MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume 8, Number 4, October 2011

doi:10.3934/mbe.2011.8.931

pp. 931–952

# AN SEIR EPIDEMIC MODEL WITH CONSTANT LATENCY TIME AND INFECTIOUS PERIOD

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(Communicated by Yang Kuang)

Abstract. We present a two delays SEIR epidemic model with a saturation incidence rate. One delay is the time taken by the infected individuals to become infectious (i.e. capable to infect a susceptible individual), the second delay is the time taken by an infectious individual to be removed from the infection. By iterative schemes and the comparison principle, we provide global attractivity results for both the equilibria, i.e. the disease-free equilibrium $\mathbf{E}_0$ and the positive equilibrium $\mathbf{E}_+$, which exists iff the basic reproduction number $\mathcal{R}_0$ is larger than one. If $\mathcal{R}_0 > 1$ we also provide a permanence result for the model solutions. Finally we prove that the two delays are harmless in the sense that, by the analysis of the characteristic equations, which result to be polynomial trascendental equations with polynomial coefficients dependent upon both delays, we confirm all the standard properties of an epidemic model: $\mathbf{E}_0$ is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$, while if $\mathcal{R}_0 > 1$ then $\mathbf{E}_+$ is always asymptotically stable.

## 1. Introduction.

In recent years, attempts have been made by many authors to analyse the global stability properties of delay epidemiological models with a general nonlinear infection rate. For example, Takeuchi and coworkers analysed SIR, SIS, SEIR and SEI epidemic delay models [4, 5, 6] by constructing Lyapunov functionals and thus generalizing to the delay case the general approach by Lyapunov functions proposed by Korobeinikov for non-delayed epidemic models with a very general infection rate ([7] and the references therein). Even very interesting contribution to the topic has been recently presented by Xu and Ma [12] and by Xu and Du [11], who analysed, by using iterative schemes and comparison principles, global attractivity properties of the equilibria respectively of an SEIRS delay model and of an SIR delay epidemic model, where the non-linear infection rate is the one introduced by Capasso and Serio [2] with a saturated incidence rate with respect to the infectious individuals. In [11] there is one constant delay which represents the

2000 *Mathematics Subject Classification.* Primary: 34K19, 34K20, 92D30.

*Key words and phrases.* Epidemic model, time delays, nonlinear incidence rate, delay differential equations, permanence, global attractivity, local stability analysis.

constant infectious period after which the infected individuals are removed, whereas in [12] the delay represents the constant latency time which is the time taken by an infected individual to become infectious. In [11, 12] the authors have paid attention also to the analysis of the characteristic equation. While in [12] they show that the delay is harmless in inducing stability switches that modify the standard stability behaviour of the equilibria, for the delay SIR model in [11] they confirm the local stability property only for the disease-free equilibrium (i.e. if the basic reproduction number $\mathcal{R}_0 < 1$ then it is asymptotically stable and if $\mathcal{R}_0 > 1$ it becomes unstable), whereas for the endemic equilibrium (i.e. $\mathcal{R}_0 > 1$) they refer to a result by Beretta and Kuang [1] but leave the problem open.

In our paper, we modify the SIR delay epidemic model presented in [11] by introducing the class of the exposed individuals and by considering two (constant) delays. The first delay $\tau_1$ is the constant latency time and represents the time taken by a susceptible individual that, infected at time $t$ becomes infectious $I$, i.e. capable to infect other susceptibles, but only at a time $t + \tau_1$: the infected individuals who are not yet infectious are called exposed $E$ and they stand in the exposed class for the time $\tau_1$. The second delay $\tau_2$ is the constant infectious period and represents the time necessary to remove the infectious individuals $I$ from the cycle of the infection. Therefore, an individual infected at time $t$ will be removed at time $t + \tau_1 + \tau_2$ and it stands in the infectious class $I$ for a time $\tau_2$. We assume that the removed individuals $R$ cannot return to the susceptible class $S$. A possible interpretation is that $\tau_2$ is the time taken by an infectious individual before presenting the symptoms of the infection, with the assumption that the infectious individuals $I$ are removed from the infection as soon as the symptoms appear. In this case the infection is transmitted by the asymptomatic infectious $I$. This could be the case, for example, of SARS where all individuals with symptoms are removed ([9] and the references therein).

According to the above remarks, our model is an SEIR model with two delays and we assume for it the same nonlinear rate of infection as in [11, 12]. With these assumptions it turns out to be a generalization of the delay SIR model in [11] and [13] with constant infectious period. Moreover in [13] the infection rate is a bilinear function in $S$ and $I$. Our model becomes coincident with the SIR model in [11] if we assume $\tau_1 = 0$, thus implying that the exposed class $E(t)$ is identically vanishing. If we further assume a bilinear infection rate in $S$ and $I$ we also obtain the model in [13]. An interesting aspect of the model is that the characteristic equation at the endemic equilibrium can be reduced to a second order trascendental polynomial equation with two delays, where the polynomial coefficients are real functions of both delays. By its analysis, we prove that, whenever it exists, the endemic equilibrium is locally asymptotically stable, that is, the delays $\tau_1$ and $\tau_2$ are harmless in inducing stability switches. Though we prove that the endemic equilibrium is globally attractive only if the sup of the incidence rate is less than the susceptibles death rate constant (as in [11]), our feeling is that we should be able to prove that, whenever it exists, the endemic equilibrium is globally asymptotically stable, for example by using the Lyapunov functional approach (see [8, Section 2.5]) but with different Lyapunov functions with respect to those considered in [4, 5, 6] since their method seems not working in our model. However, this is left as a future work.

The structure of the paper is the following: in Section 2 we introduce the model equations with their main properties and we also present all the results that we are going to prove in the paper. In Section 3 we analyse the characteristic equation at the disease-free equilibrium as well as at the endemic one. Section 4 (together with Appendix A) is devoted to a permanence result for the solutions of the model. In Section 5 we prove attractivity results for both equilibria. Conclusions are driven in Section 6.

## 2. **The model equations.**

Herefollowing we introduce the necessary notation and the model equations.

We denote by $S$ the susceptible individuals; by $E$ the exposed individuals, who have been infected and take a time $\tau_1$ to become infectious, i.e. capable to infect the susceptible individuals $S$; by $I$ the infectious individuals, capable to infect the susceptibles and who take a time $\tau_2$ to be removed from the infection; finally, $R$ denotes the removed individuals, for which we assume that they cannot return to the susceptible class because they have been "quarantined" and/or they acquire permanent immunity. It is assumed $\tau := (\tau_1, \tau_2) \in \mathbb{R}^2_+$.

As far as the parameters of the model are concerned, $\Lambda$ and $\mu_1$ are the constant recruitment and death rate, respectively, of susceptibles $S$; $\mu_2$ is the constant death rate for both exposed $E$ and infectious $I$; $\mu_3$ is the constant death rate for removed $R$; it is assumed $\Lambda \in \mathbb{R}_+$ as well as $\mu_i \in \mathbb{R}_+$, $i = 1, 2, 3$. Of course, we could assume different death rate constants for exposed $E$ and infectious $I$ individuals, but this would not change the results, while enabling us to slightly simplify the notation.

While denoting the rate of infection $f(S, I)$, we assume the structure (see [2, 11, 12])

$$f(S, I) := g(I)S \tag{1}$$

with a saturated incidence rate

$$g(I) := \frac{\beta I}{1 + \alpha I} \tag{2}$$

with respect to the number $I$ of the infectious individuals. With $\beta, \alpha \in \mathbb{R}_+$, $\beta I$ is a measure of the force of infection and $\frac{1}{1+\alpha I}$ accounts for the inhibition effect on the rate of infection when $I$ becomes large.

By assuming

(A.1) $\mu_1 = \min_{i=1,2,3}\{\mu_i\}$, $\mu_2 = \max_{i=1,2,3}\{\mu_i\}$

and by taking into account that the rate of infection at time $t$ is $g(I(t))S(t)$ and that the exposed individuals that become infectious $I$ at time $t$ are those infected at the previous time $t - \tau_1$, multiplied for the fraction $e^{-\mu_2 \tau_1}$ of the exposed survived in the time interval $[t - \tau_1, t]$, we get the evolution equation for the exposed $E(t)$. By similar arguments we can write the evolution equations also for $I(t)$ and $R(t)$, while the one for $S(t)$ is standard. Thus, the model equations are

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu_1 S(t) - g(I(t))S(t) \\ \dfrac{dE(t)}{dt} = g(I(t))S(t) - g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1} - \mu_2 E(t) \\ \dfrac{dI(t)}{dt} = g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1} \\ \qquad\qquad -g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_2 I(t) \\ \dfrac{dR(t)}{dt} = g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_3 R(t). \end{cases} \tag{3}$$

Before providing the initial conditions for system (3), we want to note that it can also be rewritten by formally integrating the delay differential equations for $E(t)$, $I(t)$ and $R(t)$ as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu_1 S(t) - g(I(t))S(t) \\ E(t) = \displaystyle\int_0^{\tau_1} g(I(t-\theta))S(t-\theta)e^{-\mu_2\theta}d\theta \\ I(t) = \displaystyle\int_{\tau_1}^{\tau_1+\tau_2} g(I(t-\theta))S(t-\theta)e^{-\mu_2\theta}d\theta \\ R(t) = R(0)e^{-\mu_3 t} \\ \qquad +e^{-\mu_2(\tau_1+\tau_2)}\displaystyle\int_0^t g(I(\theta-\tau_1-\tau_2))S(\theta-\tau_1-\tau_2)e^{-\mu_3(t-\theta)}d\theta \end{cases} \tag{4}$$

with initial conditions that, for biological reasons, are positive continuous functions $S(\theta) = \varphi_1(\theta)$ and $I(\theta) = \varphi_2(\theta)$ for $\theta \in [-(\tau_1+\tau_2), 0]$, with $S(0) > 0$, $E(0), I(0) > 0$ satisfying

$$E(0) = \int_{-\tau_1}^{0} g(\varphi_2(\theta))\varphi_1(\theta)e^{\mu_2\theta}d\theta, \quad I(0) = \int_{-(\tau_1+\tau_2)}^{-\tau_1} g(\varphi_2(\theta))\varphi_1(\theta)e^{\mu_2\theta}d\theta$$

and $R(0) \geq 0$.

2.1. **Positivity.** We see that the positivity of the above initial conditions for $S$ and $I$ in $[-(\tau_1+\tau_2), 0]$ imply positivity for all solutions $(S(t), E(t), I(t), R(t))$, $t > 0$, of system (3) or (4), simply considering recurrence arguments applied to the integral forms for $E(t)$, $I(t)$ and $R(t)$ in (4). We further note that $S(t)$ can never vanish since at each time $t > 0$ where $S(t)$ vanishes it is $\frac{dS(t)}{dt} = \Lambda > 0$.

We can also prove the following.

**Lemma 2.1.** *The compact set*

$$\Omega := \left\{(S, E, I, R) \in \mathbb{R}^4_{+0} : \frac{\Lambda}{\mu_2} \leq S + E + I + R \leq \frac{\Lambda}{\mu_1}\right\}$$

*is globally attractive and invariant for the solutions of (3).*

*Proof.* By defining $N(t) := S(t) + E(t) + I(t) + R(t)$, according to (3) and by Assumption (A.1) we get

$$\Lambda - \mu_2 N(t) \leq \frac{dN(t)}{dt} \leq \Lambda - \mu_1 N(t)$$

for $t \geq 0$ with initial condition $N(0) > 0$. Thus, we obtain

$$\left(N(0) - \frac{\Lambda}{\mu_2}\right)e^{-\mu_2 t} + \frac{\Lambda}{\mu_2} \leq N(t) \leq \left(N(0) - \frac{\Lambda}{\mu_1}\right)e^{-\mu_1 t} + \frac{\Lambda}{\mu_1}$$

for all $t \geq 0$, which proves the Lemma. □

Since $\Omega$ is a limit set for system (3), in the sequel we assume initial conditions satisfying

(A.2) $\frac{\Lambda}{\mu_2} \leq N(0) \leq \frac{\Lambda}{\mu_1}$.

2.2. **Equilibria.** From (3) it is easy to see that the Disease-Free Equilibrium (DFE) is

$$\mathbf{E}_0 := \left(\frac{\Lambda}{\mu_1}, 0, 0, 0\right)$$

which exists for all values of the parameters. As far as the interior (positive) equilibrium $\mathbf{E}_+$ is concerned, first we need to define the basic reproduction number $\mathcal{R}_0$ according to the definition in [4], extended to a delayed epidemic model.

The basic reproductive number is the mean number of secondary cases that a typical infected case will cause in a population with no immunity to the disease in the absence of interventions to control the infection:

$$\mathcal{R}_0 := P_i \frac{1}{\mu_2}\left(e^{-\mu_2\tau_1} - e^{-\mu_2(\tau_1+\tau_2)}\right)$$

where $1/\mu_2$ is the mean infection period and $e^{-\mu_2\tau_1} - e^{-\mu_2(\tau_1+\tau_2)}$ is the probability of an infected people to be in the infectious period. Thus $\frac{1}{\mu_2}\left(e^{-\mu_2\tau_1} - e^{-\mu_2(\tau_1+\tau_2)}\right)$ is the mean infectious period. $P_i$ is the initial maximum infection rate:

$$P_i = g'(0)S_0.$$

Observe that $g'(0)S_0 = \frac{\partial f(S_0,0)}{\partial I}$, where $S_0$ is the initial susceptible population i.e., according to the first equation in (3), is the equilibrium value of susceptibles in the absence of infection: $S_0 = \Lambda/\mu_1$. Therefore

$$\mathcal{R}_0 := \frac{1}{\mu_2}\frac{\partial f(S_0,0)}{\partial I}\left(e^{-\mu_2\tau_1} - e^{-\mu_2(\tau_1+\tau_2)}\right).$$

Accordingly, the interior (positive) equilibrium is

$$\mathbf{E}_+ := (S_+, E_+, I_+, R_+)$$

of components

$$\begin{cases} S_+ := \dfrac{\Lambda}{\mu_1}\dfrac{\beta+\alpha\mu_1\mathcal{R}_0}{\mathcal{R}_0(\beta+\alpha\mu_1)} \\ E_+ := \dfrac{\Lambda\beta(\mathcal{R}_0-1)}{\mu_2\mathcal{R}_0(\beta+\alpha\mu_1)}(1-e^{-\mu_2\tau_1}) \\ I_+ := \dfrac{\mu_1(\mathcal{R}_0-1)}{\beta+\alpha\mu_1} \\ R_+ := \dfrac{\Lambda\beta(\mathcal{R}_0-1)}{\mu_3\mathcal{R}_0(\beta+\alpha\mu_1)}e^{-\mu_2(\tau_1+\tau_2)}). \end{cases} \tag{5}$$

which exists iff the basic reproduction number, which according to the previous definition is

$$\mathcal{R}_0 = \frac{\beta\Lambda}{\mu_1\mu_2}e^{-\mu_2\tau_1}(1-e^{-\mu_2\tau_2}), \tag{6}$$

satisfies $\mathcal{R}_0 > 1$.

From (5) we see that $\mathbf{E}_+ \to \mathbf{E}_0$ as $\mathcal{R}_0 \to 1^+$ and that the infection rate (1) at the equilibrium $\mathbf{E}_+$ is

$$f(S_+, I_+) = \frac{\Lambda\beta(\mathcal{R}_0-1)}{\mathcal{R}_0(\beta+\alpha\mu_1)}.$$

The delay domain of existence of the positive equilibrium $\mathbf{E}_+$ requires $\mathcal{R}_0 > 1$. If we denote it by $\Omega_+$ we have:

$$\Omega_+ = \left\{\tau = (\tau_1,\tau_2) \in \mathbb{R}^2_+ : \tau_1 < h(\tau_2) := \frac{1}{\mu_2}\ln\left[\frac{\beta\Lambda}{\mu_1\mu_2}(1-e^{-\mu_2\tau_2})\right]\right\}$$

and, of course, $\mathcal{R}_0 = 1$ when $\tau_1 = h(\tau_2)$ for $(\tau_1, \tau_2) \in \mathbb{R}^2_+$. Furthermore, notice that

$$\tau_1 < \tau_1^* := \frac{1}{\mu_2} \ln\left(\frac{\beta\Lambda}{\mu_1\mu_2}\right)$$

and

$$\tau_2 > \tau_2^* := \frac{1}{\mu_2} \ln\left(\frac{\beta\Lambda}{\beta\Lambda - \mu_1\mu_2}\right)$$

hold in $\Omega_+$, i.e. $\mathbf{E}_+$ exists if the time taken to become infectious $\tau_1$ is sufficiently small and the asymptomatic infectivity period $\tau_2$ is sufficiently large, Figure 1.

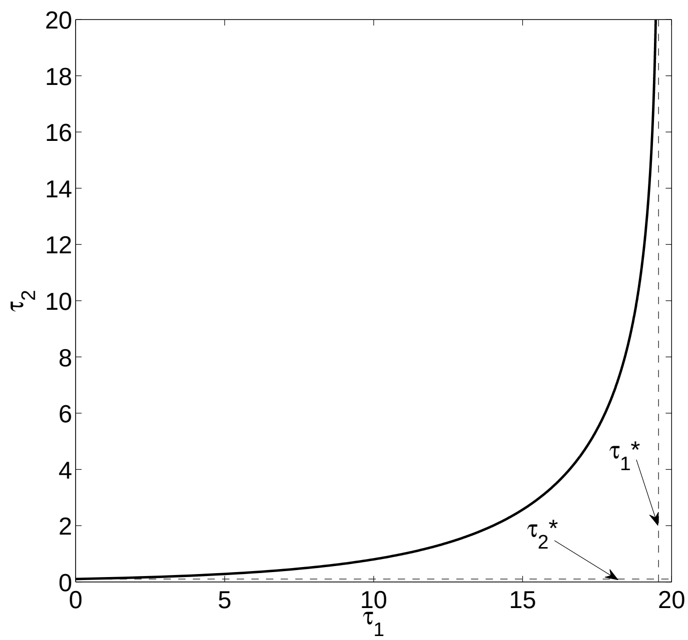


Figure 1. the delay domain of existence (above the curve) of the positive equilibrium $\mathbf{E}_+$.

The above results can be summarized in the following Lemma.

**Lemma 2.2.** *If the parameters in (3) satisfy*

*(i)* $\beta\Lambda \le \mu_1\mu_2$ *then for all delays* $(\tau_1, \tau_2) \in \mathbb{R}^2_+$ *the basic reproduction number is* $\mathcal{R}_0 < 1$ *and therefore only the DFE* $\mathbf{E}_0$ *is feasible;*

*(ii)* $\beta\Lambda > \mu_1\mu_2$ *we have two cases*
  - *a)* $\mathcal{R}_0 \le 1$ *and therefore only the DFE* $\mathbf{E}_0$ *is feasible;*
  - *b)* $\mathcal{R}_0 > 1$, *then the positive equilibrium* $\mathbf{E}_+$ *is feasibe besides* $\mathbf{E}_0$.

Concerning the equilibria of the model we give below the main results, which will be proved in the forthcoming sections. In particular, Theorems 2.4 and 2.6 about global attractivity will be proved in Section 5; Theorems 2, 2.5 and 2.8 about stability will be proved in Section 3; finally Theorem 2.7 on permanence of solutions will be proved in Section 4. About this latter we recall the following.

**Definition 2.3.** The dynamical system (3) is permanent if there exists a compact subset, say $\Sigma \subset \text{int}(R^4_{+0})$, such that $\Sigma$ is positively invariant for (3) and its solutions are ultimately bounded in $\Sigma$.

First of all, the following statement is a particular case of the forthcoming Theorem 2.4 since $\beta\Lambda \leq \mu_1\mu_2$ implies $\mathcal{R}_0 < 1$.

**Corollary 1.** *If $\beta\Lambda \leq \mu_1\mu_2$ only the DFE $\mathbf{E}_0$ is feasible and it is globally attractive, i.e. for all initial conditions the solutions of (3) satisfy*

$$\lim_{t\to\infty} (S(t), E(t), I(t), R(t)) = \left(\frac{\Lambda}{\mu_1}, 0, 0, 0\right).$$

Secondly, by assuming $\beta\Lambda > \mu_1\mu_2$ we have the following results.

**Theorem 2.4.** *If $\mathcal{R}_0 \leq 1$ the DFE $\mathbf{E}_0$ is globally attractive.*

**Corollary 2.** *If $\mathcal{R}_0 < 1$ the DFE $\mathbf{E}_0$ is locally asymptotically stable.*

**Theorem 2.5.** *If $\mathcal{R}_0 > 1$ the DFE $\mathbf{E}_0$ is unstable.*

**Theorem 2.6.** *The positive equilibrium $\mathbf{E}_+$ is globally attractive if $\alpha > \frac{\beta}{\mu_1}$.*

**Theorem 2.7.** *System (3) is permanent provided $\mathcal{R}_0 > 1$ and $\frac{\beta}{\alpha\mu_1} < \mathcal{R}_0$.*

**Theorem 2.8.** *Whenever it exists, the equilibrium $\mathbf{E}_+$ is locally asymptotically stable.*

**Remark 1.** Notice that the condition $\alpha > \frac{\beta}{\mu_1}$ in Theorem 2.6 can also be read as $\sup_{I>0} g(I) < \mu_1$.

## 3. **The reduced system and the characteristic equation.**

In this section we want to prove Theorems 2.5 and 2.8 by the help of the characteristic equation at $\mathbf{E}_0$ and $\mathbf{E}_+$, respectively. In particular, we assume the global attractivity results of Theorems 2.4 and 2.6, which will be proved in Section 5, to hold true.

Herefollowing we denote by $p := (\Lambda, \alpha, \beta, \mu_1, \mu_2, \mu_3, \tau_1, \tau_2)$ the vector of all real parameters of model (3). They belong either to

$$\Gamma := \left\{ p \in \mathbb{R}^8_+ : \frac{\beta\Lambda}{\mu_1\mu_2} > 1 \right\}$$

when we are dealing with the characteristic equation at $\mathbf{E}_0$ or to

$$\Gamma_+ := \left\{ p \in \mathbb{R}^6_+ \times \Omega_+ : \frac{\beta\Lambda}{\mu_1\mu_2} > 1 \right\}$$

when we are dealing with the characteristic equation at $\mathbf{E}_+$, respectively.

Since in (3) the evolution equations for $S(t)$ and $I(t)$ do not contain the variables $E(t)$ and $R(t)$, in order to compute the characteristic equation at any equilibrium $\mathbf{E} = (S^*, E^*, I^*, R^*)$ it is sufficient to consider the characteristic equation of the reduced system

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu_1 S(t) - g(I(t))S(t) \\ \dfrac{dI(t)}{dt} = g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1} \\ \qquad -g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_2 I(t) \end{cases} \tag{7}$$

at $\mathbf{E} = (S^*, I^*)$. In fact, it is easy to check that the characteristic roots, i.e. the solutions of the characteristic equation, for the complete system (3) are either the negative ones $\lambda = -\mu_i$, $i = 2, 3$ (due to the second and fourth equations), or given by the solutions of the characteristic equation of the reduced system (7), which reads

$$G(\lambda; p) = 0 \tag{8}$$

where

$$\begin{aligned} G(\lambda; p) := {} & (\mu_2 + \lambda)(\mu_1 + g(I^*) + \lambda) - (\mu_1 + \lambda)g'(I^*)S^* e^{-(\mu_2+\lambda)\tau_1} \\ & + (\mu_1 + \lambda)g'(I^*)S^* e^{-(\mu_2+\lambda)(\tau_1+\tau_2)} \end{aligned} \tag{9}$$

with $p \in \Gamma$ or $p \in \Gamma_+$ according to which equilibrium is considered.

Thus, at $\mathbf{E}_+$ the characteristic equation is given by (8) whose associated characteristic function (9) has, according to (5), coefficients

$$\begin{cases} g(I^*) = \dfrac{\beta\mu_1(\mathcal{R}_0 - 1)}{\beta + \alpha\mu_1\mathcal{R}_0} \\ g'(I^*)S^* = \dfrac{\beta\Lambda}{\mu_1} \dfrac{\beta + \alpha\mu_1}{\mathcal{R}_0(\beta + \alpha\mu_1\mathcal{R}_0)} \end{cases} \tag{10}$$

and $p \in \Gamma_+$.

Similarly, at the DFE $\mathbf{E}_0$, the coefficients in (9) are

$$\begin{cases} g(I^*) = 0 \\ g'(I^*)S^* = \dfrac{\beta\Lambda}{\mu_1}. \end{cases}$$

and so, for $p \in \Gamma$,

$$G(\lambda; p) = (\mu_1 + \lambda)\left[\lambda + \mu_2 - \frac{\beta\Lambda}{\mu_1} e^{-(\mu_2+\lambda)\tau_1} + \frac{\beta\Lambda}{\mu_1} e^{-(\mu_2+\lambda)(\tau_1+\tau_2)}\right] = 0.$$

Thus, beyond the characteristic roots $\lambda = -\mu_i$, $i = 1, 2, 3$, the other roots are solutions of

$$\Delta(\lambda; p) = 0 \tag{11}$$

with

$$\Delta(\lambda; p) := \lambda + \mu_2 - \frac{\beta\Lambda}{\mu_1} e^{-(\mu_2+\lambda)\tau_1} + \frac{\beta\Lambda}{\mu_1} e^{-(\mu_2+\lambda)(\tau_1+\tau_2)} \tag{12}$$

for $p \in \Gamma$. We can notice that a root of (11) is $\lambda = -\mu_2$, while $\lambda = 0$ is another root when $\mathcal{R}_0 = 1$.

*Proof of Theorem 2.5.* Consider $\Delta(\lambda; p)$ in (12) as a continuous function of the real variable $\lambda$ and observe that, by (6),

$$\begin{aligned} \Delta(0; p) &= \mu_2 - \left(\frac{\beta\Lambda}{\mu_1} e^{-\mu_2\tau_1} - \frac{\beta\Lambda}{\mu_1} e^{-\mu_2(\tau_1+\tau_2)}\right) \\ &= \mu_2 \left[1 - \frac{\beta\Lambda}{\mu_1\mu_2} e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2})\right] \\ &= \mu_2(1 - \mathcal{R}_0). \end{aligned}$$

Then $\mathcal{R}_0 > 1$ implies $\Delta(0; p) < 0$ and, being $\Delta(\lambda; p) \to +\infty$ as $\lambda \to +\infty$ for all $p \in \Gamma$, there is at least one positive root of (11). □

*Proof of Corollary 2.* As already anticipated, at the DFE $\mathbf{E}_0$, the solutions of (8) are either the real negative characteristic roots $\lambda = -\mu_i$, $i = 1, 2, 3$, or the solutions of (11). Let $\lambda = \alpha + \mathrm{i}\omega$ be any of these latter. We have to prove that $\mathcal{R}_0 < 1$ implies $\alpha < 0$.

The real part of (12) for $\lambda = \alpha + \mathrm{i}\omega$ reads

$$\alpha = -\mu_2 + \frac{\beta\Lambda}{\mu_1} e^{-(\mu_2+\alpha)\tau_1} \cos(\omega\tau_1) - \frac{\beta\Lambda}{\mu_1} e^{-(\mu_2+\alpha)(\tau_1+\tau_2)} \cos(\omega(\tau_1 + \tau_2)).$$

If we assume $\mathcal{R}_0 = 1$, i.e. $\mu_2 = \frac{\beta\Lambda}{\mu_1} e^{-\mu_2\tau_1} - \frac{\beta\Lambda}{\mu_1} e^{-\mu_2(\tau_1+\tau_2)}$, then the previous equation becomes

$$\begin{aligned}\alpha = &-\frac{\beta\Lambda}{\mu_1} e^{-\mu_2\tau_1} \left[1 - e^{-\alpha\tau_1}\cos(\omega\tau_1)\right] \\ &+\frac{\beta\Lambda}{\mu_1} e^{-\mu_2(\tau_1+\tau_2)} \left[1 - e^{-\alpha(\tau_1+\tau_2)}\cos(\omega(\tau_1+\tau_2))\right] \le 0.\end{aligned}$$

Since at $\mathcal{R}_0 = 1$ the DFE $\mathbf{E}_0$ is globally attractive by virtue of Theorem 2.4, then all the characteristic roots $\lambda$ have $\alpha \le 0$.

Assume instead $\mathcal{R}_0 < 1$, i.e. $\mu_2 > \frac{\beta\Lambda}{\mu_1} e^{-\mu_2\tau_1} - \frac{\beta\Lambda}{\mu_1} e^{-\mu_2(\tau_1+\tau_2)}$. This inequality implies

$$\begin{aligned}\alpha < &-\frac{\beta\Lambda}{\mu_1} e^{-\mu_2\tau_1} \left[1 - e^{-\alpha\tau_1}\cos(\omega\tau_1)\right] \\ &+\frac{\beta\Lambda}{\mu_1} e^{-\mu_2(\tau_1+\tau_2)} \left[1 - e^{-\alpha(\tau_1+\tau_2)}\cos(\omega(\tau_1+\tau_2))\right] \le 0.\end{aligned}$$

thus proving the local asymptotic stability. □

Before going through the proof of Theorem 2.8, we recall that the characteristic roots at $\mathbf{E}_+$ are either the real negative ones $\lambda = -\mu_i$, $i = 2, 3$, or the solutions of (8) with (9) having coefficients (10). We notice that (8) is a second order trascendental polynomial equation in $\lambda$ of the kind

$$P_0(\lambda; p) + P_1(\lambda; p)e^{-\lambda\tau_1} + P_2(\lambda; p)e^{-\lambda(\tau_1+\tau_2)} = 0$$

where the polynomial coefficients in $P_l(\lambda; p)$, $l = 0, 1, 2$, are continuously differentiable real functions with respect to $p \in \Gamma_+$, and moreover they depend on both delays $\tau = (\tau_1, \tau_2)$. Then, the following (standard) properties hold:

(i) in any open set $D$ of the complex plane $\mathbb{C}$, the complex function $G : D \times \Gamma_+ \to \mathbb{C}$ is analytic;
(ii) since the polynomial coefficients are real, for any $\omega \in \mathbb{R}_+$ the symmetry property $G(\mathrm{i}\omega, p) = 0 \Leftrightarrow G(-\mathrm{i}\omega, p) = 0$ holds for all $p \in \Gamma_+$;
(iii) there exist a principal part, i.e.

$$\deg(P_0(\lambda; p)) > \max\left\{\deg(P_1(\lambda; p)), \deg(P_2(\lambda; p))\right\}.$$

These properties together with Rouché's Theorem imply that the number of zeros of $G(\lambda; p)$ in $\mathbb{C}_+$ (i.e. the right-hand side of $\mathbb{C}$) can change only if a root appears on or crosses the imaginary axis. Furthermore, for any $(\lambda, p) \in D \times \Gamma_+$ the function (9) satisfies the Implicit Function Theorem extended to complex-valued functions (e.g. [10, Theorem A.3, p.152]), This ensures the continuous dependence of the roots $\lambda$ of (8) upon $p$.

*Proof of Theorem 2.8.* $\mathbf{E}_+$ exists if and only if $\tau = (\tau_1, \tau_2) \in \Omega_+$. As previously observed, the associated characteristic roots $\lambda$ depend continuously on all parameters

$p \in \Gamma_+$ and their multiplicities in $\mathbb{C}_+$ can change only if at least one root appears on or crosses the imaginary axis.

By Theorem 2.6 we know that the global attractivity of $\mathbf{E}_+$ holds if $\frac{\beta}{\alpha} < \mu_1$. Then all roots $\lambda$ satisfy $\Re(\lambda) \leq 0$ for all $\tau \in \Omega_+$ if $\frac{\beta}{\alpha} < \mu_1$.

Our aim here is to prove that for all $p \in \Gamma_+$ the characteristic roots $\lambda$ cannot reach the imaginary axis. This implies that if $\frac{\beta}{\alpha} < \mu_1$ then all charactreristic roots $\lambda$ satisfy $\Re(\lambda) < 0$ for all delays $\tau \in \Omega_+$, and that all the characteristic roots remain with $\Re(\lambda) < 0$ for all $p \in \Gamma_+$.

Let us first notice from (8) that, for all $p \in \Gamma_+$,

$$G(0;p) = \frac{\mu_2\mu_1(\beta + \alpha\mu_1)}{\beta + \alpha\mu_1\mathcal{R}_0}(\mathcal{R}_0 - 1) > 0$$

since $\mathbf{E}_+$ exists iff $\mathcal{R}_0 > 1$. Thus, we exclude that $\lambda = 0$ is a root. Therefore, thanks to the simmetry property (ii), it remains to prove that $G(\mathrm{i}\omega, p) \neq 0$ for $\omega \in \mathbb{R}_+$ and $p \in \Gamma_+$. Let then $\lambda = \mathrm{i}\omega$, $\omega \in \mathbb{R}_+$, which turns (8) into

$$\begin{aligned}(\mu_2 + \mathrm{i}\omega)(\mu_1 + g(I^*) + \mathrm{i}\omega) \;-\; &(\mu_1 + \mathrm{i}\omega)g'(I^*)S^*e^{-(\mu_2+\mathrm{i}\omega)\tau_1} \\ +\; &(\mu_1 + \mathrm{i}\omega)g'(I^*)S^*e^{-(\mu_2+\mathrm{i}\omega)(\tau_1+\tau_2)} = 0\end{aligned} \tag{13}$$

For the sake of simplicity, set $c_1 = c_1(\omega) := \cos(\omega\tau_1)$, $c_2 = c_2(\omega) := \cos(\omega\tau_2)$ and $c_{12} = c_{12}(\omega) := \cos(\omega(\tau_1 + \tau_2))$ and, similarly, $s_1$, $s_2$ and $s_{12}$ for the sinus. Further, let be $e_1 := e^{-\mu_2\tau_1}$ and $e_2 := e^{-\mu_2\tau_2}$. By using Euler's formula and by separating real and imaginary parts in (13) we get that if $\omega$ satisfies (13) then it must satisfy

$$\begin{cases} \mu_2(\mu_1 + g(I^*)) - \omega^2 = g'(I^*)S^*e_1[\mu_1c_1 + \omega s_1 - e_2(\mu_1c_{12} + \omega s_{12})] \\ \omega(\mu_2 + \mu_1 + g(I^*)) = g'(I^*)S^*e_1[\omega c_1 - \mu_1 s_1 - e_2(\omega c_{12} - \mu_1 s_{12})]. \end{cases}$$

If there exists $\omega$ satisfying both the above equations, then it must also satisfy the following one obtained by squaring and summing them member to member (of course, the condition is necessary but not sufficient):

$$\begin{aligned}[\mu_2(\mu_1 + g(I^*)) \;-\; &\omega^2]^2 + [\omega(\mu_2 + \mu_1 + g(I^*))]^2 \\ =\; &[g'(I^*)S^*e_1]^2\big\{[\mu_1c_1 + \omega s_1 - e_2(\mu_1c_{12} + \omega s_{12})]^2 \\ &+[\omega c_1 - \mu_1 s_1 - e_2(\omega c_{12} - \mu_1 s_{12})]^2\big\}.\end{aligned} \tag{14}$$

As for the left-hand side of (14) we obtain

$$\begin{aligned}[\mu_2(\mu_1 + g(I^*)) \;-\; &\omega^2]^2 + [\omega(\mu_2 + \mu_1 + g(I^*))]^2 \\ =\; &\mu_2^2(\mu_1 + g(I^*))^2 + \omega^4 - 2\omega^2\mu_2(\mu_1 + g(I^*)) \\ &+\omega^2\left[\mu_2^2 + (\mu_1 + g(I^*))^2 + 2\mu_2(\mu_1 + g(I^*))\right] \\ =\; &\mu_2^2(\mu_1 + g(I^*))^2 + \omega^4 + \omega^2\left[\mu_2^2 + (\mu_1 + g(I^*))^2\right] \\ =\; &(\mu_2^2 + \omega^2)\left[(\mu_1 + g(I^*))^2 + \omega^2\right].\end{aligned}$$

As for the right-hand side of (14), by neglecting the costant $[g'(I^*)S^*e_1]^2$ at a first time, we obtain

$$\begin{aligned}[\mu_1c_1 + \omega s_1 \;-\; &e_2(\mu_1c_{12} + \omega s_{12})]^2 + [\omega c_1 - \mu_1 s_1 - e_2(\omega c_{12} - \mu_1 s_{12})]^2 \\ =\; &[\mu_1^2c_1^2 + \omega^2s_1^2 + e_2^2\mu_1^2c_{12}^2 + e_2^2\omega^2s_{12}^2 + 2\mu_1\omega c_1s_1 \\ &-2\mu_1^2e_2c_1c_{12} - 2\mu_1\omega e_2c_1s_{12} - 2\mu_1\omega e_2s_1c_{12} - 2\omega^2e_2s_1s_{12} \\ &+2\mu_1\omega e_2^2c_{12}s_{12}] + [\omega^2c_1^2 + \mu_1^2s_1^2 + e_2^2\omega^2c_{12}^2 + e_2^2\mu_1^2s_{12}^2 \\ &-2\mu_1\omega c_1s_1 - 2\omega^2e_2c_1c_{12} + 2\mu_1\omega e_2c_1s_{12} + 2\mu_1\omega e_2s_1c_{12} \\ &-2\mu_1^2e_2s_1s_{12} - 2\mu_1\omega e_2^2c_{12}s_{12}] \\ =\; &(\mu_1^2 + \omega^2)(c_1^2 + s_1^2) + e_2^2(\mu_1^2 + \omega^2)(c_{12}^2 + s_{12}^2) \\ &-2\mu_1^2e_2(c_1c_{12} + s_1s_{12}) - 2\omega^2e_2(c_1c_{12} + s_1s_{12}).\end{aligned}$$

Since $c_1^2 + s_1^2 = 1$, $c_{12}^2 + s_{12}^2 = 1$ and $c_1 c_{12} + s_1 s_{12} = c_2$, the latter reads

$$[\mu_1 c_1 + \omega s_1 - e_2(\mu_1 c_{12} + \omega s_{12})]^2 + [\omega c_1 - \mu_1 s_1 - e_2(\omega c_{12} - \mu_1 s_{12})]^2 = (\mu_1^2 + \omega^2)(1 + e_2^2 - 2e_2 c_2).$$

Therefore (14) becomes

$$(\mu_2^2 + \omega^2)[(\mu_1 + g(I^*))^2 + \omega^2] = [g'(I^*)S^* e_1]^2(\mu_1^2 + \omega^2)(1 + e_2^2 - 2e_2 c_2). \qquad (15)$$

Now, by substituting (6), (10) and $c_2 = \cos(\omega\tau_2)$ in (15) we get

$$(\mu_2^2 + \omega^2)\left[\left(\mu_1 + \frac{\beta\mu_1(\mathcal{R}_0 - 1)}{\beta + \alpha\mu_1\mathcal{R}_0}\right)^2 + \omega^2\right] = \left[\frac{\beta\Lambda}{\mu_1}\frac{\beta + \alpha\mu_1}{\mathcal{R}_0(\beta + \alpha\mu_1\mathcal{R}_0)}e_1\right]^2 (\mu_1^2 + \omega^2)(1 + e_2^2 - 2e_2\cos(\omega\tau_2))$$

that, after some manipulations, can be rewritten as

$$\left[1 + \left(\frac{\omega}{\mu_2}\right)^2\right]\frac{\mu_1^2\mathcal{R}_0^2 + \omega^2\left(\frac{\beta+\alpha\mu_1\mathcal{R}_0}{\beta+\alpha\mu_1}\right)^2}{\mu_1^2 + \omega^2} = \frac{1 + e_2^2 - 2e_2\cos(\omega\tau_2)}{(1 - e_2)^2}. \qquad (16)$$

Let us now define the functions

$$H(\omega) := \frac{1 + e_2^2 - 2e_2\cos(\omega\tau_2)}{(1 - e_2)^2}$$

for the right member of (16) and

$$K(\omega) := L(\omega)\frac{\mu_1^2\mathcal{R}_0^2 + \omega^2\left(\frac{\beta+\alpha\mu_1\mathcal{R}_0}{\beta+\alpha\mu_1}\right)^2}{\mu_1^2 + \omega^2}$$

with

$$L(\omega) := 1 + \left(\frac{\omega}{\mu_2}\right)^2$$

for the left one. It is not difficult to see that in $\Omega_+$ where $\mathcal{R}_0 > 1$, it always holds $K(\omega) > L(\omega)$. If we are able to prove that $L(\omega) > H(\omega)$, then the equation $K(\omega) = H(\omega)$, i.e. (16), can never be verified in $\Omega_+$. This leads to the absurd that $\mathrm{i}\omega$ can solve the characteristic equation (8).

To this aim let us observe that $H$ is periodic with period $2\pi/\tau_2$ and always assume values in $\left[1, \left(\frac{1+e_2}{1-e_2}\right)^2\right]$, with $H(0) = 1$ and $H(\pi/\tau_2) = \left(\frac{1+e_2}{1-e_2}\right)^2$. Further $H'(\omega) = \frac{2\tau_2 e_2 \sin(\omega\tau_2)}{(1-e_2)^2}$, whereas $L(\omega)$ is trivially increasing with $L'(\omega) = 2\omega/\mu_2^2$. Hence $H'(0) = L'(0) = 0$, and both functions start from 1 in $\omega = 0$ (remember that, by simmetry, we are analysing only the case $\omega \geq 0$) with horizontal tangent to grow up to a maximum in $\omega = \pi/\tau_2$ for $H$ and indefinitely for $L$. Since $L''(\omega) = \frac{2}{\mu_2^2}$ is constant, whereas $H''(\omega) = \frac{2\tau_2^2 e_2 \cos(\omega\tau_2)}{(1-e_2)^2}$ is decreasing for $\omega \in [0, \pi/\tau_2]$, with $H''(0) = \frac{2\tau_2^2 e_2}{(1-e_2)^2}$, a sufficient condition to ensure that $L(\omega) > H(\omega)$ always holds is $L''(0) > H''(0)$. Now, the difference $L''(0) - H''0) = \frac{2}{\mu_2^2} - \frac{2\tau_2^2 e_2}{(1-e_2)^2}$ is, unless for multiplicative constants, a function of the parameters product $\mu_2\tau_2$. In particular, $\frac{\mu_2^2}{2}[L''(0) - H''(0)] = 1 - \frac{(\mu_2\tau_2)^2 e_2}{(1-e_2)^2} = \phi(\mu_2\tau_2)$. By the change of variable $x = \mu_2\tau_2$ we obtain $\phi(x) = 1 - \frac{x^2 e^{-x}}{(1-e^{-x})^2}$, which is asymptotically increasing from zero to 1 in $(0, +\infty)$, Figure 2. Since $x = 0$ is excluded in $\Omega_+$, we obtain $\phi(x) > 0$ for any $x > 0$

and, therefore, $L''(0) - H''(0) > 0$ which in sequence implies $L(\omega) > H(\omega)$, $K(\omega) > H(\omega)$ and the absurd $G(\mathrm{i}\omega, p) = 0$, $p \in \Gamma_+$, from which the thesis follows. □

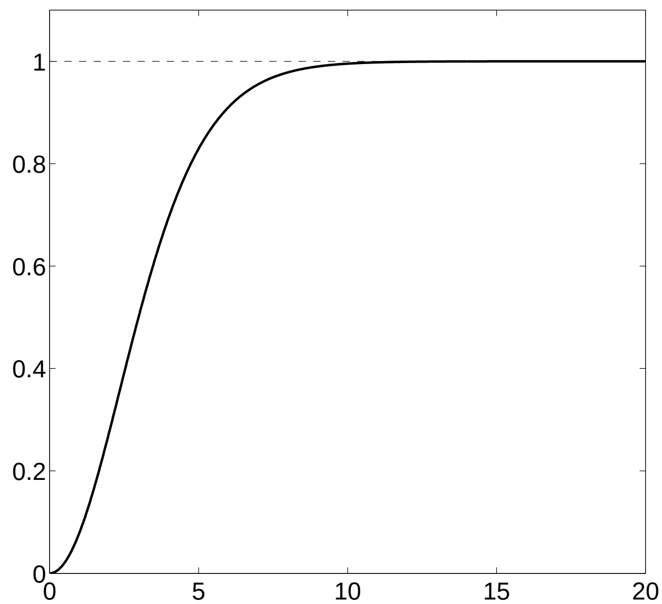


FIGURE 2. the function $\phi$ used in the proof of Theorem 2.8.

4. **Permanence of the solutions.** In this and in the forthcoming section we make use of the following Lemmas (e.g. [3, pp.83-84]).

**Lemma 4.1** (Fatou Lemma)**.** *Let* $\{f_n\}_{n\in\mathbb{N}_0}$ *be a measurable sequence of non-negative functions defined on a measurable set* $\Omega$*. Then*

$$\int_\Omega \liminf_{n\to+\infty} f_n dx \le \liminf_{n\to+\infty} \int_\Omega f_n dx.$$

**Lemma 4.2** (Inverse Fatou Lemma)**.** *Let* $\{f_n\}_{n\in\mathbb{N}_0}$ *be a measurable sequence of functions defined on a measurable set* $\Omega$*. If there exists a non-negative integrable function* $g$ *defined on* $\Omega$ *and such that* $f_n \le g$ *on* $\Omega$ *for all* $n$*, then*

$$\limsup_{n\to+\infty} \int_\Omega f_n dx \le \int_\Omega \limsup_{n\to+\infty} f_n dx.$$

About the permanence of the solutions of system (3), we first prove the following result.

**Lemma 4.3.** *Under Assumtion (A.2), if* $\mathcal{R}_0 > 1$ *then all solutions of (3) are ultimately bounded in the compact set*

$$\Omega_S := \left\{(S, E, I, R) \in \mathbb{R}^4_{+0} : S + E + I + R \le \frac{\Lambda}{\mu_1}, S \ge \nu_S\right\}$$

*where*

$$\nu_S = \frac{\Lambda}{\mu_1}\frac{\mu_1\alpha\mathcal{R}_0}{\alpha\mu_1\mathcal{R}_0+\beta(\mathcal{R}_0-1)}.$$

*Proof.* From (A.2) and the positivity of the solutions we have

$$\limsup_{t\to+\infty} S(t) \le \frac{\Lambda}{\mu_1}. \tag{17}$$

By (2), the third equation in (4) and Lemma 4.2 we obtain

$$\limsup_{t\to+\infty} I(t) \le \int_{\tau_1}^{\tau_1+\tau_2} \beta \limsup_{t\to+\infty}\left(\frac{I(t-\theta)S(t-\theta)}{1+\alpha I(t-\theta)}\right) e^{-\mu_2\theta}d\theta. \tag{18}$$

From (17) and (18) it follows that

$$\limsup_{t\to+\infty} I(t) \le \frac{\limsup_{t\to+\infty} I(t)}{1+\alpha\limsup_{t\to+\infty} I(t)}\mathcal{R}_0.$$

Being $\limsup_{t\to+\infty} I(t) > 0$ (if not, i.e. $\lim_{t\to+\infty} I(t) = 0$, then the global attractivity of the DFE $\mathbf{E}_0$ follows from Theorem 2.4 in contradiction to the existence of $\mathbf{E}_+$), then

$$\limsup_{t\to+\infty} I(t) \le \frac{\mathcal{R}_0-1}{\alpha}.$$

By the first equation in (3) and this latter inequality, for sufficiently large times we get

$$\frac{dS(t)}{dt} \ge \Lambda - \left[g\left(\frac{\mathcal{R}_0-1}{\alpha}\right)+\mu_1\right]S(t)$$

and, by the comparison principle,

$$\liminf_{t\to+\infty} S(t) \ge \frac{\Lambda}{\mu_1}\frac{\mu_1}{g\left(\frac{\mathcal{R}_0-1}{\alpha}\right)+\mu_1} =: \nu_S$$

which completes the proof. □

Since the solutions of (3) are ultimately bounded in $\Omega_S$, without loss of generality we assume that the initial conditions belong to $\Omega_S$. Now we are ready to prove Theorem 2.7.

*Proof of Theorem 2.7.* Since (3) is ultimately bounded in $\Omega_S$, it is sufficient to prove that there exist positive constants $\nu_I$, $\nu_E$ and $\nu_R$ such that

$$\liminf_{t\to+\infty} I(t) \ge \nu_I, \quad \liminf_{t\to+\infty} E(t) \ge \nu_E, \quad \liminf_{t\to+\infty} R(t) \ge \nu_R.$$

The key point is to prove the first inequality, from which the other two easily follow. We start by proving that $\liminf_{t\to+\infty} I(t) > 0$. According to Lemma 4.3 we therefore give continuous and positive initial conditions $S(\theta) = \varphi_1(\theta) \ge \nu_S$ and $I(\theta) = \varphi_2(\theta) \ge \varepsilon_I > 0$ for $\theta \in \Omega_0 := [-(\tau_1+\tau_2), 0]$, where we have set

$$\varepsilon_I := \min_{\theta\in\Omega_0} I(\theta). \tag{19}$$

Denoted by $\Sigma_n := [n\tau_1, (n+1)\tau_1]\,, n \in \mathbb{N}_0$, we consider the positive real axis of times $\mathbb{R}_{+0}$ as covered by the union of these infinitely many intervals $\Sigma_n$ in such a way that

$$\mathbb{R}_{+0} = \bigcup_{n=0}^{+\infty} \Sigma_n.$$

By the third equation in (4) we can write

$$I(t) = \int_{-(\tau_1+\tau_2)}^{-\tau_1} g(I(t+u))S(t+u)e^{\mu_2 u}du \tag{20}$$

and we see that $t \in \Sigma_n$ implies $t+u \in \Omega_n := [(n-1)\tau_1 - \tau_2, n\tau_1]$ since $u \in [-(\tau_1+\tau_2), -\tau_1]$. With $k$ depending upon the value of $\tau_2$ in relation to that of $\tau_1$ (see Appendix A), we have

$$\begin{cases} \Omega_n \subseteq \bigcup_{j=0}^{k} \Sigma_{n-j} \\ \Omega_n \cap \Sigma_{n-j} \neq \phi,\ j = 0, 1, \ldots, k,\ n \geq k. \end{cases} \tag{21}$$

From (20) we have that if $t \in \Sigma_n$ then

$$I(t) \geq g\left(\min_{t+u\in\Omega_n} I(t+u)\right) \frac{\nu_S}{\mu_2} e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2})$$

from which, by defining

$$\Phi(\mathcal{R}_0) := \frac{\beta\nu_S}{\mu_2} e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2}) = \frac{\mu_1\alpha\mathcal{R}_0^2}{\alpha\mu_1\mathcal{R}_0 + \beta(\mathcal{R}_0 - 1)}, \tag{22}$$

we obtain

$$I(t) \geq \frac{g\left(\min_{t+u\in\Omega_n} I(t+u)\right)}{\beta} \Phi(\mathcal{R}_0). \tag{23}$$

Denoting by $\mathbf{I}_n := \min_{t\in\Sigma_n} I(t)$ and by $\varepsilon_n$ a constant lower bound for $I(t)$ over $\Sigma_n$ such that $\mathbf{I}_n \geq \varepsilon_n$, $n \in \mathbb{N}_0$, we can define the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ of lower bounds for $I(t)$

over $\Sigma_n$ according to (19), (21), (22) and (23):

$$\begin{aligned}
t\in\Sigma_0\ ,\ I(t) &\geq \frac{g\left(\min\limits_{t+u\in\Omega_0} I(t+u)\right)}{\beta}\Phi(\mathcal{R}_0) = \frac{g\left(\varepsilon_I\right)}{\beta}\Phi(\mathcal{R}_0) =: \varepsilon_0\\
t\in\Sigma_1\ ,\ I(t) &\geq \frac{g\left(\min\limits_{t+u\in\Omega_1} I(t+u)\right)}{\beta}\Phi(\mathcal{R}_0)\\
&\geq \frac{g\left(\min\left\{\varepsilon_I,\varepsilon_0\right\}\right)}{\beta}\Phi(\mathcal{R}_0) =: \varepsilon_1\\
\dots\\
t\in\Sigma_n\ ,\ I(t) &\geq \frac{g\left(\min\limits_{t+u\in\Omega_n} I(t+u)\right)}{\beta}\Phi(\mathcal{R}_0)\\
&\geq \frac{g\left(\min\left\{\mathbf{I}_{n-1},\mathbf{I}_{n-2},\dots,\mathbf{I}_{n-k}\right\}\right)}{\beta}\Phi(\mathcal{R}_0)\\
&\geq \frac{g\left(\min\left\{\varepsilon_{n-1},\varepsilon_{n-2},\dots,\varepsilon_{n-k}\right\}\right)}{\beta}\Phi(\mathcal{R}_0) =: \varepsilon_n
\end{aligned} \tag{24}$$

for $n\in\mathbb{N}_0$ such that $n\geq k$. Of course, $t\to+\infty$ iff $n\to+\infty$.

Now let us notice that the following inequalities are equivalent:

$$\begin{cases}\varepsilon_I\leq\varepsilon_0\Leftrightarrow\varepsilon_I\leq\varepsilon_c\\ \varepsilon_I>\varepsilon_0\Leftrightarrow\varepsilon_I>\varepsilon_c.\end{cases} \tag{25}$$

where

$$\varepsilon_c := \frac{1}{\alpha}\left[\Phi(\mathcal{R}_0)-1\right] = \frac{\mu_1(\mathcal{R}_0-1)\left(\mathcal{R}_0-\frac{\beta}{\alpha\mu_1}\right)}{\alpha\mu_1\mathcal{R}_0+\beta(\mathcal{R}_0-1)}$$

by (22). This shows that $\varepsilon_c>0$ if $\mathcal{R}_0>1$ and $\frac{\beta}{\alpha\mu_1}<\mathcal{R}_0$. According to (24) and (25), for the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ we have the following results:

(i) if $\varepsilon_I\leq\varepsilon_c$ then $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ is non-decreasing, i.e. $\varepsilon_n\geq\varepsilon_0>0$ for all $n\in\mathbb{N}_0$;

(ii) if $\varepsilon_I>\varepsilon_c$ then $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ is strictly decreasing, i.e. $\varepsilon_n>\varepsilon_{n+1}$ for all $n\in\mathbb{N}_0$.

Now we prove that in case (ii) the strictly decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ must satisfy $\lim\limits_{n\to+\infty}\varepsilon_n\geq\varepsilon_c$. If not (i.e. if $\lim\limits_{n\to+\infty}\varepsilon_n<\varepsilon_c$), there exists $n^*\in\mathbb{N}_0$ such that $\varepsilon_{n^*}<\varepsilon_c=\frac{1}{\alpha}[\Phi(\mathcal{R}_0)-1]$, which implies $\varepsilon_{n^*}<\frac{g(\varepsilon_{n^*})}{\beta}\Phi(\mathcal{R}_0)$. Since $\{\varepsilon_n\}$ is strictly decreasing and $\varepsilon_{n^*+1}=\frac{g(\varepsilon_{n^*})}{\beta}\Phi(\mathcal{R}_0)$, we get to the contradiction $\varepsilon_{n^*}<\varepsilon_{n^*+1}$. Therefore, we can conclude that in either case (i) and (ii)

$$\lim_{n\to+\infty}\varepsilon_n>0\Rightarrow\liminf_{t\to+\infty}I(t)>0 \tag{26}$$

holds.

Now, from (4), (20) and by Lemma 4.1 it follows

$$\begin{aligned}
\liminf_{t\to+\infty}I(t) &= \liminf_{t\to+\infty}\int_{-(\tau_1+\tau_2)}^{-\tau_1} g(I(t+u))S(t+u)e^{\mu_2 u}du\\
&\geq \frac{\beta\liminf\limits_{t\to+\infty}I(t)}{1+\alpha\liminf\limits_{t\to+\infty}I(t)}\liminf_{t\to+\infty}S(t).\frac{1}{\mu_2}e^{-\mu_2\tau_1}(1-e^{-\mu_2\tau_2}).
\end{aligned}$$

This latter, together with Lemma 4.3 and (22), provides

$$\liminf_{t\to+\infty} I(t) \geq \frac{\liminf_{t\to+\infty} I(t)}{1+\alpha \liminf_{t\to+\infty} I(t)} \Phi(\mathcal{R}_0)$$

and, by (26),

$$1+\alpha \liminf_{t\to+\infty} I(t) \geq \Phi(\mathcal{R}_0),$$

i.e.

$$\liminf_{t\to+\infty} I(t) \geq \frac{1}{\alpha}[\Phi(\mathcal{R}_0)-1] = \varepsilon_c =: \nu_I$$

which, again by Lemma 4.3, let the last differential equation in (3) imply that, for a sufficiently large time $T>0$,

$$\frac{dR(t)}{dt} \geq e^{-\mu_2(\tau_1+\tau_2)} g(\nu_I)\nu_S - \mu_3 R(t)$$

for all $t>T$. Then, the comparison principle implies

$$\liminf_{t\to+\infty} R(t) \geq \frac{e^{-\mu_2(\tau_1+\tau_2)} g(\nu_I)\nu_S}{\mu_3} =: \nu_R.$$

Finally, by applying Lemma 4.1 to the second equation in (4), i.e.

$$E(t) = \int_{-\tau_1}^{0} g(I(t+u))S(t+u)e^{\mu_2 u} du,$$

we obtain

$$\liminf_{t\to+\infty} E(t) \geq \frac{g(\nu_I)\nu_S(1-e^{-\mu_2\tau_1})}{\mu_2} =: \nu_E$$

which completes the proof of the permanence of system (3). □

5. **Global attractivity results.** In this section we present the proofs of Theorem 2.4 on the global attractivity of the DFE $\mathbf{E}_0$ when $\mathcal{R}_0 \leq 1$ and of Theorem 2.6 on the global attractivity of the positive equilibrium $\mathbf{E}_+$ when $\frac{\beta}{\alpha} < \mu_1$. The proofs are performed with the help of Lemmas 4.1 and 4.2 of Section 4 and by using comparison arguments which are close to those alredy used in the recent paper [11] by Xu and Du on a SIR model with one delay, i.e. constant infectious period. We extend their results to the SEIR model with two delays (3). In particular, in the proof of Theorem 2.6 we introduce the explicit dependence of the sequences of upper and lower bounds upon the basic reproduction number $\mathcal{R}_0$, then proving the decreasing strict monotonicity of the upper bounds and the increasing one for the lower bounds. Hence, herefollowing we recall the key points of the proofs leaving most of the details to the appropriate references in the above mentioned paper.

*Proof of Theorem 2.4.* By the structure of the model equations in (4) we see that if for all initial conditions we prove that

$$\lim_{t\to+\infty} I(t) = 0, \tag{27}$$

then it is easy to prove (see [11, Theorem 4.2, p.14]) that

$$\lim_{t\to+\infty} S(t) = \frac{\Lambda}{\mu_1}$$

and

$$\lim_{t\to+\infty} R(t) = \lim_{t\to+\infty} E(t) = 0$$

and, therefore,

$$\lim_{t\to+\infty} (S(t), E(t), I(t), R(t)) = \mathbf{E}_0.$$

By virtue of the positivity of the solutions, to prove (27) it is sufficient to prove that $\limsup\limits_{t\to+\infty} I(t) = 0$ whenever $\mathcal{R}_0 \le 1$. To this end, from the third equation in (4) and by the use of Lemma 4.2 we have

$$\begin{aligned}
\limsup_{t\to+\infty} I(t) &= \limsup_{t\to+\infty} \int_{\tau_1}^{\tau_1+\tau_2} g(I(t-\theta))S(t-\theta)e^{-\mu_2\theta}d\theta \\
&\le \int_{\tau_1}^{\tau_1+\tau_2} \frac{\beta \limsup\limits_{t\to+\infty} I(t)}{1+\alpha \limsup\limits_{t\to+\infty} I(t)} \limsup_{t\to+\infty} S(t)e^{-\mu_2\theta}d\theta \\
&\le \frac{\Lambda\beta}{\mu_1} \frac{\limsup\limits_{t\to+\infty} I(t)}{1+\alpha \limsup\limits_{t\to+\infty} I(t)} \int_{\tau_1}^{\tau_1+\tau_2} e^{-\mu_2\theta}d\theta \\
&= \frac{\limsup\limits_{t\to+\infty} I(t)}{1+\alpha \limsup\limits_{t\to+\infty} I(t)} \mathcal{R}_0.
\end{aligned}$$

Then

$$\frac{\limsup\limits_{t\to+\infty} I(t)}{1+\alpha \limsup\limits_{t\to+\infty} I(t)} \left[\mathcal{R}_0 - 1 - \alpha \limsup_{t\to+\infty} I(t)\right] \ge 0. \tag{28}$$

In this latter, the left-hand side is negative for $\mathcal{R}_0 \le 1$, unless for requiring $\limsup\limits_{t\to+\infty} I(t) = 0$, which satisfies (28) with the equality sign. This proves the assertion. □

*Proof of Theorem 2.6.* Let us denote

$$\begin{aligned}
\overline{\mathbf{S}} &:= \limsup_{t\to+\infty} S(t), & \overline{\mathbf{I}} &:= \limsup_{t\to+\infty} I(t), & \overline{\mathbf{R}} &:= \limsup_{t\to+\infty} R(t),\\
\underline{\mathbf{S}} &:= \liminf_{t\to+\infty} S(t), & \underline{\mathbf{I}} &:= \liminf_{t\to+\infty} I(t), & \underline{\mathbf{R}} &:= \liminf_{t\to+\infty} R(t).
\end{aligned}$$

We have to prove that

$$\overline{\mathbf{S}} = \underline{\mathbf{S}} = S_+, \quad \overline{\mathbf{I}} = \underline{\mathbf{I}} = I_+, \quad \overline{\mathbf{R}} = \underline{\mathbf{R}} = R_+.$$

We proceed by constructing sequences $\{\overline{S}_n\}_{n\in\mathbb{N}_0}$, $\{\overline{I}_n\}_{n\in\mathbb{N}_0}$ and $\{\overline{R}_n\}_{n\in\mathbb{N}_0}$ of upper bounds

$$\overline{\mathbf{S}} \le \overline{S}_n, \quad \overline{\mathbf{I}} \le \overline{I}_n, \quad \overline{\mathbf{R}} \le \overline{R}_n$$

which are strictly decreasing, and sequences $\{\underline{S}_n\}_{n\in\mathbb{N}_0}$, $\{\underline{I}_n\}_{n\in\mathbb{N}_0}$, $\{\underline{R}_n\}_{n\in\mathbb{N}_0}$ of lower bounds

$$\underline{S}_n \le \underline{\mathbf{S}}, \quad \underline{I}_n \le \underline{\mathbf{I}}, \quad \underline{R}_n \le \underline{\mathbf{R}}$$

which are strictly increasing, satisfying

$$\lim_{n\to+\infty} \underline{S}_n = S_+ = \lim_{n\to+\infty} \overline{S}_n,$$

$$\lim_{n\to+\infty} \underline{I}_n = I_+ = \lim_{n\to+\infty} \overline{I}_n$$

and

$$\lim_{n\to+\infty} \underline{R}_n = R_+ = \lim_{n\to+\infty} \overline{R}_n.$$

Since $n \to +\infty$ implies that $t \to +\infty$, then:

$$\lim_{t\to+\infty} (S(t), E(t), I(t), R(t)) = \mathbf{E}_+.$$

In order to construct the above sequences (for the details see [11, Theorem 4.1, p.11]) we start with

$$\limsup_{t\to+\infty} S(t) \leq \frac{\Lambda}{\mu_1} =: \overline{S}_1. \tag{29}$$

From the third equation in (4) it follows

$$\limsup_{t\to+\infty} I(t) \leq \int_{\tau_1}^{\tau_1+\tau_2} \frac{\beta \limsup_{t\to+\infty} I(t)}{1+\alpha \limsup_{t\to+\infty} I(t)} \, \overline{S}_1 e^{-\mu_2\theta} d\theta$$

and we obtain

$$\overline{I}_1 = \frac{1}{\alpha}(\mathcal{R}_0 - 1). \tag{30}$$

Then, from the first equation in (4),

$$\frac{dS(t)}{dt} \geq [\Lambda - g(\overline{I}_1)\overline{S}_1] - \mu_1 S(t)$$

we obtain

$$\underline{S}_1 = \frac{1}{\mu_1}[\Lambda - g(\overline{I}_1)\overline{S}_1]. \tag{31}$$

Again from the third equation in (4) and by the permanence result which ensures $\liminf_{t\to\infty} I(t) > 0$, we have

$$\liminf_{t\to\infty} I(t) \geq \int_{\tau_1}^{\tau_1+\tau_2} \frac{\beta \liminf_{t\to\infty} I(t)}{1+\alpha \liminf_{t\to\infty} I(t)} \underline{S}_1 \, e^{-\mu_2\theta} d\theta$$

from which we obtain

$$\underline{I}_1 = \frac{1}{\alpha}\left(\frac{\underline{S}_1}{\overline{S}_1}\mathcal{R}_0 - 1\right). \tag{32}$$

Again, from the first equation in (4),

$$\frac{dS(t)}{dt} \leq [\Lambda - g(\underline{I}_1)\underline{S}_1] - \mu_1 S(t)$$

we obtain

$$\overline{S}_2 = \frac{1}{\mu_1}[\Lambda - g(\underline{I}_1)\underline{S}_1]. \tag{33}$$

Of course, from the last equation in (3) we get

$$\begin{cases} \limsup_{t\to+\infty} R(t) \leq \dfrac{1}{\mu_3}[e^{-\mu_3(\tau_1+\tau_2)} g(\overline{I}_n)\overline{S}_n] = \overline{R}_n \\ \liminf_{t\to+\infty} R(t) \geq \dfrac{1}{\mu_3}[e^{-\mu_3(\tau_1+\tau_2)} g(\underline{I}_n)\underline{S}_n] = \underline{R}_n \end{cases} \tag{34}$$

for $n \geq 1$.

By iterating the above procedure (29)-(34), we obtain six sequences $\{\overline{S}_n\}$, $\{\underline{S}_n\}$, $\{\overline{I}_n\}$, $\{\underline{I}_n\}$, $\{\overline{R}_n\}$ and $\{\underline{R}_n\}$, $n \in \mathbb{N}$, with elements

$$\begin{cases} \overline{S}_n &= \dfrac{1}{\mu_1}\left(\Lambda - \dfrac{\beta \underline{S}_{n-1}\underline{I}_{n-1}}{1+\alpha \underline{I}_{n-1}}\right),\ n \geq 2, \\ \underline{S}_n &= \dfrac{1}{\mu_1}\left(\Lambda - \dfrac{\beta \overline{S}_n \overline{I}_n}{1+\alpha \overline{I}_n}\right),\ n \geq 1, \end{cases} \tag{35}$$

where $\overline{S}_1$ is given by (29) and, for $n \geq 1$,

$$\begin{cases} \overline{I}_n &= \dfrac{1}{\alpha}\left(\mathcal{R}_0 \dfrac{\overline{S}_n}{\overline{S}_1} - 1\right) \\ \underline{I}_n &= \dfrac{1}{\alpha}\left(\mathcal{R}_0 \dfrac{\underline{S}_n}{\overline{S}_1} - 1\right) \end{cases} \tag{36}$$

$$\begin{cases} \overline{R}_n &= \dfrac{1}{\mu_3}\left(e^{-\mu_3(\tau_1+\tau_2)} \dfrac{\beta \overline{S}_n \overline{I}_n}{1+\alpha \overline{I}_n}\right) \\ \underline{R}_n &= \dfrac{1}{\mu_3}\left(e^{-\mu_3(\tau_1+\tau_2)} \dfrac{\beta \underline{S}_n \underline{I}_n}{1+\alpha \underline{I}_n}\right). \end{cases} \tag{37}$$

Notice that

$$\begin{cases} \underline{S}_1 &= \overline{S}_1\left(1 - \dfrac{\beta}{\alpha\mu_1}\dfrac{\mathcal{R}_0 - 1}{\mathcal{R}_0}\right) > 0 \\ \underline{I}_1 &= \dfrac{1}{\alpha}\left(\dfrac{\underline{S}_1}{\overline{S}_1}\mathcal{R}_0 - 1\right) = \dfrac{1}{\alpha}(\mathcal{R}_0 - 1)\left(1 - \dfrac{\beta}{\alpha\mu_1}\right) > 0 \end{cases}$$

since $\mathcal{R}_0 > 1$ and $\frac{\beta}{\alpha\mu_1} < 1$. Furthermore we notice that if the sequence $\overline{S}_n$ is strictly decreasing, in the sequel $\overline{S}_n \downarrow$, the first in (36) implies $\overline{I}_n \downarrow$ and by the first of (37) $\overline{R}_n \downarrow$. If $\overline{S}_n \downarrow$ and $\overline{I}_n \downarrow$, then $\underline{S}_n \uparrow$ and this in turn implies $\underline{I}_n \uparrow$ and, by the second of (37), even $\underline{R}_n \uparrow$. About the sequence $\{\overline{S}_n\}_{n\in\mathbb{N}_0}$, by using (35) and (36), we get the recurrence formula:

$$\overline{S}_{n+1} = \overline{S}_1\left(1 - \frac{\beta}{\alpha\mu_1}\right)\left(1 + \frac{\beta}{\alpha\mu_1\mathcal{R}_0}\right) + \left(\frac{\beta}{\alpha\mu_1}\right)^2 \overline{S}_n,\ n = 1, 2, \ldots, \tag{38}$$

through which

$$\overline{S}_{n+1} - \overline{S}_n = \left(\frac{\beta}{\alpha\mu_1}\right)^2 (\overline{S}_n - \overline{S}_{n-1}), n = 2, 3, \ldots.$$

Then it is sufficient to prove that $\overline{S}_2 - \overline{S}_1 < 0$ to prove that $\overline{S}_{n+1} - \overline{S}_n < 0$ for all $n \geq 1$. To check this, we take (38) when $n = 1$ to obtain

$$\overline{S}_2 = \overline{S}_1\left(1 - \frac{\beta}{\alpha\mu_1}\right)\left(1 + \frac{\beta}{\alpha\mu_1\mathcal{R}_0}\right) + \left(\frac{\beta}{\alpha\mu_1}\right)^2 \overline{S}_1.$$

By the hypotheses $1 - \frac{\beta}{\alpha\mu_1} > 0$ and $\mathcal{R}_0 > 1$ we get

$$\overline{S}_2 < \overline{S}_1\left[1 - \left(\frac{\beta}{\alpha\mu_1}\right)^2\right] + \left(\frac{\beta}{\alpha\mu_1}\right)^2 \overline{S}_1 = \overline{S}_1.$$

Thus $\{\overline{S}_n\}_{n\in\mathbb{N}_0}$ is strictly decreasing and lower bounded. Therefore there exists $\lim\limits_{n\to+\infty} \overline{S}_n$ and we can compute it by (38) obtaining

$$\lim_{n\to+\infty} \overline{S}_n = \frac{\beta + \alpha\mu_1\mathcal{R}_0}{\mathcal{R}_0(\beta + \alpha\mu_1)}\frac{\Lambda}{\mu_1} = S_+.$$

(Notice instead that if $\frac{\beta}{\alpha\mu_1} > 1$, then $\overline{S}_n \uparrow$). From (36) it follows that

$$\lim_{n\to+\infty} \overline{I}_n = \frac{1}{\alpha}\left[\frac{\mathcal{R}_0}{\overline{S}_1}\frac{\beta+\alpha\mu_1\mathcal{R}_0}{\mathcal{R}_0(\beta+\alpha\mu_1)}\overline{S}_1 - 1\right] = \frac{\mu_1(\mathcal{R}_0-1)}{\beta+\alpha\mu_1} = I_+.$$

From (37) we can compute

$$\lim_{n\to+\infty} \underline{S}_n = \frac{1}{\mu_1}[\Lambda - g(I_+)S_+] = S_+. \tag{39}$$

Therefore

$$\overline{\mathbf{S}} = \underline{\mathbf{S}} = S_+ \Leftrightarrow \lim_{t\to+\infty} S(t) = S_+. \tag{40}$$

From (36) and (39) we obtain

$$\lim_{n\to+\infty} \underline{I}_n = \frac{1}{\alpha}\left(\frac{S_+}{\overline{S}_1}\mathcal{R}_0 - 1\right) = I_+. \tag{41}$$

Therefore

$$\overline{\mathbf{I}} = \underline{\mathbf{I}} = I_+ \Leftrightarrow \lim_{t\to+\infty} I(t) = I_+. \tag{42}$$

Now, from (37) and thanks to (40) and (42) we can easily prove

$$\overline{\mathbf{R}} = \underline{\mathbf{R}} = R_+ = \frac{1}{\mu_3}[e^{-\mu_3(\tau_1+\tau_2)}g(I_+)S_+] \Leftrightarrow \lim_{t\to+\infty} R(t) = R_+. \tag{43}$$

Finally, from the second equation in (4) and by Lemmas 4.1 and 4.2 we get

$$\begin{aligned}\liminf_{t\to+\infty} E(t) &\geq \int_0^{\tau_1} g(\liminf_{t\to+\infty} I(t-\theta))\liminf_{t\to+\infty} S(t-\theta)e^{-\mu_2\theta}d\theta \\ &= \frac{1}{\mu_2}g(I_+)S_+(1-e^{-\mu_2\tau_1}) \\ &= E_+\end{aligned}$$

and, similarly,

$$\limsup_{t\to+\infty} E(t) \leq \frac{1}{\mu_2}g(I_+)S_+(1-e^{-\mu_2\tau_1}) = E_+.$$

Therefore

$$\overline{\mathbf{E}} = \underline{\mathbf{E}} = E_+ \Leftrightarrow \lim_{t\to+\infty} E(t) = E_+,$$

completing the proof together with (40), (42) and (43) □

6. **Conclusions.** As already pointed out in the Introduction, though the model equations (3) are delay differential equations with delay dependent parameters, the delays just influence the existence delay domain $\Omega_+$ of the positive equilibrium $\mathbf{E}_+$, but are harmless to induce stability switches for example from asymptotic stability to instability within $\Omega_+$ (Theorem 2.8).

We think that the limitation in Theorem 2.6 to the global attractivity of $\mathbf{E}_+$ is only a technical result of the approach followed and that, perhaps, a different approach to the global asymptotic stability of $\mathbf{E}_+$ by Lyapunov functionals should be possible.

As far as the DFE equilibrium $\mathbf{E}_0$ is concerned, we see that all the classical results of the epidemic models hold true. If the basic reproduction number satisfies $\mathcal{R}_0 \leq 1$ then $\mathbf{E}_0$ is globally attractive (Theorem 2.4) and also locally asymptotically stable if $\mathcal{R}_0 < 1$ (Corollary 2), whereas if $\mathcal{R}_0 > 1$ then $\mathbf{E}_0$ becomes unstable (Theorem 2.5).

The epidemic model could also be improved for two aspects:

- by generalizing the infection rate, for example according to [7];
- by allowing the removed people to be reinfected, for example with a delay, thus generalizing the SEIRS model in [12].

However, this is left as a future work.

**Appendix** A. **Appendix to Section 4.** Assume $t \in \Sigma_n = [n\tau_1, (n+1)\tau_1]$ for all $n \in \mathbb{N}_0$. Then $t + u \in \Omega_n$ where $\Omega_n = [(n-1)\tau_1 - \tau_2, n\tau_1]$. For example assume that $\tau_2$ is such that $2\tau_1 < \tau_2 \le 3\tau_1$. Then we have

$$\Omega_0 = [-(\tau_1 + \tau_2), 0] \tag{44}$$

and

$$\Omega_1 = [-\tau_2, \tau_1] = [-\tau_2, 0] \cup [0, \tau_1] \subset \Omega_0 \cup \Sigma_0$$

which are true for any value of $\tau_2$, whereas according to (44)

$$\Omega_2 = [\tau_1 - \tau_2, 2\tau_1] = [\tau_1 - \tau_2, 0] \cup [0, \tau_1] \cup [\tau_1, 2\tau_1] \subset \Omega_0 \cup \Sigma_0 \cup \Sigma_1,$$

$$\Omega_3 = [2\tau_1 - \tau_2, 3\tau_1] = [2\tau_1 - \tau_2, 0] \cup [0, \tau_1] \cup [\tau_1, 2\tau_1] \cup [2\tau_1, 3\tau_1] \subset \Omega_0 \cup \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$$

and, in general, for all $n \ge 4$

$$\begin{aligned} \Omega_n = & [(n-1)\tau_1 - \tau_2, (n-3)\tau_1] \cup [(n-3)\tau_1, (n-2)\tau_1] \cup [(n-2)\tau_1, (n-1)\tau_1] \\ & \cup [(n-1)\tau_1, n\tau_1] \subseteq \Sigma_{n-4} \cup \Sigma_{n-3} \cup \Sigma_{n-2} \cup \Sigma_{n-1} \end{aligned}$$

where $\Omega_n \cap \Sigma_{n-k} \ne \phi$, $k = 1, 2, 3, 4$.

Then, according to (24), the sequence of lower bounds $\{\varepsilon_n\}$ for $I(t)$ on $\Sigma_n$ will be

$$\begin{aligned} & \varepsilon_I \\ \varepsilon_0 &= \frac{g(\varepsilon_I)}{\beta} \Phi(\mathcal{R}_0) \\ \varepsilon_1 &= \frac{1}{\beta} g(\min\{\varepsilon_I, \varepsilon_0\}) \Phi(\mathcal{R}_0) \\ \varepsilon_2 &= \frac{1}{\beta} g(\min\{\varepsilon_I, \varepsilon_0, \varepsilon_1\}) \Phi(\mathcal{R}_0) \\ \varepsilon_3 &= \frac{1}{\beta} g(\min\{\varepsilon_I, \varepsilon_0, \varepsilon_1, \varepsilon_2\}) \Phi(\mathcal{R}_0) \\ & \cdots \\ \varepsilon_n &= \frac{1}{\beta} g(\min\{\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_{n-4}\}) \Phi(\mathcal{R}_0), \ n \ge 4. \end{aligned} \tag{45}$$

Now assume

(i) $\varepsilon_I \le \varepsilon_0 \Leftrightarrow \varepsilon_I \le \varepsilon_c$. Then (45) implies:

$$\begin{aligned} \varepsilon_I &\le \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 \\ &\le \varepsilon_4 = \frac{1}{\beta} g(\min\{\varepsilon_3, \varepsilon_2, \varepsilon_1, \varepsilon_0\}) \Phi(\mathcal{R}_0) \le \cdots \\ &\le \varepsilon_n, \ n \ge 4, \end{aligned}$$

i.e. the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ is non-decreasing;

(ii) $\varepsilon_I > \varepsilon_0 \Leftrightarrow \varepsilon_I > \varepsilon_c$. Then (45) implies:

$$\begin{aligned} \varepsilon_I &> \varepsilon_0 > \varepsilon_1 = \tfrac{1}{\beta} g(\varepsilon_0) \Phi(\mathcal{R}_0) > \varepsilon_2 = \tfrac{1}{\beta} g(\varepsilon_1) \Phi(\mathcal{R}_0) > \cdots \\ &> \varepsilon_n = \frac{1}{\beta} g(\varepsilon_{n-1}) \Phi(\mathcal{R}_0), \ n \ge 4, \end{aligned}$$

i.e. the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}_0}$ is strictly decreasing.

Of course, this kind of arguments can be applied and hold true for any positive value of $\tau_2$.

### REFERENCES


[1] E. Beretta and Y. Kuang, *Geometric stability switch criteria in delay differential systems with delay dependent parameters,* SIAM J. Math. Anal., **33** (2002), 1144–1165.

[2] V. Capasso and G. Serio, *A generalization of the Kermack-McKendric deterministic epidemic model,* Math. Biosci., **42** (1978), 43–61.

[3] M. Giaquinta and G. Modica, "Mathematical Analysis. An Introduction to Functions of Several Variables," Birkhauser Boston, Inc., Boston, MA, 2009.

[4] G. Huang and Y. Takeuchi, *Global analysis on delay epidemiological dynamic models with nonlinear incidence,* J. Math. Biol., **63** (2011), 125–139.

[5] G. Huang, Y. Takeuchi and W. Ma, *Lyapunov functionals for delay differential equations model of viral infection,* SIAM J. Appl. Math., **70** (2010), 2693–2708.

[6] G. Huang, Y. Takeuchi, W. Ma and D. Wei, *Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate,* Bull. Math. Biol., **72** (2010), 1192–1207.

[7] A. Korobeinikov, *Global properties of infectious disease models with nonlinear incidence,* Bull. Math. Biol., **69** (2007), 1871–1886.

[8] Y. Kuang, "Delay Differential Equations with Application in Population Dynamics," Dynamics in Science and Engineering, Academic Press, New York, 1993.

[9] M. A. Safi and A. B. Gumel, *Global asymptotic dynamics of a model of quarantine and isolation,* Discrete Contin. Dyn. S., **14** (2010), 209–231.

[10] H. L. Smith, "An Introduction to Delay Differential Equations with Applications to the Life Sciences," Texts in Applied Mathematics, Springer, New York, 2011.

[11] R. Xu and Y. Du, *A delayed SIR epidemic model with saturation incidence and constant infectious period,* J. Appl. Math. Comp., **35** (2010), 229–250.

[12] R. Xu and Z. Ma, *Global stability of a delayed SEIRS epidemic model with saturation incidence rate*, Nonlinear Dynam., **61** (2010), 229–239.

[13] F. Zhang, Z. Li and F. Zhang, *Global stability of an SIR epidemic model with constant infectious period,* Appl. Math. Comput, **199** (2008), 285–291.


Received September 22, 2010; Accepted March 28, 2011.

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