STABILITY OF A DELAY EQUATION ARISING FROM A JUVENILE-ADULT MODEL

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(Communicated by Yang Kuang)

Abstract. We consider a delay equation that has been formulated from a juvenile-adult population model. We give respective conditions on the vital rates to ensure local stability of the positive equilibrium and global stability of the trivial equilibrium. We also show that under certain conditions the equation undergoes a Hopf bifurcation. We then study global asymptotic stability and present bifurcation diagrams for two special cases of the model.

1. **Introduction.** In this paper, we consider the following delay equation arising from a juvenile-adult population model that we developed in [2]:

$$\varphi'(t) = e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}\beta(\varphi(t-\bar{a}))\varphi(t-\bar{a}) - \mu(\varphi(t))\varphi(t) \qquad \text{for } t > \bar{a}, \qquad (1.1)$$

where $\nu \in L^\infty(0,\bar{a})$ is nonnegative, β is nonnegative and continuously differentiable with $\beta' \leq 0$, and μ is nonnegative and continuously differentiable with $\mu' \geq 0$. We establish stability results for equation (1.1) and study the existence of local Hopf bifurcation. We also investigate global asymptotic stability and present bifurcation diagrams for two special cases of the model.

Over the past few years, the following delay equation has been extensively studied in the literature:

$$y'(t) = \sigma f(y(t-\tau)) - \gamma y(t), \qquad (1.2)$$

where σ and γ are positive constants. When f takes certain special forms, (1.2) represents several biological models. For example, when f takes the forms

$$f = \frac{\delta}{\delta + y^n(t-\tau)} \quad \text{and} \quad f = \frac{\delta y(t-\tau)}{\delta + y^n(t-\tau)} \qquad (1.3)$$

equation (1.2) was proposed by Mackey and Glass [5, 18] to describe a physiological control system, where $y(t)$ denotes the density of mature cells in blood circulation at time t, and τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstreams. When f takes the form

$$f = e^{-\delta y(t-\tau)} \qquad (1.4)$$

equation (1.2) was proposed by Wazewska-Czyzewska and Lasota [24] to describe the survival of red blood cells in animals, where $y(t)$ denotes the number of red

blood cells at time t, and τ is the time required to produce a red blood cell. When f takes the form

$$f = y(t-\tau)e^{-\delta y(t-\tau)} \tag{1.5}$$

equation (1.2) was proposed by Gurney *et al.* [10] to describe the dynamics of Nicholson's blowflies, where $y(t)$ is the population of sexually mature adults, and τ is the time for all eggs to develop into sexually mature adults. Moreover, many results have been established on the global attractivity of the positive equilibrium and on the oscillatory behavior of the solution for (1.2) with f of the forms (1.3) and (1.4) [6, 7, 11, 12, 16, 17, 25]. On the other hand, when f satisfies conditions such as $\xi f(\xi) < 0$ for $\xi \neq 0$, f is bounded from above or below, and $f'(\xi) < 0$ for all ξ, a Poincaré-Bendixson theorem was established by Walther [22] and the stability analysis for periodic solutions was conducted by Skubachevskii and Walther [20, 21]. For details, the reader is referred to [15, 23] and the references cited therein.

It is also worth mentioning that in [13] a system of delay equations was derived from the McKendrick age-structured population model and stability results were established. Moreover, when the competition parameter is zero, the system reduces to (1.2) with $f = \beta(y(t-\tau))y(t-\tau)$. Meanwhile, a special case of (1.1) was studied in [4] with ν, μ being positive constants and $\beta(\varphi) = be^{-c\varphi}$, where b and c are positive constants.

There are two main differences between equations (1.1) and (1.2). First, in equation (1.1) the first term on the right-hand side contains $e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}$ which results in the dependence of the positive equilibrium on the delay parameter $\bar{a}$. Second, in equation (1.1) the second term on the right-hand side contains a function $\mu(\varphi(t))$, while it is a positive constant γ in equation (1.2). Due to the different structure of equation (1.1), the situation becomes more complicated, and certain techniques used in the above mentioned papers seem not applicable to (1.1).

The paper is organized as follows. In Section 2 we formulate equation (1.1) from a juvenile-adult population model. In Section 3 we give respective conditions on the vital rates in the model to ensure local stability of the positive equilibrium and global stability of the trivial equilibrium. We also show that under certain conditions the equation undergoes a Hopf bifurcation. In Section 4 we establish global asymptotic stability results and present bifurcation diagrams for two special cases of the model. Finally in Section 5 we make some concluding remarks.

2. **Derivation of the delay equation.** Motivated by an amphibian population of Green Treefrogs (*Hyla cinerea*), we recently developed a juvenile-adult population model (see [2]). We assumed that juveniles live in an environment with abundant resources and thus do not compete, while adults live in an environment with limiting resource and thus competition between them takes place. We then obtained the following system of partial differential equations:

$$
\begin{aligned}
&J_t(a,t) + J_a(a,t) + \nu(a)J(a,t) = 0, && 0 < a < \bar{a}, && 0 < t < T, \\
&A_t(x,t) + (g(x,t)A(x,t))_x + \mu(\varphi(t))A(x,t) = 0, && \underline{x} < x < \bar{x}, && 0 < t < T, \\
&J(0,t) = \beta(\varphi(t))\varphi(t), && && 0 < t < T, \\
&g(\underline{x},t)A(\underline{x},t) = J(\bar{a},t), && && 0 < t < T, \\
&J(a,0) = J_0(a), && 0 \le a \le \bar{a}, \\
&A(x,0) = A_0(x), && \underline{x} \le x \le \bar{x},
\end{aligned}
\tag{2.1}
$$

where $J(a,t)$ and $A(x,t)$ denote the density of juveniles of age a and adults of size x at time t, respectively, $\bar{a}$ denotes the age at which juveniles metamorphose into adults of minimum size $\underline{x}$, and $\bar{x}$ denotes the maximum size of adults. The function $\varphi(t) = \int_{\underline{x}}^{\bar{x}} A(x,t)dx$ is the total population of adults. The parameters ν and μ are mortality rates for juveniles and adults, respectively. The functions g and β are the growth and reproduction rates for adults, respectively.

Throughout the discussion we assumed that the parameters in (2.1) satisfy the following assumptions:

(A1) $g \in C^1([\underline{x}, \bar{x}] \times [0, T])$. Furthermore, $g(x,t) > 0$ for $(x,t) \in [\underline{x}, \bar{x}] \times [0, T]$ and $g(\bar{x}, t) = 0$ for $t \in [0, T]$.

(A2) $\nu \in L^\infty(0, \bar{a})$ is nonnegative.

(A3) μ is nonnegative and continuously differentiable with $\mu' \geq 0$.

(A4) β is nonnegative and continuously differentiable with $\beta' \leq 0$.

(A5) $J_0 \in L^\infty(0, \bar{a})$ is nonnegative.

(A6) $A_0 \in L^\infty(\underline{x}, \bar{x})$ is nonnegative.

In [2], we established the global existence and uniqueness results for model (2.1). Actually, $J(a,t)$ can be represented as follows:

$$J(a,t) = J_0(a-t)e^{-\int_0^t \nu(a-t+\tau)d\tau} \quad \text{if } t \leq a, \tag{2.2}$$

$$J(a,t) = \beta(\varphi(t-a))\varphi(t-a)e^{-\int_{t-a}^t \nu(a-t+\tau)d\tau} \quad \text{if } t > a. \tag{2.3}$$

Integrating the second equation of (2.1) in x from $\underline{x}$ to $\bar{x}$ and making use of (2.3), we have

$$\begin{aligned}
\varphi'(t) &= J(\bar{a}, t) - \mu(\varphi(t))\varphi(t) \\
&= e^{-\int_{t-\bar{a}}^t \nu(\bar{a}-t+\tau)d\tau}\beta(\varphi(t-\bar{a}))\varphi(t-\bar{a}) - \mu(\varphi(t))\varphi(t) \qquad \text{for } t > \bar{a},
\end{aligned} \tag{2.4}$$

which, by a variable change, gives (1.1).

3. **Stability and Hopf bifurcation.** In this section we study the stability of the positive and trivial equilibria and the existence of local Hopf bifurcation for equation (1.1). We first discuss the local stability of the positive equilibrium. To this end, we impose the following additional assumption on the parameters.

(A7) There exists a unique positive constant φ^* such that

$$e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}\beta(\varphi^*) - \mu(\varphi^*) = 0.$$

Clearly, φ^* is the positive equilibrium of equation (1.1), and the linearization of (1.1) about φ^* is given by

$$\varphi'(t) = e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)]\varphi(t-\bar{a}) - [\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]\varphi(t). \tag{3.1}$$

Then the characteristic equation of (3.1) is as follows:

$$\lambda + \mu'(\varphi^*)\varphi^* + \mu(\varphi^*) - e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)]e^{-\bar{a}\lambda} = 0. \tag{3.2}$$

As is well known, equation (3.2) has infinitely many characteristic roots λ [3], and the local stability of φ^* is determined by the sign of the real parts of λ: If all the roots have negative real parts, then φ^* is locally stable. For this purpose, we first introduce a result due to Hayes (cf. p. 444 of [3]).

Lemma 3.1. *All the roots of $pe^z + q - ze^z = 0$, where p and q are real, have negative real parts if and only if*

(a) $p < 1$, and

(b) $p < -q < \sqrt{\theta_0^2 + p^2}$,

where θ_0 is the root of $\theta = p \tan \theta$ for $0 < \theta < \pi$.

We are then in a position to establish the following result.

Theorem 3.2. *The positive equilibrium φ^* of equation (1.1) is locally stable if and only if*

$$e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)] > [\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]\left(\frac{1}{\cos\theta_0}\right), \qquad (3.3)$$

where θ_0 is the root of $\theta = -\bar{a}[\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]\tan\theta$ for $\pi/2 < \theta < \pi$.

Proof. We first rewrite (3.2) as

$$-\bar{a}[\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]e^{\bar{a}\lambda} + \bar{a}e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)] - \bar{a}\lambda e^{\bar{a}\lambda} = 0. \quad (3.4)$$

We then set $z = \bar{a}\lambda$ to find that

$$p = -\bar{a}[\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)] \quad \text{and} \quad q = \bar{a}e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)]. \qquad (3.5)$$

In view of assumption (A3), part (a) of Lemma 3.1 is satisfied, and the first inequality in part (b) requires that

$$\mu'(\varphi^*)\varphi^* + \mu(\varphi^*) > e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)],$$

which is valid by (A3), (A4), and (A7). The second inequality in (b) of Lemma 3.1 requires that

$$-\bar{a}e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)] < \sqrt{\theta_0^2 + \bar{a}^2[\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]^2}, \qquad (3.6)$$

where $\theta_0 = -\bar{a}[\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]\tan\theta_0$. Since $-\bar{a}[\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)] < 0$, $\pi/2 < \theta_0 < \pi$, and hence (3.6) yields (3.3). $\qquad\square$

Note that taking (A4) into account, (A7) implies that $e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}\beta(0) > \mu(0)$. If this relation is reversed, one can naturally expect that the trivial equilibrium of equation (1.1) is stable.

Theorem 3.3. *Suppose that $e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}\beta(0) < \mu(0)$. Then the trivial equilibrium of equation (1.1) is globally stable.*

Proof. Introduce a Lyapunov function

$$V(t, v) = v^2(t) + \mu(0)\int_{t-\bar{a}}^t v^2(\tau)d\tau, \qquad t \geq \bar{a}. \qquad (3.7)$$

If $\varphi(t)$ is the solution of (1.1), then

$$\begin{aligned}
\tfrac{d}{dt}V(t, \varphi) &= 2\varphi(t)\left[e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}\beta(\varphi(t-\bar{a}))\varphi(t-\bar{a}) - \mu(\varphi(t))\varphi(t)\right] \\
&\quad + \mu(0)\varphi^2(t) - \mu(0)\varphi^2(t-\bar{a}) \\
&\leq 2e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}\beta(0)\varphi(t)\varphi(t-\bar{a}) - \mu(0)\varphi^2(t) - \mu(0)\varphi^2(t-\bar{a}) \\
&\leq \left[e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}\beta(0) - \mu(0)\right][\varphi^2(t) + \varphi^2(t-\bar{a})].
\end{aligned} \qquad (3.8)$$

$$\square$$

We now show that under certain conditions equation (1.1) undergoes a Hopf bifurcation. For simplicity, we introduce the following notations.

$$\sigma = \mu'(\varphi^*)\varphi^* + \mu(\varphi^*), \quad \gamma = e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)], \quad (3.9)$$

and

$$\Lambda = -(1 + \bar{a}\sigma)\frac{d\sigma}{d\bar{a}} + \left(\frac{\sigma}{\gamma} + a\gamma\right)\frac{d\gamma}{d\bar{a}} + \gamma^2 - \sigma^2. \quad (3.10)$$

Theorem 3.4. *Suppose that*

$$e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}[\beta'(\varphi^*)\varphi^* + \beta(\varphi^*)] \le [\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]\left(\frac{1}{\cos\theta_0}\right), \quad (3.11)$$

where θ_0 is the root of $\theta = -\bar{a}[\mu'(\varphi^*)\varphi^* + \mu(\varphi^*)]\tan\theta$ for $\pi/2 < \theta < \pi$. If $i\sqrt{\gamma^2 - \sigma^2}$ is a purely imaginary root of the characteristic equation (3.2) and if $\Lambda \ne 0$, then equation (1.1) undergoes a Hopf bifurcation.

Proof. With the notations in (3.9), the characteristic equation (3.2) takes the form:

$$\lambda + \sigma - \gamma e^{-\bar{a}\lambda} = 0. \quad (3.12)$$

Clearly, $i\sqrt{\gamma^2 - \sigma^2}$ is a purely imaginary root of (3.12) if and only if

$$i\sqrt{\gamma^2 - \sigma^2} + \sigma - \gamma\cos\left(\bar{a}\sqrt{\gamma^2 - \sigma^2}\right) + i\gamma\sin\left(\bar{a}\sqrt{\gamma^2 - \sigma^2}\right) = 0,$$

that is,

$$\sqrt{\gamma^2 - \sigma^2} = -\gamma\sin\left(\bar{a}\sqrt{\gamma^2 - \sigma^2}\right) \quad \text{and} \quad \sigma = \gamma\cos\left(\bar{a}\sqrt{\gamma^2 - \sigma^2}\right). \quad (3.13)$$

We then differentiate (3.12) with respect to $\bar{a}$ to find

$$\frac{d\lambda}{d\bar{a}} + \frac{d\sigma}{d\bar{a}} - \frac{d\gamma}{d\bar{a}}e^{-\bar{a}\lambda} + \gamma\lambda e^{-\bar{a}\lambda} + \bar{a}\gamma e^{-\bar{a}\lambda}\frac{d\lambda}{d\bar{a}} = 0,$$

which, together with (3.12) gives

$$\frac{d\lambda}{d\bar{a}} = \frac{1}{1 + \bar{a}(\lambda + \sigma)}\left[-\frac{d\sigma}{d\bar{a}} + \frac{d\gamma}{d\bar{a}}e^{-\bar{a}\lambda} - (\lambda + \sigma)\lambda\right]. \quad (3.14)$$

Thus, at $i\sqrt{\gamma^2 - \sigma^2}$ we have

$$Re\frac{d\lambda}{d\bar{a}} = \frac{1}{1 + 2\bar{a}\sigma + \bar{a}^2\gamma^2}\left[(1 + \bar{a}\sigma)\left(-\frac{d\sigma}{d\bar{a}} + \frac{d\gamma}{d\bar{a}}\cos\left(\bar{a}\sqrt{\gamma^2 - \sigma^2}\right) + \gamma^2 - \sigma^2\right)\right.$$
$$\left. + \bar{a}\sqrt{\gamma^2 - \sigma^2}\left(\frac{d\gamma}{d\bar{a}}\sin\left(\bar{a}\sqrt{\gamma^2 - \sigma^2}\right) - \sigma\sqrt{\gamma^2 - \sigma^2}\right)\right]$$
$$= \frac{1}{1 + 2\bar{a}\sigma + \bar{a}^2\gamma^2}\Lambda \ne 0.$$
$$(3.15)$$

□

Remark 1. The condition $\Lambda \ne 0$ cannot be improved. For example, if we choose $\nu(\eta) = 0.1202\sqrt{\bar{\eta}}$, $\beta(\varphi) = 68/(1 + \varphi^4)$, and $\mu(\varphi) = 0.1$, we find that when $\bar{a} = 16.6072$, the condition (3.11) is satisfied and $\Lambda = -0.0372$. On the other hand, if we select $\nu(\eta) = 32.4004\eta$, $\beta(\varphi) = 100e^{-0.51\varphi^2}$, and $\mu(\varphi) = \varphi$, we find that at $\bar{a} = 0.1424$, (3.11) holds and $\Lambda = 118.405$.

4. **Global stability and bifurcation diagram.** In this section, we establish global asymptotic stability results and present bifurcation diagrams for two special cases of the model (1.1). We first consider the following case:

$$\varphi'(t) = \sigma e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} \frac{\varphi(t-\bar{a})}{1+\varphi^n(t-\bar{a})} - \gamma\varphi(t), \tag{4.1}$$

where σ, γ are positive constants and $n > 1$. Such an equation is closely related to the model (1.2) with f taking the form

$$f = \frac{y(t-\tau)}{1+y^n(t-\tau)}. \tag{4.2}$$

In [14] Kuang showed that if

$$\sigma > \gamma \quad \text{and} \quad \frac{\sigma}{4n}(n-1)^2 < \gamma, \tag{4.3}$$

then the unique positive equilibrium y^* of (1.2) with f given by (4.2) is globally asymptotically stable. Consequently, one can see that if

$$\sigma e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} > \gamma \quad \text{and} \quad \frac{\sigma}{4n}(n-1)^2 e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} < \gamma, \tag{4.4}$$

then the positive equilibrium φ^* of (4.1) is globally asymptotically stable.

To draw a bifurcation diagram for (4.1), we choose $\nu(\eta) = 0.1202\sqrt{\bar{\eta}}$, $\sigma = 68$, $n = 4$, and $\gamma = 0.1$. The diagram is given in Figure 1. This figure shows that for $\bar{a} < 6.8031$ and $\bar{a} > 16.6072$, the population converges to a steady state. This steady state is the extinction one if $\bar{a} > 18.8510$. For $6.8031 < \bar{a} < 16.6072$, the population converges to a periodic orbit. Therefore, the values $\bar{a} = 6.8031$ and $\bar{a} = 16.6072$ are bifurcation points.
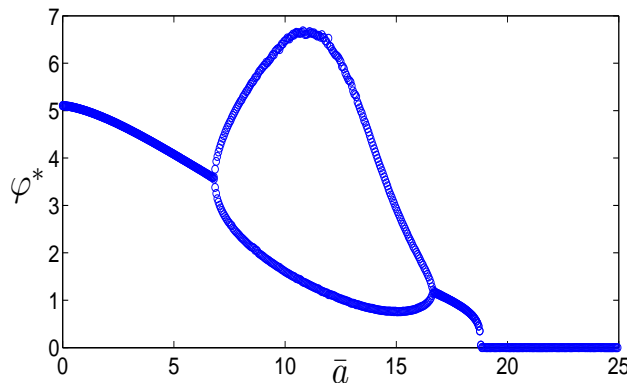

FIGURE 1. Bifurcation diagram for equation (4.1)

We then consider the following case:

$$\varphi'(t) = \beta e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} e^{-\delta\varphi^n(t-\bar{a})} \varphi(t-\bar{a}) - \mu\varphi^2(t), \tag{4.5}$$

where β, δ, μ are positive constants and $n \geq 1$.

Theorem 4.1. *Assume the condition*

$$\frac{1}{4}\beta n^2 \delta^2 \left(\frac{2n-1}{n\delta}\right)^{\frac{2n-1}{n}} e^{-\frac{2n-1}{n}} e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} < \mu. \tag{4.6}$$

Then the positive equilibrium φ^* of (4.5) is globally asymptotically stable.

Proof. Let $f(\varphi) = \beta e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} e^{-\delta\varphi^n}\varphi$ and $g(\varphi) = \mu\varphi^2$. We find that f is strictly increasing in $[0, \varphi_M)$ and strictly decreasing in (φ_M, ∞), where $\varphi_M = (1/n\delta)^{\frac{1}{n}}$, and $\lim_{\varphi \to +\infty} f(\varphi) = 0$. If $\varphi^* < \varphi_M$, in view of Theorem 4.3 of [14], it suffices to show that

$$-f'(\varphi_1) < g'(\varphi_2), \tag{4.7}$$

where $\varphi_1 \in (\varphi_M, \infty)$, $\varphi_2 \in (0, \varphi_M)$, and $f(\varphi_1) = g(\varphi_2)$.

Clearly, inequality (4.7) is satisfied if we require

$$n\beta\delta e^{-\delta\varphi_1^n}\varphi_1^n e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} < 2\mu\varphi_2. \tag{4.8}$$

On the other hand, from the equation $f(\varphi_1) = g(\varphi_2)$ we have

$$\beta\delta e^{-\delta\varphi_1^n}\varphi_1 e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} = \mu\varphi_2^2,$$

or equivalently,

$$\varphi_2 = \sqrt{\frac{\beta}{\mu}}\varphi_1 e^{-\frac{1}{2}\delta\varphi_1^n} e^{-\frac{1}{2}\int_0^{\bar{a}} \nu(\eta)d\eta}. \tag{4.9}$$

Substituting (4.9) into (4.8) yields

$$\beta n^2 \delta^2 e^{-\delta\varphi_1^n}\varphi_1^{2n-1} e^{-\int_0^{\bar{a}} \nu(\eta)d\eta} < 4\mu. \tag{4.10}$$

The left-hand side of (4.10) attains its maximum at $\bar{\varphi} = [(2n-1)/n\delta]^{\frac{1}{n}}$, and thus the condition (4.6) follows.

If $\varphi^* \geq \varphi_M$, the result can be established in a similar manner. $\square$

To draw a bifurcation diagram for (4.5), we choose $\nu(\eta) = 32.4004\eta$, $\beta = 100$, $\delta = 0.51$, $n = 2$, and $\mu = 1$. The diagram is given in Figure 2. This figure shows that the points $\bar{a} = 0.1424$ and $\bar{a} = 0.2980$ are bifurcation points. If $\bar{a}$ is between these points, then the population converges to a periodic orbit; otherwise the population converges to a steady state, and this steady steady is the trivial one if $\bar{a} > 0.7610$.

Remark 2. From the two diagrams, one can see that due to the presence of $e^{-\int_0^{\bar{a}} \nu(\eta)d\eta}$, the global asymptotic stability results are not optimal in the sense that the conditions (4.4) and (4.6) do not seem to cover all the values of $\bar{a}$ for which the numerical results indicate global asymptotic stability.

5. **Concluding remarks.** In this paper, we studied the dynamical behavior of a delay differential equation arising from a juvenile-adult population model that was developed to describe the dynamics of an amphibian population of Green Treefrogs (*Hyla cinerea*) [2]. With the exception of some information about the reproduction rate of an adult Green Treefrog, little is known about the survivorship rates of tadpoles and frogs for this population [1, 19]. Experimental studies indicate that tadpole survivorship can be highly variable depending on both the predator community and the type of available refugia [8, 9]. Furthermore, the age of tadpoles at which metamorphosis occurs also varies and can range from around 5 to 8 weeks depending on several environmental factors.
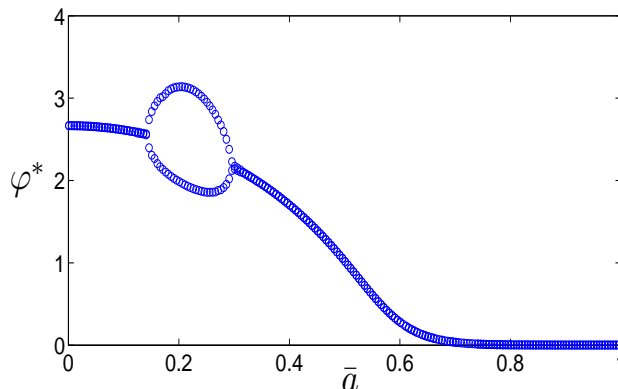
FIGURE 2. Bifurcation diagram for equation (4.5)

Density dependence in adult mortality or birth rates is also unknown for these populations. Therefore, in our above analysis, we used special cases which include the well-known forms of Beverton-Holt and Ricker to represent the density dependence in the birth rate and used constant or linear (logistic type) density dependence in the mortality rate. In [1], the authors developed a discrete model and used Beverton-Holt density dependence in the survivorship rates to describe this population. Such a model was fitted to a two year data set (2004 and 2005).

The biological interpretation of the above analysis and the bifurcation diagrams in particular is that the time when the metamorphosis process occurs affects the dynamical behavior of the population. Especially, if metamorphosis occurs very quickly or very slowly, then one may expect the population to converge to a steady state, which can be the extinction one as shown in Figures 1-2; while if metamorphosis is in an intermediate range, then oscillatory behavior of the population can be observed.

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