MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume 6, Number 4, October 2009

doi:10.3934/mbe.2009.6.683

pp. 683–700

EVOLUTION OF LOTKA-VOLTERRA PREDATOR-PREY SYSTEMS UNDER TELEGRAPH NOISE

P. Auger†

IRD, UMI 209, UMMISCO, IRD France Nord, F-93143, Bondy, France
UPMC Univ Paris 06, UMI 209, UMMISCO, F-75005, Paris, France
University Cadi Ayyad, LMDP, Marrakech, Morocco

N. H. Du♮ and N. T. Hieu ‡

Faculty of Mathematics, Mechanics and Informatics
Vietnam National University, Hanoi, Vietnam
334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam

(Communicated by Azmy S. Ackleh)

Abstract. In this paper we study a Lotka-Volterra predator-prey system with prey logistic growth under the telegraph noise. The telegraph noise switches at random two prey-predator models. The aim of this work is to determine the subset of omega-limit set of the system and show out the existence of a stationary distribution. We also focus on persistence of the predator and thus we look for conditions that allow persistence of the predator and prey community. We show that the asymptotic behaviour highly depends on the value of some constant λ which is useful to make suitable predictions about the persistence of the system.

1. Introduction.

The dynamics of predator-prey systems have been investigated very largely in the frame of deterministic models, Murray, (1989) [11], Edelstein-Keshet (1998) [7]. In Bazykin (1998) [5], one can find a review of most classical deterministic predator- prey models. The two variables are the prey $x(t)$ and predator $y(t)$ densities at time t. The classical form of a predator-prey model is the following one:

$$\begin{cases} \frac{dx}{dt} = f(x) - h(x,y)y \\ \frac{dy}{dt} = \widetilde{e}h(x,y)y - \mu y, \end{cases}$$

where the function $f(x)$ is the natural growth function of the prey, $h(x,y)y$ is the capture term; $\widetilde{e}$ is a positive prey biomass into predator biomass conversion parameter. μ is the natural mortality rate for predators. $h(x,y)$ is the so-called functional response, i.e., the prey density captured per unit of time and per unit of predator density. In the classical Lotka-Volterra model, it is assumed that the functional response is type I, i.e., depending only on the prey density and linear,

2000 *Mathematics Subject Classification.* 34C12, 60H10, 92D25.
Key words and phrases. Lotka-Volterra predator-prey systems, carrying capacity, telegraph noise, stationary distribution.
This work was done under the support of Vietnam National University, Hanoi, Project QGTD 08-02.
♮ Corresponding author.

i.e., $h(x) = qx$ where q is a positive constant which is called the catch-ability. It is also usual to assume that prey grows logistically leading to the model:

$$\begin{cases} \frac{dx}{dt} = x\left(r\left(1 - \frac{x}{K}\right) - qy\right) = x\left(a - bx - cy\right) \\ \frac{dy}{dt} = \left(\widetilde{e}qx - \mu\right)y = \left(-d + ex\right)y. \end{cases}$$

Where r is the growth rate of the prey and K its carrying capacity. For the sake of simplification, in the next sections, we use the model under the form involving the parameter set (a, b, c, d, e) and the links with ecological parameters $(r, K, q, \mu, \widetilde{e})$ is not given here because it is obvious. A simple mathematical analysis of this Lotka-Volterra model with prey logistic growth shows that two cases can occur according to parameters values, Bazykin (1998) [5]:

- if $K < \frac{\mu}{\widetilde{e}q}$, the predator cannot invade and goes extinct while the prey density asymptotically tends to its carrying capacity;
- if $K > \frac{\mu}{\widetilde{e}q}$, the predator-prey community is persistent and prey, predator asymptotically coexist at constant equilibrium densities.

This classical Lotka-Volterra model assumes that species live in a constant environment. However, it is clear that it is not the case in reality and that it is important to take into account the variability of the environment which may have important consequences on the dynamics and persistence of a predator-prey community. The variability of the environment may be expressed under the stochastic factors. For the stochastic Lotka-Volterra equation, there is not too much in mathematical literature, and almost nothing in statistical inference. Here, we mention one of the first attempts in this direction, the very interesting paper of Arnold et al. [4] in which the authors used the theory of Brownian motion processes and the related white noise models to study the sample paths of the equation. For the branching models in a varying environment, we can refer to [2, 3, 12]. A systematic review has been given in [1]. In the simplest case, one might consider that environmental conditions can switch between two states, a hot and cold one, a dry state and wet one. Thus, we can suppose there is a telegraph noise affecting on the model in the form of switching between two-element set, $E = \{1, 2\}$. With different states, the coefficients of model are different. The stochastic displacement of environmental conditions provokes model to change from the system in state one to the system in state two and vice versa. When the carrying capacity of environment is absent ($K = \infty$), the telegraph noise does the model chaotically. It can not be permanent if the positive rest points of two deterministic systems do not coincide [13, 10]. In this paper we study this model with carrying capacity of environment ($K < \infty$) where the dynamics of the system is quite different. The predator may be extinct when one deterministic system has only non positive rest point. However, if two deterministic systems have positive rest points, it is proved that the model will be permanent with probability 1. Moreover, we show the existence of stationary distribution of solution in this case.

The paper has 6 sections. Section 2 details model and gives some properties of the boundary equations. In Section 3, dynamic behavior of the solution is studied. Subsets of ω-limit set are also described for each case. It is shown that the ω-limit sets include every orbit starting at a point on the curves linking two rest points of the subsystems. In Section 4, it proves the existence of invariant measures. In the section 5 and last section, some computational results illustrate the behavior of Lotka-Volterra systems under telegraph noise and the research work is discussed and summarized.

2. **Preliminary.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space satisfying the general hypotheses [8] and $(\xi_t)_{t\geq 0}$ be a Markov process, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in the set of two elements, say $E = \{1, 2\}$. Suppose that (ξ_t) has the transition intensities $1 \xrightarrow{\alpha} 2$ and $2 \xrightarrow{\beta} 1$ with $\alpha > 0, \beta > 0$. The process (ξ_t) has a unique stationary distribution

$$p = \lim_{t\to\infty} \mathbb{P}\{\xi_t = 1\} = \frac{\beta}{\alpha+\beta};\ q = \lim_{t\to\infty} \mathbb{P}\{\xi_t = 2\} = \frac{\alpha}{\alpha+\beta}. \tag{2.1}$$

The trajectories of (ξ_t) are piecewise-constant, cadlag functions. Let

$$0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n < ... \tag{2.2}$$

be its jump times. Put

$$\sigma_1 = \tau_1 - \tau_0,\ \sigma_2 = \tau_2 - \tau_1, ..., \sigma_n = \tau_n - \tau_{n-1} ... \tag{2.3}$$

$\sigma_1 = \tau_1$ is the first exile from the initial state, σ_2 is the time the process (ξ_t) spends in the state into which it moves from the first state... It is known that $(\sigma_k)_{k=1}^{\infty}$ are independent in the condition of given sequence $(\xi_{\tau_k})_{k=1}^{\infty}$ (see [8, vol. 2, pp. 217]). Note that if ξ_0 is given then ξ_{τ_n} is known since the process (ξ_t) takes only two values. Hence, $(\sigma_k)_{n=1}^{\infty}$ is a sequence of conditionally independent random variables, valued in $[0, \infty)$. Moreover, if $\xi_0 = 1$ then σ_{2n+1} has the exponential density $\alpha 1_{[0,\infty)} \exp(-\alpha t)$ and σ_{2n} has the density $\beta 1_{[0,\infty)} \exp(-\beta t)$. Conversely, if $\xi_0 = 2$ then σ_{2n} has the exponential density $\alpha 1_{[0,\infty)} \exp(-\alpha t)$ and σ_{2n+1} has the density $\beta 1_{[0,\infty)} \exp(-\beta t)$ (see [8, vol. 2, pp. 217]). Here $1_{[0,\infty)} = 1$ for $t \geq 0$ ($= 0$ for $t < 0$).

Denote $\mathcal{F}_0^n = \sigma(\tau_k : k \leq n)$; $\mathcal{F}_n^\infty = \sigma(\tau_k - \tau_n : k > n)$. We see that $\mathcal{F}_0^n$ is independent of $\mathcal{F}_n^\infty$ for any $n \in \mathbb{N}$ in the condition that ξ_0 given.

We consider the Lotka-Volterra predator-prey system described by the equation

$$\begin{cases} \dot{x} = x\left(a(\xi_t) - b(\xi_t)x - c(\xi_t)y\right), \\ \dot{y} = y\left(-d(\xi_t) + e(\xi_t)x\right), \end{cases} \tag{2.4}$$

where $g : E \to (0, \infty)$ for $g = a, b, c, d, e$. The noise (ξ_t) intervenes virtually into the equation (2.4), it makes a switching between the deterministic system

$$\begin{cases} \dot{x}_1(t) = x_1(t)(a_1 - b_1 x_1(t) - c_1 y_1(t)), \\ \dot{y}_1(t) = y_1(t)(-d_1 + e_1 x_1(t)), \end{cases} \tag{2.5}$$

and the deterministic one

$$\begin{cases} \dot{x}_2(t) = x_2(t)(a_2 - b_2 x_2(t) - c_2 y_2(t)), \\ \dot{y}_2(t) = y_2(t)(-d_2 + e_2 x_2(t)), \end{cases} \tag{2.6}$$

where $g_i = g(i)$ for $i = 1, 2$ and $g = a, b, c, d, e$.

Since (ξ_t) takes values in a two-element set E, if the solution of the system (2.4) satisfies the system (2.5) on the interval (τ_{n-1}, τ_n), then it must satisfy the system (2.6) on the interval (τ_n, τ_{n+1}) and vice versa. Therefore, $(x(\tau_n), y(\tau_n))$ is the switching point which plays the terminal point of one system and simultaneously the initial condition of the other. The relationship between the system (2.5) and the system (2.6) will determine the trajectory behavior of the system (2.4).

The case where $b_1 = b_2 = 0$ has been studied in [13] where it has shown that the dynamics of the solution is very chaotic. In this paper, we focus only on the case $b_1 > 0$ and $b_2 > 0$.

It is well-known that the systems (2.5) and (2.6) respectively have the rest points

$$x_i^* = \frac{d_i}{e_i}, \quad y_i^* = \frac{a_i e_i - b_i d_i}{c_i e_i}, \qquad i = 1, 2, \tag{2.7}$$

and their global dynamics depend on these rest points. Concretely, if $y_i^* > 0$ then the ith-rest point is asymptotically stable, i.e., $\lim_{t\to\infty}(x_i(t), y_i(t)) = (x_i^*, y_i^*)$ when $x_i(0) > 0$, $y_i(0) > 0$. If $y_i^* \le 0$ then $\lim_{t\to\infty}(x_i(t), y_i(t)) = \big(\frac{a_i}{b_i}, 0\big)$ for $i = 1, 2$.

The behavior of two boundary equations is easy to be studied. In the case where the prey is absent, the quantity $v(t)$ of predator at the time t satisfies the equation $\dot{v} = -d(\xi_t)v$. Thus, $v(t)$ decreases exponentially to 0. Similarly, without the predator, the quantity $u(t)$ of the prey at the time t satisfies the logistic equation

$$\dot{u} = u\big(a(\xi_t) - b(\xi_t)u\big), \quad u(0) > 0. \tag{2.8}$$

To simplify notations, we put

$$h_1 = h_1(u) = u(a_1 - b_1 u), \ h_2 = h_2(u) = u(a_2 - b_2 u),$$
$$I = [u_*, u^*] \text{ where } u_* = \min\{a_1/b_1, a_2/b_2\} \text{ and } u^* = \max\{a_1/b_1, a_2/b_2\}.$$

It is known that (see [6]) $(\xi_t, u(t))$ is a Markov process with the infinitesimal operator

$$\begin{cases} Lf(1,u) = -\alpha(f(1,u) - f(2,u)) + h_1(u)\frac{d}{du}f(1,u), \\ Lf(2,u) = \beta(f(1,u) - f(2,u)) + h_2(u)\frac{d}{du}f(2,u), \end{cases}$$

with $f(i,x)$ to be a function defined on $E \times (0,\infty)$, continuously differentiable in x. The stationary density (μ_1, μ_2) of $(\xi_t, u(t))$ can be found from the Fokker-Planck equation

$$\begin{cases} -\alpha\mu_1(u) + \beta\mu_2(u) - \frac{d}{du}[h_1\mu_1(u)] = 0, \\ \alpha\mu_1(u) - \beta\mu_2(u) - \frac{d}{du}[h_2\mu_2(u)] = 0. \end{cases} \tag{2.9}$$

This equation has a unique solution

$$\mu_1(u) = \frac{F(u)}{h_1(u)}\left[X(u_0) + \beta m \int_{u_0}^{u} \frac{1}{F(x)h_2(x)}\,dx\right], \tag{2.10}$$

$$\mu_2(u) = \frac{F(u)}{h_2(u)}\left[m - X(u_0) + \alpha m \int_{u_0}^{u} \frac{1}{F(x)h_1(x)}\,dx\right], \tag{2.11}$$

where the constants m and $X(u_0)$ are chosen such that

$$\mu_1(u) \ge 0, \mu_2(u) \ge 0, \quad \int_I \big(p\mu_1(u) + q\mu_2(u)\big)du = 1.$$

Further,

$$\liminf_{t\to\infty} u(t) = u_*; \quad \limsup_{t\to\infty} u(t) = u^*, \tag{2.12}$$

and by the law of large numbers, for any continuous function $f : E \times \mathbb{R} \to \mathbb{R}$ with

$$\int_I \big(pf(1,u)\mu_1(u) + qf(2,u)\mu_2(u)\big)du < \infty,$$

we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\xi_s, u(s))\,ds = \int_I \big(pf(1,u)\mu_1(u) + qf(2,u)\mu_2(u)\big)du. \tag{2.13}$$

In fact, we can calculate the explicit formula for the stationary densities $\mu_1(u)$, $\mu_2(u)$ but in practice, it is not useful. To study some their properties, we had better use the simulation method.

3. **Dynamic behavior of the solutions.** Let $(x_0, y_0) \in \mathbb{R}^2_+$. Denote by $(x(t, x_0, y_0), y(t, x_0, y_0))$ the solution of (2.4) satisfying the initial condition $(x(0, x_0, y_0), y(0, x_0, y_0)) = (x_0, y_0)$. For the sake of simplification, we write $(x(t), y(t))$ for $(x(t, x_0, y_0), y(t, x_0, y_0))$ if there is no confusion. Denote $g_{\min} = \min(g_1, g_2)$, $g_{\max} = \max(g_1, g_2)$ for $g = a, b, c, d, e$. A function f defined on $[0, \infty)$ is said to be ultimately bounded above (respectively, ultimately bounded below) by a if $\limsup_{t\to\infty} f(t) < a$ (respectively, $\liminf_{t\to\infty} f(t) > a$).

Proposition 3.1. *The system (2.4) is dissipative.*

Proof. From the first equation of the system (2.4) we see that whenever $x(t) \geq \overline{x} =: \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\}$, $x(t)$ is decreasing in t. Therefore, $x(t)$ is ultimately bounded above by $\overline{x}$. That is, $x(t) < \overline{x}\ \forall\, t > t_0$ for some $t_0 > 0$. Denote $\overline{y} = \max\{\frac{2a_1}{c_1}, \frac{2a_2}{c_2}\}$ and $k = -\max\{\frac{2e_1}{c_1}, \frac{2e_2}{c_2}\}$. Since $-c_i + \frac{a_i - b_i x}{y} < 0$ when $0 < x < \overline{x}$ and $y > \overline{y}$,

$$y(-d_i + e_i x) < kx(a_i - b_i x - c_i y)\ \ \forall\, 0 < x < \overline{x}, y > \overline{y}, i = 1, 2.$$

This means that the vector fields on the straight segment AB joining two points $A = (\overline{x}, \overline{y})$ and $B = (0, \overline{y} - k\overline{x})$ direct into the domain H limited by the straight lines $x = 0, x = \overline{x}, y = 0$ and the segment AB. Further, it is easy to see that there exists a $t_0 > 0$ such that $(x(t_0), y(t_0)) \in H$ which implies $(x(t), y(t)) \in H$ for any $t > t_0$. Thus, we conclude that the system (2.4) is dissipative. The proof is complete. □

Proposition 3.2. $\limsup_{t\to\infty} x(t) \geq \min\{x_1^*, x_2^*, \frac{a_1}{b_1}, \frac{a_2}{b_2}\}$.

Proof. Suppose in the contrary that $\limsup_{t\to\infty} x(t) < \min\{x_1^*, x_2^*, \frac{a_1}{b_1}, \frac{a_2}{b_2}\}$. With this assumption, there are $\delta > 0, \varepsilon > 0$ and $t_1 > 0$ satisfying $b_i\delta - c_i\varepsilon > 0, i = 1, 2$ and $x(t) < \min\{x_1^*, x_2^*, \frac{a_1}{b_1}, \frac{a_2}{b_2}\} - \delta$ for all $t \geq t_1$. From the second equation of the system (2.4), it follows that $y(t)$ decreases exponentially in t $(t \geq t_1)$. Therefore, there is $t_2 > t_1$ such that $y(t) < \varepsilon$ for any $t > t_2$. Hence, $a(\xi_t) - b(\xi_t)x(t) - c(\xi_t)y(t) > a(\xi_t) - b(\xi_t)\big(\frac{a(\xi_t)}{b(\xi_t)} - \delta\big) - c(\xi_t)\varepsilon > 0$ for any $t > t_2$. This implies that $\lim_{t\to\infty} x(t) = \infty$ which is impossible since $x(t)$ is bounded. □

Proposition 3.3. *There exists a positive number $x_{\min}$, independent from the choice of $(x_0, y_0) \in \mathbb{R}^2_+$, such that $x(t)$ is ultimately bounded below by $x_{\min}$. This means that there is $t_3 > 0$ such that $x(t) \geq x_{\min}$ for all $t \geq t_3$.*

Proof. Denote $y_{\max} = \overline{y} - k\overline{x}$ where $k, \overline{y}, \overline{x}$ are mentioned in the proposition 3.1. By virtue of the proposition 3.2, there exists $t_3 > 0$ such that $x(t_3) > \frac{1}{2}\min\{x_1^*, x_2^*, \frac{a_1}{b_1}, \frac{a_2}{b_2}\}$. Let $0 < \varepsilon \leq \frac{1}{2}\min\{x_1^*, x_2^*, \frac{a_1}{b_1}, \frac{a_2}{b_2}\}$ such that $\delta_1 = -d_{\min} + \varepsilon e_{\max} < 0$. If $x(t) \geq \varepsilon$ for $t > t_3$ then the proposition is proved. Otherwise, $x(t) < \varepsilon$ for a $t > t_3$. Let $h_1 = \inf\{s > t_3 : x(s) < \varepsilon\}$. We see that if $x(t) \leq \varepsilon$ for $t \in (h_1, h_2)$ then $\dot{y} = y(-d + ex) \leq y(-d_{\min} + e_{\max}\varepsilon) = \delta_1 y$ for all $t \in (h_1, h_2)$ which implies that

$$y(t) \leq y(h_1)\exp\{\delta_1(t - h_1)\} \leq y_{\max}\exp\{\delta_1(t - h_1)\}, \quad \forall t \in (h_1, h_2).$$

Hence,

$$\dot{x} = x(a - bx - cy) \geq x(a_{\min} - b_{\max}x - c_{\max}y_{\max}\exp\{\delta_1(t - h_1)\}), \quad \forall t \in (h_1, h_2). \tag{3.1}$$

Putting

$$n(t) = \int_{h_1}^{t} (a_{\min} - c_{\max}y_{\max}\exp\{\delta_1(s - h_1)\})ds, \quad N(t) = \int_{h_1}^{t} \exp\{n(s)\}ds,$$

by comparison theorem we get

$$x(t) \geq \frac{\varepsilon \exp\{n(t)\}}{1+\varepsilon b_{\max} N(t)}, \quad \forall t \in (h_1, h_2).$$

Let

$$\alpha = \min_{t>h_1} \frac{\varepsilon \exp\{n(t)\}}{1+\varepsilon b_{\max} N(t)} > 0.$$

It is clear that α does not depend on $(x(0), y(0))$ and h_1. Let $x_{\min} = \min\{\alpha, \varepsilon\}$ we see that $x(t) > x_{\min}$ for all $t > t_3$. The proof is complete. □

Denote

$$\lambda := \int_I \big(p(-d_1 + e_1 u)\mu_1(u) + q(-d_2 + e_2 u)\mu_2(u)\big) du. \tag{3.2}$$

Proposition 3.4.

a) *If $\lambda > 0$ then there is a positive number δ such that $\limsup_{t\to\infty} y(t, x_0, y_0) > \delta$ with probability 1.*

b) *If $\lambda < 0$ we have $\lim_{t\to\infty} y(t, x_0, y_0) = 0$ with probability 1.*

Proof. Let $u(t)$ be the solution of (2.8) with $u(0) = x(0)$. By virtue of the inequality

$$\dot{x} = x(a - bx - cy) \leq x(a - bx)$$

and the comparison theorem we have $u(t) \geq x(t)$ for any $t \geq 0$.
a) Suppose in the contrary that for any $\delta > 0$, there is a set $B_\delta \in \mathcal{F}$ with $P(B_\delta) > 0$ and $\limsup_{t\to\infty} y(t,\omega) < \delta$ for any $\omega \in B_\delta$. By the proposition 3.3, $\liminf_{t\to\infty} x(t,\omega) \geq x_{\min}$ a.s. This means that there is $t_4 > 0$ such that $0 < \frac{c(\xi_t(\omega))y(t,\omega)}{x(t,\omega)} < m_1\delta$ for any $t > t_4$ and $\omega \in B_\delta$ where $m_1 = \dfrac{c_{\max}}{x_{\min}}$. Putting $z = \frac{1}{x} - \frac{1}{u}$, we obtain $\dot{z} = -az + \frac{cy}{x}$. Therefore,

$$\begin{aligned} z(t) &= e^{-A(t)}(z(t_4) + \int_{t_4}^t e^{A(s)} \frac{c(\xi_s)y(s)}{x(s)}\, ds) \\ &< e^{-A(t)}(z(t_4) + m_1\delta \int_{t_4}^t e^{A(s)}\, ds), \quad A(t) = \int_{t_4}^t a(\xi_s) ds. \end{aligned}$$

This inequality implies that $\limsup_{t\to\infty} z(t) < m_2\delta$ where $m_2 = m_1/a_{\min}$. Hence, it is easy to see that there is a constant m_3 such that $\limsup_{t\to\infty} \mid e_{\max}(x(t,x_0,y_0) - u(t,x_0,y_0)) \mid < m_3\delta$. Further, from the equality

$$\frac{\dot{y}(t)}{y(t)} = -d(\xi_t) + e(\xi_t)x(t) = -d(\xi_t) + e(\xi_t)u(t) + e(\xi_t)(x(t) - u(t)),$$

it follows that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\ln y(t)-\ln y(0)}{t}\\
&=\limsup_{t\to\infty}\frac{1}{t}\Big(\int_{t_4}^{t}(-d(\xi_s)+e(\xi_s)u(s)+e(\xi_s)(x(s)-u(s)))\,ds\Big)\\
&\geq\liminf_{t\to\infty}\frac{1}{t}\Big(\int_{t_4}^{t}(-d(\xi_s)+e(\xi_s)u(s)+e(\xi_s)(x(s)-u(s)))\,ds\Big)\\
&\geq\liminf_{t\to\infty}\frac{1}{t}\int_{t_4}^{t}(-d(\xi_s)+e(\xi_s)u(s))\,ds+\liminf_{t\to\infty}\frac{1}{t}\Big(\int_{t_4}^{t}e(\xi_s)(x(s)-u(s))\,ds\Big)\\
&\geq\liminf_{t\to\infty}\frac{1}{t}\int_{t_4}^{t}(-d(\xi_s)+e(\xi_s)u(s))\,ds-m_3\delta.
\end{aligned}$$

for any $\omega\in B_\delta$. Using (2.13) we have

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\int_{t_4}^{t}(-d(\xi_s)+e(\xi_s)u(s))\,ds=\\
&\int_I\big(p(-d_1+e_1u)\mu_1(u)+q(-d_2+e_2u)\mu_2(u)\big)du=\lambda>0\quad a.s.
\end{aligned}$$

Let $\delta<\frac{\lambda}{2m_3}$, we have $\limsup_{t\to\infty}\frac{\ln y(t)-\ln y(0)}{t}>\lambda-\frac{\lambda}{2}>0$. This is a contradiction because $\limsup_{t\to\infty}\frac{\ln y(t)-\ln y(0)}{t}\leq 0$ for $\omega\in B_\delta$. Thus, $\limsup_{t\to\infty}y(t,x_0,y_0)>\delta>0$ $a.s.$

b) Suppose that $\lambda<0$. From above we see that

$$\frac{\dot y(t)}{y(t)}=-d(\xi_t)+e(\xi_t)x(t)\leq -d(\xi_t)+e(\xi_t)u(t).$$

Similar a) we obtain

$$\limsup_{t\to\infty}\frac{\ln y(t)-\ln y(0)}{t}\leq\limsup_{t\to\infty}\frac{1}{t}\Big(\int_{t_4}^{t}(-d(\xi_s)+e(\xi_s)u(s))\,ds\Big)=\lambda<0,$$

which implies that $\limsup_{t\to\infty}y(t)=0$. The proof is complete. □

Corollary 3.5. *If $\lambda>0$ then (2.4) is persistent with probability 1.*

Proof. By combining the propositions 3.1 and 3.4. □

We give the explicit formula to calculate λ.
Denote $\lambda_i=\frac{a_i}{b_i}-\frac{d_i}{e_i}, i=1,2$.

Proposition 3.6.

$$\lambda=e_1\lambda_1p+e_2\lambda_2q. \tag{3.3}$$

Proof. Without loss of generality, we suppose that $\xi_0=1$ with probability 1. By the proposition 3.4 we have

$$\lambda=\lim_{n\to\infty}\frac{1}{\tau_{2n}}\int_0^{\tau_{2n}}(-d(\xi_s)+e(\xi_s)u(s))ds\quad(a.s).$$

From the relation

$$\begin{aligned}-d(\xi_s)+e(\xi_s)u(s) &= -d(\xi_s)+\frac{e(\xi_s)}{b(\xi_s)}a(\xi_s)-\frac{e(\xi_s)}{b(\xi_s)}(a(\xi_s)-b(\xi_s)u(s))\\ &= -d(\xi_s)+\frac{e(\xi_s)}{b(\xi_s)}a(\xi_s)-\frac{e(\xi_s)}{b(\xi_s)}\frac{\dot u(s)}{u(s)}\\ &= -d(\xi_s)+\frac{e(\xi_s)}{b(\xi_s)}a(\xi_s)+\Big(\frac{e_2}{b_2}-\frac{e_1}{b_1}\Big)\frac{\dot u(s)}{u(s)}1_{\{\xi_s=1\}}-\frac{e_2}{b_2}\frac{\dot u(s)}{u(s)},\end{aligned}$$

it follows that

$$\begin{aligned}&\int_0^{\tau_{2n}}(-d(\xi_s)+e(\xi_s)u(s))ds\\ &=\int_0^{\tau_{2n}}(-d(\xi_s)+\frac{e(\xi_s)}{b(\xi_s)}a(\xi_s))ds+\int_0^{\tau_{2n}}\left(\Big(\frac{e_2}{b_2}-\frac{e_1}{b_1}\Big)\frac{\dot u(s)}{u(s)}1_{\{\xi_s=1\}}-\frac{e_2}{b_2}\frac{\dot u(s)}{u(s)}\right)ds\\ &=\int_0^{\tau_{2n}}(-d(\xi_s)+\frac{e(\xi_s)}{b(\xi_s)}a(\xi_s))ds+\left(\frac{e_2}{b_2}-\frac{e_1}{b_1}\right)\sum_{k=0}^{n-1}\ln\frac{u(\tau_{2k+1})}{u(\tau_{2k})}-\frac{e_2}{b_2}\ln\frac{u(\tau_{2n})}{u(0)}.\end{aligned}$$

Applying the law of large numbers we obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\tau_{2n}}\int_0^{\tau_{2n}}(-d(\xi_s)+\frac{e(\xi_s)}{b(\xi_s)}a(\xi_s))ds\\ &\qquad=(-d_1+\frac{e_1}{b_1}a_1)p+(-d_2+\frac{e_2}{b_2}a_2)q=e_1\lambda_1p+e_2\lambda_2q\quad (a.s).\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda &= \lim_{n\to\infty}\frac{1}{\tau_{2n}}\int_0^{\tau_{2n}}(-d(\xi_s)+e(\xi_s)u(s))ds\\ &= e_1\lambda_1p+e_2\lambda_2q+\lim_{n\to\infty}\frac{1}{\tau_{2n}}\left\{\left(\frac{e_2}{b_2}-\frac{e_1}{b_1}\right)\sum_{k=0}^{n-1}\ln\frac{u(\tau_{2k+1})}{u(\tau_{2k})}-\frac{e_2}{b_2}\ln\frac{u(\tau_{2n})}{u(0)}\right\}.\end{aligned}$$

Because $u(t)$ is bounded above and below by positive constants we get

$$\lim_{n\to\infty}\frac{1}{\tau_{2n}}\frac{e_2}{b_2}\ln\frac{u(\tau_{2n})}{u(0)}=0.$$

Moreover, since

$$u(\tau_{2k+2})=\frac{u(\tau_{2k})\exp\{a_1\sigma_{2k+1}+a_2\sigma_{2k+2}\}}{1+u(\tau_{2k})[(\frac{b_1}{a_1}-\frac{b_2}{a_2})\exp\{a_1\sigma_{2k+1}\}+\frac{b_2}{a_2}\exp\{a_1\sigma_{2k+1}+a_2\sigma_{2k+2}\}]},$$

it follows that $(u_{2k})_{k\in\mathbb{N}}$ is a Markov process with the transition operator

$$\begin{aligned}P_1f(u) &= \mathbb{E}\{f(u_{2k+2})\mid u_{2k}=u\}\\ &= \mathbb{E}\left\{\frac{u\exp\{a_1\sigma_{2k+1}+a_2\sigma_{2k+2}\}}{1+u[(\frac{b_1}{a_1}-\frac{b_2}{a_2})\exp\{a_1\sigma_{2k+1}\}+\frac{b_2}{a_2}\exp\{a_1\sigma_{2k+1}+a_2\sigma_{2k+2}\}]}\right\}\\ &= \mathbb{E}\left\{\frac{u\exp\{a_1\sigma_1+a_2\sigma_2\}}{1+u[(\frac{b_1}{a_1}-\frac{b_2}{a_2})\exp\{a_1\sigma_1\}+\frac{b_2}{a_2}\exp\{a_1\sigma_1+a_2\sigma_2\}]}\right\},\end{aligned}$$

where f is a bounded, continuous function defined on $\mathbb{R}_+$. Similarly, $(u_{2k+1})_{k\in\mathbb{N}}$ is a Markov process with the transition operator

$$\begin{aligned}P_2 f(u) &= \mathbb{E}\{f(u_{2k+3}) \mid u_{2k+1} = u\}\\ &= \mathbb{E}\left\{\frac{u\exp\{a_1\sigma_{2k+3} + a_2\sigma_{2k+2}\}}{1 + u[(\frac{b_1}{a_1} - \frac{b_2}{a_2})\exp\{a_1\sigma_{2k+3}\} + \frac{b_2}{a_2}\exp\{a_1\sigma_{2k+3} + a_2\sigma_{2k+2}\}]}\right\}\\ &= \mathbb{E}\left\{\frac{u\exp\{a_1\sigma_1 + a_2\sigma_2\}}{1 + u[(\frac{b_1}{a_1} - \frac{b_2}{a_2})\exp\{a_1\sigma_1\} + \frac{b_2}{a_2}\exp\{a_1\sigma_1 + a_2\sigma_2\}]}\right\}.\end{aligned}$$

Hence, by the law of large numbers

$$\lim_{n\to\infty}\frac{1}{\tau_{2n}}\sum_{k=0}^{n-1}\ln u(\tau_{2k+1}) = \lim_{n\to\infty}\frac{1}{\tau_{2n}}\sum_{k=0}^{n-1}\ln u(\tau_{2k}).$$

By combining these results we obtain (3.3). The proposition is proved. □

Next part we will describe subsets of $\omega-$limit set of the system (2.4). Let $(x_1(t,x,y),\ y_1(t,x,y))$ be the solution of (2.5) and $(x_2(t,x,y), y_2(t,x,y))$ be the solution of (2.6) starting in the point $(x,y)\in\mathbb{R}^2_+$. Denote by $\mathcal{U}_\varepsilon(x,y)$ the ε- neighborhood of (x,y) and $M_i = (x_i^*, \max\{y_i^*, 0\})$ for $i = 1,2$. Let $K\subset\mathbb{R}^2_+$ be a compact set.

Let $\omega(x_0,y_0)$ be the ω-limit set of the solution $(x(t,x_0,y_0), y(t,x_0,y_0))$ of the system (2.4).

Lemma 3.7. *For any $\delta_1 > 0$, there is a $T_1 = T_1(\delta_1) > 0$ such that if $(x_0,y_0)\in K$ then $(x_1(t,x_0,y_0), y_1(t,x_0,y_0))\in\mathcal{U}_{\delta_1}(M_1)$ and $(x_2(t,x_0,y_0),\ y_2(t,x_0,y_0))\in\mathcal{U}_{\delta_1}(M_2)$ for any $t\geq T_1$.*

Proof. Consider the system (2.5), if $(x,y)\in K$ then $\lim_{t\to\infty}(x_1(t,x,y), y_1(t,x,y)) = M_1$. Therefore, there exists a T_{xy} such that

$$(x_1(t,x,y), y_1(t,x,y))\in U_{\delta_1/2}(M_1)\quad\text{for all } t\geq T_{xy}.$$

By the continuity of the solutions in the initial conditions, there is a neighborhood U_{xy} of (x,y) such that for any $(u,v)\in U_{xy}$ we have

$$(x_1(t,u,v), y_1(t,u,v))\in U_{\delta_1}(M_1)\quad\text{for all } t\geq T_{xy}+1.$$

Since K is compact and the family $\{U_{xy} : (x,y)\in K\}$ is an open covering of K, by Heine-Borel lemma, there is a finite subfamily, namely $\{U_{x_iy_i}, i = 1,2,...,n\}$, which covers K. Let $T_1' = \max_{1\leq i\leq n}\{T_{x_iy_i}+1\}$. We see that if $(x_0,y_0)\in K$ then $(x_1(t,x_0,y_0), y_1(t,x_0,y_0))\in\mathcal{U}_{\delta_1}(M_1)$ for any $t\geq T_1'$.

Similarly, we can choose $T_1'' > 0$ such that if $(x_0,y_0)\in K$ then $(x_2(t,x_0,y_0),$ $y_2(t,x_0,y_0))\in\mathcal{U}_{\delta_1}(M_2)$ for any $t\geq T_1''$. By putting $T_1 = \max\{T_1', T_1''\}$ the lemma is proved. □

Lemma 3.8. *For any $\delta_3 > 0$, there exists a $\delta_2 > 0$ such that if $d((x_0,y_0),(u,v)) < \delta_2$, $(x_0,y_0)\in K$ and $(u,v)\in K$ then*

$$d((x_1(t,x_0,y_0), y_1(t,x_0,y_0)); (x_1(t,u,v), y_1(t,u,v))) < \delta_3 \text{ for all } t\geq 0,$$
$$d((x_2(t,x_0,y_0), y_2(t,x_0,y_0)); (x_2(t,u,v), y_2(t,u,v))) < \delta_3 \text{ for all } t\geq 0.$$

where $d(A,B)$ denote the distance between two points A and B.

Proof. This lemma follows from the lemma 3.7 and the continuous dependence of the solutions on the initial data. □

Let γ_1 (respectively, γ_2) be the orbit of the solution of (2.5) (respectively, of (2.6)) with the initial point M_2 (respectively, with the initial point M_1).

Lemma 3.9. *For any $\delta_5 > 0$ and $(\overline{x}, \overline{y}) \in \gamma_2$, there exists a $\delta_4 > 0, T_2 > 0$ and $t_2 > 0$ such that if $d((u,v), M_1) < \delta_4$ then $d((x_2(t,u,v), y_2(t,u,v)), (\overline{x}, \overline{y})) < \delta_5$ for all $T_2 \le t \le T_2 + t_2$.*
A similar formulation for the solution curves γ_1 of (2.5) is valid.

Proof. This lemma follows from the continuous dependence of the solutions on the initial data. □

Without loss of generality, suppose that $\xi_0 = 1$ with probability 1.

Theorem 3.10.
a) Suppose that $\lambda > 0$

1. *If $y_1^* > 0$ and $y_2^* > 0$, both the positive orbit γ_1 of the solution $(x_1(t, x_2^*, y_2^*), y_1(t, x_2^*, y_2^*))$ of the system (2.5); γ_2 of the solution $(x_2(t, x_1^*, y_1^*), y_2(t, x_1^*, y_1^*))$ of system (2.6) are subsets of $\omega(x_0, y_0)$. Moreover,*
2. *Any positive orbit $\overline{\gamma}_2$ of the solution $(x_2(t, \overline{x}, \overline{y}), y_2(t, \overline{x}, \overline{y}))$ of the system (2.6), starting in a point $(\overline{x}, \overline{y}) \in \gamma_1$ at $t = 0$, is a subset of $\omega(x_0, y_0)$.*
   Similarly, any positive orbit $\widetilde{\gamma}_1$ of the solution $(x_1(t, \widetilde{x}, \widetilde{y}), y_1(t, \widetilde{x}, \widetilde{y}))$ of the system (2.5), starting in a point $(\widetilde{x}, \widetilde{y}) \in \gamma_2$ at $t = 0$, is a subset of $\omega(x_0, y_0)$.
3. *If $y_1^* > 0$ and $y_2^* < 0$, then we have a similar result as in 1.; provided that (x_2^*, y_2^*) is replaced by $(\frac{a_2}{b_2}, 0)$ and γ_1 is replaced by closure of $\widetilde{\gamma}_1$ - says $\widehat{\gamma}_1$. Concurrently, $\gamma^u \subset \omega(x_0, y_0)$ with γ^u to be the ω-limit set of $(u(t), 0)$, here $u(t)$ is the solution of the system (2.8).*

b) If $\lambda < 0$, $y_1^* < 0$ and $y_2^* < 0$ then $\gamma^u \equiv \omega(x_0, y_0)$.

Proof. Let

$$x_n = x(\tau_n, x_0, y_0), \quad y_n = y(\tau_n, x_0, y_0),$$

$$\mathcal{F}_0^n = \sigma(\tau_k : k \le n), \quad \mathcal{F}_n^\infty = \sigma(\tau_k - \tau_n : k > n).$$

We see that (x_n, y_n) is $\mathcal{F}_0^n-$adapted. Moreover, if ξ_0 is given then $\mathcal{F}_n^\infty$ is independent of $\mathcal{F}_0^n$.
a) From the proposition 3.1, there are non-random constants $\Delta > 0$ and t_0 such that $x(t,x,y) < \Delta$ and $y(t,x,y) < \Delta$ for any $t \ge t_0$ with probability 1. Without loss of generality, we can suppose that $t_0 = 0$. Since $\lambda > 0$, by virtue of the proposition 3.4, there is $\delta > 0$ such that $x_{2k} \ge \delta$ and $y_{2k} \ge \delta$ for infinitely many time. Let $K = [\delta, \Delta] \times [\delta, \Delta]$. We construct a sequence

$$\begin{aligned}
\eta_1 &= \inf\{2k : x_{2k} \ge \delta,\ y_{2k} \ge \delta\},\\
\eta_2 &= \inf\{2k > \eta_1 : x_{2k} \ge \delta;\ y_{2k} \ge \delta\},\\
&\dots\\
\eta_n &= \inf\{2k > \eta_{n-1} : x_{2k} \ge \delta;\ y_{2k} \ge \delta\}\dots,
\end{aligned}$$

$\eta_1 < \eta_2 < ... < \eta_k < ...$ is a sequence of $\mathcal{F}_0^n-$ stopping times (see [8]). Moreover, $\{\eta_k = n\} \in \mathcal{F}_0^n$ for any k, n. Thus, the event $\{\eta_k = n\}$ is independent of $\mathcal{F}_n^\infty$. Because $\lambda > 0$ it yields $\eta_n < \infty$ a.s. for any n. With $\delta_1, T_1 > 0$ in the lemma 3.7 we put

$$A_k = \{\sigma_{\eta_k+1} \ge T_1\}, \quad k = 1, 2, ...$$

We see that

$$\mathbf{P}(A_k) = \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1\} = \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1 \mid \eta_k = 2n\}\mathbf{P}\{\eta_k = 2n\}$$
$$= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+1} \geq T_1 \mid \eta_k = 2n\}\mathbf{P}\{\eta_k = 2n\} = \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+1} \geq T_1\}\mathbf{P}\{\eta_k = 2n\}$$
$$= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_1 \geq T_1\}\mathbf{P}\{\eta_k = 2n\} = \mathbf{P}\{\sigma_1 \geq T_1\} > 0.$$

Similarly,

$$\mathbf{P}(A_k \cap A_{k+1}) = \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_{k+1}+1} \geq T_1\}$$
$$= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_{k+1}+1} \geq T_1 \mid \eta_k = 2l, \eta_{k+1} = 2n\}$$
$$\times \mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\}$$
$$= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{2l+1} \geq T_1, \sigma_{2n+1} \geq T_1 \mid \eta_k = 2l, \eta_{k+1} = 2n\}\mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\}$$
$$= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{2n+1} \geq T_1\}\mathbf{P}\{\sigma_{2l+1} \geq T_1 \mid \eta_k = 2l, \eta_{k+1} = 2n\}$$
$$\times \mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\}$$
$$= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_1 \geq T_1\}\mathbf{P}\{\sigma_{2l+1} \geq T_1 \mid \eta_k = 2l, \eta_{k+1} = 2n\}\mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\}$$
$$= \mathbf{P}\{\sigma_1 \geq T_1\} \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{2l+1} \geq T_1 \mid \eta_k = 2l, \eta_{k+1} = 2n\}\mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\}$$
$$= \mathbf{P}\{\sigma_1 \geq T_1\} \sum_{l=0}^{\infty} \mathbf{P}\{\sigma_{2l+1} \geq T_1 \mid \eta_k = 2l\}\mathbf{P}\{\eta_k = 2l\} = \mathbf{P}\{\sigma_1 \geq T_1\}^2 \ldots$$

Hence

$$\mathbf{P}(A_k \cup A_{k+1}) = 1 - (1 - \mathbf{P}\{\sigma_1 \geq T_1\})^2.$$

Continuing this way we obtain

$$\mathbf{P}\Big(\bigcup_{i=k}^{n} A_i\Big) = 1 - (1 - \mathbf{P}\{\sigma_1 \geq T_1\})^{n-k+1}.$$

Thus,

$$\mathbf{P}\{\sigma_{\eta_k+1} \geq T_1 \quad i.o. \quad k\} = \mathbf{P}\Big(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i\Big) = \lim_{k \to \infty} \mathbf{P}\Big(\bigcup_{i=k}^{\infty} A_i\Big) = 1.$$

From $x_{\eta_k} > \delta$ and $y_{\eta_k} > \delta$, it follows that if $\sigma_{\eta_k+1} > T_1$ then $(x_{\eta_k+1}, y_{\eta_k+1}) \in \mathcal{U}_{\delta_1}(x_1^*, y_1^*)$ by the lemma 3.7. This relation says that there are infinitely many n satisfying $(x_{2n+1}, y_{2n+1}) \in \mathcal{U}_{\delta_1}(x_1^*, y_1^*)$. Hence, $(x_1^*, y_1^*) \in \omega(x_0, y_0)$. Similarly, $(x_2^*, y_2^*) \in \omega(x_0, y_0)$.

Consider a point $(\overline{x}, \overline{y}) \in \gamma_2$. By virtue of the lemmas 3.8 and 3.9, for any neighborhood $\mathcal{U}_{\delta_3}$ of $(\overline{x}, \overline{y})$ there exists $T_2, t_2, \delta_2 > 0$ such that if $(u, v) \in \mathcal{U}_{\delta_2}(x_1^*, y_1^*)$

then $(x_2(t,u,v), y_2(t,u,v)) \in \mathcal{U}_{\delta_3}(\overline{x},\overline{y}), \forall t \in [T_2, T_2+t_2]$. Let $T_1 > 0$ be a number mentioned in the lemma 3.7 with $\delta_1 = \delta_2$. Put

$$B_k = \{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_k+2} \in [T_2, T_2+t_2]\}, \quad k = 1, 2, ...$$

Then,

$$\begin{aligned}
\mathbf{P}(B_k) &= \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_k+2} \in [T_2, T_2+t_2]\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_k+2} \in [T_2, T_2+t_2] \mid \eta_k = 2n\} \mathbf{P}\{\eta_k = 2n\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+1} \geq T_1, \sigma_{2n+2} \in [T_2, T_2+t_2] \mid \eta_k = 2n\} \mathbf{P}\{\eta_k = 2n\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+1} \geq T_1, \sigma_{2n+2} \in [T_2, T_2+t_2]\} \mathbf{P}\{\eta_k = 2n\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\} \mathbf{P}\{\eta_k = 2n\} \\
&= \mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\} > 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbf{P}(B_k \cap B_{k+1}) \\
&= \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_k+2} \in [T_2, T_2+t_2], \sigma_{\eta_{k+1}+1} \geq T_1, \sigma_{\eta_{k+1}+2} \in [T_2, T_2+t_2]\} \\
&= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_k+2} \in [T_2, T_2+t_2], \sigma_{\eta_{k+1}+1} \geq T_1, \sigma_{\eta_{k+1}+2} \in \\
&\qquad [T_2, T_2+t_2] \mid \eta_k = 2l, \eta_{k+1} = 2n\} \mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\} \\
&= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{2l+1} \geq T_1, \sigma_{2l+2} \in [T_2, T_2+t_2], \sigma_{2n+1} \geq T_1, \sigma_{2n+2} \in [T_2, T_2+t_2] \mid \\
&\qquad \mid \eta_k = 2l, \eta_{k+1} = 2n\} \mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\} \\
&= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{2n+1} \geq T_1, \sigma_{2n+2} \in [T_2, T_2+t_2]\} \mathbf{P}\{\sigma_{2l+1} \geq T_1, \\
&\qquad \sigma_{2l+2} \in [T_2, T_2+t_2] \mid \eta_k = 2l, \eta_{k+1} = 2n\} \mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\} \\
&= \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\} \mathbf{P}\{\sigma_{2l+1} \geq T_1, \sigma_{2l+2} \in [T_2, T_2+t_2] \mid \\
&\qquad \mid \eta_k = 2l, \eta_{k+1} = 2n\} \mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\} \\
&= \mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\} \sum_{0 \leq l < n < \infty} \mathbf{P}\{\sigma_{2l+1} \geq T_1, \sigma_{2l+2} \in [T_2, T_2+t_2] \mid \\
&\qquad \mid \eta_k = 2l, \eta_{k+1} = 2n\} \mathbf{P}\{\eta_k = 2l, \eta_{k+1} = 2n\} \\
&= \mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\} \sum_{l=0}^{\infty} \mathbf{P}\{\sigma_{2l+1} \geq T_1, \sigma_{2l+2} \in [T_2, T_2+t_2] \mid \\
&\qquad \mid \eta_k = 2l\} \mathbf{P}\{\eta_k = 2l\} \\
&= (\mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\})^2 ...
\end{aligned}$$

Thus,

$$\mathbf{P}(B_k \cup B_{k+1}) = 1 - (1 - \mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\})^2.$$

Continuing this way we have

$$\mathbf{P}\Big(\bigcup_{i=k}^{n} B_i\Big) = 1-(1-\mathbf{P}\{\sigma_1 \geq T_1, \sigma_2 \in [T_2, T_2+t_2]\})^{n-k+1}.$$

Hence,

$$\mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_k+2} \in [T_2, T_2+t_2] \text{ i.o. } k\} = \mathbf{P}\Big(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} B_i\Big) = \lim_{k\to\infty}\mathbf{P}\Big(\bigcup_{i=k}^{\infty} B_i\Big) = 1.$$

This means that $(\overline{x}, \overline{y}) \in \omega(x_0, y_0)$. Therefore, $\gamma_2 \subset \omega(x_0, y_0)$. Similarly, $\gamma_1 \subset \omega(x_0, y_0)$. Thus, we get *1.*

To prove *2.*, let $(\widetilde{x}, \widetilde{y}) \in \gamma_2$ and $\widetilde{\gamma}_1$ be the orbit of the solution of (2.5) starting in $(\widetilde{x}, \widetilde{y})$. We consider $(\widetilde{u}, \widetilde{v}) \in \widetilde{\gamma}_1$. By the lemmas 3.8 and 3.9, for any neighborhood $\mathcal{U}_{\delta_6}(\widetilde{u}, \widetilde{v})$, there exists $T_3, t_3, T_4, t_4, \delta_4, \delta_5 > 0$ such that if $(u, v) \in \mathcal{U}_{\delta_4}(x_1^*, y_1^*)$ and $h \in [T_3, T_3+t_3]$, we have $((x_2(h, u, v), y_2(h, u, v)) \in \mathcal{U}_{\delta_5}(\widetilde{x}, \widetilde{y})$. Moreover, $(x_1(t, \overline{u}, \overline{v}), y_1(t, \overline{u}, \overline{v})) \in \mathcal{U}_{\delta_6}(\widetilde{u}, \widetilde{v})$ for any $t \in [T_4, T_4+t_4]$, where $(\overline{u}, \overline{v}) = ((x_2(h, u, v), y_2(h, u, v))$. Let $T_1 > 0$ be a number mentioned in the lemma 3.7 with $\delta_1 = \delta_4$. By the same way as above we obtain

$$\mathbf{P}\{\sigma_{\eta_k+1} \geq T_1, \sigma_{\eta_k+2} \in [T_3, T_3+t_3], \sigma_{\eta_k+3} \in [T_4, T_4+t_4]; \quad i.o. \quad k > 0\} = 1.$$

This implies that $(\widetilde{u}, \widetilde{v}) \in \omega(x_0, y_0)$ and $\widetilde{\gamma}_1 \subset \omega(x_0, y_0)$. Similarly, $\overline{\gamma}_2 \subset \omega(x_0, y_0)$.

To get *3.* we note that $(x_1^*, y_1^*) \in \omega(x_0, y_0)$, $\gamma_2 \subset \omega(x_0, y_0)$, $\widetilde{\gamma}_1 \subset \omega(x_0, y_0)$ by *1.* and *2.* Concurrently, $(\frac{a_2}{b_2}, 0)$ belongs to the closure of γ_2 thus $(\frac{a_2}{b_2}, 0) \in \omega(x_0, y_0)$ and $\widehat{\gamma}_1$ is the closure of $\widetilde{\gamma}_1$ then $\widehat{\gamma}_1 \subset \omega(x_0, y_0)$. By the similar ways, we can show that $\big[(\frac{a_1}{b_1}, 0), (\frac{a_2}{b_2}, 0)\big] := \gamma^u$ is the ω-limit set of $(u(t), 0)$.

Consider any $(\widetilde{x}, \widetilde{y}) \in \gamma^u$. For any neighborhood $\mathcal{V}_{\varepsilon_2}(\widetilde{x}, \widetilde{y})$, there exists $T_5, t_5, \varepsilon_1 > 0$ such that if $(u, v) \in \mathcal{V}_{\varepsilon_1}(\frac{a_2}{b_2}, 0)$ then $(x_1(t, u, v), y_1(t, u, v)) \in \mathcal{V}_{\varepsilon_2}(\widetilde{x}, \widetilde{y})$, $\forall t \in [T_5, T_5+t_5]$. We set

$$\begin{aligned}
\rho_1 &= \inf\{2k : (x_{2k}, y_{2k}) \in \mathcal{V}_{\varepsilon_1}(\frac{a_2}{b_2}, 0)\},\\
\rho_2 &= \inf\{2k > \rho_1 : (x_{2k}, y_{2k}) \in \mathcal{V}_{\varepsilon_1}(\frac{a_2}{b_2}, 0)\},\\
&\dots\\
\rho_n &= \inf\{2k > \rho_{n-1} : (x_{2k}, y_{2k}) \in \mathcal{V}_{\varepsilon_1}(\frac{a_2}{b_2}, 0)\}\dots
\end{aligned}$$

Since $(\frac{a_2}{b_2}, 0) \in \omega(x_0, y_0)$, it follows that $\rho_n < \infty$ a.s. for any n. By similar way, we get $(\widetilde{x}, \widetilde{y}) \in \omega(x_0, y_0)$ and $\gamma^u \subset \omega(x_0, y_0)$.

b) By the proposition 3.4, if $\lambda < 0$, we get $\lim_{t\to\infty} y(t, x_0, y_0) = 0$. Therefore, behavior of the system (2.4) is described by system (2.8). The proof of the theorem 3.10 is complete. □

4. **The existence of invariant measures.** In this section, we study the existence of an invariant measure for Markov process $(x(t), y(t), \xi(t))$. From now on, we assume $y_1^* > 0, y_2^* > 0$.

Proposition 4.1. *There exists a non-random positive number $y_{\min} > 0$ such that $y(t)$ is ultimately bounded below by $y_{\min}$ for all $t \geq 0$.*

Proof. Without loss of generality, suppose that $x_1^* \le x_2^*$. From the assumption $y_1^* > 0, y_2^* > 0$ and proposition 3.6, it is easy to see that $\lambda > 0$. Let $\varepsilon_0 > 0$ be a positive number satisfying $\varepsilon_1 = \min\{a_1 - b_1 x_1^* - c_{\max}\varepsilon_0, a_2 - b_2 x_1^* - c_{\max}\varepsilon_0\} > 0$. By the theorem 3.10, the solution $(x(t), y(t))$ visits any neighborhood of the rest point (x_1^*, y_1^*). Therefore, we can suppose that $x(t) \ge x_{\min}\ \forall\, t \ge 0$ and $\varepsilon_0 < y(0) < y_1^*$.

Let $x(t) < x_1^*$ and $y(t) \le \varepsilon_0$ for $h_0 \le t \le h_1$. From the inequality

$$\dot{x}(t) = x(t)(a - bx(t) - cy(t)) \ge \varepsilon_1 x(t), \quad t \in (h_0, h_1),$$

it follows that

$$x(t) \ge x(h_0)\exp\{\varepsilon_1(t - h_0)\} \ge x_{\min}\exp\{\varepsilon_1(t - h_0)\}. \tag{4.1}$$

Denote $T = \frac{1}{\varepsilon_1}\ln\frac{x_1^*}{x_{\min}}$ and $\varepsilon_2 = \varepsilon_0\exp\{-d_{\min}T\}$. In the case where $y(t) \ge \varepsilon_0$ for any $t > 0$ we choose $y_{\min} = \varepsilon_0$ to prove the proposition. Otherwise, there is a $h_2 > 0$ such that $y(h_2) = \varepsilon_0$. Let $\tau = \inf\{t > h_2 : x(t) = x_1^*\}$ (with the convention $\inf\emptyset = h_2$). By virtue of (4.1) we see that $\tau \le h_2 + T$. Therefore, from $\dot{y}(t) = y(t)(-d + cx(t)) \ge -d_{\min}y(t)$, it follows that $y(\tau) \ge \varepsilon_0\exp\{-d_{\min}T\} = \varepsilon_2$.

Let $(x_2(t), y_2(t))$ be the solution of (2.6) satisfying $x_2(0) = x_{\min}$, $y_2(0) = \varepsilon_2$. Put $y_{\min} = \inf\{y_2(t) : t > 0\}$. It is easy to see that $y_{\min} = y_2(\tau^*) > 0$ where $\tau^* = \inf\{s > 0 : x_2(s) = x_2^*\}$. We show that $y(t) \ge y_{\min}$ for any $t \ge h_2$. Indeed, $y(t)$ is increasing whenever $x(t) \ge x_2^*$. In this case $y(t) > y_{\min}$. If $x(t) < x_1^*$ we get $y(t) \ge y(\tau) \ge \varepsilon_2 > y_{\min}$. We consider the case $x_1^* \le x(t) \le x_2^*$. Let $\gamma = \{(x(t), y(t)) : t \ge h_2\}$ and $\gamma_2 = \{(x_2(t), y_2(t)) : 0 < t < \tau^*\}$. We see that when $\xi(t) = 1$ then $\dot{y}(t) > 0$ because $x(t) > x_1^*$. If $\xi(t) = 2$ then $\dot{y}(t) = \dot{y}_2(t)$. This implies that γ lies above the solution curve γ_2. It means that $y(t) > y_{\min}$. The proof is complete. □

Proposition 4.2. *The system (2.4) is permanent.*

Proof. The proof follows from the propositions 3.1, 3.3 and 4.1. □

Theorem 4.3. *For the Markov process $(x(t), y(t), \xi_t)_{t\ge 0}$, there exists a stationary distribution.*

Proof. For the sake of simplicity, we denote $z(t) = (x(t), y(t), \xi_t)$. Let $P(z, t, A)$ is homogeneous stochastically continuous Feller transition function of the Markov process $(z(t))_{t\ge 0}$ where $z \in \mathbb{R}^2_+$, i.e., $P(z, t, A) = P\{z(t) \in A \mid z(0) = z\}$. For $r > 0$, let $\mathcal{U}_r = \{(x, y) \in \mathbb{R}^2_+ : \frac{1}{r} \le x \le r; \frac{1}{r} \le y \le r\}$ and $\overline{\mathcal{U}}_r = \mathcal{U}_r^c \times \mathbb{E}$, where $\mathcal{U}_R^c$ is the complement of $\mathcal{U}_r$ in $\mathbb{R}^2_+$ and $E = \{1, 2\}$. For r large enough, the set U_r contains the set $\{(x, y) \in H : x \ge x_{\min}, y \ge y_{\min}\}$, where H is the set mentioned in the proposition 3.1. Therefore, by virtue of the proposition 4.2, if r is large enough then $\lim_{t\to\infty} P(z_0, t, \overline{\mathcal{U}}_r) = 0$. Hence,

$$\lim_{r\to\infty}\varliminf_{T\to\infty}\frac{1}{T}\int_0^T P(z_0, t, \overline{\mathcal{U}}_r)dt = 0.$$

By the theorem (see [9, pp.72]), Markov process $(z(t))_{t\ge 0}$ has a stationary distribution. The proof is complete. □

5. **Computational results.** In this section, we present some numerical simulations. As the first example, we consider the case $a_1 = 4.2, b_1 = 1, c_1 = 1.1, d_1 = 5, e_1 = 1.8, a_2 = 6, b_2 = 0.8, c_2 = 1.3, d_2 = 9.4, e_2 = 2.5, x(0) = 4, y(0) = 4.3, \alpha = 0.6, \beta = 0.8$, where $y_1^* > 0, y_2^* > 0$. The number of switching is $n = 300$. The individual sample paths in the figure 1 illustrate the $\omega-$limit set in the theorem

3.10 in section 3 and the proposition 4.2; the theorem 4.3 in the section 4. The figure 2 shows the oscillations of the population sizes $x(t)$ and $y(t)$.

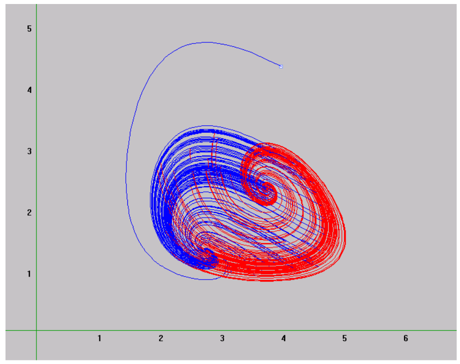

FIGURE 1. Orbit of system in case $y_1^* > 0, y_2^* > 0$. The solution moves between the two rest points $(2.77, 1.29)$ and $(3.76, 2.30)$ as switching occurs.

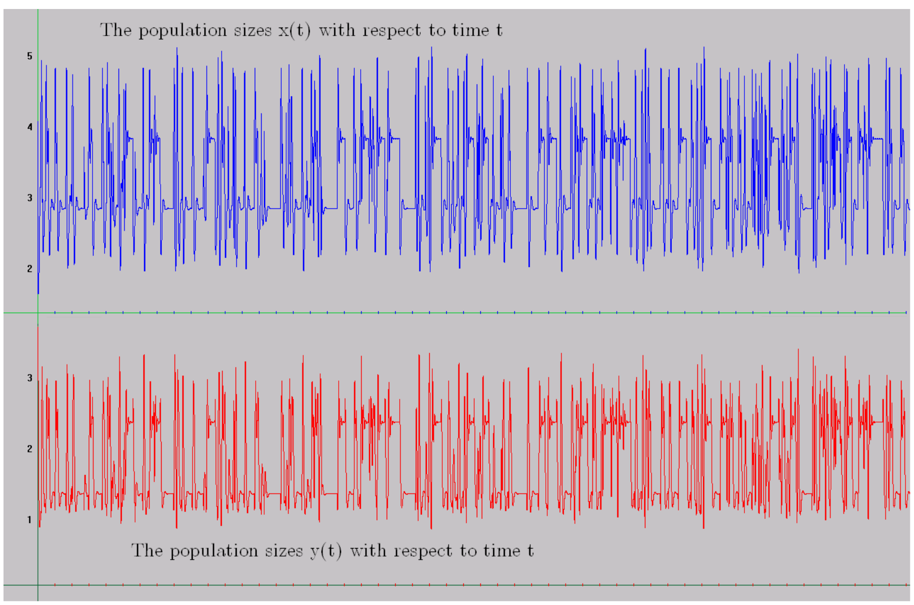

FIGURE 2. The oscillations of $x(t)$ and $y(t)$ in case $y_1^* > 0, y_2^* > 0$.

The next examples concern with the numerical solutions of systems where $y_1^* > 0, y_2^* < 0$. On the case A of the figure 3, we compute with $a_1 = 7.2, b_1 = 2.1, c_1 = 0.8, d_1 = 3.2, e_1 = 1.6, a_2 = 6.3, b_2 = 1.1, c_2 = 2.5, d_2 = 5.8, e_2 = 0.9, x(0) = 0.8, y(0) = 3.3, \alpha = 0.5, \beta = 0.4$ and the number of switching n=500. In this case $\lambda \approx 0.66 > 0$. As is seen $\limsup_{t\to\infty} y(t) > 0$ but $\liminf_{t\to\infty} y(t) = 0$.

For the case B, the parameters are $a_1 = 4.2, b_1 = 0.9, c_1 = 0.4, d_1 = 6.5, e_1 = 1.8, a_2 = 5.8, b_2 = 2, c_2 = 1.5, d_2 = 7.3, e_2 = 1.1, x(0) = 4.5, y(0) = 2.6, \alpha = 0.3, \beta = 0.6$, number of switching $n = 300$. Since $\lambda \approx -0.1 < 0$, it is seen that

$\lim_{t\to\infty} y(t) = 0$ (according to the proposition 3.4). Thus the ω-limit set of all solutions starting in int $\mathbb{R}^2_+$ is the segment $[(4.7, 0); (2.9, 0)]$ (see the theorem 3.10).

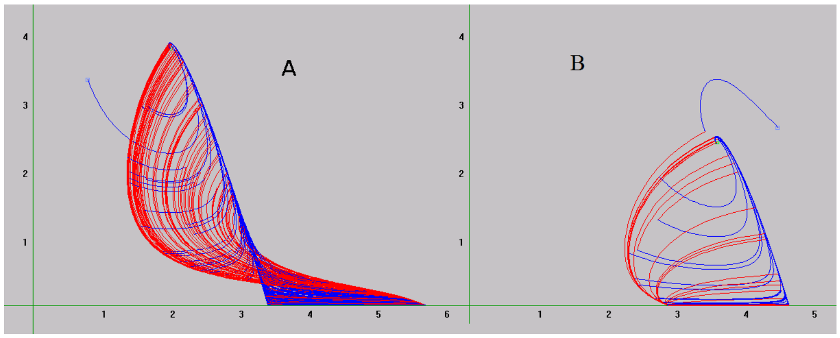

FIGURE 3. Orbit of system in case $y_1^* > 0, y_2^* < 0$.

We sketch the oscillations of $x(t)$ and $y(t)$ in these cases in the figure 4 and 5.

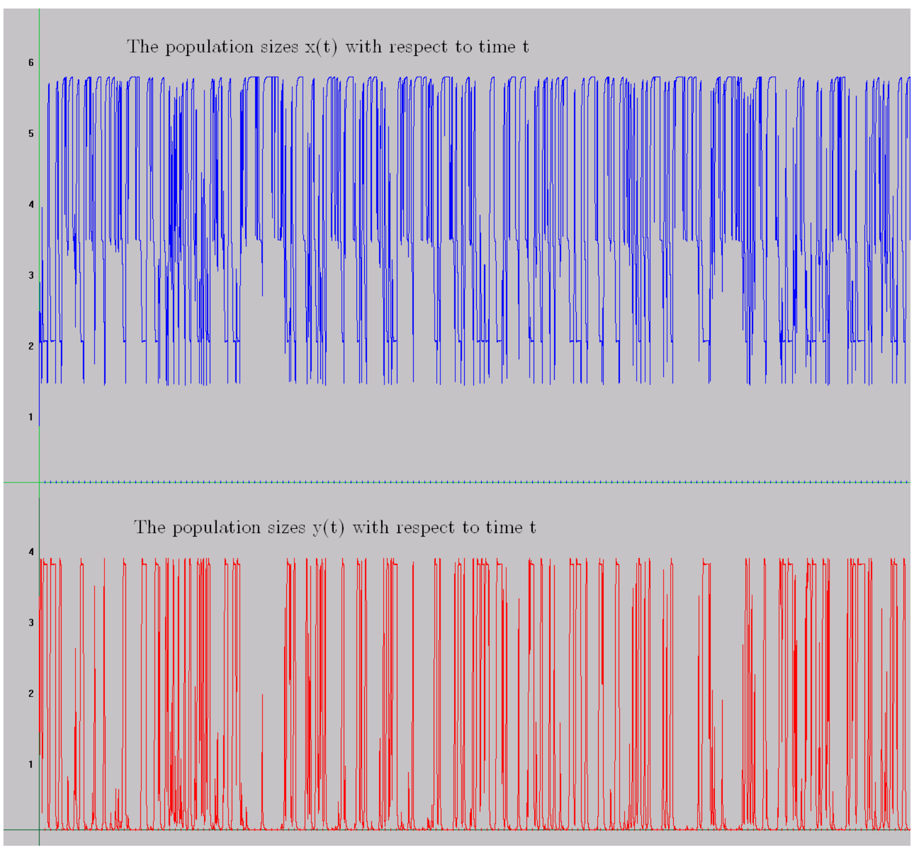

FIGURE 4. The oscillations of $x(t)$ and $y(t)$ in case A.

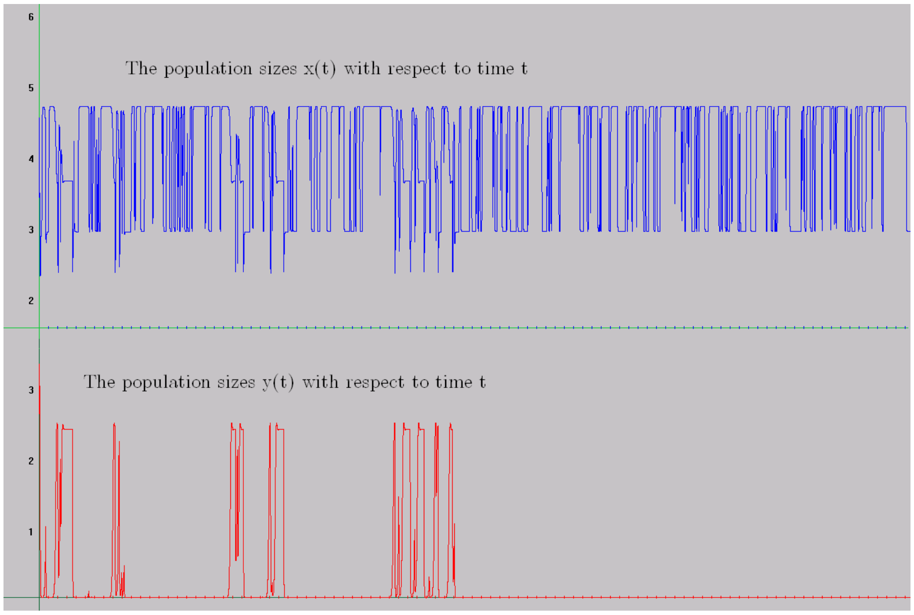

FIGURE 5. The oscillations of $x(t)$ and $y(t)$ in case B.

6. **Discussion and conclusion.** This work provides some results about the asymptotic behavior of a system of two coupled deterministic predator-prey models switching at random. The mathematical analysis presented in this model shows that according to the value of some number λ, one can make suitable predictions about the asymptotic behavior of the overall predator-prey system.

The formula for the value λ is explicitly computed. This factor plays an important role in practice because by analyzing the coefficients, we understand the behavior of the systems. However, as is seen in (3.3), the coefficient c is absent in this formula. The matter of fact is that when the catch ability c is great, the density $x(t)$ of the prey is reduced which implies that the density $y(t)$ of the predator decreases sharply because they are "hungry." In this situation, the prey has a chance to be recovered. A similar situation occurs when c is small. Therefore, the persistence of the system (2.4) does not depend on c.

We consider an ecology system where there are two species related by predator-prey relation. Suppose that the evolution of every species depends on the quantity of rainfall for every period. If the rainfall is sufficient (good state), the catch ability of the predator is good and the quantity of every species asymptotes to the positive values (the prey and predator co-exist). Whenever the rainfall is small (bad state), the hunting potential of the predator becomes very weak and the amount of predator gets smaller with increasing of time (the predator vanishes). Suppose that the rainfall is in a stationary regime (switching stationarily between dry season and rainfall one). If the two states are good, i.e., both $y_1^* > 0$ and $y_2^* > 0$, although the quantity of two species is chaotic, but the system is still permanent. Consequently, none of species is extinct. When there is at least a system having the bad state, i.e., either $y_1^* < 0$ or $y_2^* < 0$ we see that $\liminf_{t\to\infty} y(t) = 0$. Depending on the sign of the value λ, the quantity of the predator $y(t)$ can be recovered or not. In case $\lambda > 0$ we have $\limsup_{t\to\infty} y(t) > 0$, i.e., the amount of the predator is recovered (of course in the rainfall season). If $\lambda < 0$ we have $\lim_{t\to\infty} y(t) = 0$, i.e., the predator vanishes.

However, in reality, when the amount of a species is smaller than a threshold then in fact we consider this species disappears. Thus, the estimate $\liminf_{t\to\infty} y(t) = 0$ tells us that in a predator-prey system developing under the influence of random environment, if there is at least a bad situation, the predator must be vanished in this system. **This conclusion warns us to have a timely decision to protect species in our eco-system.**

The figures 1 and 3 (case A) suggest that when $\lambda > 0$ the dynamics of the predator-prey system leads to a nice attractor. Moreover, in this case it seems that the system (2.4) is asymptotically stable and the stationary distribution of the solutions is unique. So far these are still open problems. As a perspective, it seems interesting to extend the previous work to more realistic functional response such as the Holling II functional response: $h(x) = \frac{ax}{1+bx}$ which exhibits a saturation effect at large prey densities.

Acknowledgment. This paper was partially done while N.T. Hieu was in International Center for Theoretical Physics (ICTP), Strada Costiera 11, 34014 Trieste, Italy under the support of Federation Scheme. The authors are grateful to the anonymous referees for their many helpful comments and suggestions.

REFERENCES

[1] L. J. S. Allen, "An Introduction to Stochastic Processes with Applications to Biology," Pearson Education Inc., Upper Saddle River, New Jersey 2003.

[2] L. J. S. Allen and E. J. Allen, *A comparison of three different stochastic population models with regard to persistence time*, Theoretical Population Biology, **64** (2003), 439–449.

[3] E. J. Allen, L. J. S. Allen and H. Schurz, *A comparison of persistence-time estimation for discrete and continuous population models that include demographic and environ- mental variability*, J. Mathematical Biosciences, **196** (2005), 14–38.

[4] L. Arnold, W. Horsthemke and J. W. Stucki, *The influence of external real and white noise on the Lotka-Volterra model*, Biom. Journal, **21** (1979), 451–471.

[5] A. D. Bazykin, "Nonlinear Dynamics of Interacting Populations," Singapore: World Scientific, 1998.

[6] N. H. Du, R. Kon, K. Sato and Y. Takeuchi, *Dynamical behavior of Lotka - Volterra competition systems: Non autonomous bistable case and the effect of telegraph noise*, J. Comput. Appl. Math, **170** (2004), 399–422.

[7] L. Edelstein-Keshet, "Mathematical Models in Biology," Random House, New York, 1988.

[8] I. I Gihman and A. V. Skorohod, "The Theory of Stochastic Processes," Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[9] R. Z. Hasminskij, "Stochastic Stability of Differential Equations," Sijthoff Noordhoff, 1980.

[10] Q. Luo and X. Mao, *Stochastic population dynamics under regime switching*, J. Math. Anal. Appl, **334** (2007), 69–84.

[11] J. D. Murray, "Mathematical Biology," Springer Verlag, Heidelberg, 1989.

[12] S. Sathananthan, *Stability analysis of a stochastic logistic model*, Math. Comput. Modelling, **38** (2003), 585–593.

[13] Y. Takeuchi, N. H. Du, N. T. Hieu and K. Sato, *Evolution of predator-prey systems described by a Lotka - Volterra equation under random environment*, J. Math. Anal. Appl, **323** (2006), 938–957.

Received April 3, 2009; Accepted June 5, 2009.

E-mail address: ♮: dunh@vnu.edu.vn
E-mail address: †: Pierre.Auger@bondy.ird.fr
E-mail address: ‡: hieungt@vnu.edu.vn