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NOVEL STABILITY RESULTS FOR TRAVELING WAVEFRONTS IN AN AGE-STRUCTURED REACTION-DIFFUSION EQUATION

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Abstract. For a time-delayed reaction-diffusion equation of age-structured single species population, the linear and nonlinear stability of the traveling wavefronts were proved by Gourley [4] and Li-Mei-Wong [8] respectively. The stability results, however, assume the delay-time is sufficiently small. We now prove that the wavefronts remain stable even when the delay-time is arbitrarily large. This essentially improves the previous stability results obtained in [4, 8]. Finally, we point out that this novel stability can be applied to the age-structured reaction-diffusion equation with a more general maturation rate.

1. **Introduction and main results.** Subsequent to [8], in this paper we consider a model of population for a single species with age-structure

$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + \alpha e^{-\gamma\tau} v(x,t-\tau) - \beta v^2, \qquad t \in [0,\infty),\ x \in R, \tag{1}$$

with an initial value condition

$$v(x,s) = v_0(x,s), \quad s \in [-\tau, 0], \tag{2}$$

where $v(x,t)$ denotes the total population of mature species after the mature age $\tau > 0$ at time t and location x, $d > 0$ is the spatial diffusion rate of the mature species, α and β both are positive constants, the terms $\alpha e^{-\gamma\tau} v(x,t-\tau)$ and βv^2 represent the birth rate and the maturation rate of the mature population, respectively. For more details, we refer to [1]-[8].

Notice that equation (1) has two constant equilibria

$$v_- = 0 \quad \text{and} \quad v_+ = \frac{\alpha}{\beta} e^{-\gamma\tau}.$$

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We assume that the initial data satisfies

$$v_0(x,s) \to v_\pm, \quad s \in [-\tau, 0] \quad \text{as } x \to \pm\infty, \tag{3}$$

A traveling wavefront connecting the constant states $v_\pm$ is the monotone solution of equation (1) in the form of $\phi(x+ct)$, where c is the wave speed. Namely, $\phi(x+ct)$ satisfies the following ordinary differential equation

$$\begin{cases} d\phi''(\xi) - c\phi'(\xi) + \alpha e^{-\gamma\tau}\phi(\xi - c\tau) - \beta\phi^2(\xi) = 0, \\ \phi(-\infty) = 0 = v_-, \qquad \phi(\infty) = (\alpha/\beta)e^{-\gamma\tau} = v_+, \end{cases} \tag{4}$$

where $\xi = x + ct$, and $' = \frac{d}{d\xi}$.

The existence of the traveling wavefronts was proved by Al-Omari and Gourley [3]. That is, there exists a minimum speed $c_0 = c_0(\tau)$, also called the critical speed, such that for all $c > c_0$, the traveling wavefront $\phi(x+ct)$ uniquely exists (up to shift), where the minimum speed $c_0 = c_0(\tau)$ is determined by

$$F_{c_0}(\lambda_{c_0}) = G_{c_0}(\lambda_{c_0}), \qquad F'_{c_0}(\lambda_{c_0}) = G'_{c_0}(\lambda_{c_0}). \tag{5}$$

Here

$$F_c(\lambda) = 2\alpha e^{-\gamma\tau}e^{-\lambda c\tau/2}, \qquad G_c(\lambda) = c\lambda - \frac{1}{2}d\lambda^2. \tag{6}$$

Namely, (c_0, λ_{c_0}) is the tangent point of $F_c(\lambda)$ and $G_c(\lambda)$, and c_0 is the solution of the following implicit equation

$$\alpha \exp\left(1 - \gamma\tau - \frac{c_0^2\tau}{2d} - \frac{1}{2d}\sqrt{4d^2 + c_0^4\tau^2}\right) = \frac{1}{c_0^2\tau^2}\Big(-2d + \sqrt{4d^2 + c_0^4\tau^2}\Big), \tag{7}$$

which implies $c_0^2 < 4\alpha d e^{-\gamma\tau}$ (see the explanation in [8]).

It can be also seen from the graphs of $F_c(\lambda)$ and $G_c(\lambda)$ showed in [8] that, when $c = c_0$, then $F_{c_0}(\lambda_{c_0}) = G_{c_0}(\lambda_{c_0})$; while, when $c > c_0$, then $F_c(\lambda_{c_0}) < G_c(\lambda_{c_0})$, namely,

$$2\alpha e^{-\gamma\tau}e^{-\lambda_{c_0}c\tau/2} < c\lambda_{c_0} - \frac{1}{2}d\lambda_{c_0}^2. \tag{8}$$

The linear stability of traveling wavefronts was proved by Gourley [4], and further extended by Li-Mei-Wong [8] for the nonlinear stability. But both need to restrict the delay-time τ to be small enough.

Let $\tilde v = v - \phi$. From (1), the nonlinear perturbation $\tilde v$ for v around the wavefront ϕ satisfies

$$\frac{\partial\tilde v}{\partial t} - d\frac{\partial^2\tilde v}{\partial x^2} + \alpha e^{-\gamma\tau}\tilde v(x, t-\tau) + 2\beta\phi\tilde v + \beta\tilde v^2 = 0. \tag{9}$$

If one drops the nonlinear term $\beta\tilde v^2$, one gets a linear perturbation of v around ϕ as follows

$$\frac{\partial\tilde v}{\partial t} - d\frac{\partial^2\tilde v}{\partial x^2} + \alpha e^{-\gamma\tau}\tilde v(x, t-\tau) + 2\beta\phi\tilde v = 0. \tag{10}$$

In [4], when the delay-time τ is small enough such that

$$4\alpha\tau e^{-\gamma\tau} < \cosh^{-1}(2), \tag{11}$$

and the initial perturbation $\tilde v_0(x,s) := v_0(x,s) - \phi(x+cs)$ $(s \in [-\tau, 0])$ decays as fast as

$$|\tilde v_0(x,s)| = |v_0(x,s) - \phi(x+cs)| = O(1)e^{-\lambda_{c_0}|x|/2}, \quad \text{as } x \to -\infty, \tag{12}$$

where the wavefront $\phi(x+ct)$ is slower with a small speed

$$c_0 < c < \sqrt{\frac{d\cosh^{-1}(2)}{\tau}}\ , \tag{13}$$

by using the weighted energy method (see also [12, 13]), Gourley proved that the wavefront $\phi(x+ct)$ is linearly stable, namely, the linear perturbation of (10) satisfies

$$\sup_{x\in R}|\tilde{v}(x,t)| = O(1)e^{-\mu t}, \quad t>0. \tag{14}$$

Furthermore, under the same conditions as requested in [4], by using the weighted energy method with the comparison principle together, Li-Mei-Wong [8] extended Gourley's result to the nonlinear stability, namely, (14) holds also for the nonlinear perturbation $\tilde{v}$ of (9). However, the stability of traveling wavefronts for arbitrarily large delay-time τ still keeps open, but more significant and challenging as we know. To solve this problem is our main purpose of this note.

An extensive examination of (2.15)-(2.17) in [4] shows that, the restriction (11) on the small delay-time τ is only requested in the proof of $B(x)>0$ as $x\in(x_0,\infty)$, where $x_0 \gg 1$ is sufficiently large (see pp.263 in [4]). However, as we know, when $x=+\infty$, the wavefront $\phi(x+ct)$ is just equal to the equilibrium v_+, and v_+ is a stable node of equation (1). This should be one advantage for us in the proof of stability. In fact, the difficulty for the wave stability is only caused by the non-stable node $v_-=0$ as $x\to-\infty$ for $\phi(x+ct)$. That is why we need to construct a weight function as $e^{\lambda_{c_0}|x|}$ for $x\to-\infty$ to overcome such a difficulty in the proof by the energy method. Based on the above consideration, by constructing a new weight function, and technically treating the convergence of the solution $v(x,t)$ of (1) to the wavefront $\phi(x+ct)$ at $x=+\infty$, here, we can further prove the stability of the traveling wavefronts for all delay-time τ (no matter it is large or small) and for all wavefronts $\phi(x+ct)$ with $c>c_0$ (no matter their speed is large or small). A similar result for the Nicholson's blowflies equation with a local birth rate function or a nonlocal birth rate function has been also obtained in [10, 11].

Notations. Before we state our new stability, we introduce some notations. In what follows, $C>0$ denotes a generic constant, while $C_i>0$ $(i=0,1,2,\cdots)$ represents a specific constant. Let I be an interval, typically $I=\mathbf{R}$. $L^2(I)$ is the space of the square integrable functions on I, and $H^k(I)$ $(k\ge 0)$ is the Sobolev space of the L^2-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}f$, $i=1,\cdots,k$, also belong to $L^2(I)$. $L^2_w(I)$ represents the weighted L^2-space with the weight $w(x)>0$ and its norm is defined by

$$\|f\|_{L^2_w} = \Big(\int_I w(x)f(x)^2dx\Big)^{1/2}.$$

$H^k_w(I)$ is the weighted Sobolev space with the norm

$$\|f\|_{H^k_w} = \Big(\sum_{i=0}^k\int_I w(x)\big|\frac{d^i}{dx^i}f(x)\big|^2dx\Big)^{1/2}.$$

Let $T>0$ and let $\mathbf{B}$ be a Banach space, we denote by $C^0([0,T];\mathbf{B})$ the space of the $\mathbf{B}$-valued continuous functions on $[0,T]$, and $L^2([0,T];\mathbf{B})$ as the space of the $\mathbf{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathbf{B}$-valued functions on $[0,\infty)$ are defined similarly.

Our main result is as follows.

Theorem 1.1. *Let $w(x) > 0$ be a weight function given by*

$$w(x) = e^{-\lambda_{c_0} x}, \tag{15}$$

where $\lambda_{c_0} > 0$ is specified in (5). For a given traveling wavefront $\phi(x + ct)$ with speed $c > c_0$, if the initial datum satisfies

$$v_0(x, s) - \phi(x + cs) \in C^0([-\tau, 0]; H^1_w(R)), \tag{16}$$

then the unique solution $v(x, t)$ of the Cauchy problem (1) and (2) exists globally

$$v(x, t) - \phi(x + ct) \in C^0([0, \infty); H^1_w(R)) \cap L^2([0, \infty); H^2_w(R))$$

and it converges to the traveling wavefront $\phi(x + ct)$ time-asymptotically

$$\sup_{x\in R} |v(x, t) - \phi(x + ct)| \le Ce^{-\mu t}, \quad 0 \le t \le \infty \tag{17}$$

for some positive constant μ.

Remark 1. This novel stability result for the specific maturation rate βv^2 in Eq. (1) can be also applied to a general maturation rate $m(v)$. For details, we refer to the last section of the present paper, where some of typical examples for $m(v)$ are also given.

2. **Proof of Theorem 1.1.** We are going to prove the new stability of the traveling wavefronts by the weighted energy method with the comparison principle together, which was used in our previous paper [8], and initially in [9].

For given traveling wavefront $\phi(x + ct)$ with $c > c_0$, we define

$$\begin{cases} v_0^+(x, s) := \max\{v_0(x, s), \phi(x + cs)\}, \\ v_0^-(x, s) := \min\{v_0(x, s), \phi(x + cs)\}, \end{cases} \quad \text{for } (x, s) \in R \times [-\tau, 0], \tag{18}$$

so

$$v_0^-(x, s) \le v_0(x, s) \le v_0^+(x, s) \quad \text{for } (x, s) \in R \times [-\tau, 0] \tag{19}$$

$$v_0^-(x, s) \le \phi(x + cs) \le v_0^+(x, s) \quad \text{for } (x, s) \in R \times [-\tau, 0]. \tag{20}$$

Denote $v^+(x, t)$ and $v^-(x, t)$ as the corresponding solutions of equations (1) and (2) with respect to the above mentioned initial data $v_0^+(x, s)$ and $v_0^-(x, s)$; i.e.,

$$\begin{cases} \dfrac{\partial v^\pm}{\partial t} - d\dfrac{\partial^2 v^\pm}{\partial x^2} + \beta(v^\pm)^2 = \alpha e^{-\gamma\tau} v^\pm(x, t - \tau), & (x, t) \in R \times R_+ \\ v^\pm(x, s) = v_0^\pm(x, s), & x \in R, \ s \in [-\tau, 0]. \end{cases} \tag{21}$$

By the Comparison Principle (see Lemma 3.2 in [8]), we have

$$v^-(x, t) \le v(x, t) \le v^+(x, t) \quad \text{for } (x, t) \in R \times R_+, \tag{22}$$

$$v^-(x, t) \le \phi(x + ct) \le v^+(x, t) \quad \text{for } (x, t) \in R \times R_+. \tag{23}$$

Now we are going to prove the new stability in three steps.

Step 1: The convergence of $v^+(x, t)$ to $\phi(x + ct)$

Let $\xi := x + ct$ and

$$\begin{cases} z(\xi, t) := v^+(x, t) - \phi(x + ct), \\ z_0(\xi, s) := v_0^+(x, s) - \phi(x + cs), \end{cases} \tag{24}$$

then by (20) and (23), we have

$$z(\xi, t) \ge 0, \qquad z_0(\xi, s) \ge 0. \tag{25}$$

Since $v^+(x,t)$ and $\phi(x+ct)$ both satisfy equation (1), it can be verified that $z(\xi,t)$ satisfies

$$\begin{cases} \dfrac{\partial z}{\partial t} + c\dfrac{\partial z}{\partial \xi} - d\dfrac{\partial^2 z}{\partial \xi^2} - \alpha e^{-\gamma\tau} z(\xi - c\tau, t-\tau) + 2\beta\phi(\xi)z + \beta z^2 = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\xi,t) \in R \times R_+, \\ z(\xi,s) = z_0(\xi,s), \qquad\qquad\qquad\qquad\qquad\quad (\xi,s) \in R \times [-\tau, 0]. \end{cases} \tag{26}$$

Multiplying (26) by $e^{2\mu t}w(\xi)z(\xi,t)$, we obtain

$$\begin{aligned} &\Big(\frac{1}{2}e^{2\mu t}wz^2\Big)_t + e^{2\mu t}\Big(\frac{1}{2}cwz^2 - dwzz_\xi\Big)_\xi + de^{2\mu t}wz_\xi^2 + de^{2\mu t}w'zz_\xi \\ &-\mu e^{2\mu t}wz^2 - \frac{1}{2}ce^{2\mu t}\frac{w'}{w}wz^2 - \alpha e^{-\gamma\tau}e^{2\mu t}w(\xi)z(\xi,t)z(z-c\tau,t-\tau) \\ &+2\beta e^{2\mu t}w\phi z^2 + \beta e^{2\mu t}wz^3 = 0. \end{aligned} \tag{27}$$

By the Cauchy-Schwarz inequality, we have

$$|de^{2\mu t}w'zz_\xi| \le de^{2\mu t}wz_\xi^2 + \frac{d}{4}e^{2\mu t}\Big(\frac{w'}{w}\Big)^2 wz^2,$$

and dropping the positive term $\beta e^{2\mu t}wz^3$ (i.e., the last term in (27)), because $z(\xi,t) \ge 0$ (see (25)), we have

$$\begin{aligned} &\Big(\frac{1}{2}e^{2\mu t}wz^2\Big)_t + e^{2\mu t}\Big(\frac{1}{2}cwz^2 - dwzz_\xi\Big)_\xi \\ &-\mu e^{2\mu t}wz^2 - \frac{1}{2}ce^{2\mu t}\frac{w'}{w}wz^2 - \frac{d}{4\eta_1}e^{2\mu t}wz^2 \\ &-\alpha e^{-\gamma\tau}e^{2\mu t}w(\xi)z(\xi,t)z(z-c\tau,t-\tau) + 2\beta e^{2\mu t}w\phi z^2 \le 0. \end{aligned} \tag{28}$$

As exactly showed in [4, 8], by integrating (28) with respect to (ξ,t) over $R \times [0,t]$, we further obtain

$$\begin{aligned} &e^{2\mu t}\|z(t)\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu s}B(\mu,\xi)w(\xi)z^2(\xi,s)\ d\xi ds \\ &\le \|z(0)\|_{L_w^2}^2 + \frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\int_{-\tau}^0\int_R e^{2\mu s}w(\xi+c\tau)z_0^2(\xi,s)\ d\xi ds, \end{aligned} \tag{29}$$

where (see (2.6) and (2.7) in [4], and (43)-(45) in [8], respectively)

$$B(\mu,\xi) := B(\xi) - 2\mu - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\frac{w(\xi+c\tau)}{w(\xi)}\Big(e^{2\mu\tau}-1\Big), \tag{30}$$

$$B(\xi) := -c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2}\Big(\frac{w'(\xi)}{w(\xi)}\Big)^2 + 4\beta\phi(\xi) - \eta_2\alpha e^{-\gamma\tau} - \frac{1}{\eta_2}\alpha e^{-\gamma\tau}\ \frac{w(\xi+c\tau)}{w(\xi)}, \tag{31}$$

$$\eta_2 := e^{-\lambda_{c_0}c\tau/2}. \tag{32}$$

Since $w(\xi)=e^{-\lambda_{c_0}\xi}$, $\eta_2=e^{-\lambda_{c_0}c\tau/2}$ and $\phi(\xi)\geq 0$, it can be easily verified that

$$\begin{aligned} B(\xi) &= c\lambda_{c_0}-\frac{1}{2}d\lambda_{c_0}^2+4\beta\phi(\xi)-\eta_2\alpha e^{-\gamma\tau}-\frac{1}{\eta_2}\alpha e^{-\gamma\tau}e^{-\lambda_{c_0}c\tau} \\ &= c\lambda_{c_0}-\frac{1}{2}d\lambda_{c_0}^2+4\beta\phi(\xi)-2\alpha e^{-\gamma\tau}e^{-\lambda_{c_0}c\tau/2} \\ &\geq c\lambda_{c_0}-\frac{1}{2}d\lambda_{c_0}^2-2\alpha e^{-\gamma\tau}e^{-\lambda_{c_0}c\tau/2} \\ &=: C_1>0, \quad [\text{ by } (8)]. \end{aligned} \tag{33}$$

Let $\mu_1>0$ be the unique solution to the following equation

$$C_1-2\mu_1-\frac{1}{\eta_2}\alpha e^{-\gamma\tau}e^{-\lambda_{c_0}c\tau}(e^{2\mu_1\tau}-1)=0. \tag{34}$$

Thus, for $0<\mu<\mu_1$, from (30) and (33), we obtain

$$\begin{aligned} B(\mu,\xi) &= B(\xi)-2\mu-\frac{1}{\eta_2}\alpha e^{-\gamma\tau}e^{-\lambda_{c_0}c\tau}(e^{2\mu\tau}-1) \\ &\geq C_1-2\mu-\frac{1}{\eta_2}\alpha e^{-\gamma\tau}e^{-\lambda_{c_0}c\tau}(e^{2\mu\tau}-1) \\ &> 0. \end{aligned} \tag{35}$$

Applying (35) to (29), and dropping the positive term $\int_0^t\int_R e^{2\mu s}B(\mu,\xi)w(\xi)z^2(\xi,s)\,d\xi ds$, we get the first energy estimate

$$\begin{aligned} e^{2\mu t}\|z(t)\|_{L_w^2}^2 &\leq \|z(0)\|_{L_w^2}^2+\frac{1}{\eta_2}\alpha e^{-\gamma\tau+2\mu\tau}\int_{-\tau}^0\int_R e^{2\mu s}w(\xi+c\tau)z_0^2(\xi,s)\,d\xi ds \\ &\leq \|z(0)\|_{L_w^2}^2+C\int_{-\tau}^0\|z_0(s)\|_{L_w^2}^2\,ds. \end{aligned} \tag{36}$$

Similarly, differentiating (26) with respect to ξ and multiplying it by $e^{2\mu t}w(\xi)z_\xi(\xi,t)$, then by integrating the resultant equation with respect to (ξ,t) over $R\times[0,t]$, and applying (36), we obtain

$$e^{2\mu t}\|z_\xi(t)\|_{L_w^2}^2\leq C\Big(\|z(0)\|_{H_w^1}^2+\int_{-\tau}^0\|z_0(s)\|_{H_w^1}^2\,ds\Big). \tag{37}$$

Thus, summing (36) and (37) gives the basic energy estimate as follows.

Lemma 2.1. *It holds that*

$$\|z(t)\|_{H_w^1}^2\leq Ce^{-2\mu t}\Big(\|z(0)\|_{H_w^1}^2+\int_{-\tau}^0\|z_0(s)\|_{H_w^1}^2\,ds\Big). \tag{38}$$

Notice that, we cannot have the embedding result $H_w^1(R)\hookrightarrow C(R)$, because of the shortage $w(\xi)\to 0$ as $\xi\to\infty$. However, for any given sufficiently large number $\xi_0\gg 1$, we have $w(\xi)\geq e^{-\lambda_{c_0}\xi_0}$ for $\xi\in(-\infty,\xi_0]$, and

$$H_w^1((-\infty,\xi_0])\hookrightarrow H^1((-\infty,\xi_0])$$

with

$$\|z(t)\|_{H^1((-\infty,\xi_0])}\leq e^{\lambda_{c_0}\xi_0}\|z(t)\|_{H_w^1((-\infty,\xi_0])}.$$

Thus, the Sobolev's embedding theorem further implies

$$H_w^1((-\infty,\xi_0])\hookrightarrow H^1((-\infty,\xi_0])\hookrightarrow C((-\infty,\xi_0])$$

and

$$\sup_{\xi\in(-\infty,\xi_0]} |z(\xi,t)| \le C_2\|z(t)\|_{H^1_w((-\infty,\xi_0])}, \tag{39}$$

where $C_2 = C_2(\xi_0)$. Combining (39) and (38) yields the following convergence.

Lemma 2.2. *It holds that*

$$\sup_{\xi\in(-\infty,\xi_0]} |z(\xi,t)| \le C_3 e^{-\mu t}, \quad t>0 \tag{40}$$

for a sufficiently large number $\xi_0 \gg 1$ and a positive constant $C_3 = C_3(\xi_0)$.

Now we are going to prove the convergence of $z(\xi,t)$ at $\xi=\infty$.

Lemma 2.3. *It holds that*

$$\lim_{\xi\to\infty} |z(\xi,t)| \le Ce^{-\mu_2 t}, \quad t>0, \tag{41}$$

where $\mu_2 := \alpha e^{-\gamma\tau}$.

Proof. From equation (26) and by dropping the positive term βz^2, we have

$$\frac{\partial z}{\partial t} + c\frac{\partial z}{\partial \xi} - d\frac{\partial^2 z}{\partial \xi^2} - \alpha e^{-\gamma\tau} z(\xi - c\tau, t-\tau) + 2\beta\phi(\xi)z \le 0.$$

Taking limits to the above inequality as $\xi\to\infty$, and noting that $z_\xi(\infty,t)=0$ and $z_{\xi\xi}(\infty,t)=0$ because of the boundedness of $z(\xi,t)$ for all $\xi\in R$, and $\phi(\infty)=v_+$, we immediately obtain

$$\frac{d}{dt}z(\infty,t) - \alpha e^{-\gamma\tau} z(\infty, t-\tau) + 2\beta v_+ z(\infty,t) \le 0.$$

Integrating the above inequality over $[0,t]$ with respect to t yields

$$z(\infty,t) - \alpha e^{-\gamma\tau}\int_0^t z(\infty, s-\tau)ds + 2\beta v_+\int_0^t z(\infty,s)ds \le z(\infty,0) = z_0(\infty) = 0. \tag{42}$$

Here we used (3) and (4) to get $z(\infty,0)=z_0(\infty)=0$. Notice also that, by the change of variables,

$$\begin{aligned}\alpha e^{-\gamma\tau}\int_0^t z(\infty,s-\tau)ds &= \alpha e^{-\gamma\tau}\int_{-\tau}^{t-\tau} z(\infty,s)ds \\ &\le \alpha e^{-\gamma\tau}\int_0^t z(\infty,s)ds + \alpha e^{-\gamma\tau}\int_{-\tau}^0 z_0(\infty,s)ds. \end{aligned} \tag{43}$$

Substituting (43) to (42), and noting $v_+ = \frac{\alpha}{\beta}e^{-\gamma\tau}$, we then have

$$z(\infty,t) + \alpha e^{-\gamma\tau}\int_0^t z(\infty,s)ds \le \alpha e^{-\gamma\tau}\int_{-\tau}^0 z_0(\infty,s)ds.$$

Since $z(\infty,t)\ge 0$, by Grownwall's inequality, we prove

$$(0\le) z(\infty,t) \le C_4 e^{-\mu_2 t},$$

where $C_4 := \alpha e^{-\gamma\tau}\int_{-\tau}^0 z_0(\infty,s)ds$ and $\mu_2 = \alpha e^{-\gamma\tau}$. Thus, the proof of the lemma is completed. □

Finally, combining Lemma 2.2 and Lemma 2.3, we prove the convergence of $z(\xi,t)$ in the whole space $(-\infty,\infty)$ as follows.

Lemma 2.4. *It holds that*

$$\sup_{x\in R} |v^+(x,t) - \phi(x+ct)| = \sup_{\xi\in R} |z(\xi,t)| \le Ce^{-\mu t}, \quad t > 0 \tag{44}$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

Step 2: The convergence of $v^-(x,t)$ **to** $\phi(x+ct)$

Lemma 2.5. *It holds that*

$$\sup_{x\in R} |v^-(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \qquad t \ge 0 \tag{45}$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

Proof. Let $z(\xi,t) = \phi(x+ct) - v^-(x,t)$, $\xi = x+ct$, and $z_0(\xi,s) = \phi(x+cs) - v_0^-(x,s)$; then $z(\xi,t)$ satisfies the Cauchy problem (26). As shown in Step 1, we can similarly prove Lemma 2.5. The detail is omitted. □

Step 3: The convergence of $v(x,t)$ **to** $\phi(x+ct)$

Lemma 2.6. *It holds*

$$\sup_{x\in R} |v(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \qquad t \ge 0 \tag{46}$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

Proof. Since $v_0^-(x,s) \le v_0(x,s) \le v_0^+(x,s)$ for $(x,s) \in R \times [-\tau, 0]$, by using the comparison principle showed in [8], the corresponding solutions satisfy

$$v^-(x,t) \le v(x,t) \le v^+(x,t), \qquad (x,t) \in R \times R_+.$$

Thanks to Lemmas 2.4 and 2.5, we have the following convergence results:

$$\sup_{x\in R} |v^-(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \qquad \sup_{x\in R} |v^+(x,t) - \phi(x+ct)| \le Ce^{-\mu t}.$$

Then, by using the squeeze technique, we finally prove

$$\sup_{x\in R} |v(x,t) - \phi(x+ct)| \le Ce^{-\mu t}.$$

The proof is complete. □

3. **Concluding remark.**

In this section, we extend the wave stability result obtained before for equation (1) with a specific maturation rate $m(v) = \beta v^2$ to a general case. Namely, we consider

$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + \alpha e^{-\gamma\tau} v(x, t-\tau) - m(v), \qquad t \in [0,\infty),\ x \in R, \tag{47}$$

with an initial value condition

$$v(x,s) = v_0(x,s), \quad s \in [-\tau, 0], \tag{48}$$

where $m(v)$ is a more general maturation rate satisfying the following hypotheses

(H) There exist $v_- = 0$ and $v_+ > 0$ as two constant equilibria for equation (47) such that $\alpha e^{-\gamma\tau} v_\pm - m(v_\pm) = 0$, and satisfy $m(0) = 0$, $m'(0) = 0$, $m'(v_+) > \alpha e^{-\gamma\tau}$, and $m'(v) > 0$, $m''(v) > 0$ for $0 = v_- < v < v_+$.

As some important and typical examples, we may take $m(v)$ as

$$m_1(v) := \beta v^p, \quad \text{for } \beta > 0,\ p > 1,$$

with $v_- = 0$ and $v_+ = \left(\frac{\alpha}{\beta} e^{-\gamma\tau}\right)^{\frac{1}{p-1}}$. Here, $m_1(v)$ satisfies the condition (H).

Or, we may take

$$m_2(v) := \beta v^p e^{av}, \quad \text{for } \beta > 0,\ p > 1,\ a > 0,$$

with only two equilibria $v_- = 0$ and $v_+ > 0$ such that $v_+^{p-1} e^{av_+} = \frac{\alpha}{\beta} e^{-\gamma\tau}$ (the increasing monotonicity of $v^{p-1}e^{av}$ determines a unique $v_+ > 0$). Note also that, both βv^p and e^{av} are increasing and concave upward, so $m_2(v)$ satisfies (H), namely, $m_2(0) = 0$, $m_2'(0) = 0$, $m_2'(v_+) > \alpha e^{-\gamma\tau}$, and $m_2'(v) > 0$, $m_2''(v) > 0$ for $v_- = 0 < v \le v_+$.

We may also take

$$m_3(v) := \frac{\beta v^p}{1 - av}, \quad \text{for } \beta > 0,\ p > 1,\ a > 0,$$

with only two constant equilibria $v_- = 0$ and $v_+ > 0$ such that $\beta v_+^{p-1} = \alpha e^{-\gamma\tau}(1 - av_+)$. It can be easily seen that such a $v_+ > 0$ is uniquely determined by the increasing monotonicity of the function βv^{p-1} and the decreasing monotonicity of the function $\alpha e^{-\gamma\tau}(1 - av)$. It can be also verified directly that $m_3(v)$ satisfy the conditions given in (H), namely, $m_3(0) = 0$, $m_3'(0) = 0$, $m_3'(v_+) > \alpha e^{-\gamma\tau}$, and $m_3'(v) > 0$, $m_3''(v) > 0$ for $0 = v_- < v < v_+$.

Thus, by the method of upper-lower solutions as showed in [3], we can similarly prove the existence of the traveling wavefront $\phi(x+ct)$ for equation (47) connecting the state constants $v_\pm$ given in the hypotheses (H), where the speed $c > c_0 > 0$, and c_0 is the critical wave speed and exactly same to what we obtained in (5). Furthermore, we can check out that, under the same sufficient conditions in Theorem 1.1, the wave stability stated in Theorem 1.1 is also true for the equation (47).

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