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FEEDBACK STABILIZATION FOR A CHEMOSTAT WITH DELAYED OUTPUT

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Abstract. We apply basic tools of control theory to a chemostat model that describes the growth of one species of microorganisms that consume a limiting substrate. Under the assumption that available measurements of the model have fixed delay $\tau > 0$, we design a family of feedback control laws with the objective of stabilizing the limiting substrate concentration in a fixed level. Effectiveness of this control problem is equivalent to global attractivity of a family of differential delay equations. We obtain sufficient conditions (upper bound for delay $\tau > 0$ and properties of the feedback control) ensuring global attractivity and local stability. Illustrative examples are included.

1. Introduction.

The chemostat is a continuous bioreactor with constant volume V, which is used to culture microorganisms for experimental and industrial purposes (simulation of aquatic ecosystems, wastewater treatment, production of cellular mass,etc). It contains one species of microorganism that consumes a limiting substrate. The evolution of microorganisms and substrate is described by the ODE system [1],[18],[20]:

$$\begin{cases} \dot{s}(t) = Ds_{in} - Ds(t) - \alpha f(s(t))x(t), \\ \dot{x}(t) = x(t)f\big(s(t)\big) - Dx(t), \\ s(0) \geq 0 \quad \text{and} \quad x(0) > 0, \end{cases} \tag{1}$$

where $s(t)$ denotes the concentration of limiting substrate at time t and $x(t)$ denotes the biomass density of the species of microorganism at time t. $f(s)$ represents the per capita growth rate of nutrient of the microorganism and so $\alpha > 0$ is a yield constant related with conversion rate of substrate in new biomass. Limiting substrate is pumped into the chemostat at rate $F > 0$ with concentration $s_{in} > 0$ and the mixing of substrate/biomass is pumped out of the chemostat also at rate $F > 0$. The constant $D = F/V$ is called the dilution rate.

In this work, we assume that the function $f \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the following conditions **(F)**:

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(F1) Either f is strictly increasing or unimodal (*i.e.* it has at most one critical point $s_{\max} < s_{in}$ which is a maximum). Moreover, f is always positive, bounded, $f(0) = 0$ and $f'(0) > 0$.

(F2) If f is strictly increasing then it is concave. Moreover, If f is unimodal then it is concave in an interval $[0, s_c)$, with $s_c > s_{\max}$.

Conditions **(F)** are satisfied for several functions describing the microbial growth, for example (see e.g., [1],[22])

$$f_1(s) = \frac{\mu_{\max} s}{k_s + s}, \quad f_2(s) = \frac{\mu_{\max} s}{k_s + s + \frac{s^2}{k_i}} \quad \text{and} \quad f_3(s) = \frac{\mu_{\max} s}{k_s + s^2}, \tag{2}$$

where $\mu_{\max}$, k_s and k_i are positive parameters. It is straightforward to verify that f_1 (also called Michaelis-Menten function) is strictly increasing and concave. Moreover, the functions f_2 and f_3 are unimodal with one inflection point $s_c > s_{\max}$.

In this paper we follow an idea developed in [1],[6],[5],[23],[17] and consider system (1) as a *single input single output* system (see e.g., [12]), that means a structure with three elements:

(1) a *Plant* defined by the chemostat model;
(2) a *Output* $y(t)$, given by the measurements that we are able to carry out in the chemostat;
(3) an *Input* or control variable given by some parameters of the model that are susceptible to being modified externally. For example the input concentration of substrate $s_{in} > 0$ or the dilution rate $D > 0$.

Systems in which input is a function of the output are called *closed–loop* or *feedback* control systems. In this article, we will consider system (1) as a feedback control system under the following hypothesis:

(H1) (Input hypothesis) The dilution rate D is the feedback control variable.
(H2) (Output hypothesis) The only output available is described by:

$$y(t) = s(t - \tau), \quad \tau > 0. \tag{3}$$

The requirement of a nonnegative feedback control comes from the fact that the control variable (dilution rate) represents an input flow. So, it has to be nonnegative to have a physical meaning. In general, it is assumed that outputs are available online from the plant. Nevertheless, time delays between inputs and outputs are common phenomena in industrial processes and biological systems [6],[15]. Motivated by this fact, we introduce assumption **(H2)**.

We will design a feedback control law with the goal of stabilizing the substrate concentration at a given level s^*, more formally:

> **Control Problem (CP)**: Given a constant $s^* \in (0, \min\{s_{\max}, s_{in}\})$ satisfying the inequality $f(s^*) < f(s_{in})$, design a dilution rate now defined as function of $y(t)$ that stabilizes the system (1)–(3) with respect to this reference value,i.e., $\lim\limits_{t\to+\infty} s(t) = s^*$.

In this feedback control framework, dilution rate D is now a positive function dependent on $s(t-\tau)$ and (1) becomes a system of differential delay equations. Hence, control problem **(CP)** is equivalent to finding sufficient conditions to ensure global attractivity of a system of differential delay equations. For a rigorous presentation of the differential delay equations theory, we refer to [2],[8] and [13].

Remark 1. There are no restrictions for s^* when f is strictly increasing. On the other hand, inequality $f(s^*) < f(s_{in})$ imply that **(CP)** is well defined only for a subset $(0, s_{\max})$ when f is unimodal.

1.1. **Motivation.** It is clear that by choosing $D = f(s^*)$, the problem is solved immediately when f is increasing and is solved for a set of initial conditions (s_0, x_0) when f is unimodal (see e.g., [1],[20]). Nevertheless, the introduction of a feedback control law can improve the performance and efficiency of the bioprocesses with respect to this "fixed dilution" approach. To emphasize the relevance of this problem, we want to refer to two concrete applications (a more exhaustive numerical explanation will be given in Section 5, see also [7]):

Example 1: We can consider the chemostat as a depollution device. This process consists of a chemostat in which toxic contaminants (e.g., phenol, toluene) are pumped into it with a fixed concentration s_{in} higher than an acceptable level $s^+ > s^*$ fixed by environmental authorities. For example, $s^+ = 0.30$mg/L is used to avoid pollution in the water (see e.g., [7] for more details).

This chemostat also contains a microorganism (e.g., *Pseudomonas Putida*) that can resist the adverse effects of organic solvents and is capable of decontamining the tank because it is able to utilize the toxic contaminants as limiting substrate.

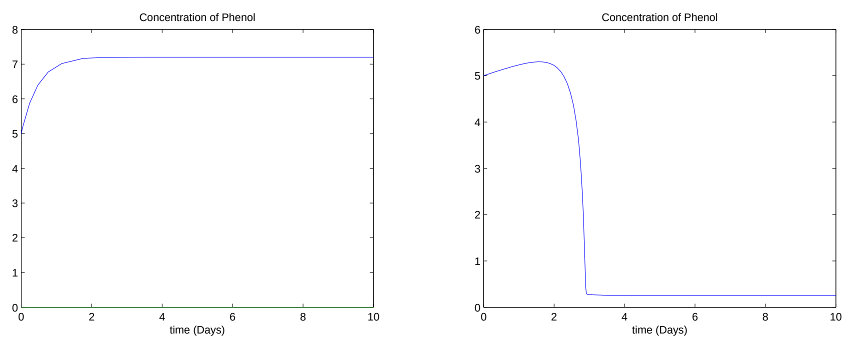

FIGURE 1. Phenol concentration ($s(0) = 5$ mg/L and $x(0) = 0.001$ mg/L). Notice the difference between asymptotic behaviors: depollution process fails using fixed dilution (left) while feedback stabilization in $s^* = 0.25$ mg/L is achieved (right).

The introduction of a feedback control law can drastically modify the outputs of the model. Figure 1 shows the concentration of phenol (which is pumped into the chemostat with concentration $s_{in} = 7.2$ mg/L): in the left, the depollution process is carried on by using a fixed dilution $D = f(s^*)$, which fails (washout of *P. Putida* and $\lim_{t\to+\infty} s(t) = s_{in}$) for a set of initial conditions. In the right, we introduce a feedback control law (satisfying properties described in the next section but supposing that the measurements are available online) and phenol concentration converges to s^* for all initial conditions and the depollution goal is obtained. Although this control approach is effective, we are interested to know the robustness if we account for the delay in the measurements.

Example 2: Chemostat is employed to study phytoplankton in a simulated marine environment: indeed, several features of marine environments such as light intensity and pH and temperature, can be reproduced externally. Moreover the use of the chemostat makes it possible to reproduce several levels s^* of limiting substrate; consequently we can study the metabolism of phytoplanktonic algae.

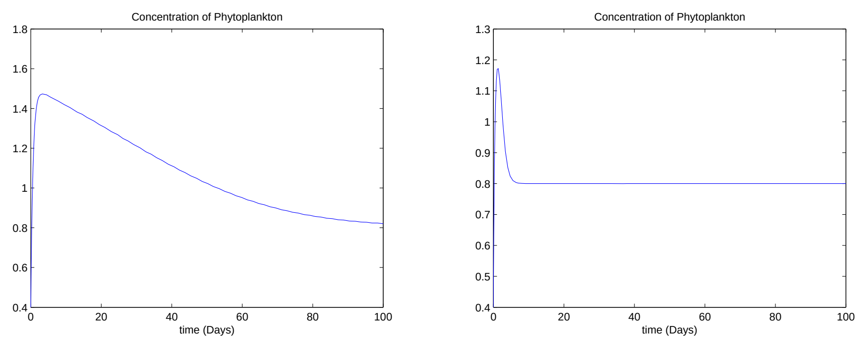

FIGURE 2. Nitrate concentration in the phytoplankton culture ($s(0) = 0.4$ μatg/L and $x(0) = 0.5$ μatg/L): use of fixed dilution (left) and use of a feedback control (right). Notice the difference (days) between the speed of convergence to 0.8 μatg/L (microatoms grams by liter).

Figure 2 shows the concentration of nitrate in a culture of *Dunaniella Tertiolecta*. Notice that the use of a feedback control law (satisfying properties described in the next section but supposing the absence of delays on the measurements) can drastically improve the speed of the convergence (with respect a fixed dilution strategy) toward the wanted level s^*. Nevertheless, as it has been pointed out in [15], there exist delays in the measure of substrate. Hence, we are interested to know the robustness of control laws if we account for these delays.

1.2. **Some related results.** Control problem **(CP)** has been proposed in [6, Ch.6], where some linear feedback control laws which stabilize locally the output at s^* are proposed. Some related control problems for the chemostat have been considered in [7] where **(H1)** is assumed, but function f and output (available online) have uncertainties. In [17], an open loop control is considered by supposing **(H1)**, but the goal is to build an open loop periodic input leading to a globally attractive periodic output.

A control problem for a competition model (two species) is studied in [23] where **(H1)** is assumed and $y(t) = (x_1(t-\tau), x_2(t-\tau)) \in \mathbb{R}^2_+$. Necessary and sufficient conditions ensuring local stability and bifurcation analysis are presented.

The paper is organized as follows: a family of feedback control laws is proposed in Section 2. A result of local stability is presented in Section 3 and two results of global stability (delay independent and delay dependent) are presented in Section 4. The examples considered in this introduction are revisited in Section 5.

2. **Feedback control law.** Let us build the family of feedback control laws:

$$D\big(y(t)\big) = h\big(s^* - s(t-\tau)\big) \tag{4}$$

where the function $h \in C^2(\mathbb{R}, \mathbb{R}_+)$ satisfies the following properties:

(P1) h is increasing, bounded, strictly positive and $h(0) = f(s^*)$.

(P2) The value s^* is the only root of the equation $h(s^* - s) - f(s) = 0$ over the interval $(0, s_{in})$.

(P3) h has a unique inflection point at 0.

Remark 2. If $\tau = 0$, then **(P1)**–**(P2)** ensure the solution of problem **(CP)**. Notice that if f is described by an increasing function, then property **(P2)** is automatically satisfied. Finally **(P1)** and **(P3)** imply that h is convex in the negative real axis and concave in the positive real axis.

We replace D in the system (1) by the feedback control law (4), the closed-loop system becomes:

$$\begin{cases} \dot{s}(t) = h(s^* - s(t-\tau))(s_{in} - s(t)) - \alpha f\big(s(t)\big)x(t), \\ \dot{x}(t) = x(t)f\big(s(t)\big) - h\big(s^* - s(t-\tau)\big)x(t), \\ x(0) > 0, \quad 0 \le s(\theta) = \varphi_1(\theta) \le s_{in} \quad \text{for any } \theta \in [-\tau, 0], \end{cases} \tag{5}$$

where φ_1 is a nonnegative continuous function bounded above on the interval $[-\tau, 0]$.

By using **(P1)**–**(P2)**, we can prove that the equilibria of system (5) are given by $E_0 = (s_{in}, 0)$ and $E_1 = (s^*, \alpha^{-1}[s_{in} - s^*])$.

Let us define

$$C = C\big([-\tau, 0], \mathbb{R}^2\big) \quad \text{and} \quad C_+ = C\big([-\tau, 0], \mathbb{R}^2_+\big)$$

the Banach space of scalar continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^2$ and the cone of nonnegative continuous functions, respectively. C is equipped with the supremum norm and C_+ becomes a complete metric space (C_+, d) under the induced metric.

The initial conditions of (5) are in the space $C_+ \times \mathbb{R}_+$ and can be included in the space $X = C_+ \times C_+$. Global existence and uniqueness of the solutions of system (5) can be easily proved (see e.g., [8, Ch.2]) and consequently, these define a continuous semiflow $\phi\colon \mathbb{R}_+ \times X \mapsto X$ satisfying $\phi(0, x_0) = x_0$ and $\phi_{t+s}(x_0) = \phi(t+s, x_0) = \phi(t, \phi(s, x_0))$.

As we pointed out in the introduction, **(CP)** will be solved if we build a function h satisfying **(P)** and find sufficient conditions for global attractivity of solution E_1 of system (5).

3. Local stability results.

Theorem 3.1. *Let f and h be functions satisfying conditions* **(F)** *and* **(P)***:*

(i) If $f'(s^*) \ge h'(0)$, then the critical point E_1 of system (5) is locally asymptotically stable independently of delay τ.

(ii) If $f'(s^*) < h'(0)$, then the critical point E_1 of system (5) is locally asymptotically stable for any $\tau \in [0, \bar{\tau})$ where $\bar{\tau}$ is defined as follows

$$\bar{\tau} = \begin{cases} \dfrac{\arccos\left(\dfrac{|f'(s^*)|}{h'(0)}\right)}{[s_{in} - s^*]\sqrt{h'(0)^2 - f'(s^*)^2}} & \textit{if} \quad f'(s^*) < 0 \\[2ex] \dfrac{\pi - \arccos\left(\dfrac{f'(s^*)}{h'(0)}\right)}{[s_{in} - s^*]\sqrt{h'(0)^2 - f'(s^*)^2}} & \textit{if} \quad f'(s^*) > 0 \end{cases} \tag{6}$$

(iii) The critical point E_0 of system (5) is unstable.

(iv) If $f'(s^*) < h'(0)$, then a Hopf bifurcation occurs at E_1 for $\tau = \bar{\tau}$.

Proof. First, we make the change of variables $(s, x) \to (v, u) = (s_{in} - s - \alpha x, s - s^*)$ and linearize the new system around $(0, 0)$ obtaining

$$\dot{z}(t) = Az(t) + Bz(t - \tau) \tag{7}$$

where $z(t) = \begin{pmatrix} v(t) & u(t) \end{pmatrix}^T$ and the matrices A and B are

$$A = \begin{bmatrix} -h(0) & 0 \\ -f(s^*) & -f'(s^*)(s_{in} - s^*) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -h'(0)(s_{in} - s^*) \end{bmatrix}.$$

By properties **(P)** we can verify that the matrix $A+B$ is stable, hence the system (7) is stable when $\tau = 0$. Now, we have that $(0, 0)$ is a locally asymptotically stable critical point of system if all roots of

$$\det(sI - A - Be^{-s\tau}) = \big(s + h(0)\big)\big(s + f'(s^*)[s_{in} - s^*] + h'(0)[s_{in} - s^*]e^{-s\tau}\big) = 0$$

have negative real part (see e.g., [2] for details).

As $h(0) < 0$, we only need to study the roots of the characteristic equation

$$p(s) + q(s)e^{-s\tau} = 0, \tag{8}$$

where $p(s) = s + f'(s^*)[s_{in} - s^*]$ and $q(s) = h'(0)[s_{in} - s^*]$.

All roots of equation (8) have negative real part when $\tau = 0$ and move continuously to the right in the complex plane when τ increases (see e.g., [4]). Consequently, the linear system becomes unstable if and only if for some $\bar{\tau} > 0$, there exists an imaginary solution of equation (8).

Let $i\mathbb{R}$ be the set of purely imaginary numbers. If $i\omega_c \in i\mathbb{R}$ is a root of equation (8), then it follows that $|p(i\omega_c)|^2 - |q(i\omega_c)|^2 = 0$, which implies

$$\omega_c = [s_{in} - s^*]\sqrt{h'(0)^2 - f'(s^*)^2}.$$

If $f'(s^*) > h'(0)$, it follows that (8) cannot have roots in $i\mathbb{R}$, which implies that all roots have negative real part for any $\tau > 0$ and statement (i) follows.

If $f'(s^*) < h'(0)$, by taking real and imaginary parts of (8) at $s = j\omega_c$ it is easy to see that

$$\omega_c = h'(0)[s_{in} - s^*]\sin(\omega_c \tau), \tag{9}$$

$$\cos\left(\tau[s_{in} - s^*]\sqrt{h'(0)^2 - f'(s^*)^2}\right) = -\frac{f'(s^*)}{h'(0)}. \tag{10}$$

As $\omega_c > 0$, equations (9)–(10) imply that for $k = 0, 1, \ldots$

$$\omega_c \in \begin{cases} (2k\pi, \pi/2 + 2k\pi) & \text{when} \quad f'(s^*) < 0 \\ (\pi/2 + 2k\pi, \pi + 2k\pi) & \text{when} \quad f'(s^*) > 0. \end{cases}$$

Hence, statement (ii) follows by using identity $\arccos(-x) = \pi - \arccos(x)$ when $f'(s^*) > 0$.

Second, we make the change of variables $(s, x) \to (v, u) = (s_{in} - s - \alpha x, s - s_{in})$ and linearize the new system around $(0, 0)$ obtaining

$$\dot{z}(t) = Az(t) + Bz(t - \tau), \tag{11}$$

where $z(t) = \begin{pmatrix} v(t) & u(t) \end{pmatrix}^T$ and the square matrices A and B are defined as follows

$$A = \begin{bmatrix} -h(s^* - s_{in}) & 0 \\ -f(s_{in}) & f(s_{in}) - h(s^* - s_{in}) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By properties **(P)** it follows that $f(s_{in}) > h(s^* - s_{in})$, hence $(s_{in}, 0)$ is unstable and statement (iii) follows.

Finally, let s be a root of equation (8). Notice that implicit function theorem and equation (8) imply

$$\frac{ds}{d\tau} = \frac{sq(s)e^{-s\tau}}{p'(s) + q'(s)e^{-s\tau} + \tau p(s)} = -s\left[\frac{p'(s)}{p(s)} - \frac{q'(s)}{q(s)} + \tau\right]^{-1}.$$

Let $W(\omega_c) = SgnRe\left\{\frac{ds}{d\tau}\right\}$ at $s = i\omega_c$. As $q'(s) = 0$ and $p'(s) = 1$ it follows that

$$W(\omega_c) = -SgnRe\left\{i\omega_c\left[\frac{1}{p(i\omega_c)} + \tau\right]^{-1}\right\} = -SgnRe\left\{i\omega_c\left[\frac{1}{p(i\omega_c)} + \tau\right]\right\}$$

since $Sgn(Re\{z\}) = Sgn(Re\{z^{-1}\})$. Thus

$$W(\omega_c) = SgnIm\left\{\frac{1}{\omega_c}\left[\frac{1}{i\omega_c + f'(s^*)[s_{in} - s^*]}\right]\right\} = Sgn\frac{1}{\omega_c^2 + f'(s^*)^2[s_{in} - s^*]^2} > 0.$$

Finally, by Hopf bifurcation theorem (see e.g.,[8, Ch.11]) we can see that a family of periodic solutions bifurcate from E_1 at $\tau = \bar{\tau}$ and (iv) follows. □

Remark 3. Let us recall that the control function h has to be designed for us, then local stability can be always be obtained with reasonable choices of h. Moreover, the method employed to determine $\bar{\tau}$ and $W(\omega_c)$ has been developed and generalized for systems of n differential delay equations in [16].

4. **Global stability results.** In order to shorten the statement of theorems, we will first state some notations to be employed. By properties **(F)** and **(P)**, there exists an interval $I \subset (0, \min\{s_{\max}, s_{in}\})$ containing s^* such that the map $t \mapsto f^{-1}\big(h(s^* - s)\big) = \rho(s)$ is well defined and $\rho\colon I \mapsto I$ follows. Moreover $\rho^n(r) = \underbrace{\rho \circ \ldots \circ \rho(r)}_{n \quad \text{times}}$.

Remark 4. The following properties of ρ are elementary:

(i) ρ is decreasing and has a fixed point at s^*.
(ii) If $f'(s^*) > h'(0)$ ($f'(s^*) < h'(0)$) then $|\rho'(s^*)| < 1$ (resp. $|\rho'(s^*)| > 1$).
(iii) There is a unique number $l_1 \in (0, s^*)$ satisfying $l_1 = \rho(s_{in})$.
(iv) If the inequality

$$h(s^* - l_1) < f\big(\min\{s_{\max}, s_{in}\}\big) = \begin{cases} f(s_{in}) & \text{if} \quad f \quad \text{is increasing,} \\ f(s_{\max}) & \text{if} \quad f \quad \text{is unimodal.} \end{cases} \tag{12}$$

is satisfied, then there is a number $L_1 \in (s_*, \min\{s_{\max}, s_{in}\})$ satisfying $\rho(l_1) = L_1$, which means $h(s^* - l_1) = f(L_1)$.

Theorem 4.1 (Delay independent results)**.** *Let f and h be functions satisfying conditions* **(F)** *and* **(P)***. Assume that inequalities $f'(s^*) \geq h'(0)$ and (12) are satisfied. If*

$$|\rho(s) - s^*| < |s - s^*| \quad \text{for any} \quad s \in [l_1, L_1] \tag{13}$$

is satisfied, then the critical point E_1 of system (5) is globally attractive and delay independent.

Theorem 4.2. *Let f and h be functions satisfying conditions* **(F)** *and* **(P)***. Moreover, inequality $f'(s^*) < h'(0)$ and equation (6) are satisfied. If either*

(i) *If inequality* (12) *is satisfied and delay* $\tau > 0$ *satisfies*

$$\tau < \frac{1}{[s_{in} - l^*][f'(l^*) + h'(0)]}, \tag{14}$$

where $[l^*, L^*] \subset [l_1, L_1]$ *are defined as follows:*

$$\lim_{n\to+\infty} \rho^{2n}(l_1) = l^* \quad \text{and} \quad \lim_{n\to+\infty} \rho^{2n}(L_1) = L^*.$$

(ii) *If inequality* (12) *is not satisfied but delay* $\tau > 0$ *satisfies*

$$\tau < \frac{1}{[s_{in} - l_1][f'(l_1) + h'(0)]}, \tag{15}$$

then the critical point E_1 *of system* (5) *is globally attractive.*

Remark 5. Once again, we recall that the control function h is designed for us and equations (12)–(13) can be verified with reasonable choices of h. Indeed these assumptions can be interpreted geometrically (in term of graphs of ρ).

The proof of theorems will be divided into two steps. First, we will prove that (under some compactness and persistence properties of the semiflow) the asymptotic behavior of system (5) is equivalent to the asymptotic behavior of a scalar differential delay equation. Second, we will find sufficient conditions ensuring global attractivity of the scalar equation.

4.1. **Reduction of system.** It is straightforward to prove that under the transformations $(x, s) \to (v, s) = (s_{in} - s - \alpha x, s)$, the system (5) is equivalent to:

$$\begin{cases} \dot{s}(t) = \big\{s_{in} - s(t)\big\}\big\{h(s^* - s(t-\tau)) - f\big(s(t)\big)\big\} - v(t)f\big(s(t)\big), \\ \dot{v}(t) = -h\big(s^* - s(t-\tau)\big)v(t), \\ v(0) \geq 0, \quad 0 \leq s(\theta) = \varphi_1(\theta) \quad \text{for any } \theta \in [-\tau, 0]. \end{cases} \tag{16}$$

The following results are standard in chemostat and differential delay equations literature. For convenience, proofs are provided in the appendix.

Lemma 4.3. *The solutions of system* (5) *define a semiflow* ϕ_t *on* X*, which has the following properties:*

(i) $v_t = s_{in} - s_t - \alpha x_t \to 0$ *when* $t \to +\infty$ *(here,* $s_t = s(t+\theta)$*,*$x_t = x(t+\theta)$ *with* $\theta \in [-\tau, 0]$*).*

(ii) *There exists a global attractor set* $\mathcal{A} \subset X$ *for the semiflow* ϕ_t*. That means, a set* $\mathcal{A}$ *maximal compact invariant that attracts each bounded set in* X*.*

(iii) *There exists a number* $\delta > 0$ *such that for any solution of system* (5) *with initial condition* $\varphi = (\varphi_1, \varphi_2)$*, there exists a number* $T_0(\varphi)$ *such that* $s(t) < s_{in} - \delta$ *and* $x(t) > \delta$ *when* $t > T_0$*.*

Proof. See Appendix. □

By Lemma 4.3, we have that $E_0 = (s_{in}, 0)$ is a repeller. This is equivalent to saying that the biomass $x(t)$ is uniformly persistent, i.e., there exists a number $\delta_0 > 0$ (independent of initial conditions) such that $\liminf_{t\to+\infty} x_t > \delta_0$ (see e.g., [20]). Hence, without loss of generality we will assume only initial conditions $(s_t, x_t) \in \mathcal{A}$ such that $s_t < s_{in}$.

It is easy to see that local (global) stability of critical point E_1 of system (5) is equivalent to local (global) stability of critical point $\tilde{E}_1 = (s^*, 0)$ of system (16).

From Lemma 4.3, we can deduce that $v(t) \to 0$ when $t \to +\infty$, hence it interests to us ask if the asymptotical behavior of the scalar differential delay equation:

$$\begin{cases} \dot{s}(t) \ = \big\{s_{in} - s(t)\big\}\big\{h(s^* - s(t-\tau)) - f\big(s(t)\big)\big\}, \\ s(\theta) = \varphi(\theta) \geq 0 \quad \text{for any } \theta \in [-\tau, 0] \end{cases} \tag{17}$$

is related to the asymptotic behavior of system (16). An affirmative answer is given by the following result:

Lemma 4.4. *If $\tilde{E}_1$ is a locally asymptotically stable solution of system (16) and the trivial solution $s \equiv s^*$ of the differential delay equation (17) is globally attractive, then $\tilde{E}_1$ is a globally attractive solution of system (16).*

Proof. Without loss of generality, we suppose that initial conditions are in $\mathcal{A}$. Let $\vec{u}_0 \in \Omega$ be an initial condition of system (16) where Ω is an open subset of $\mathcal{A}$. Let ϕ_t be the semiflow defined by (16). By Lemma 4.3, we have that ϕ_t is positively invariant in the compact set $\mathcal{A} \subset X$.

Let $(\bar{w}, \bar{v}) \in \omega(\vec{u}_0)$, the ω–limit of $\vec{u}_0$ defined by

$$\omega(\vec{u}_0) = \Big\{(\hat{u}, \hat{v}) \in \mathcal{A} \colon \exists t_n \to +\infty \quad \text{such that} \lim_{n \to +\infty} \phi_{t_n}(\vec{u}_0) = (\hat{u}, \hat{v})\Big\}.$$

By Lemma 4.3, one easily checks that $\lim\limits_{t \to +\infty} v(t) = 0$. Consequently, it is straightforward to verify that $\bar{v} = 0$.

Let us define $\vec{u}_1 = (\bar{w}, 0) \in \omega(\vec{u}_0)$. As $\omega(\vec{u}_0)$ is an invariant set (see *e.g.*[9]), we have that $\phi_t(\vec{u}_1) \in \omega(\vec{u}_0)$ for any $t \geq 0$. By hypothesis, it follows that $\lim\limits_{t \to +\infty} \phi_t(\vec{u}_1) = (s^*, 0)$, which implies $(s^*, 0) \in \omega(\vec{u}_0)$. Using this fact, combined with local asymptotically stability of $\tilde{E}_1$, we can conclude that $\phi_t(\vec{u}_0)$ enters the basin of attraction of $(s^*, 0)$ in a finite time and the Lemma follows. ☐

The idea behind the proof is now clear: as local asymptotic stability of solution $(s^*, 0)$ from equation (16) is ensured by Theorem 3.1, it follows by Lemma 4.4 that sufficient conditions for global attractivity of s^* in (17) hold also for E_1 in (5).

Notice that equation (17) can be related to equation

$$\dot{s}(t) = s(t)\big\{p(s(t-\tau)) - g(s(t))\big\}$$

that is studied in [13, Ch.4], where g is increasing and unbounded whereas function p is either strictly increasing (with $p(s) \to 0$ when $s \to +\infty$) or unimodal. Sufficient conditions ensuring global stability are presented. In spite of f and g does not satisfy these assumptions, we apply some ideas presented in [13] and combine them with one-dimensional map techniques in order to prove global stability of equation (17).

4.2. **Some properties of equation (17).** Without loss of generality, we will assume that initial conditions φ of equation (17) satisfy $||\varphi||_\infty < s_{in}$. It is straightforward to prove that $||s_t||_\infty < s_{in}$ for any $t > 0$ and that s_{in} is an unstable equilibria of (17).

Lemma 4.5. *If $s(t)$ is non–oscillatory with respect to s^* (* i.e., *there is a number $T > 0$ such that $s(t) - s^*$ has constant sign for $t > T + \tau$), it follows that $\lim\limits_{t \to +\infty} s(t) = s^*$.*

Proof. We will prove that there is a number $\bar{T} \geq T$ such that $s(t)$ is monotone for any $t > \bar{T}$.

First, assume that $s(t) < s^*$ for any $t > T + \tau$. As f is always increasing in $(0, s^*)$, it follows that

$$h(s^* - s(t-\tau)) - f(s(t)) > h(0) - f(s(t)) = f(s^*) - f(s(t)) > 0$$

and $\dot{s}(t) > 0$ for any $t > T + \tau$.

Second, assume that $s(t) > s^*$ for any $t > T + \tau$. Notice that

$$h(s^* - s(t-\tau)) - f(s(t)) < h(0) - f(s(t)) = f(s^*) - f(s(t)) < 0$$

is always verified when f is increasing. Moreover, it is still verified when f is unimodal and $f(s^*) < f(s_{in})$. Hence, $\dot{s}(t) < 0$ for any $t > T + \tau$.

Now, without loss of generality, we assume that $s(t) > s^*$ and $\dot{s}(t) < 0$ for any $t > \bar{T} + \tau$ (the other case can be proved similarly). Then, it follows that

$$\lim_{t \to +\infty} s(t) = l \geq s^* \quad \text{and} \quad \lim_{t \to +\infty} \dot{s}(t) = 0. \tag{18}$$

If $l > s^*$, then by using **(P2)** we can prove that

$$\lim_{t \to +\infty} \dot{s}(t) = (s_{in} - l)\big\{h(s^* - l) - f(l)\big\} < 0$$

obtaining a contradiction with (18), hence $l = s^*$ and the Lemma follows. □

By Lemma 4.5, we have only to consider the case when solutions of equation (17) are oscillatory with respect to s^*. This means, there exists a sequence $\{v_n\} \to +\infty$ when $n \to +\infty$ satisfying $s(v_n) = 0$ for any integer $n > 1$.

If the solution $s(t)$ is oscillatory, we can assume that

$$\liminf_{t \to +\infty} s(t) = m \leq s^* \leq M = \limsup_{t \to +\infty} s(t). \tag{19}$$

It is straightforward to prove that $0 \leq m$ and $M \leq s_{in}$. By the fluctuation lemma (see e.g., [10, Lemma 4.2]) there exist two sequences of real numbers $\{t_n\}$,$\{r_n\} \to +\infty$ when $n \to +\infty$ such that for any integer $n \geq 1$ it follows that:

$$\dot{s}(t_n) = 0 \quad i.e. \quad h(s^* - s(t_n - \tau)) - f(s(t_n)) = 0, \tag{20}$$

$$\dot{s}(r_n) = 0 \quad i.e. \quad h(s^* - s(r_n - \tau)) - f(s(r_n)) = 0, \tag{21}$$

$$\lim_{n \to +\infty} s(t_n) = \lim_{n \to +\infty} M_n = M \quad \text{and} \quad \lim_{n \to +\infty} s(r_n) = \lim_{n \to +\infty} m_n = m. \tag{22}$$

Without loss of generality, we can suppose that $s(t_n) > s^*$ and $s(r_n) < s^*$ for any integer $n \geq 1$. Furthermore, by equations (20)–(22) combined with properties of sequences $s(t_n)$,$s(r_n)$ and functions f,h for any integer $n \geq 1$ it follows that:

$$h(s^* - m) > h(s^* - s(t_n - \tau)) = f(M_n),$$

$$h(s^* - M) < h(s^* - s(r_n - \tau)) = f(m_n).$$

Finally, letting $n \to \infty$ we obtain that

$$f(M) \leq h(s^* - m) \quad \text{and} \quad h(s^* - M) \leq f(m). \tag{23}$$

By (23), combined with the fact that h is increasing, $M < s_{in}$ and $h(s^* - s_{in}) = f(l_1)$, we can conclude that

$$f(l_1) < h(s^* - M) \leq f(m)$$

as $m \leq s^* < s_{\max}$ and f is strictly increasing in $(0, s_{\max})$, we have that $l_1 \leq m$.

4.3. **Proof of Theorem 4.1.** The idea of the proof is the following: by using $l_1 \leq m$, we will prove that $M < L_1$ and build a one dimensional discrete dynamical system

$$u_{n+1} = \rho(u_n) \quad \text{with} \quad \rho\colon [l_1, L_1] \to [l_1, L_1]. \tag{24}$$

Following some ideas developed in [14] and references therein, we will verify that the attractor of map (24) gives upper and lower bounds for the unknown constants m and M defined above.

By equation (23) combined with $l_1 \leq m$ we have that

$$f(L_1) = h(s^* - l_1) > h(s_{in} - m) \geq f(M).$$

As f is increasing in $(0, \min\{s_{\max}, s_{in}\})$ and $L_1 < \min\{s_{\max}, s_{in}\}$, the last inequality implies that $M \leq L_1$.

Moreover, let $l_2 = \rho(L_1)$, which means $f(l_2) = h(s^* - L_1)$. By $\min\{s_{\max}, s_{in}\} > L_1 \geq M$ and equation (23), it follows that

$$h(s^* - s_{in}) \leq h(s^* - \min\{s_{\max}, s_{in}\}) < h(s^* - L_1) < h(s^* - M) \leq f(m)$$

and using this inequality combined with equalities stated above it follows

$$f(l_1) \leq f(l_2) \leq f(m),$$

which implies that $l_2 \in [l_1, m]$ and $\rho(L_1) = l_2 \geq l_1$.

Hence, we have that the map $\rho\colon [l_1, L_1] \mapsto [l_1, L_1]$ is well defined, decreasing, and has a unique fixed point s^* that is locally stable because $f'(s^*) > h'(0)$. By using these facts combined with $[m, M] \subset [l_1, L_1]$ and equation (23) we can prove that

$$[m, M] \subseteq \rho\big([m, M]\big) \subseteq \ldots \subseteq \rho^k\big([m, M]\big) \subseteq [l_1, L_1]. \tag{25}$$

By (13) it follows that $|\rho'(s)| < 1$ for any $s \in [l_1, L_1]$. Hence, s^* is a globally stable fixed point of ρ which implies:

$$[m, M] \subseteq \lim_{k \to +\infty} \rho^k\big([m, M]\big) = s^*,$$

hence $m = M = s^*$ and the Theorem follows by using Lemma 4.4. □

Following the lines of the proof but dropping inequality (13), other results can be obtained.

Corollary 1. *If equation* (12) *is satisfied, f is a Michaelis–Menten function and h has negative Schwarz derivative* (*i.e.* $Sh = h'''/h' - 3/2(h''/h')^2 < 0$) *then the critical point E_1 of system* (16) *is globally attractive and delay independent.*

Proof. We know that $\rho\colon [l_1, L_1] \mapsto [l_1, L_1]$ is decreasing with s^* a unique fixed point that is locally stable. If we prove that $(S\rho) < 0$ for any $s \in [l_1, L_1]$, then Proposition 3.3 from [14] implies that s^* is a global attractor of the map ρ and letting $k \to \infty$ in equation (25) implies that $m = M = s^*$.

It is straightforward to prove that $(Sf)(s) = (Sf^{-1})(s) = 0$. Moreover, by the formula for the Schwarzian derivative of the composition of two C^3 functions (see e.g., [14]), we have

$$(S\rho)(s) = (Sh)(s^* - s) < 0$$

and the result follows. □

Remark 6. We point out that this last result is not arbitrary because the functions h satisfying **(P)** and $(Sh) < 0$ can be designed as follows:

$$h(s) = f(s^*) + kg(s), \quad \text{with} \quad 0 < k < f(s^*),$$

where g (with $g(0) = g''(0) = 0$ and $(Sg) < 0$) is a differentiable approximation of the saturation function with slope $1/a > 0$:

$$sat(t) = \begin{cases} -1 & \text{if} \quad r < -1 \\ t/a & \text{if} \quad t \in [-1, 1] \\ 1 & \text{if} \quad t > 1, \end{cases}$$

and is employed in several control designs. For example, $g(t) = \pi/2 \arctan(t)$ and $g(t) = \tanh(t)$ approximates $sat(t)$ and have negative Schwarzian derivative.

4.4. **Proof of Theorem 4.2.** We will give the proof separately for cases (i) and (ii).

Case (i): As inequality (12) is verified, we can follow the lines of the precedent proof and build the map $\rho\colon [l_1, L_1] \mapsto [l_1, L_1]$, which now has an unstable fixed point at s^* because $f'(s^*) < h'(0)$.

Following the lines of precedent proof, we can prove by induction that the sequences $l_i = \rho(L_{i-1})$ and $L_i = \rho(l_i)$ satisfy

$$l_1 \leq \ldots \leq l_i \leq l_{i+i} \leq m \quad \text{and} \quad M \leq L_{i+1} \leq L_i \leq \ldots \leq L_1, \tag{26}$$

and we can conclude that

$$[m, M] \subset \lim_{k \to +\infty} \rho^k\big([l_1, L_1]\big) = [l^*, L^*] = I^*.$$

By definition of m and M, it follows that the attractor of equation (17) is a subset of $[m, M] \subseteq [l^*, L^*]$. Hence, without loss of generality, we only consider initial conditions in I^* which contains $[m, M]$.

Let us recall that we are only considering solutions $s(t)$ of equation (17) that are oscillatory about s^*. By fluctuation Lemma we have that given a number $T > 0$, there exists numbers r_n,$t_n > T$ satisfying equations (20)–(22) such that $s(r_n) < s^* < s(t_n)$, hence we can conclude that

$$h(s^* - s(r_n - \tau)) = f(s(r_n)) < f(s^*) = h(0),$$
$$h(s^* - s(t_n - \tau)) = f(s(t_n)) > f(s^*) = h(0),$$

as h is an increasing function, we have that $s(r_n - \tau) > s^* > s(t_n - \tau)$. Hence, there exists two numbers θ,$\theta' \in (-\tau, 0)$ satisfying

$$s(r_n + \theta) = s(t_n + \theta') = s^*. \tag{27}$$

Let us define

$$u = \limsup_{t \to +\infty} |s(t) - s^*|.$$

If $u > 0$, by using equation (14) we can conclude that there exists $\varepsilon > 0$ such that

$$(u - \varepsilon) > \tau[s_{in} - l^*][f(l^*) + h(0)](u + \varepsilon). \tag{28}$$

Let $T_0 > T$ such that, for $t \geq T_0 - 2\tau$ the following inequality

$$|s^* - s(t)| < u + \varepsilon$$

is satisfied.

First, let us assume that there is a number $r_n > T_0$ such that

$$s^* - s(r_n) > u - \varepsilon.$$

Notice that

$$s^* - s(r_n) = \int_{r_n + \theta}^{r_n} \{s_{in} - s(t)\}\{h(0) - h\big(s^* - s(t - \tau)\big)\}\, dt$$

$$+\int_{r_n+\theta}^{r_n}\{s_{in}-s(t)\}\{f\big(s(t)\big)-f(s^*)\}\,dt$$
$$\leq \tau[s_{in}-l^*]\Big\{\max_{\xi\in I^*}|h'(\xi)|+\max_{\xi\in I^*}|f'(\xi)|\Big\}(u+\varepsilon).$$

As f is concave in $(0, s_{\max})$ and h has an inflection point at 0,it follows that:

$$u-\varepsilon<[s_{in}-l^*][f(l^*)+h(0)]\tau(u+\varepsilon),$$

which contradicts equation (28), hence $u = 0$.

Now, if we assume that there is a number $t_n > T_0$ such that

$$s(t_n)-s^*>u-\varepsilon,$$

the proof runs as before, hence $u = 0$ and the Theorem follows.

Case (ii): If f equation (12) is not satisfied, we cannot use the map ρ to find bounds for m and M. Nevertheless, we replace I^* by $[l_1, s_{in}]$ and the proof runs as before. □

5. **Numerical examples.** In this section we revisit the examples mentioned in the introduction.

5.1. **Depollution of phenol in the water.** We consider biological degradation of phenol in the water by using *Pseudomonas putida* with growth described by

$$f(s)=\mu_{\max}\frac{s}{k_s+s^2},$$

where the parameters are defined below (see also [22]):

Parameter	Value	Units
$\mu_{\max}$	15.96	Day^{-1}
k_s	1.82	mg/L
s_{in}	7.2	mg/L
α	1	non dimensional

Our goal is to stabilize the phenol concentration around $s^* = 0.25$mg/L. Hence, we build the feedback control law:

$$D(y(t))=2.11952+2\tanh\big(\eta[s^*-s(t-\tau)]\big). \tag{29}$$

We verify that **(P1)**–**(P2)** are satisfied. By Theorem 3.1 it follows that E_1 is a locally stable and delay–independent solution of system (5) if and only if $\eta \leq 3.957566$.

First, let us assume $\eta = 3$. We verify that $f(l_1) = h(s^* - s_{in})$ for $l_1 = 0.0136311$ and $3.3397547 = h(s^* - l_1) < f(s_{\max}) = 5.8438016$. Moreover,$f(L_1) = h(s^* - l_1)$ is satisfied by $L_1 = 0.4172869$.

We can see with the help of a computer that equation (13) is satisfied and by Theorem 4.1, it follows that the feedback control law (29) with $\eta = 3$ stabilizes the phenol concentration at $s^* = 0.25$mg/L independently of delay.

Figure 3 shows numerical simulations for phenol concentration (carried out with MATLAB DDE23 [19]) for $\tau = 0.5$ and $\tau = 3.5$ and initial condition $(\varphi_1, \varphi_2) = (5, 0.001)$.

Second, let us assume $\eta = 5$. By Theorem 3.1, it follows that critical point E_1 is locally asymptotically stable for any $\tau < \bar{\tau} \approx 0.058483$ days.

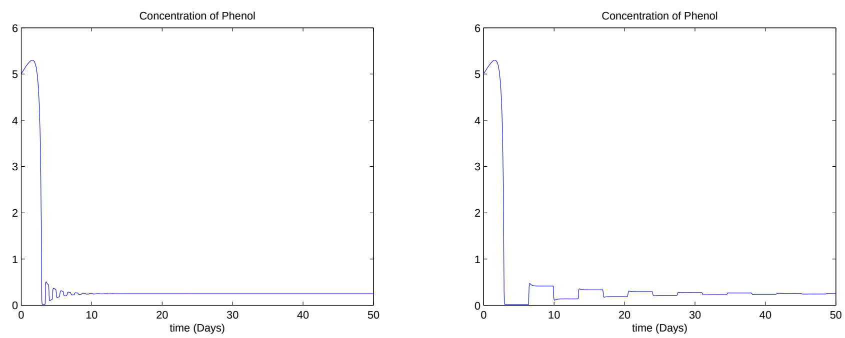

FIGURE 3. Output of system ($\eta = 3$): $\tau = 0.5$ (left) and $\tau = 3.5$ (right).

We verify that $l_1 = 0.013631$ and $L_1 = 0.486552$ and the map ρ^2 has two new stable fixed points $l^* = 0.066998$ and $L^* = 0.45246$ in $[l_1, L_1] \setminus \{s^*\}$. By statement (i) of Theorem 4.2, it follows that if

$$\tau < \frac{1}{[s_{in} - l^*][f'(l^*) + h'(0)]} \approx 0.007495 \quad \text{days}$$

then the feedback control law (29) with $\eta = 5$ stabilizes the phenol concentration at $s^* = 0.25$mg/L.

Figure 4 shows numerical simulations of phenol concentration for $\eta = 5$ and initial condition $(\varphi_1, \varphi_2) = (5, 0.001)$. The figure shows that our delay bound is conservative. Indeed, our result says that if $\tau < 0.007495$ then E_1 is globally stable while figure 4 suggest that if E_1 is locally stable then is globally stable. This kind of result is usual in scalar delay equations (Wright's conjecture, Smith's conjecture [13],[21]) and has triggered a considerable quantity of research with the goal of understanding the relationships between global and local stability.

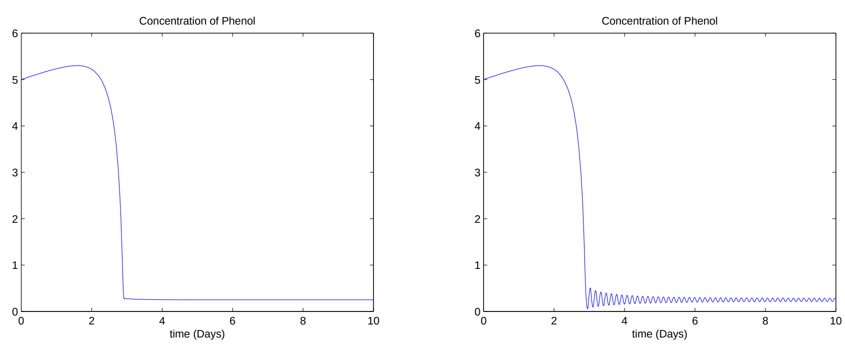

FIGURE 4. Output of system ($\eta = 5$): $\tau = 0.008$ (left) and $\tau = 0.058$ (right)

5.2. **Culture of phytoplankton.** We will consider *Dunaniella tertiolecta* growth in a chemostat with nitrate as the limiting substrate. We will work with a growth function given by the Michaelis-Menten function

$$f(s) = \mu_{\max} \frac{s}{k_s + s},$$

where the parameters are shown below (see also [3],[24] for more details):

Parameter	Value	Units
$\mu_{\max}$	1.6	Day^{-1}
k_s	0.02	μatg/L
s_{in}	2	μatg/L
α^{-1}	1	nondimensional

Our goal is to stabilize the nitrate concentration in a neighborhood of $s^* = 0.8\mu$atg/L, for this task we build the feedback control law:

$$D(y(t)) = 1.561 + \tanh\left(\eta[s^* - s(t-\tau)]\right), \quad \eta > 0. \tag{30}$$

We verify that **(P1)**–**(P2)** are satisfied. By Theorem 3.1 it follows that critical point $(s^*, \alpha^{-1}[s_{in} - s^*])$ of system (5) is locally stable and delay independent if and only if $\eta \leq 0.0475907$.

First, let us assume that $\eta = 0.04$. We verify that $f(l_1) = h(s^* - s_{in})$ is satisfied with $l_1 = 0.348$ and $1.5798 = h(s^* - l_1) < f(s_{in}) = 1.58416$, which means that inequality (12) is satisfied. Finally as $(Sh)(r) = -2$, it follows by Corollary 1 that the feedback control (30) stabilizes the output $y(t)$ in s^* for any $\tau > 0$.

Figure 5 shows numerical simulations for phytoplankton concentration with delays $\tau = 1.8$ and $\tau = 5$ and considering initial condition $(\varphi_1, \varphi_2) = (0.4, 0.5)$.

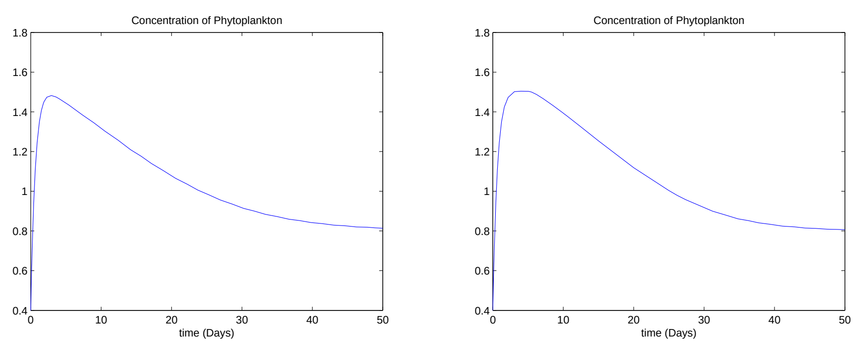

Figure 5. Output of system ($\eta = 0.04$): $\tau = 1.8$ (left) and $\tau = 5$ (right)

Second, we will assume that $\eta = 1$: By using Theorem 3.1, it follows that critical point is locally stable for any $\tau < \bar{\tau} \approx 1.3502$ days.

Notice that $f(l_1) = h(s^* - s_{in})$ for $l_1 = 0.0166687$ and inequality (12) is not satisfied. We can verify that $f'(l_1) = 23.799902$ and $h'(0) = 1$.

By statement (ii) of Theorem 4.2 it follows that if

$$\tau < \frac{1}{[s_{in} - l_1][f'(l_1) + h'(0)]} \approx 0.02033 \quad \text{days}$$

then the feedback control law (30) with $\eta = 1$ stabilizes the phytoplankton concentration at $s^* = 0.8\mu$atg/L.

Notice that for this initial condition (and others not described in the article), an observation of Figure 6 shows that the same remarks stated for the precedent example (conservativeness of delay bound and relationship with local stability) are valid also for this case.

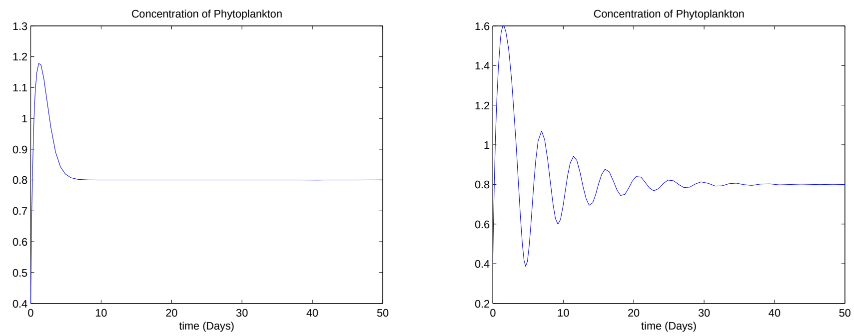

FIGURE 6. Output of system ($\eta = 1$): $\tau = 0.02$ (left) and $\tau = 1.1$ (right)

6. **Discussion.** We consider the simplest chemostat model as an *Input–Output* system and consider dilution rate as the feedback control variable whereas we suppose that measurements of outputs are available with delay. We build a family of control laws which stabilize asymptotically the output in a reference level s^*. The control model is described by a system of differential delay equations with a unique steady state and we prove that it is globally attractive and locally asymptotically stable.

Theorems 4.1 and 4.2 give sufficient conditions for global attractivity, delay-independent (Theorem 4.1) and delay-dependent (Theorem 4.2). In the examples presented in the previous section we determined that our sufficient conditions for delay-dependent stability are conservative in comparison with numerical results, which suggests that our delay margins (14) and (15) can be improved. The improvement of these conditions is a future research direction. Other potential research directions are to study **(CP)** problem under some uncertainty assumptions stated in [7] and the stability of periodic solutions issued from the Hopf bifurcation (Theorem 3.1) that have been observed numerically.

7. **Appendix: Proof of Lemma 1.**

7.1. **Technical results.** To prove Lemma 4.3, we need the following definitions and results:

Definition 7.1. ([9, chapt.3]) Let ϕ_t be a semiflow defined in a complete metric space (X, d). The semiflow ϕ_t is:

(a) *Point dissipative* on X if there exists a bounded set B that attracts each point of X (*i.e.* $\lim_{t\to+\infty} d(x, B) = x$ for any $x \in X$).

(b) *Conditionally completely continuous* for $t \geq t_1$ if, for each $t \geq t_1$ and each bounded set $B \subset X$ for which $\phi_s(B)$ (with $s \in [0, t]$) is bounded, we have that $\phi_t(B)$ is precompact for any $t > t_1$.

(c) *Completely continuous* for $t \geq t_1$ if it is conditionally completely continuous and, for each $t \geq 0$, the set $\phi_s(B)$ (with $s \in [0, t]$) is bounded.

Proposition 1 (Theorem 3.4.8 [9])**.** *If a semiflow is point dissipative and conditionally completely continuous then there exists a maximal set invariant, attractive and compact.*

Definition 7.2 ([11])**.** Let ϕ_t be a semiflow defined in a compact metric space (X,d) and let $X_0 \subset X$ be a closed and invariant set. An application $P\colon X \mapsto \mathbb{R}$ is an *Average Lyapunov function* if satisfies

(a) $P(u) > 0$ for $u \in X \setminus X_0$ and $P(u) = 0$ for $u \in X_0$.
(b) $\dot{P} = \Psi(u)P$ with $\Psi\colon X \mapsto \mathbb{R}$ continuous.

Proposition 2 (Corollary 2 [11])**.** *Let P be an Average Lyapunov function and let Λ*

$$\Lambda = \Big\{ r_i \in X_0 \colon \phi_t(r_i) = r_i \quad \text{for any} \quad t \in \mathbb{R} \Big\}.$$

If $\lim\limits_{t\to+\infty} \phi_t(u) = r_i$ and $\Psi(r_i) > 0$ for any $u \in X_0$ and $r_i \in \Lambda$, then X_0 is a repeller.

7.2. **Proof of Lemma 4.3.** First, we take some initial condition (φ_1, φ_2) satisfying:

$$|s_{in} - \varphi_1(\theta) - \alpha\varphi_2(\theta)| \leq K \quad \text{for any } \theta \in [-\tau, 0]$$

for some $K > 0$. Moreover, let us build the functional $v(t) = s_{in} - s(t) - \alpha x(t)$, where (s_t, x_t) is a solution of the system (5). It is straightforward to prove that $v(t)$ satisfies the following differential equation:

$$\begin{cases} \dot{v}(t) = -h\big(s^* - s(t-\tau)\big)v(t), & \text{for } t > 0 \\ v(\theta) = \eta(\theta) = \varphi_1(\theta) + \alpha\varphi_2(\theta) - s_{in}, & \theta \in [-\tau, 0]. \end{cases}$$

It is a simple exercise to prove that for any $t \geq 0$ it follows that:

$$|v(t)| = |s_{in} - \varphi_1(0) - \alpha\varphi_2(0)| \exp\Big(-\int_0^t h(s^* - s(r-\tau))\, dr \Big).$$

By using **(P1)**, we can prove that there exists a constant $h_{\min} = \min\{h(s^* - u)\colon u \in \mathbb{R}\} > 0$ satisfying:

$$||s_{in} - s_t - \alpha x_t||_\infty \leq K e^{-h_{\min} t} \quad \text{for any } t > 0 \text{ and } \theta \in [-\tau, 0]. \tag{31}$$

Now, letting $t \to \infty$ we have that for any initial condition (φ_1, φ_2) it follows that $\lim\limits_{t\to+\infty} d\big(\phi_t(\varphi_1,\varphi_2), K_0\big) = 0$, where the bounded set K_0 is defined by:

$$K_0 = \Big\{ (\varphi_1, \varphi_2) \in C_+ \times C_+ \colon \varphi_1 + \alpha\varphi_2 = s_{in} \Big\},$$

which implies statement (i) and point dissipativity.

Second, we will prove that ϕ_t is completely continuous for any $t > \tau$ and statement (ii) will be a consequence of Prop.1. Indeed, we take any initial condition (φ_1, φ_2) in a bounded set $B \subset X$. We will see that the orbits of system (5) form a precompact set for any $t \geq \tau$.

By using point dissipativity properties, we define the constants K_1 and K_2:

$$K_1 = \sup_{t\geq 0} \Big\{ ||s_t||_\infty \colon s_0 = \varphi_1 \in B \Big\} \quad \text{and} \quad K_2 = \sup_{t \geq 0} \Big\{ ||x_t||_\infty \colon x_0 = \varphi_2 \in B \Big\}.$$

Notice that, the set $\phi_t(B)$ is equicontinuous for any $t \geq \tau$. Indeed, there exists a number $\delta(\varepsilon) = \min\{\varepsilon/C_1, \varepsilon/C_2\}$ where C_1,C_2 are defined by:

$$C_1 = \max_{|u| \leq K_1} \{ h(s^* - u)s_{in} + \alpha f(u) K_2 \} \quad \text{and} \quad C_2 = K_2 \max_{|u|\leq K_1} \{ f(u) - h(s^* - u) \}$$

such that for any pair $\theta',\theta'' \in [-\tau,0]$ satisfying $|\theta'-\theta''|<\delta$, we have $|s_t(\theta')-s_t(\theta'')|<\varepsilon$ and $|x_t(\theta')-x_t(\theta'')|<\varepsilon$. Hence, by Arzelà–Ascoli Theorem, it follows that $\phi_t(B)$ is precompact for $t\geq\tau$, which implies that ϕ_t is completely continuous and statement (ii) follows by Proposition 1.

Hereafter and without loss of generality, we can assume in this proof that the initial conditions of the system (5) are in the compact set $\mathcal{A}$.

Let us define the subset $\mathcal{A}_0=\big\{(\varphi_1,\varphi_2)\in\mathcal{A}\colon \varphi_2=0\big\}$ and notice that the set $\mathcal{A}_0$ is closed and positively invariant under the semiflow ϕ_t. We will prove that $\mathcal{A}_0$ is a repeller by using Proposition 2.

Let us build the functional $P\colon\mathcal{A}\mapsto\mathbb{R}$ defined by $P(\phi_t(\vec{\varphi}))=x_t(0)$. This functional satisfies the following properties:

(a) $P(\phi_t(\vec{\varphi}))\equiv 0$ if $\vec{\varphi}\in\mathcal{A}_0$ and $P(\phi_t(\vec{\varphi}))>0$ if $\vec{\varphi}\in\mathcal{A}\setminus\mathcal{A}_0$.

(b) $\dot{P}=\Psi(\phi_t(\vec{\varphi}))P$ where $\Psi\colon\mathcal{A}\mapsto\mathbb{R}$ is a continuous function defined by:

$$\Psi(\phi_t(\vec{\varphi}))=f(s_t(0))-h\big(s^*-s_t(-\tau)\big).$$

(c) By using **(P1)**–**(P2)** combined with $\Psi(\phi_t(E_0))=f(s_{in})-h(s^*-s_{in})>0$ and the fact that for any initial condition in $\mathcal{A}_0$, we can conclude that there exists a number $\rho>0$ such that $||s_t-s_{in}||_\infty\leq|\varphi_0-s_{in}|e^{-\rho t}$ for any $t>0$ and it follows that $\lim\limits_{t\to+\infty}(s_t,x_t)=E_0$.

Notice that properties (a)-(b) imply that P is an average Lyapunov function and the sets $\mathcal{A}$,$\mathcal{A}_0$ (with $\Lambda=\{E_0\}$) satisfy the properties of Proposition 2. Hence, it follows that $\mathcal{A}_0$ is a repeller set and statement (iii) follows $\square$.

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