MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume 7, Number 1, January 2010

doi:10.3934/mbe.2010.7.51

pp. 51–66

DYNAMICS OF AN SIS REACTION-DIFFUSION EPIDEMIC MODEL FOR DISEASE TRANSMISSION

Wenzhang Huang

Department of Mathematics, Shanghai Normal University
Shanghai, 200234 China
and
Department of Mathematical Sciences, University of Alabama in Huntsville
Huntsville, AL 35899, USA

Maoan Han

Department of Mathematics, Shanghai Normal University
Shanghai, 200234 China

Kaiyu Liu

College of Mathematics and Econometrics, Hunan University
Changsha, Hunan 410082, China

Dedicated to Horst R. Thieme on the Occasion of his 60th Birthday

Abstract. Recently an SIS epidemic reaction-diffusion model with Neumann (or no-flux) boundary condition has been proposed and studied by several authors to understand the dynamics of disease transmission in a spatially heterogeneous environment in which the individuals are subject to a random movement. Many important and interesting properties have been obtained: such as the role of diffusion coefficients in defining the reproductive number; the global stability of disease-free equilibrium; the existence and uniqueness of a positive endemic steady; global stability of endemic steady for some particular cases; and the asymptotical profiles of the endemic steady states as the diffusion coefficient for susceptible individuals is sufficiently small. In this research we will study two modified SIS diffusion models with the Dirichlet boundary condition that reflects a hostile environment in the boundary. The reproductive number is defined which plays an essential role in determining whether the disease will extinct or persist. We have showed that the disease will die out when the reproductive number is less than one and that the endemic equilibrium occurs when the reproductive number is exceeds one. Partial result on the global stability of the endemic equilibrium is also obtained.

1. **Introduction.** The SIS models provide essential frames in studying the dynamics of disease transmission in the filed of theoretical epidemiology. The understanding of dynamics of SIS models in a homogeneous media, in which models are described by systems of ordinary differential equations, is quite complete, in particular, due to the work of Lajmanovich and York [7]. To summarize, the dynamics

2000 *Mathematics Subject Classification.* Primary: 35K57, 35B35; Secondary: 92D25.

Key words and phrases. epidemic model, reaction-diffusion equations, stability.

The third author is currently visiting Department of Mathematical Sciences, UAH, AL 35899, USA.

of disease transmission is governed by a reproductive number R_0. The disease will become extinct if $R_0 < 1$ and disease persists if $R_0 > 1$. To be more specific, when $R_0 > 1$, the population converges to a unique endemic steady state. This result later was extended to a single group SIS age-structured model by Busenberg, Iannelli and Thieme [3] and to a multi-group SIS age-structure model by Feng, Huang and Chavez [5] . Recently Allen, Bolker, Lou, and Nevai [2] proposed a basic SIS model to investigate the impact of spatial heterogeneity of environment and movement of individuals on the persistence and extinction of a disease. Their model is described by a system of reaction-diffusion equations

$$\begin{aligned} \frac{\partial S}{\partial t} &= d_S \Delta S - \beta(x)\frac{SI}{S+I} + \gamma(x)I, \qquad x \in \Omega, \quad t > 0, \\ \frac{\partial I}{\partial t} &= d_I \Delta I + \beta(x)\frac{SI}{S+I} - \gamma(x)I, \qquad x \in \Omega, \quad t > 0, \end{aligned} \tag{1.1}$$

with a no-flux boundary condition (or Neumann boundary condition)

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{1.2}$$

Here $S(x,t)$ and $I(x,t)$ denote the densities of susceptible and infected individuals at location x and time t; d_S and d_I are positive diffusion coefficients for the susceptible and infected populations, $\beta(x)$ and $\gamma(x)$ denote respectively the rate of disease transmission and the rate of recovery from the infectives; $\Omega \subset \mathbb{R}^m$ is an open region in space and n is the outward normal vector in the boundary of Ω. The reproductive number R_0 has been defined in [2] such that when $R_0 < 1$, the population density $(S(t,\cdot), I(t,\cdot))$ converges to a unique disease free equilibrium $(S_0, 0)$, and when $R_0 > 1$, there exists a unique positive endemic equilibrium (S^*, I^*). In addition, [2] has showed a very interesting result on the asymptotic behavior of endemic equilibrium as the diffusion coefficient $d_S \to 0$. The analysis of the stability of endemic equilibrium for model (1.1)-(1.2) has been conducted most recently by Peng and Liu [10]. With some additional restrictions, namely when the diffusion coefficients are equal or $\beta/\gamma =$ a constant (in this case the endemic state is a constant), [10] has obtained the global stability of the endemic equilibrium when $R_0 > 1$.

In this paper we will consider a similar SIS reaction-diffusion model to (1.1) but instead, we are interested in the Dirichlet boundary condition:

$$S(t,x) = I(t,x) = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{1.3}$$

This boundary condition reflects an environment where the boundary of region Ω is hostile for the survival of population. Extremely cold or hot temperature, the lack of any supporting resource, etc. give a few examples (see [8] for more examples).

A modification of (1.1) is necessary under the boundary condition (1.3). To see this, suppose that $(S(t,x), I(t,x))$ is a positive solution of (1.1). Then we have

$$\frac{\partial S(t,x)}{\partial n} \le 0, \quad \frac{\partial I(t,x)}{\partial n} \le 0, \quad x \in \partial\Omega, \quad t > 0.$$

Adding two equations in (1.1), integrating the sum over the region Ω and with the use of the divergence theorem we obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega [S(t,x)+I(t,x)]dx &= \int_\Omega \big[d_S\Delta S(t,x)+d_I\Delta I(t,x)\big]dx \\ &= \int_{\partial\Omega} \big[d_S\frac{\partial S(t,x)}{\partial n}+d_I\frac{\partial I(t,x)}{\partial n}\big]dx \\ &\le 0, \qquad t>0.\end{aligned} \tag{1.4}$$

From the inequality (1.4) it follows that the total population is decreasing. In fact, it is not difficult to show that $\int_\Omega[S(t,x)+I(t,x)]dx \to 0$ as $t\to+\infty$. Thus the dynamics of the model is trivial, which is caused by the population loss in the boundary of the region and the diffusion process. To have a more meaningful model we must add an additional growth term, other than the growth due to the recovery from infectives, to the model to balance the population decay in the boundary. In this paper we shall study the model in which we suppose the disease is not inheritable. We consider two cases separately. For the first case we suppose the population in the region is demographicly stable. Thus we may assume that the growth rate for susceptible is independent of the population density and the model takes the form

$$\begin{aligned}\frac{\partial S}{\partial t} &= d_S\Delta S-\beta(x)\frac{SI}{S+I}+\gamma(x)I+\Lambda(x), && x\in\Omega, \quad t>0,\\ \frac{\partial I}{\partial t} &= d_I\Delta I+\beta(x)\frac{SI}{S+I}-\gamma(x)I, && x\in\Omega, \quad t>0,\end{aligned} \tag{1.5}$$

where $\Lambda(x)$ is positive and continuous on $\overline{\Omega}$ which represents the growth rate for the new born, etc.. The existence and uniqueness of solutions to the initial value problem to equation (1.5) with the boundary condition (1.3) follow from the standard theory for the parabolic equations. By using the maximum principle one can also show that, if the initial distributions $S(0,\cdot)$ and $I(0,\cdot)$ are positive, then $S(t,x)$ and $I(t,x)$ are bounded and positive for all $t>0$ and $x\in\Omega$.

The choice of the growth function Λ in equation (1.5) may be oversimplified because it depends only on the space variable x but not the population density in this location. An alternative model in which the growth depends on the population sizes can be formulated as

$$\begin{aligned}\frac{\partial S}{\partial t} &= d_S\Delta S-\beta(x)\frac{SI}{S+I}+\gamma(x)I+k(S+I)-\nu S(S+I), && x\in\Omega, \quad t>0,\\ \frac{\partial I}{\partial t} &= d_I\Delta I+\beta(x)\frac{SI}{S+I}-\gamma(x)I-\nu I(S+I), && x\in\Omega, \quad t>0.\end{aligned} \tag{1.6}$$

Here k, ν are positive constants, $k(S+I)$ is the birth rate for susceptible population, and $\nu S(S+I)$ and $\nu I(S+I)$ denote the death rates for susceptible and infected populations respectively. [We take the above form for the simplicity. More generally we can let $k=k(x)$ and $\nu=\nu(x)$ be positive functions.] We choose square term for the death rate to reflect a common fact that death rate increases when population size increases.

The reproductive number for both models are defined that governs the stability of disease free equilibrium steady state and the existence of an endemic equilibrium state. For a particular case in which the diffusion coefficients for two populations are identical, we also showed the global stability of the endemic equilibrium state when the reproductive number exceeds one. However, the stability of endemic equilibrium state remains unsolved if two diffusion coefficients are not equal.

This paper is organized as follows. In Section 2 we define the reproductive number and show that the disease free equilibrium is globally stable for the model (1.5), and is locally stable for the model (1.6) when the reproductive number is less than one. Section 3 is devoted to studying the existence and uniqueness of an endemic equilibrium of the model (1.5) if the reproductive number is larger than one. In particular, we prove the global stability of an endemic equilibrium for both models if the diffusion coefficients d_S and d_I are identical. Concluding remarks are given in Section 4.

2. **Disease-free equilibrium and its stability.**

Let us first consider the system

$$\begin{aligned} \frac{\partial S}{\partial t} &= d_S \Delta S - \beta(x)\frac{SI}{S+I} + \gamma(x)I + \Lambda(x), && x \in \Omega, \quad t > 0, \\ \frac{\partial I}{\partial t} &= d_I \Delta I + \beta(x)\frac{SI}{S+I} - \gamma(x)I, && x \in \Omega, \quad t > 0, \\ & S(t,x) = I(t,x) = 0, && x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{2.1}$$

Throughout the paper we suppose β, γ, and Λ are positive and continuous functions on $\overline{\Omega}$ and $\partial\Omega$ is C^2 smooth.

Our main interest of this section is the existence, uniqueness and stability of the disease-free equilibrium. A disease free equilibrium is a time independent solution of the form $(S_0, 0)$, where $S_0(x) > 0$ for $x \in \Omega$. For this purpose let us first introduce some notations. For a closed linear operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$, where $D(A)$ is the domain of A, the spectral spread $\mathbf{s}(A)$ of A is defined by

$$\mathbf{s}(A) = \sup\{\Re\lambda : \lambda \in \sigma_p(A)\},$$

where σ_p denotes the point spectrum of A. We let

$$H_0^1(\Omega) = \left\{ \phi \in L^2(\overline{\Omega}) : \frac{\partial \phi}{\partial x_j} \in L^2(\Omega),\ j = 1, \cdots, m,\ \phi(x) = 0,\ x \in \partial\Omega \right\}.$$

For $\phi \in L^2(\Omega)$, let

$$\|\phi\| = \|\phi\|_{L^2(\Omega)}.$$

For a positive constant d and a function $\mu \in C(\overline{\Omega}, \mathbb{R})$, we let $d\Delta + \mu$ be the operator defined by

$$[d\Delta + \mu]\phi(x) = d\Delta\phi(x) + \mu(x)\phi(x), \quad x \in \Omega$$

with $D(d\Delta + \mu) = H_0^2$.

We need the following lemma. The proof of this lemma can be found in the indicated references.

Lemma 2.1. *Let d be a positive constant and $\mu \in C(\overline{\Omega}, \mathbb{R})$. Then*

(1) (Theorem 1, p.301 in [12]) Let

$$-\lambda^* = \min\left\{ \int_\Omega [d|\nabla\phi(x)|^2 - \mu(x)\phi^2(x)]dx : \phi \in H_0^1(\Omega),\ \int_\Omega \phi^2(x) = 1 \right\}.$$

Then $\mathbf{s}(d\Delta+\mu)=\lambda^*$ *and* λ^* *is the largest eigenvalue of the operator* $d\Delta+\mu$ *under the Dirichlet boundary condition on* $\partial\Omega$ *and the corresponding eigenfunction is strictly positive. In particular, by setting* $\mu=0$*, we deduce that*

$$\mathbf{s}(d\Delta)<0.$$

(2) [9] *Let* $T(t) : L^2(\Omega)\to L^2(\Omega)$*,* $t\geq 0$*, be the semigroup generated by the operator* $d\Delta+\mu$ *and let* $\lambda^*=\mathbf{s}(d\Delta+\mu)$*. Then there is a constant* $C>0$ *such that*

$$\|T(t)\|\leq Ce^{\lambda^* t},\quad \text{i.e.}\quad \|T(t)\phi\|\leq Ce^{\lambda^* t}\|\phi\|,\quad \phi\in L^2(\Omega),\quad t\geq 0,$$

where $\|T(t)\|$ *is the operator norm* $L^2(\Omega)$*.*

(3) *(Theorem 1, P.318, [12]) For any* $\rho>\lambda^*$ *and any function* $h\in L^2(\Omega)$*, the equation*

$$d\Delta\phi+[\mu-\rho]\phi=-h$$

has a unique solution $\phi\in H_0^2(\Omega)$ *and* $\phi\in C^2(\Omega)$ *if* $h\in C(\overline{\Omega})$*. Moreover, by the strong maximum principle, if* h *is positive, then* $\phi(x)>0$ *all* $x\in\Omega$*. In particular, there is a unique function* $S_0(x)$*, which is strictly positive for* $x\in\Omega$*, such that*

$$d_S\Delta S_0+\Lambda=0,\quad S_0(x)=0,\quad x\in\partial\Omega.$$

As an immediate consequence of Lemma 2.1 (3) we have

Corollary 2.2. *The equation (2.1) has a unique disease free equilibrium* $(S_0,0)$*, where* S_0 *is defined in (3) of Lemma 2.1.*

Now let us turn to define the reproductive number R_0 that governs the stability of the disease free equilibrium. For (2.1), the corresponding reproductive number R_0 actually can be defined in the same way as in [2],

$$R_0=\sup\left\{\frac{\int_\Omega \beta(x)\phi^2(x)dx}{\int_\Omega\left[d_I|\nabla\phi(x)|^2+\gamma(x)\phi^2(x)\right]dx}\ :\ \phi\in H_0^1(\Omega),\ \phi\neq 0\right\}. \tag{2.2}$$

With the above defined reproductive number, we have

Lemma 2.3.

$$R_0<1\,(>1)\qquad\text{if and only if}\qquad \mathbf{s}(d_I\Delta+\beta-\gamma)<0\,(>0).$$

One is able to directly verify this lemma so that we shall omit its proof.

By the maximum principle for parabolic systems, if $S(0,\cdot)$ and $I(0,\cdot)$ are positive, then both $S(t,x)$ and $I(t,x)$ are positive and bounded for $x\in\Omega$ and $t>0$ whenever the solution exists. It therefore follows from the standard theory for semi-linear parabolic systems that $(S(t,x),I(t,x))$ actually is a classical solution that exists for all $t>0$ [4].

We are ready to establish the following global stability for the disease free equilibrium.

Theorem 2.4. *For equation (2.1), if* $R_0<1$*, then all its nonnegative solutions converge to the disease free equilibrium* $(S_0,0)$ *as time goes to* $+\infty$*.*

Proof. Suppose that $R_0<1$. We will use the comparison principle to show that $I(t,\cdot)\to 0$ as $t\to\infty$. First we observe from the second equation of (2.1) that

$$\frac{\partial I}{\partial t}\leq d_I\Delta I+(\beta(x)-\gamma(x))I,\quad x\in\Omega\text{ and }t>0$$

since $I(t,x)$ is nonnegative. Next, let $u(t,x)$ with $u(0,x) = I(0,x)$ be the solution of the linear equation

$$\frac{\partial u}{\partial t} = d_I \Delta u + (\beta(x) - \gamma(x))u, \quad x \in \Omega \text{ and } t > 0,$$

and $u(x,t) = 0$ for $x \in \partial\Omega$ and $t > 0$. By the comparison principle, $0 \leq I(t,x) \leq u(t,x)$ for all $t \geq 0$ and $x \in \Omega$. Let $U(t)$ be the semigroup generated by the operator $d_I\Delta + \beta - \gamma$. Now $R_0 < 1$ implies that $\lambda^* = \mathbf{s}(d_I\Delta + \beta - \gamma) < 0$. It follows from Part (2) of Lemma 2.1 that there is a constant $M > 0$ such that

$$\|I(t,\cdot)\| \leq \|u(t,\cdot)\| = \|U(t)u(0,\cdot)\| \leq Me^{\lambda^* t}\|I(0,\cdot)\| \to 0 \quad \text{as} \quad t \to \infty. \tag{2.3}$$

We now show that $S(t,\cdot)$ tends to S_0 as $t \to \infty$. Notice that $d_S\Delta S_0 = -\Lambda$. Thus we can rewrite the first equation of (2.1) as

$$\frac{\partial(S - S_0)}{\partial t} = d_S\Delta(S - S_0) + \left[\gamma - \frac{\beta S}{S + I}\right] I, \quad x \in \Omega \text{ and } t > 0. \tag{2.4}$$

(2.3) yields that

$$\|\big[\gamma - \frac{\beta S(t,\cdot)}{S(t,\cdot) + I(t,\cdot)}\big] I(t,\cdot)\| \leq C_1 e^{-bt}, \quad t > 0, \quad b = -\lambda^* = -\mathbf{s}(d_I\Delta + \beta - \gamma) \tag{2.5}$$

for some positive constants C_1. Let $T(t) : L^2(\Omega) \to L^2(\Omega)$, $t \geq 0$ be the semigroup generated by the operator $d_S\Delta$. Then, since $\mathbf{s}(d_S\Delta) < 0$, there is a constant $C > 0$ such that

$$\|T(t)\| \leq Ce^{-at}, \quad t \geq 0, \quad a = -\mathbf{s}(d_S\Delta).$$

Applying the variation-of-constant formula to (2.4) and with the use of (2.5) we arrive at

$$\begin{aligned} \|S(t,\cdot) - S_0\| \quad &\leq \|T(t)S(0,\cdot)\| + \int_0^t \|T(t-s)\big[\gamma - \frac{\beta S(t,\cdot)}{S(t,\cdot) + I(t,\cdot)}\big] I(t,\cdot)\| ds \\ &\leq Ce^{-at}\|S(0,\cdot)\| + CC_1 e^{-at}\textstyle\int_0^t e^{(a-b)s} ds \\ &\to 0 \qquad \text{as} \quad t \to \infty. \end{aligned} \tag{2.6}$$

It follows that $S(t,\cdot) \to S_0$ as $t \to \infty$. $\square$

We remark that, by the above proof, the convergence is in $L^2(\Omega)$. However, it is clear that the convergence holds in $L^p(\Omega)$ for and $p > 1$. Hence, with the use of the regularity theorem we actually are able to show that the convergence is uniform.

Now we turn to study the system

$$\begin{aligned} &\frac{\partial S}{\partial t} = d_S\Delta S - \beta(x)\frac{SI}{S + I} + \gamma(x)I + k(S + I) - \nu S(S + I), && x \in \Omega, \quad t > 0, \\ &\frac{\partial I}{\partial t} = d_I\Delta I + \beta(x)\frac{SI}{S + I} - \gamma(x)I - \nu I(S + I), && x \in \Omega, \quad t > 0, \\ &S(t,x) = I(t,x) = 0, && x \in \partial\Omega. \end{aligned} \tag{2.7}$$

It is obvious that $(S_0^*, 0)$ is a disease free equilibrium if and only if S_0^* is a positive solution of the equation

$$d_S \Delta S + S(k - \nu S) = 0, \quad x \in \Omega, \qquad S(x) = 0, \quad x \in \partial\Omega. \tag{2.8}$$

The following proposition is well known (see Proposition 7.7, p.332 in [12]).

Proposition 2.5. *Equation (2.8) has a positive solution S_0^* if and only if $\mathbf{s}(d_S\Delta + k) > 0$. Moreover, the positive solution S_0^* is unique and it is strictly positive on Ω.*

Hence we always assume that $\mathbf{s}(d_S\Delta + k) > 0$. Thus the system (2.7) has a unique disease free equilibrium $(S_0^*, 0)$. For the system (2.7) we define the reproductive number R_0 by

$$R_0 = \sup\left\{ \frac{\int_\Omega \beta(x)\phi^2(x)dx}{\int_\Omega \left[d_I|\nabla\phi(x)|^2 + [\gamma(x) + \nu S_0^*(x)]\phi^2(x)\right]dx} :\ \phi \in H_0^1(\Omega),\, \phi \neq 0 \right\}. \tag{2.9}$$

With the above defined reproductive number, similar to Lemma 2.3, we have

Lemma 2.6.

$$R_0 < 1\,(> 1) \qquad \text{if and only if} \qquad \mathbf{s}(d_I\Delta + \beta - \gamma - \nu S_0^*) < 0\,(> 0).$$

Theorem 2.7. *For the system (2.7), the disease free equilibrium $(S_0^*, 0)$ is local asymptotically stable If $R_0 < 1$. It is unstable if $R_0 > 1$.*

Proof. A direct computation shows that the linearization of (2.7) at the disease free equilibrium $(S_0^*, 0)$ is

$$\begin{aligned} &\frac{\partial S}{\partial t} = d_S\Delta S + (k - 2\nu S_0^*)S - (\beta - \gamma - k + \nu S_0^*)I, && x \in \Omega, \quad t > 0,\\ &\frac{\partial I}{\partial t} = d_I\Delta I + [\beta - \gamma - \nu S_0^*]I, && x \in \Omega, \quad t > 0,\\ &S(t,x) = I(t,x) = 0, && x \in \partial\Omega. \end{aligned} \tag{2.10}$$

Since the linear system (2.10) generates an analytic semigroup, the local stability of the disease free equilibrium $(S_0^*, 0)$ can be determined by the corresponding eigenvalues that are determined by the following system:

$$\begin{aligned} &d_S\Delta\phi + (k - 2\nu S_0^*)\phi - (\beta - \gamma - k + \nu S_0^*)\psi = \lambda\phi, && x \in \Omega, \quad t > 0,\\ &d_I\Delta\psi + [\beta - \gamma I - \nu S_0^*]\psi = \lambda\psi && x \in \Omega, \quad t > 0,\\ &\phi(x) = \psi(x) = 0, && x \in \partial\Omega. \end{aligned} \tag{2.11}$$

First we notice that $d_S\Delta S_0^* + (k - \nu S_0^*)S_0^* = 0$ and $S_0^*(x) > 0$ for $x \in \Omega$ imply that $\mathbf{s}(d_S\Delta + k - \nu S_0^*) = 0$. It follows that

$$\mathbf{s}(d_S\Delta + k - 2\nu S_0^*) < \mathbf{s}(d_S\Delta + k - \nu S_0^*) = 0. \tag{2.12}$$

Now suppose that $R_0 < 1$ and λ is an eigenvalue with $\psi \neq 0$. Then $\Re\lambda < 0$ by Lemma 2.6. If $\psi = 0$, then $\phi \neq 0$ and

$$d_S\Delta\phi + (k - 2\nu S_0^*)\phi = \lambda\phi.$$

Thus (2.12) yields that $\Re\lambda < 0$. Hence all eigenvalues of (2.11) have the negative real part. This implies that $(S_0^*, 0)$ is locally asymptotically stable. Next suppose that $R_0 > 1$. Then there is a $\lambda_0 > 0$ and $\psi_0 \neq 0$ such that

$$d_I\Delta\psi + [\beta - \gamma I - \nu S_0^*]\psi = \lambda_0\psi.$$

Rewrite the first equation in (2.11) with $\lambda = \lambda_0$ as

$$d_S\Delta\phi + (k - 2\nu S_0^* - \lambda_0)\phi = (\beta - \gamma - k + \nu S_0^*)\psi_0. \tag{2.13}$$

$\lambda_0 > 0$ implies that $\mathbf{s}(d_S\Delta + k - 2\nu S_0^* - \lambda_0) < 0$. Hence (2.13) has a unique solution ϕ_0 satisfying $\phi_0(x) = 0$ for $x \in \Omega$. That is, $\lambda_0 > 0$ is an eigenvalue. Therefore the equilibrium $(S_0^*, 0)$ is unstable. $\square$

3. **The existence of endemic equilibrium.** An endemic equilibrium of equation (2.1) is a positive solution (S^*, I^*) of the following elliptic system

$$\begin{aligned} 0 \quad &= d_s\Delta S - \left[\beta\tfrac{S}{S+I} - \gamma\right]I + \Lambda, && x \in \partial\Omega, \\ 0 \quad &= d_I\Delta I + \left[\beta\tfrac{S}{S+I} - \gamma\right]I, && x \in \partial\Omega, \\ &S(x) = I(x) = 0, && x \in \partial\Omega. \end{aligned} \tag{3.1}$$

Here both S^* and I^* are strictly positive on Ω. By adding two equations in (3.1) we obtain the equivalent system

$$\begin{aligned} &\Delta(d_S S + d_I I) + \Lambda(x) = 0, && x \in \Omega, \\ &0 = d_I\Delta I + [\beta\frac{S}{S+I} - \gamma]I, && x \in \Omega, \\ &0 = I(x,t), && x \in \partial\Omega. \end{aligned} \tag{3.2}$$

The first equation in (3.2) yields that $S + \frac{d_I}{d_S}I = S_0$, where S_0 is the unique positive function satisfying $d_S\Delta S_0 = -\Lambda$. Thus we can express S as

$$S = \frac{d_S S_0 - d_I I}{d_S}. \tag{3.3}$$

Substituting the above equality into the second equation of (3.2) we obtain the equation for I as

$$d_I\Delta I + \left[\beta - \gamma - \beta\frac{d_S I}{d_S S_0 + (d_S - d_I)I}\right]I = 0. \tag{3.4}$$

Theorem 3.1. *Suppose that $R_0 > 1$. Then (3.2) has a unique nonnegative solution (S^*, I^*) such that $S^*, I^* \in C^2(\overline{\Omega})$ and $I^* > 0$. Furthermore, $S^*(x) > 0$, and $0 < I^*(x) < \dfrac{d_S}{d_I}S_0(x)$ for $x \in \Omega$.*

Proof. The proof of this Lemma is essentially the same as the proof of Lemma 3.3 in [2]. In view of (3.3), (S^*, I^*) is a positive solution of (3.2) if and only if I^* is a positive solution of (3.4) with $I^*(x) < \dfrac{d_S}{d_I}S_0(x)$ for $x \in \Omega$ and $S^* = \dfrac{d_S S_0 - d_I I^*}{d_S}$. So let us consider the boundary value problem

$$G(I) = d_I\Delta I + If(x, I) = 0 \quad \text{for} \quad x \in \Omega \quad \text{and} \quad I(x) = 0, \quad \text{for} \quad x \in \partial\Omega, \tag{3.5}$$

where

$$f(x,u) = \beta(x) - \gamma(x) - \beta(x)\frac{d_S u}{d_S S_0 + (d_S - d_I)u}, \quad x \in \Omega, \quad u \in [0, \frac{d_S}{d_I} S_0(x)].$$

Since (3.5) is a scalar equation, the existence of desirable positive solution of (3.5) will follow if we can construct a sub-solution $\underline{I}$ and a supper-solution $\overline{I}$ such that $0 < \underline{I} \le \overline{I} \le \dfrac{d_S}{d_I} S_0$. Recall that $R_0 > 1$ implies that

$$\lambda^* = \mathbf{s}(d_I \Delta + \beta - \gamma) > 0$$

is the largest eigenvalue of the operator $d_I\Delta + \beta - \gamma$ and the corresponding eigenfunction $\phi^*(x)$ is strictly positive for $x \in \Omega$. We now show that $\underline{I} = \epsilon\phi^*$ and $\overline{I} = \frac{d_S}{d_I} S_0$ are sub-solution and super-solutions for (3.5) if $\epsilon > 0$ is sufficiently small. Note that $d_I \Delta\phi^* + [\beta - \gamma]\phi^* = \lambda^*\phi^*$. Upon a direct substitution we obtain

$$\begin{aligned} G(\underline{I}) &= d_I \Delta(\epsilon\phi^*) + \epsilon\phi^* f(x, \epsilon\phi^*) \\ &= \epsilon\Big[d_I\Delta\phi^* + \phi^*(\beta - \gamma) - \epsilon\beta\frac{d_S[\phi^*]^2}{d_S S_0 + \epsilon(d_S - d_I)\phi^*}\Big] \\ &= \epsilon\phi^*\Big[\lambda^* - \epsilon\beta\frac{d_S[\phi^*]^2}{d_S S_0 + \epsilon(d_S - d_I)\phi^*}\Big]. \end{aligned} \tag{3.6}$$

$\lambda^* > 0$ immediately implies that $G(\underline{I}) > 0$ if ϵ is small enough, so that $\underline{I}$ gives a sub-solution of (3.5). Next, since

$$\begin{aligned} G(\overline{I}) &= d_I\Delta(\overline{I}) + \overline{I} f(x, \overline{I}) \\ &= d_I\Delta(\overline{I}) + \overline{I}\left[\beta(x)\left(1 - \frac{d_S\overline{I}}{d_S S_0 + (d_S - d_I)\overline{I}}\right) - \gamma(x)\right] \\ &= -\Lambda(x) - \gamma\overline{I} \end{aligned}$$

is negative on Ω, and $\overline{I}(x,t) = 0$ on $\partial\Omega$, it follows that $\overline{I}$ is a super-solution of (3.5). Also, it is obvious that $\underline{I} \le \overline{I}$ on Ω if ϵ is chosen sufficiently small. We conclude from the above remarks that there must be an I^*, with $\underline{I} < I^* < \overline{I}$, satisfying (3.5). This proves the existence of positive solution (S^*, I^*) of (3.2).

Observe that $f(x,u)$ is strictly decreasing with respect to u for $u \in [0, \dfrac{d_S S_0(x)}{d_I}]$. Thus, the uniqueness of the positive solution of equation (3.2) can be proved following exactly the same argument given in the proof of Lemma 3.3 in [2]. □

Although we have showed the existence of a unique endemic equilibrium for equation (2.1) when $R_0 > 1$, the analysis of the stability of endemic equilibrium is far from trivial, even for the local stability. For equation (2.7), it is possible to prove the existence of an endemic equilibrium if the corresponding reproductive number $R_0 > 1$. The uniqueness of the endemic equilibrium remains unsolved if the diffusion coefficients d_S and d_I are not equal. However, when the diffusion coefficients d_S and d_I are identical, we can show that the endemic equilibrium actually is globally stable for both equation (2.1) and equation (2.7).

In the rest of the paper we are going to establish the following theorem.

Theorem 3.2. *For equation (2.1) or equation (2.7), if $d_S = d_I$ and the corresponding reproductive number $R_0 > 1$, then all positive solutions converge to a* *unique endemic equilibrium* (S^*, I^*) *as time goes to infinity. That is, the endemic equilibrium* (S^*, I^*) *is globally stable.*

We shall prove this theorem only for equation (2.7). The proof for equation (2.1) will be more straightforward than the proof for equation (2.7) because (2.1) can be transformed to a monotone system. Now in equation (2.7) let $d = d_S = d_I$ and $N(t, x) = S(t, x) + I(t, x)$. Then $N(t, x)$ and $I(t, x)$ satisfy the system

$$\begin{aligned} \frac{\partial N}{\partial t} &= d\Delta N + N(k - \nu N), & x \in \Omega, \quad t > 0, \\ \frac{\partial I}{\partial t} &= d\Delta I + \big[\beta - \gamma - \nu N - \beta \frac{I}{N}\big] I, & x \in \Omega, \quad t > 0, \\ & N(t,x) = I(t,x) = 0, & x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{3.7}$$

Notice that (3.7) is not a monotone system if the constant ν is large. Also Dirichlet boundary condition makes the solution vanishes in the boundary, the comparison principle is not immediately valid for our case. Instead, we must use a more sophisticated comparison argument motivated by the ideas in [6]. We need a few auxiliary results before proceeding to the proof of this theorem.

Notice that, for a nonnegative solution $(S(t, x), I(t, x))$ of equation (2.7), if $I(0, x)$ is positive, then the solution is strictly positive for all $t > 0$ and $x \in \Omega$ by strong maximum principle. Hence, if $(N(t, x), I(t, x))$ is solution of equation (3.7) with $0 < I(0, \cdot) \le N(0, \cdot)$, then

$$0 < I(t, x) < I(t, x) + S(t, x) = N(t, x)$$

for all $t > 0$ and $x \in \partial\Omega$.

Lemma 3.3. *(Theorem 1.2, P.297 [6]) Let* $N(t, \cdot)$ *be a solution of the first equation of (3.7). If* $0 \le N(0, \cdot) \ne 0$*, then* $N(t, x) \to S^*(x)$ *as* $t \to \infty$ *uniformly for* $x \in \Omega$*, where* S^* *is the unique positive solution of*

$$d\Delta S + S(k - \nu S), \quad x \in \Omega, \qquad S(x) = 0, \quad x \in \partial\Omega.$$

Lemma 3.4. *Let* R_0 *be the reproductive number corresponding to equation (2.7). If* $R_0 > 1$*, then there is a* $c^* > 0$ *such that for each* c *with* $|c| \le c^*$*, ther equation*

$$d\Delta I + \big[\beta - \gamma - \nu S^* + c - \beta \frac{I}{S^*}\big] I = 0, \quad x \in \Omega, \qquad I = 0, \quad x \in \partial\Omega \tag{3.8}$$

has a unique positive solution I_c^**. Moreover,* $I_c^* \to I^*$ *as* $c \to 0$*, where* (S^*, I^*) *is the unique endemic equilibrium of equation (3.7).*

Proof. By Lemma 2.6, $R_0 > 1$ implies that $\mathbf{s}(d\Delta + \beta - \gamma - \nu S^*) > 0$. Hence there is a $c^* > 0$ such that $\mathbf{s}(d\Delta + \beta - \gamma - \nu S^* + c - \nu S^*) > 0$ for all c with $|c| \le c^*$. One therefore is able to show the existence of a unique positive solution I_c^* of above equation by the same argument used the the proof of Theorem 3.1. The last conclusion follows easily from the continuity. ◻

Lemma 3.5. *Let* $0 \ne c \in [-c^*, c^*]$ *and* $(N(t, x), I(t, x))$ *be a positive solution of equation (3.7) such that* $S^*(x) - |c|/\nu \le N(t, x) \le S^*(x) + |c|/\nu$ *for all* $t \ge 0$*. Let* $(N_c(t, \cdot), I_c(t, \cdot))$ *be a nonnegative solution of the system*

$$\begin{aligned}\frac{\partial N}{\partial t} &= d\Delta N + N(k-\nu N), \quad x\in\Omega, \quad t>0,\\ \frac{\partial I}{\partial t} &= d\Delta I + \big[\beta-\gamma-\nu S^* + c - \beta\frac{I}{N}\big]I, \quad x\in\Omega, \quad t>0, \\ & N(t,x)=I(t,x)=0, \quad x\in\partial\Omega, \quad t>0.\end{aligned} \tag{3.9}$$

If $c<0$, $N_c(0,\cdot)\le N(0,\cdot)$, $I_c(0,\cdot)\le I(0,\cdot)$, (respectively $c>0$ and $N_c(0,\cdot)\ge N(0,\cdot)$, $I_c(0,\cdot)\ge I(0,\cdot)$,) then

$$N_c(t,\cdot)\le N(t,\cdot), \quad I_c(t,\cdot)\le I(t,\cdot), \qquad t\ge 0.$$

(respectively

$$N_c(t,\cdot)\ge N(t,\cdot), \quad I_c(t,\cdot)\ge I(t,\cdot), \qquad t\ge 0.)$$

Proof. Lemma 3.5 follows easily from the comparison argument. □

Lemma 3.6. *(Remark in P.124, [11]) Let $x_0\in\partial\Omega$ and $D=\Omega\cap B_r(x_0)$, where $B_r(x_0)$ is a ball of radius $r>0$ and center x_0. Suppose that a function*

$$u\in C^1([T_1,T_2]: C(\overline{D})\cap C([T_1,T_2]: C^2(D)\cap C^1(\overline{D}),$$

and u satisfies

(i) $u(T_2,x_0)=0$ and $u(t,x)>0$ for all $(t,x)\in[T_1,T_2]\times D$,

(ii) $d\Delta u - b(x)u - \dfrac{\partial u}{\partial t}\le 0$ in $[T_1,T_2]\times D$.

Then

$$\frac{\partial u(T_2,x_0)}{\partial n}>0,$$

here $b(x)$ is continuous on $\overline{\Omega}$ and n is the inward normal vector of $\partial\Omega$ at x_0.

As a consequence of above lemma we immediately have the following

Corollary 3.7. *For equation (3.9) with $c<0$, let $(N_c(t,x), I_c(t,x))$ be a solution with $0<I_c(0,\cdot)\le N_c(0,\cdot)$. Then for any $t>0$, $N_c(t,x)$ and $I_c(t,x)$ are positive for all $x\in\Omega$. In addition,*

$$\frac{\partial N_c}{\partial n}>0, \quad \frac{\partial I_c}{\partial n}>0 \quad \text{for all } x_0\in\partial\Omega. \tag{3.10}$$

In particular, for the positive equilibrium (S^*, I_c^*) of equation (3.9) we have

$$\frac{\partial S^*(x_0)}{\partial n}>0, \quad \frac{\partial I_c^*(x_0)}{\partial n}>0 \quad \text{for all } x_0\in\partial\Omega. \tag{3.11}$$

Proof. The strict positivity of $N_c(t,x)$ and $I_c(t,x)$ for $t>0$ and $x\in\Omega$ follows directly from the strong maximum principle for parabolic equations. We claim that $I_c(t,x)/N_c(t,x)\le 1$ for all $t>0$. To see this, let $S_c(t,x)=N_c(t,x)-I_c(t,x)$. Then $S_c(0,\cdot)\ge 0$ and $S_c(t,x)$ satisfies the equation

$$\frac{\partial S}{\partial t} = d\Delta S_c - [\beta-\gamma-\nu S^*+c-\beta\frac{I_c}{S_c+I_c}]I_c + k(S_c+I_c) - \nu S_c(S_c+I_c), \quad t>0, \quad x\in\Omega.$$

With the use of maximum principle, together the positivity of $I_c(t,x)$ and $c<0$ one easily concludes that $S_c(t,x)>0$ for all $t>0$ and $x\in\Omega$. Hence $I_c(t,x)/N_c(t,x)\le 1$ for all $t>0$ and $x\in\Omega$. Now for any $0<T_1<T_2$, by the regularity,

$$N_c, I_c\in C^1([T_1,T_2]: C(\overline{\Omega})\cap C([T_1,T_2]: C^2(\Omega)\cap C^1(\overline{\Omega}).$$

Since $\dfrac{I_c(t,x)}{N_c(t)} \leq 1$ and $S^*(x)$ is bounded, we can pick a sufficiently large number α such that

$$\alpha + \beta - \gamma - \nu S^* - \beta \frac{I_c(t,x)}{N_c(t,x)} > 0, \quad x \in \Omega. \tag{3.12}$$

(3.12) yields that

$$d\Delta I_c - \alpha I_c - \frac{\partial I_c}{\partial t} = -\big[\alpha + \beta - \gamma - \nu S^* - \beta \frac{I_c}{N_c}\big] I_c < 0, \quad x \in \Omega.$$

Therefore, the above inequality and Lemma 3.6 yield that

$$\frac{\partial I_c}{\partial n} > 0, \quad x_0 \in \partial\Omega.$$

The proof for the function $N_c(t,x)$ is the same. □

Lemma 3.8. *Consider the equation*

$$\begin{aligned} \frac{\partial u}{\partial t} &= d\Delta u + \mu u + h(t,x) \qquad x \in \Omega, \\ u(t,x) &= 0, \qquad\qquad\qquad\qquad x \in \Omega, \end{aligned} \tag{3.13}$$

where $\mu(x)$ is continuous on $\overline{\Omega}$, $h(t,x)$ is continuous and bounded on $[0,t_0] \times \Omega$ for some positive constant t_0. Suppose that $\mathbf{s}(d\Delta + \mu) = 0$ and let ϕ^* be the strictly positive eigenfunction corresponding to the zero eigenvalue of the operator $d\Delta + \mu$. If

$$h(t,x) \geq 0, \quad t \in [0,t_0], \quad x \in \Omega,$$

and $u(t,x)$ is a solution of (3.11) with $u(0,\cdot) = \phi^*$, then

$$u(t,\cdot) \geq u(0,\cdot), \quad t \in [0,t_0].$$

Proof. Let $T(t)$ be the semigroup generated by the corresponding linear system

$$\begin{aligned} \frac{\partial v}{\partial t} &= d\Delta v + \mu v, \quad x \in \Omega, \\ v(t,x) &= 0, \qquad x \in \Omega. \end{aligned} \tag{3.14}$$

Then it is known that $T(t)\phi^* = \phi^*$ for all $t \geq 0$ and $T(t)$ is a positive operator. That is, $T(t)\eta \geq 0$ if $\eta \in L^2(\Omega)$ is nonnegative. Applying the variation-of-constant formula to (3.13) we obtain that

$$\begin{aligned} u(t,\cdot) &= T(t)\phi^* + \int_0^t T(t-s)h(s,\cdot)ds \\ &\geq \phi^*, \qquad t \in [0,t_0]. \end{aligned}$$

□

Recall that $R_0 > 1$ implies

$$\mathbf{s}(d\Delta + \beta - \gamma - \nu S^*) > 0.$$

Moreover we have assumed

$$\mathbf{s}(d\Delta + k) > 0.$$

Hence it is evident that there are two positive constants α_1 and α_2 such that

$$\mathbf{s}(d\Delta + k - \alpha_1) = 0, \quad \mathbf{s}(d\Delta + \beta - \gamma - \nu S^* - \alpha_2) = 0. \tag{3.15}$$

We let ϕ_1^* and ϕ_2^* be strictly positive eiegnfunctions corresponding to the zero eigenvalue of the operator $d\Delta + k - \alpha_1$ and $d\Delta + \beta - \gamma - \alpha_2$, respectively. That is,

$$d\Delta\phi_1^* + (k-\alpha_1)\phi_1^* = 0, \qquad d\Delta\phi_2^* + (\beta - \gamma - \nu S^* - \alpha_2)\phi_1^* = 0.$$

Lemma 3.9. *Let $c < 0$ be fixed and let $(N_c(t,x), I_c(t,x))$ be a solution of equation (3.9) with $0 < I_c(0,\cdot) \le N_c(0,\cdot)$. Then for any fixed $t > 0$, there exist positive constants a_1 and a_2 such that*

$$\frac{N_c(t,x)}{\phi_1^*(x)} > a_1, \qquad \frac{I_c(t,x)}{\phi_2^*(x)} > a_2, \qquad x \in \Omega,$$

where ϕ_1^* and ϕ_2* are positive eigenfunctions defined above.

Proof. For a fixed $t > 0$, from Corollary 3.7, the continuity of $\dfrac{\partial N_c(t,x)}{\partial x_j}$ and $\dfrac{\partial \phi^*(x)}{\partial x_j}$ on $\overline{\Omega}$, $j = 1, \cdots, m$, and compactness of $\partial\Omega$ it follows that there exist positive constants M_1, M_2 such that

$$M_1 < \frac{\partial N_c}{\partial n} \Big/ \frac{\partial \phi_1^*(x_0)}{\partial n} < M_2, \qquad x_0 \in \partial\Omega. \tag{3.16}$$

Moreover, we have

$$\frac{N_c(t,x)}{\phi_1^*(x)} > 0, \qquad x \in \Omega. \tag{3.17}$$

In addition, $N_c(x_0) = \phi_1^*(x_0) = 0$ for $x_0 \in \partial\Omega$ implies that

$$\lim_{x\in\Omega, x\to x_0} \frac{N_c(t,x)}{\phi_1^*(x)} = \frac{\partial N_c(x_0)}{\partial n} \Big/ \frac{\partial \phi_1^*(x_0)}{\partial n}. \tag{3.18}$$

(3.16)-(3.18) therefore yield that

$$a_1 \le \frac{N_c(t,x)}{\phi_1^*(x)}, \qquad x \in \Omega \tag{3.19}$$

for some positive constant a_1. Arguing in the same way we deduce that there is a positive constant a_2 such that

$$a_2 \le \frac{I_c(t,x)}{\phi_2^*(x)}, \quad x \in \Omega. \tag{3.20}$$

□

Lemma 3.10. *For each fixed $c < 0$ with $|c| < c^*$, there is an $\epsilon^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$, the solution $(N_c^\epsilon(t,x), I_c^\epsilon(t,x))$ of equation (3.9) with the initial condition*

$$N_c^\epsilon(0,\cdot) = \epsilon\phi_1^*, \qquad I_c^\epsilon(0,\cdot) = \epsilon^2\phi_2^*$$

is monotone increasing with respect to t.

Proof. Let α_i, $i = 1, 2$, be the positive numbers defined in (3.15). Then $\mathbf{s}(d\Delta + \beta - \gamma - \nu S^* - \alpha_2) = 0$ and $\mathbf{s}(d\Delta + \beta - \gamma - \nu S^* + c) > 0$ imply that $\alpha_2 + c > 0$. Hence it is apparent that there are positive constants ϵ^* and $\delta > 0$ such that

$$[\alpha_1 - \nu\epsilon^*\phi_1^*(x)] > \delta, \quad \big[\alpha_2 + c - \epsilon^*\beta(x)\frac{\phi_2^*(x)}{\phi_1^*(x)}\big] > \delta, \qquad x \in \Omega. \tag{3.21}$$

Now let $\epsilon \in (0, \epsilon^*]$ and let $(N_c^\epsilon(t,x), I_c^\epsilon(t,x))$ be defined as above. Then, since $\phi_i^* \in C_0^2(\overline{\Omega})$,

$$(N_c^\epsilon(t,x), I_c^\epsilon(t,x)) \to (\epsilon\phi_1^*(x), \epsilon^2\phi_2^*(x)) \quad \text{as } \ t \to 0$$

uniformly for $x \in \overline{\Omega}$. Hence from the continuity and the inequality (3.21) it follows that there is a $t_1 > 0$ such that

$$\begin{aligned} &[\alpha_1 - \nu N_c^\epsilon(t,x)] > \delta_1, && (t,x) \in [0,t_1] \times \Omega, \\ &\big[\alpha_2 + c - \beta(x)\frac{I_c^\epsilon(t,x)}{N_c^\epsilon(t,x)}\big] > \delta_1, && (t,x) \in [0,t_1] \times \Omega \end{aligned} \tag{3.22}$$

for some positive constant δ_1. By rewriting equation (3.9) we can check that N_c^ϵ and I_c^ϵ satisfy the system

$$\begin{aligned} \frac{\partial N}{\partial t} &= d\Delta N + (k - \alpha_1)N + h_1(t,x), && x \in \Omega, \quad t > 0, \\ \frac{\partial I}{\partial t} &= d\Delta I + [\beta - \gamma - \nu S^* - \alpha_2]I + h_2(t,x), && x \in \Omega, \quad t > 0, \\ N(t,x) &= I(t,x) = 0, && x \in \partial\Omega, \quad t > 0, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} h_1(t,x) &= [\alpha_1 - \nu N_c^\epsilon(t,x)]N_c^\epsilon(t,x), \\ h_2(t,x) &= \big[\alpha_2 + c - \beta\frac{I_c^\epsilon(t,x)}{N_c^\epsilon(t,x)}\big]I_c^\epsilon(t,x). \end{aligned}$$

The inequality (3.22) yields that both $h_1(t,x)$ and $h_2(t,x)$ are positive for $(t,x) \in [0,t_1] \times \Omega$. Therefore by Lemma 3.8 we conclude that

$$N_c^\epsilon(t,\cdot) > N_c^\epsilon(0,\cdot), \qquad I_c^\epsilon(t,\cdot) > I_c^\epsilon(0,\cdot), \qquad t \in (0,t_1]. \tag{3.24}$$

It is clear that equation (3.9) is a monotone system. Let

$$Z(t, Z_0) = (N_c(t,\cdot), I_c(t,\cdot))$$

be the solution of (3.9) with $Z(0,Z_0) = Z_0 = (N_c(0,\cdot), I_c(0,\cdot))$ and let $Z_0^\epsilon = (N_c^\epsilon(0,\cdot), I_c^\epsilon(0,\cdot))$. Then $(N_c^\epsilon(t,\cdot), I_c^\epsilon(t,\cdot)) = Z(t, Z_0^\epsilon)$. For for any $s > 0$, there are a positive integer m and a positive constant $\sigma \in (0, t_1]$ such that $s = m\sigma$. Now $Z(\sigma, Z_0^\epsilon) > Z_0^\epsilon$ by (3.24). Hence the monotonicity of the flow yields that

$$Z(t+\sigma, Z_0^\epsilon) = Z(t, Z(\sigma, Z_0^\epsilon)) \geq Z(t, Z_0^\epsilon).$$

By induction, we obtain that

$$Z(t+s, Z_0^\epsilon) = Z(t+m\sigma, Z_0^\epsilon) \geq Z(t+(m-1)\sigma, Z_0^\epsilon) \geq \cdots \geq Z(t, Z_0^\epsilon)$$

for all $t \geq 0$ and $s > 0$. So that $(N_\epsilon(t,\cdot), I_\epsilon(t,\cdot))$ is monotone increasing. □

Lemma 3.11. *Let $c > 0$ be fixed. For any $\sigma > 1$ with $\sigma k > c$, let $(N_c^M(t,\cdot), I_c^M(t,\cdot))$ be a solution of (3.9) with*

$$(N_c^M(0,x), I_c^M(0,x)) \equiv (\frac{\sigma k}{\nu}, \frac{\sigma k}{\nu}), \qquad x \in \Omega.$$

Then $(N_c^M(t,\cdot), I_c^M(t,\cdot))$ is decreasing with respect to t.

Proof. It is known that $(N_c^M(t,\cdot), I_c^M(t,\cdot))$ is a classical solution for $t > 0$. By applying the maximum principle one can easily verify that

$$N_c^M(t,x) \le \frac{\sigma k}{\nu}, \quad I_c^M(t,x) \le \frac{\sigma k}{\nu}$$

for all $t > 0$ and $x \in \Omega$. The monotone decreasing property of $(N_c^M(t,\cdot), I_c^M(t,\cdot))$ therefore follows the same argument used in the proof of Lemma 3.10. $\square$

We are now in the position to prove Theorem 3.2 for equation (3.7).

Proof. Let $(N(t,x), I(t,x))$ be a solution of (3.7) with the initial value satisfying $0 < I(0,\cdot) \le N(0,\cdot)$. Then $(N(t,x), I(t,x))$ is strictly positive for $t > 0$. By Lemma 3.3 $N(t,\cdot) \to S^*$ as $t \to \infty$. Hence any $c < 0$ with $|c| < c^*$, there is a $t_* > 0$ such that $N(t,\cdot) \ge S^* + c/\nu$ for all $t \ge t_*$. Lemma 3.9 implies that

$$\frac{N(t_*,x)}{\phi_1^*(x)} \ge a_1, \quad \frac{I(t_*,x)}{\phi_2^*(x)} \ge a_2, \qquad x \in \Omega$$

for some positive constants a_1, a_2. Let

$$0 < \epsilon < \min\{\epsilon^*, a_1, a_2, 1\},$$

where ϵ^* is defined in Lemma 3.10. Then $\epsilon < \epsilon^*$ and

$$\epsilon\phi_1^* \le N(t_*,\cdot), \quad \epsilon^2\phi_2^* \le I(t_*,\cdot). \tag{3.25}$$

Let $(N_c^\epsilon(t,\cdot), I_c^\epsilon(t,\cdot))$ be the monotone increasing solution of equation (3.9). Lemma 3.5 yields that

$$(N_c^\epsilon(t,\cdot), I_c^\epsilon(t,\cdot)) \le (N(t_*+t,\cdot), I(t_*+t,\cdot)), \quad t \ge 0. \tag{3.26}$$

Hence the monotonicity of the $(N_c^\epsilon(t,\cdot), I_c^\epsilon(t,\cdot))$ and the uniqueness of positive equilibrium (S^*, I_c^*) of equation (3.9) imply that

$$I_c^* \le \lim_{t\to\infty} \inf I(t,\cdot). \tag{3.27}$$

Next we pick $\sigma > 1$ sufficiently large such that

$$N(t^*,\cdot) \le \frac{\sigma k}{\nu}, \qquad I(t^*,\cdot) \le \frac{\sigma k}{\nu}.$$

Then

$$(N(t_*+t,\cdot), I(t_*+t,\cdot)) \le (N_{|c|}^M(t,\cdot), I_{|c|}^M(t,\cdot)), \quad t \ge 0 \tag{3.28}$$

where $(N_{|c|}^M, I_{|c|}^M)$ is the monotone decreasing solution of equation (3.9) with $c = |c|$. It follows that

$$\lim_{t\to\infty} \sup I(t,\cdot) \le I_{|c|}^*. \tag{3.29}$$

It is obvious that

$$I_c^* \to I^*, \qquad I_{|c|}^* \to I^* \quad \text{as} \quad c \to 0. \tag{3.30}$$

Noticing that $N(t,\cdot) \to S^*$ as $t \to \infty$, By (3.27), (3.29) and (3.30) we immediately deduce that

$$(N(t,\cdot), I(t,\cdot)) \to (S^*, I^*) \quad \text{as} \quad t \to \infty.$$

$\square$

4. **Conclusion.** For the model (2.1) we have shown that the reproductive number R_0 plays a key role in the control of disease spread. That is, the disease dies out when $R_0 < 1$ and persists if $R_0 > 1$. For the model (2.7) in which the growth rate depends on the population size, we have only proved the local stability of disease free equilibrium if $R_0 < 1$. We conjecture that the disease free equilibrium is globally stable. In the case where $R_0 > 1$ we have showed the existence and uniqueness of endemic equilibrium for the model (2.1). For the model (2.7), we still are able to prove the existence of an endemic equilibrium if $R_0 > 1$ by using a global bifurcation technique. However, it is unclear whether the endemic equilibrium is unique. For the special case when the two populations have the same diffusive rate, we have shown the global stability of endemic steady state for both models if $R_0 > 1$. The situation becomes more complicated if the diffusion coefficients of two populations are not identical. For this case it is even unclear whether the endemic equilibrium, whenever it exists, is locally stable. All unsolved problems mentioned above deserve further studies. In particular, it will be very important to understand how the magnitude of diffusion coefficients affects the dynamics of the disease transmission.

Acknowledgments. We would like to thank Professor Karl Hadeler for his very useful comment on the model, in particular, for his suggestion on the formulation of growth rate in the system (2.7).

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Received August 17, 2009; Accepted October 27, 2009.

E-mail address: huang@math.uah.edu
E-mail address: mahan@sjtu.edu.cn
E-mail address: kaiyuliuhnu@sina.com