$$1 = \beta_{31}\int_0^{\alpha_1} e^{-\xi}\xi\, d\xi = \beta_{31}g_1(\alpha_1)$$

where

$$g_1(\alpha) = \int_0^{\alpha} e^{-\xi}\xi\, d\xi = 1 - e^{-\alpha} - \alpha e^{-\alpha}\,.$$

If we assume

$$\beta_{13} = \frac{1}{g_1(\omega)-g_1(\alpha_2)}$$
$$\beta_{22} = \frac{1}{g_1(\alpha_2)-g_1(\alpha_1)} \tag{27}$$
$$\beta_{31} = \frac{1}{g_1(\alpha_1)}\,,$$

then indeed $x^* = (1,1,1)$ is a solution of (22).

Using $x^* = (1,1,1)$ and (26) in (23)–(24)–(25), the expressions simplify greatly yielding

$$A1 = \int_{\alpha_2}^{\omega} e^{-\xi}\left(\frac{\alpha_1^2}{2}+(1+\alpha_1)(\xi-\alpha_1)\right)d\xi$$
$$\cdot\int_{\alpha_1}^{\alpha_2} e^{-\xi}\left(\alpha_1+\frac{(\xi-\alpha_1)^2}{2}\right)d\xi\cdot\int_0^{\alpha_1} e^{-\xi}\xi\, d\xi$$
$$A2 = \int_{\alpha_2}^{\omega} e^{-\xi}(\xi-\alpha_2)\left(1-\frac{\xi-\alpha_2}{2}\right)d\xi\cdot$$
$$\cdot\int_{\alpha_1}^{\alpha_2} e^{-\xi}\left(\alpha_1+\frac{(\xi-\alpha_1)^2}{2}\right)d\xi\cdot\int_0^{\alpha_1} e^{-\xi}\left(\xi-\frac{1}{2}\xi^2\right)d\xi$$

$$A3 = \int_{\alpha_2}^{\omega} e^{-\xi}(\alpha_2-\alpha_1)\left(1+\frac{\alpha_1+\alpha_2}{2}-\xi\right)d\xi$$
$$\cdot \int_{\alpha_1}^{\alpha_2} e^{-\xi}\left(\alpha_1-\frac{\alpha_1^2}{2}-\alpha_1(\xi-\alpha_1)\right)d\xi \cdot \int_0^{\alpha_1} e^{-\xi}\xi\, d\xi\,.$$

We set $\alpha_1 = 2$, so that $\alpha_1 - \dfrac{{\alpha_1}^2}{2} = 0$ and the second integral in A_3 becomes simpler.

Suitable values for α_2 and ω are now to be found; we write $A_i = A_i(\omega,\alpha_2)$, $i = 1,2,3$, and compute

$$\lim_{\omega\to+\infty} A_1(\omega,\alpha_2) = e^{-\alpha_2}\left(3\alpha_2-1\right)\left(3e^{-2}-e^{-\alpha_2}(3-\alpha_2+\frac{\alpha_2^2}{2})\right)\left(1-3e^{-2}\right)$$
$$\lim_{\omega\to+\infty} A_2(\omega,\alpha_2) = 0$$
$$\lim_{\omega\to+\infty} A_3(\omega,\alpha_2) = e^{-\alpha_2}(\alpha_2-2)^2\left(e^{-2}+e^{-\alpha_2}-\alpha_2 e^{-\alpha_2}\right)\left(1-3e^{-2}\right),$$

uniformly for $\alpha_2 > 2$.

We set $\eta = \dfrac{1}{\omega}$ and consider the function

$$H(\eta,\alpha_2) = \begin{cases} A_1\left(\frac{1}{\eta},\alpha_2\right) - A_2\left(\frac{1}{\eta},\alpha_2\right) - A_3\left(\frac{1}{\eta},\alpha_2\right) & \eta>0\,,\ \alpha_2>2 \\ \lim_{\omega\to+\infty}\left(A_1(\omega,\alpha_2)-A_2(\omega,\alpha_2)-A_3(\omega,\alpha_2)\right) & \eta=0\,,\ \alpha_2>2 \end{cases}$$

on the domain $D = \left\{(\eta,\alpha_2)\,:\,\alpha_2>\alpha_1,\ 0\le\eta<\dfrac{1}{\alpha_2}\right\}$. H is continuous in D and

$$H(0,\alpha_2) = e^{-\alpha_2-2}(\alpha_2-2)(1-3e^{-2})(f_1(\alpha_2)-f_2(\alpha_2))$$

where

$$f_1(a) = \frac{(3a-1)}{a-2}\left(3-e^{-(a-2)}(3-a+\frac{a^2}{2})\right)$$
$$= \left(3+\frac{5}{a-2}\right)\left(3(1-e^{-(a-2)})-\frac{a}{2}e^{-(a-2)}(a-2)\right)$$
$$f_2(a) = (a-2)\left(1+e^{-(a-2)}-ae^{-(a-2)}\right).$$

Since

$$\lim_{a\to2^+} f_1(a) = 10 \qquad \lim_{a\to+\infty} f_1(a) = 9$$
$$\lim_{a\to2^+} f_2(a) = 0 \qquad \lim_{a\to+\infty} f_2(a) = +\infty$$

there exists $\alpha_2^* > 2$ solution of the equation $f_1(a)-f_2(a) = 0$. Solving numerically this equation, we find the value $\alpha_2^* \approx 12.4342$ which gives $H(0,\alpha_2^*) = 0$.

Moreover, we can find values $\alpha_2^+ < \alpha_2^-$ such that $f_1(\alpha_2^+) > f_2(\alpha_2^+)$ and thus $H(0,\alpha_2^+) > 0$, while $f_1(\alpha_2^-) < f_2(\alpha_2^-)$ and thus $H(0,\alpha_2^-) < 0$. By continuity there exists $\eta_0 > 0$ such that $H(\eta,\alpha_2^+) > 0$ and $H(\eta,\alpha_2^-) < 0$ for each $\eta\in(0,\eta_0)$. Hence, given $\eta^* = \dfrac{1}{\omega^*} \in (0,\eta_0)$, there exists $\alpha_2^{**}\in(\alpha_2^+,\alpha_2^-)$ such that $H(\eta^*,\alpha_2^{**}) = 0$. So if $\alpha_1 = 2\,,\ \alpha_2 = \alpha_2^{**}\,,\ \omega = \dfrac{1}{\eta^*}$ we have $A_1 - A_2 - A_3 = 0$, from which it follows that $\det(I - \tilde{\Phi}_x(x^*)) = 0$.

Finally setting $\beta_{13}, \beta_{22}, \beta_{31}$ from (27), we obtain $\tilde{\Phi}(x^*) = x^*$. For instance, if we take $\omega = 100$, i.e. $\eta^* = \dfrac{1}{100}$, then $\alpha_2^{**} \approx \alpha_2^* \approx 12.4342$ and the corresponding values are

$$\beta_{13} \approx 1.8702 \cdot 10^4\,, \quad \beta_{22} \approx 2.4633\,, \quad \beta_{31} \approx 1.6835\,.$$

Appendix C. Proofs of Section 6.

Lemma 2. *Let*

$$d_{11}(x) = \int_0^\alpha n(\xi) e^{-\gamma\xi} x_1 \varphi_\xi(x_1) d\xi\,;$$

$$d_{21}(x) = \int_\alpha^\omega n(\xi) e^{-\gamma\xi} x_1 \varphi_\alpha(x_1) d\xi\,;$$

$$d_{22}(x) = \int_\alpha^\omega n(\xi) e^{-\gamma\xi} x_2 e^{\alpha(x_2 - x_1)} \left(\varphi_\xi(x_2) - \varphi_\alpha(x_2)\right) d\xi\,;$$

so that

$$g_1(x) = d_{11}(x)\,, \quad g_2(x) = d_{21}(x) + d_{22}(x)\,.$$

Then, the following inequalities hold:

$$\frac{\partial g_i}{\partial x_i}(x) > 0\,, \quad i = 1, 2\,; \tag{28}$$

$$d_{ij}(x) > 0\,, \quad i = 1, 2\,,\ 1 \le j \le i \tag{29}$$

$$\frac{d_{ij}(x)}{x_j} - \frac{\partial g_i}{\partial x_j}(x) > 0\,, \quad i = 1, 2\,,\ 1 \le j \le i \tag{30}$$

$\forall\, x \in \mathrm{Int}\big(\mathbb{R}_+^2\big)$, $\forall\, 0 < \alpha < \omega\,,\ \forall\, \gamma > 0\,.$

Proof. The derivative of (17) with respect to t is given by

$$\varphi_s'(t) = -\int_0^s \tau e^{\tau(\gamma - t)} d\tau = \begin{cases} -\frac{1 + s(\gamma - t)e^{s(\gamma - t)} - e^{s(\gamma - t)}}{(\gamma - t)^2} & \gamma \ne t \\ -s^2/2 & \gamma = t. \end{cases}$$

Then

$$\frac{\partial g_1}{\partial x_1}(x) = \int_0^\alpha n(\xi) e^{-\gamma\xi} \big(\varphi_\xi(x_1) + x_1\, \varphi_\xi'(x_1)\big) d\xi\,.$$

We note that

$$\int_0^{+\infty} e^{-\gamma\xi} \left(\varphi_\xi(t) + t\varphi_\xi'(t)\right) d\xi = 0 \tag{31}$$

$\forall\, \gamma > 0$, $\forall\, t > 0\,$; for we have

$$\int_0^{+\infty} e^{-\gamma\xi} \varphi_\xi(t) d\xi = \frac{1}{\gamma t} \quad , \quad \int_0^{+\infty} e^{-\gamma\xi} \varphi_\xi'(t) d\xi = -\frac{1}{\gamma t^2}\,.$$

We now prove that

$$\int_0^\alpha e^{-\gamma\xi}\left(\varphi_\xi(x_1)+x_1\varphi_\xi'(x_1)\right)d\xi>0$$

$\forall\,\alpha>0,\ \forall\,\gamma>0.$
Let us consider the functions $u:[0,+\infty)\longrightarrow\mathbb{R}$ defined as

$$u(\alpha)=\int_0^\alpha e^{-\gamma\xi}\left(\varphi_\xi(x_1)+x_1\varphi_\xi'(x_1)\right)d\xi$$

and $v:[0,+\infty)\longrightarrow\mathbb{R}$ defined as

$$v(\xi)=\varphi_\xi(x_1)+x_1\varphi_\xi'(x_1)\,.$$

We have $v'(\xi)=e^{\xi(\gamma-x_1)}\big(1-\xi x_1\big)$, hence $v(\xi)$ is monotone increasing on $(0,1/x_1)$ and decreasing on $(1/x_1,+\infty)$; moreover $v(\xi)$ is such that $v(0)=0$ and

$$\lim_{\xi\to+\infty}v(\xi)=\begin{cases}-\infty & \text{if}\quad \gamma\ge x_1\\ \frac{1}{x_1-\gamma}\left(1-\frac{x_1}{x_1-\gamma}\right) & \text{if}\quad 0<\gamma<x_1\end{cases}.$$

Then

$$\exists\,\alpha_0>1/x_1\ \text{ s.t. } v(\xi)>0 \text{ on } (0,\alpha_0)\,,\ v(\xi)<0 \text{ on } (\alpha_0,+\infty)\,.$$

Hence we have $u(\alpha)$ monotone increasing on $(0,\alpha_0)$ and decreasing on $(\alpha_0,+\infty)$; furthermore, $u(0)=0$ and, because of (31), $\lim_{\alpha\to+\infty}u(\alpha)=0$. We can then conclude that $u(\alpha)>0\ \ \forall\,\alpha>0,\ \forall\,\gamma>0.$
$n(\xi)$, the stationary age profile of the population, is a monotone decreasing function (see for example [15]). If $\alpha\le\alpha_0$, then

$$\frac{\partial g_1}{\partial x_1}(x)=\int_0^\alpha n(\xi)e^{-\gamma\xi}v(\xi)d\xi>n(\alpha)\int_0^\alpha e^{-\gamma\xi}v(\xi)d\xi=n(\alpha)u(\alpha)>0\,.$$

Otherwise if $\alpha>\alpha_0$

$$\frac{\partial g_1}{\partial x_1}(x)=\int_0^\alpha n(\xi)e^{-\gamma\xi}v(\xi)d\xi=\int_0^{\alpha_0} n(\xi)e^{-\gamma\xi}v(\xi)\,d\xi+\int_{\alpha_0}^\alpha n(\xi)e^{-\gamma\xi}v(\xi)\,d\xi$$

$$>n(\alpha_0)\int_0^{\alpha_0}e^{-\gamma\xi}v(\xi)d\xi+n(\alpha_0)\int_{\alpha_0}^\alpha e^{-\gamma\xi}v(\xi)d\xi=n(\alpha_0)u(\alpha)>0\,.$$

This proves the first inequality in (28). Since we have

$$\frac{\partial g_2}{\partial x_2}(x)=\int_\alpha^\omega n(\xi)e^{-\gamma\xi}\left((1+\alpha x_2)\big(\varphi_\xi(x_2)-\varphi_\alpha(x_2)\big)+x_2\big(\varphi_\xi'(x_2)-\varphi_\alpha'(x_2)\big)\right)d\xi$$

$$=e^{-\alpha x_2}\int_0^{\omega-\alpha}n(\xi+\alpha)e^{-\gamma\xi}\left(\varphi_\xi(x_2)+x_2\varphi_\xi'(x_2)\right)d\xi\,,$$

the second inequality in (28) is proved in the same way.
To prove (29) and (30) we note that

$$\varphi_\xi(t)=\int_0^\xi e^{\tau(\gamma-t)}d\tau>0\quad\forall\,\xi>0\,,\ \forall\,t>0\,,\forall\,\gamma>0; \tag{32}$$

$$\varphi_\xi'(t)=-\int_0^\xi \tau\,e^{\tau(\gamma-t)}d\tau<0\quad\forall\,\xi>0\,,\ \forall\,t>0\,,\ \forall\,\gamma>0. \tag{33}$$

Then (29) follows from (32).

Let us write the inequalities in (30) explicitly:

$$\begin{aligned}
\frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x) &= -x_1 \int_0^\alpha n(\xi) e^{-\gamma\xi} \varphi'_\xi(x_1) d\xi > 0\,; \\
\frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) &= -x_1 \varphi'_\alpha(x_1) \int_\alpha^\omega n(\xi) e^{-\gamma\xi} d\xi \\
&\qquad + \alpha x_2 e^{\alpha(\gamma - x_1)} \int_0^{\omega-\alpha} n(\xi+\alpha) e^{-\gamma\xi} \varphi_\xi(x_2) d\xi > 0\,; \\
\frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x) &= -x_2 e^{\alpha(\gamma - x_1)} \int_0^{\omega-\alpha} n(\xi+\alpha) e^{-\gamma\xi} \varphi'_\xi(x_2) d\xi > 0
\end{aligned}$$

$\forall\, x \in \mathrm{Int}\big(\mathbb{R}^2_+\big)$, $\forall\, 0 < \alpha < \omega\,,\ \forall\, \gamma > 0\,.$ So the Lemma is proved. □

Proof of Proposition 2. If $x = (x_1, x_2) \in \mathrm{Int}\big(\mathbb{R}^2_+\big)$ is a fixed point of $\tilde{\Phi}$, the entries of the matrix $I - \tilde{\Phi}_x(x)$ are as follows, noting that $\dfrac{\partial d_{i1}}{\partial x_2}(x) = 0$ for $x \in \mathbb{R}^2_+$, $i = 1, 2$,

$$\begin{aligned}
\big(I - \tilde{\Phi}_x(x)\big)_{11} &= 1 - \frac{\partial \tilde{\Phi}_1}{\partial x_1}(x) = \frac{\tilde{\Phi}_1(x)}{x_1} - \frac{\partial \tilde{\Phi}_1}{\partial x_1}(x) \\
&= \beta_{11}\left(\frac{d_{11}(x)}{x_1} - \frac{\partial d_{11}}{\partial x_1}(x)\right) + \beta_{12}\left(\frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) + \frac{d_{22}(x)}{x_1}\right)\,; \\
\big(I - \tilde{\Phi}_x(x)\big)_{12} &= -\frac{\partial \tilde{\Phi}_1}{\partial x_2}(x) = -\beta_{12} \frac{\partial g_2}{\partial x_2}(x)\,; \\
\big(I - \tilde{\Phi}_x(x)\big)_{21} &= -\frac{\partial \tilde{\Phi}_2}{\partial x_1}(x) = -\beta_{21} \frac{\partial g_1}{\partial x_1}(x) - \beta_{22} \frac{\partial g_2}{\partial x_1}(x)\,; \\
\big(I - \tilde{\Phi}_x(x)\big)_{22} &= 1 - \frac{\partial \tilde{\Phi}_2}{\partial x_2}(x) = \frac{\tilde{\Phi}_2(x)}{x_2} - \frac{\partial \tilde{\Phi}_2}{\partial x_2}(x) \\
&= \beta_{21} \frac{d_{11}(x)}{x_2} + \beta_{22}\left(\frac{d_{21}(x)}{x_2} + \frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x)\right)\,.
\end{aligned}$$

We notice that we can write $\det\big(I - \tilde{\Phi}_x(x)\big)$ as the sum of four terms,

$$\det\big(I - \tilde{\Phi}_x(x)\big) = A_1 + A_2 + A_3 + A_4$$

where

$$A_1 = \beta_{11}\beta_{21}\left(\frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x)\right)\frac{d_{11}(x)}{x_2}\ ;$$

$$A_2 = \beta_{11}\beta_{22}\left(\frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x)\right)\left(\frac{d_{21}(x)}{x_2} + \frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x)\right)\ ;$$

$$A_3 = \beta_{12}\beta_{21}\left(\left(\frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) + \frac{d_{22}(x)}{x_1}\right)\frac{d_{11}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x)\frac{\partial g_1}{\partial x_1}(x)\right)\ ;$$

$$A_4 = \beta_{12}\beta_{22}\Bigg(\left(\frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) + \frac{d_{22}(x)}{x_1}\right)\left(\frac{d_{21}(x)}{x_2} + \frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x)\right) - \frac{\partial g_2}{\partial x_1}(x)\frac{\partial g_2}{\partial x_2}(x)\Bigg)\,.$$

A_1 is positive as we have

$$\left(\frac{d_{11}(x)}{x_1} - \frac{\partial d_{11}}{\partial x_1}(x)\right)\frac{d_{11}(x)}{x_2} > 0$$

because of (29) and (30); in the same way it is proved that A_2 is positive. A_3 is positive as we have:

$$\begin{aligned}\left(\frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) + \frac{d_{22}(x)}{x_1}\right)&\frac{d_{11}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x)\frac{\partial g_1}{\partial x_1}(x)\\ &> \frac{d_{22}(x)}{x_1}\frac{d_{11}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x)\frac{\partial g_1}{\partial x_1}(x)\\ = \frac{d_{11}(x)}{x_1}\left(\frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x)\right) &+ \frac{\partial g_2}{\partial x_2}(x)\left(\frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x)\right) > 0\end{aligned}$$

$\forall\, 0 < \alpha < \omega$, $\forall\, \gamma > 0$ because of the inequalities (28),(29), (30).

In the same way the positivity of A_4 is proved. □

Acknowledgments. We would like to thank the referees for their valuable comments and suggestions.

REFERENCES

[1] R. M. Anderson and R. M. May, *Vaccination against rubella and measles: Quantitative investigations of different policies*, J. Hyg. Camb., **90** (1983), 259–325.

[2] R. M. Anderson and R. M. May, "Infectious Diseases of Humans: Dynamics and Control," Oxford University Press, Oxford New York Tokyo, 1991.

[3] V. Andreasen, *The effect of age-dependent host mortality on the dynamics of an endemic disease*, Math.Biosci., **114** (1993), 5–29.

[4] V. Andreasen, *Instability in an SIR-model with age dependent susceptibility*, in "Theory of Epidemics" (eds. O. Arino, D. Axelrod, M. Kimmel and M. Langlais), Mathematical Population Dynamics, **1**, Wuerz Publ., Winnipeg, (1995), 3–14.

[5] D. Breda, M. Iannelli, S. Maset and R. Vermiglio, *Stability analysis of the Gurtin-MacCamy model*, SIAM J. Numer. Anal., **46** (2008), 980–995.

[6] D. Breda and D. Visetti, *Existence, multiplicity and stability of endemic states for an age-structured S-I epidemic model*, Math. Biosci., **235** (2012), 19–31.

[7] S. Busenberg, K. Cooke and M. Iannelli, *Endemic thresholds and stability in a class of age-structured epidemics*, SIAM J. Appl. Math., **48** (1988), 1379–1395.

[8] J. M. Cushing, Robert Costantino, Brian Dennis, Robert Desharnais and S. Henson, "Chaos in Ecology: Experimental Nonlinear Dynamics," Academic Press, 2002.

[9] K. Deimling, "Nonlinear Functional Analysis," Springer Verlag, Berlin, 1985.

[10] A. Franceschetti and A. Pugliese, *Threshold behaviour of a SIR epidemic model with age structure and immigration*, J. Math. Biol., **57** (2008), 1–27.

[11] D. Greenhalgh, *Threshold and stability results for an epidemic model with an age structured meeting-rate*, IMA Journal of Mathematics applied in Medicine and Biology, **5** (1988), 81–100.

[12] H. Guo, M. Y. Li and Z. Shuai, *Global stability of the endemic equilibrium of multigroup SIR epidemic models*, Canadian Appl. Math. Quart., **14** (2006), 259–284.

[13] H. W. Hethcote and H. R. Thieme, *Stability of the endemic equilibrium in epidemic models with subpopulations*, Math. Biosci., **75** (1985), 205–227.

[14] M. Iannelli, "Mathematical Theory of Age-Structured Population Dynamics," Giardini editori e stampatori in Pisa, Pisa, 1994.

[15] H. Inaba, *Threshold and stability results for an age-structured epidemic model*, Journal of Mathematical Biology, **28** (1990), 411–434.

[16] T. Kato, "Perturbation Theory for Linear Operators," Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag New York, Inc., New York, 1966.

[17] J. Mossong, N. Hens, M. Jit, P. Beutels, K. Auranen, R. Mikolajczyk, M. Massari, S. Salmaso, G. Scalia Tomba, J. Wallinga, J. Heijne, M. Sadkowska-Todys, M. Rosinska and W. J. Edmunds, *Social contacts and mixing patterns relevant to the spread of infectious diseases*, PlOS Medicine, **5** (2008), 381–391.

[18] H. R. Thieme, *Stability change of the endemic equilibrium in age-structured models for the spread of SIR type infectious diseases*, in "Differential Equations Models in Biology, Epidemiology and Ecology" (eds. S. Busenberg and M. Martelli) (Claremont, CA, 1990), Lecture Notes in Biomathematics, **92**, Springer, Berlin, (1991), 139–158.

[19] E. Zeidler, "Nonlinear Functional Analysis and its Applications. I. Fixed Point Theorems," Springer-Verlag, New York, 1986.

Received June 15, 2011; Accepted February 6, 2012.

E-mail address: francesc@science.unitn.it
E-mail address: pugliese@science.unitn.it
E-mail address: dimitri.breda@uniud.it