PRINCIPAL EIGENVALUE FOR AN ELLIPTIC PROBLEM WITH INDEFINITE WEIGHT ON CYLINDRICAL DOMAINS

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Abstract. This paper is concerned with an indefinite weight linear eigenvalue problem in cylindrical domains. We investigate the minimization of the positive principal eigenvalue under the constraint that the weight is bounded by a positive and a negative constant and the total weight is a fixed negative constant. Biologically, this minimization problem is motivated by the question of determining the optimal spatial arrangement of favorable and unfavorable regions for a species to survive. Both our analysis and numerical simulations for rectangular domains indicate that there exists a threshold value such that if the total weight is below this threshold value, then the optimal favorable region is a circular-type domain at one of the four corners, and a strip at the one end with shorter edge otherwise.

1. **Introduction.** Consider the following linear eigenvalue problem with indefinite weight

$$\begin{cases} \Delta\varphi + \lambda m(x)\varphi = 0 & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, n is the outward unit normal vector on $\partial\Omega$, and the weight m is a bounded measurable function satisfying

$$-1 \le m(x) \le \kappa \quad \text{on } \Omega \qquad (\kappa > 0). \tag{1.2}$$

We say that λ is a *principal eigenvalue* of (1.1) if λ has a positive eigenfunction $\varphi \in H^1(\Omega)$. It was shown by [3, 19, 13] that (1.1) has a positive principal eigenvalue if and only if the set

$$\Omega^+ = \{x \in \Omega : m(x) > 0\}$$

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has positive Lebesgue measure and

$$\int_\Omega m < 0. \tag{1.3}$$

Moreover, λ is the only positive principal eigenvalue, and it is also the smallest positive eigenvalue of (1.1).

We are interested in the dependence of the principal eigenvalue $\lambda = \lambda(m)$ on the weight m. The motivation comes from the diffusive logistic equation

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) \geq 0, \quad u(x,0) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases} \tag{1.4}$$

where $u(x,t)$ represents the density of a species at location x and time t, the no-flux boundary condition means that no individuals cross the boundary of the habitat Ω, and ω is a positive parameter. The weight m represents the intrinsic growth rate of species: it is positive in the favorable part of habitat (Ω_+) and negative in the unfavorable one ($\Omega_- = \{x \in \Omega : m(x) < 0\}$). The integral $\int_\Omega m$ measures the total resources in a spatially heterogeneous environment.

The logistic equation (1.4) plays an important role in studying the effects of dispersal and spatial heterogeneity in population dynamics (see, e.g. [4, 6, 7] and references therein). It is known that if $\omega \leq \lambda(m)$, then $u(x,t) \to 0$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for all nonnegative and nontrivial initial data (i.e., the species goes to extinction); if $\omega > \lambda(m)$, then $u(x,t) \to u^*(x)$ uniformly in $\overline{\Omega}$ as $t \to \infty$, where u^* is the unique positive steady solution of (1.4) in $W^{2,q}(\Omega)$ for every $q > 1$, i.e., the species survives. We are particularly concerned with the effects of spatial variation in the environment on species extinctions. In this connection, let $m_0 < 1$ be a positive constant and assume

(A1) m satisfies (1.2), Ω^+ has positive measure, and $\int_\Omega m \leq -m_0|\Omega|$.

Since the species can be maintained if and only if $\omega > \lambda(m)$, we see that the smaller $\lambda(m)$ is, the more likely the species can survive. In this connection, the following question was raised and addressed by Cantrell and Cosner in [4, 5]: *Among all functions $m(x)$ that satisfy* (A1), *which $m(x)$ will yield the smallest principal eigenvalue $\lambda(m)$?* From the biological point of view, finding such a minimizing function $m(x)$ is equivalent to determining the optimal spatial arrangement of the favorable and unfavorable parts of the habitat for species to survive [4, 5]. This issue is important for public policy decisions on conservation of species with limited resources. We also refer to [2, 7, 8, 12, 15, 20] and references therein for related work on spatial arrangement of resources and habitat fragmentation.

Given $m_0 < 1$ and $\kappa > 0$, we define

$$\mathcal{M} = \{m \in L^\infty(\Omega) : m(x) \text{ satisfies (A1)}\}, \tag{1.5}$$

and set

$$\lambda_{inf} := \inf_{m \in \mathcal{M}} \lambda(m).$$

The following result was established in [17]:

Theorem A. *The infimum λ_{inf} is attained by some $m \in \mathcal{M}$. Moreover, if $\lambda(m) = \lambda_{inf}$, then m can be represented as $m(x) = \kappa\chi_E - \chi_{\Omega \setminus E}$ a.e. in Ω for some measurable set $E \subset \Omega$.*

In particular, Theorem A implies that the global minimizers of $\lambda(m)$ in $\mathcal{M}$ must be of the "bang-bang" type. When $N = 1$ and Ω is an interval, a complete characterization of all global minimizers of $\lambda(m)$ in $\mathcal{M}$ is also given in [17].

Theorem B. *Suppose that $N = 1$, $\Omega = (0,1)$, and set $c = (1 - m_0)/(1 + \kappa)$. Then $\lambda(m) = \lambda_{inf}$ for some function $m \in \mathcal{M}$ if and only if $m = \kappa\chi_E - \chi_{\Omega \setminus E}$ a.e. in $(0,1)$, where either $|E \cap (0,c)| = c$ or $|E \cap (1-c,1)| = c$.*

Theorem B implies that when Ω is an interval, then there are exactly two global minimizers of $\lambda(m)$ (up to change of a set of measure zero). For the logistic model, this means that a single favorable region at one of the two ends of the whole habitat provides the best opportunity for the species to survive.

A major open problem is the characterization of the optimal set E in $\mathcal{M}$ for *higher-dimensional domains*. In this paper, we focus on the case where Ω is a cylindrical domain given by

$$\Omega := (0,1) \times D \subset \mathbb{R}^N, \quad N \geq 2,$$

where D is a bounded domain in $\mathbb{R}^{N-1}$ with smooth boundary ∂D. As we shall see in later discussions, even for this simple-looking case, determining the shape of the optimal set E is fairly nontrivial.

Let Ω_0^+ and Ω_0^- be subsets of Ω defined by

$$\Omega_0^+ := (0,c) \times D, \qquad \Omega_0^- := (c,1) \times D$$

with a parameter $c \in (0,1)$, and set

$$m(x,y) := \begin{cases} \kappa & \text{if } (x,y) \in \Omega_0^+, \\ -1 & \text{if } (x,y) \in \Omega_0^- \end{cases}$$

where $x \in (0,1)$ and $y \in D$. Note that (1.3) is equivalent to

$$0 < c < c^* := \frac{1}{\kappa + 1}.$$

As we will see later, the principal eigenvalue is uniquely determined by

$$\sqrt{\kappa}\tan\sqrt{\lambda\kappa}\,c = \tanh\sqrt{\lambda}(1-c). \tag{1.6}$$

Our interest is whether or not λ is locally minimal with respect to perturbation of Ω_0^+.

Let μ be the smallest positive eigenvalue of

$$\begin{cases} \Delta_y V + \mu V = 0 & \text{in } D, \\ \dfrac{\partial}{\partial n_y} V = 0 & \text{on } \partial D. \end{cases} \tag{1.7}$$

For each $c \in (0, c^*)$, we can define a positive number $\mu_c \in (\lambda\kappa, \infty)$ uniquely by

$$\sqrt{-\lambda\kappa + \mu_c}\tanh\sqrt{-\lambda\kappa + \mu_c}\,c + \sqrt{\lambda + \mu_c}\tanh\sqrt{\lambda + \mu_c}\,(1-c)$$
$$= (\kappa + 1)\sqrt{\lambda/\kappa}\cot\sqrt{\lambda\kappa}\,c. \tag{1.8}$$

Now our main analytical result can be stated as follows.

Theorem 1.1. *If $\mu < \mu_c$, then λ is not locally minimal.*

As a special case, for rectangular domain we have

Corollary 1.2. *Suppose* $\Omega = (0,1) \times (0,b)$. *If* $b > \pi/\sqrt{\mu_c}$, *then* λ *is not locally minimal. In particular, the strip at the end with much longer edge can not be an optimal favorable region.*

When $\mu > \mu_c$, we expect that λ is locally minimal, at least in a sufficiently wide class of perturbations; see Section 6 for more details. Since μ is large when D is small, we conjecture that λ is locally minimal for a thin cylinder. It seems that λ is a global minimum for a sufficiently thin cylinder.

Remark.

1. Since $\mu_c > \lambda\kappa$ and $\lambda \to \infty$ as $c \to 0$, we have $\mu_c \to \infty$ as $c \to 0$. So another implication of Theorem 1.1 is that if the area of the favorable region is small, then a strip at any end of the rectangular domain can not be an optimal favorable region.
2. Since $\lambda \to 0$ as $c \to c^*$, we have $\mu_c \to \mu^*$ as $c \to c^*$, where μ^* is defined by

$$\sqrt{\mu^*} \tanh \sqrt{\mu^*} \, \frac{\kappa}{\kappa+1} + \sqrt{\mu^*} \tanh \sqrt{\mu^*} \, \frac{1}{\kappa+1} = \frac{(\kappa+1)^2}{\kappa}. \tag{1.9}$$

3. It seems that μ_c is strictly decreasing in c. This means that if $\mu \le \mu^*$, then λ is not locally minimal for any $c \in (0, c^*)$. In other words, if Ω is a fat cylinder, then λ is not locally minimal regardless of the choice of c. See the remark at the end of Section 4.

Numerical simulations in Section 7 on unit square indicate that the shape of the optimal Ω^+ depends crucially on the value of m_0. More precisely, it seems that there exists a threshold value $m_0^* \in (0,1)$ such that if $m_0 < m_0^*$ then Ω^+ is a stripe, and if $m_0 > m_0^*$ then Ω^+ is a circular type domain at one of the four corners. For general rectangular domains, simulation shows that the optimal Ω^+ is either circular type domains located at one of the four corners or a strip at the one end with shorter edge. The numerical scheme developed in Section 7 is based upon the projection gradient method, and it is general enough to handle both topological changes of Ω^+ and general domains with nontrivial topology.

2. **One-dimensional problem.** In this section, we summarize the results for the one-dimensional problem:

$$\begin{cases} \varphi'' + \lambda m(x)\varphi = 0, & 0 < x < 1, \\ \varphi'(0) = \varphi'(1) = 0, \end{cases} \tag{2.1}$$

where

$$m(x) := \begin{cases} \kappa & \text{if } 0 < x < c, \\ -1 & \text{if } c < x < 1. \end{cases}$$

It was shown in [17] that such $m(x)$ is a global minimizer of the principal eigenvalue.

By (2.1), we may write φ as

$$\varphi(x) = \begin{cases} A\cos\sqrt{\lambda\kappa}\,x, & 0 < x < c, \\ B\cosh\sqrt{\lambda}(x-1), & c < x < 1. \end{cases} \tag{2.2}$$

From the continuity of φ and φ' at $x = c$, we have

$$A \cos \sqrt{\lambda \kappa}\, c = B \cosh \sqrt{\lambda}(1-c),$$
$$-A\sqrt{\lambda \kappa} \sin \sqrt{\lambda \kappa}\, c = -B\sqrt{\lambda} \sinh \sqrt{\lambda}(1-c).$$

For $(A, B) \neq (0, 0)$, we obtain the characteristic equation (1.6). Since $\kappa c < 1 - c$, the principal eigenvalue $\lambda > 0$ is uniquely determined from (1.6).

3. **Formal expansion.** We perturb the problem (1.1) as follows. Let $g : D \to \mathbb{R}$ be any L^2 function satisfying

$$\int_D g(y) dy = 0, \tag{3.1}$$

and define a perturbed domain

$$\Omega_\varepsilon^+ := \{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 < x < c + \varepsilon g(y),\ y \in D\},$$
$$\Omega_\varepsilon^- := \{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} : c + \varepsilon g(y) < x < 1,\ y \in D\},$$

where ε is a small parameter. Then we set

$$m_\varepsilon(x, y) := \begin{cases} \kappa & \text{if } (x, y) \in \Omega_\varepsilon^+, \\ -1 & \text{if } (x, y) \in \Omega_\varepsilon^-, \end{cases}$$

and consider the perturbed problem

$$\begin{cases} \Delta \varphi_\varepsilon + \lambda_\varepsilon m_\varepsilon(x)\varphi_\varepsilon = 0 & \text{in } \Omega, \\ \dfrac{\partial \varphi_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The main goal of this section is to find a formal asymptotic expansion of λ_ϵ for ϵ positive small and gain some insight. To this end, we expand λ_ε and φ_ε formally as

$$\lambda_\varepsilon = \lambda + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + O(\varepsilon^3),$$
$$\varphi_\varepsilon(x, y) = \varphi(x) + \varepsilon\varphi_1(x, y) + \varepsilon^2\varphi_2(x, y) + O(\varepsilon^3), \tag{3.3}$$

where (φ, λ) is an eigenpair of the one-dimensional problem (2.1). We substitute (3.3) into the weak form

$$-\int_\Omega \nabla\varphi_\varepsilon \cdot \nabla\psi + \lambda_\varepsilon \int_\Omega m_\varepsilon \varphi_\varepsilon \psi = 0 \quad \text{for any } \psi \in C^1(\overline{\Omega}), \tag{3.4}$$

and compute ε^0-, ε^1- and ε^2-order terms.

The first term in the left-hand side of (3.4) is written as follows

$$\int_\Omega \nabla\varphi_\varepsilon \cdot \nabla\psi = \int_\Omega \nabla\varphi \cdot \nabla\psi + \varepsilon \int_\Omega \nabla\varphi_1 \cdot \nabla\psi + \varepsilon^2 \int_\Omega \nabla\varphi_2 \cdot \nabla\psi + O(\varepsilon^3). \tag{3.5}$$

For the second term, we have

$$\int_\Omega m_\varepsilon \varphi_\varepsilon \psi = \int_\Omega m\varphi_\varepsilon \psi + \int_\Omega (m_\varepsilon - m)\varphi_\varepsilon \psi.$$

Here

$$
\int_\Omega (m_\varepsilon - m)\varphi_\varepsilon \psi = (\kappa + 1) \int_D \left\{ \int_c^{c+\varepsilon g(y)} \varphi_\varepsilon \psi dx \right\} dy
$$

$$
= (\kappa + 1) \int_D \left\{ (\varphi_\varepsilon \psi)\Big|_{x=c} \cdot \varepsilon g(y) + \frac{\partial}{\partial x}(\varphi_\varepsilon \psi)\Big|_{x=c} \cdot \frac{\varepsilon^2 g^2(y)}{2} \right\} dy + O(\varepsilon^3)
$$

$$
= (\kappa + 1) \int_D \left\{ \{\varphi(c) + \varepsilon \varphi_1(c,y)\}\psi(c,y) \cdot \varepsilon g(y) \right.
$$

$$
\left. + \left(\frac{\partial \varphi}{\partial x}(c)\psi(c,y) + \varphi(c)\frac{\partial \psi}{\partial x}(c,y) \right) \cdot \frac{\varepsilon^2 g^2(y)}{2} \right\} dy + O(\varepsilon^3)
$$

$$
= (\kappa + 1) \int_D \left\{ \varepsilon \varphi(c)\psi(c,y)g(y) \right.
$$

$$
\left. + \varepsilon^2 \left\{ \varphi_1(c,y)\psi(c,y)g(y) + \frac{\partial}{\partial x}(\varphi \psi)\Big|_{x=c} \cdot \frac{g^2(y)}{2} \right\} \right\} dy + O(\varepsilon^3).
$$

Hence we have

$$
\int_\Omega m_\varepsilon \varphi_\varepsilon \psi = \int_\Omega m\varphi_\varepsilon \psi + \int_\Omega (m_\varepsilon - m)\varphi_\varepsilon \psi = \int_\Omega m\varphi\psi
$$

$$
+ \varepsilon \left[\int_\Omega m\varphi_1\psi + (\kappa + 1) \int_D \varphi(c)\psi(c,y)g(y)dy \right]
$$

$$
+ \varepsilon^2 \left[\int_\Omega m\varphi_2\psi + (\kappa + 1) \int_D \left\{ \varphi_1(c,y)\psi(c,y)g(y) + \frac{\partial}{\partial x}(\varphi\psi)\Big|_{x=c} \cdot \frac{g^2(y)}{2} \right\} dy \right]
$$

$$
+ O(\varepsilon^3). \tag{3.6}
$$

Consequently, we obtain

$$
\lambda_\varepsilon \int_\Omega m_\varepsilon \varphi_\varepsilon \psi = \lambda \int_\Omega m\varphi\psi
$$

$$
+ \varepsilon \left[\lambda_1 \int_\Omega m\varphi\psi + \lambda \left\{ \int_\Omega m\varphi_1\psi + (\kappa + 1) \int_D \varphi(c)\psi(c,y)g(y)dy \right\} \right]
$$

$$
+ \varepsilon^2 \left[\lambda_2 \int_\Omega m\varphi\psi \right.
$$

$$
+ \lambda_1 \left\{ \int_\Omega m\varphi_1\psi + (\kappa + 1) \int_D \varphi(c)\psi(c,y)g(y)dy \right\} + \lambda \int_\Omega m\varphi_2\psi \, dy
$$

$$
\left. + \lambda(\kappa + 1) \int_D \left\{ \varphi_1(c,y)\psi(c,y)g(y) + \frac{\partial}{\partial x}(\varphi\psi)\Big|_{x=c} \cdot \frac{g^2(y)}{2} \right\} dy \right]
$$

$$
+ O(\varepsilon^3).
$$

By using this and (3.5), we compare the ε^0-, ε^1- and ε^2-order terms of (3.4).

First, from ε^0-order terms, we have

$$
\int_\Omega \nabla\varphi \cdot \nabla\psi = \lambda \int_\Omega m\varphi\psi. \tag{3.7}
$$

This equality clearly holds by (1.1).

Next, from ε^1-order terms, we have

$$
\int_\Omega \nabla\varphi_1 \cdot \nabla\psi = \lambda_1 \int_\Omega m\varphi\psi + \lambda \left\{ \int_\Omega m\varphi_1\psi + (\kappa + 1) \int_D \varphi(c)\psi(c,y)g(y)dy \right\}.
$$

If we take $\psi = \varphi$, then

$$\lambda_1 \int_\Omega m\varphi^2 + \lambda(\kappa+1)\varphi(c)^2 \int_D g(y)dy = 0.$$

Hence by (3.1), we obtain $\lambda_1 = 0$ so that

$$\int_\Omega \nabla\varphi_1 \cdot \nabla\psi = \lambda\left\{ \int_\Omega m\varphi_1\psi + (\kappa+1)\int_D \varphi(c)\psi(c,y)g(y)dy \right\}. \qquad (3.8)$$

This in particular means that φ_1 must satisfy the equation

$$\begin{aligned}
\Delta\varphi_1 + \lambda\kappa\varphi_1 &= 0 \quad \text{in } (0,c)\times D, \\
\Delta\varphi_1 - \lambda\varphi_1 &= 0 \quad \text{in } (c,1)\times D,
\end{aligned} \qquad (3.9)$$

with the boundary condition

$$\frac{\partial\varphi_1}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad (3.10)$$

and the matching condition

$$\left.\frac{\partial\varphi_1}{\partial x}\right|_{x=c-}^{x=c+} = -\lambda(\kappa+1)\varphi(c)g(y), \quad y \in D. \qquad (3.11)$$

Finally, from ε^2-order terms and $\lambda_1 = 0$, we have

$$\begin{aligned}
\int_\Omega \nabla\varphi_2 \cdot \nabla\psi &= \lambda\int_\Omega m\varphi_2\psi + \lambda_2\int_\Omega m\varphi\psi \\
&+ \lambda(\kappa+1)\int_D \left\{ \varphi_1(c,y)\psi(c,y)g(y) + \left.\frac{\partial}{\partial x}(\varphi\psi)\right|_{x=c} \cdot \frac{g^2(y)}{2} \right\}dy.
\end{aligned} \qquad (3.12)$$

Again by taking $\psi = \varphi$, we obtain

$$\lambda_2\int_\Omega m\varphi^2 + \lambda(\kappa+1)\int_D \left\{ \varphi_1(c,y)\varphi(c)g(y) + 2\varphi(c)\varphi'(c)\cdot\frac{g^2(y)}{2} \right\}dy = 0.$$

Hence λ_2 must be expressed as

$$\lambda_2 = -\frac{\lambda(\kappa+1)\varphi(c)\displaystyle\int_D \left\{ \varphi_1(c,y)g(y) + \varphi'(c)g^2(y) \right\}dy}{\displaystyle\int_\Omega m\varphi^2}. \qquad (3.13)$$

The sign of λ_2 is crucial for stability. Indeed, we will show in Section 5 that the principal eigenvalue λ is NOT locally minimal if

$$I[g] := \int_D \left\{ \varphi_1(c,y)g(y) + \varphi'(c)g^2(y) \right\}dy > 0$$

for some g.

4. **Computation of λ_2.** In this section, we compute λ_2 in the case where $g(y)$ is an eigenfunction of (1.7) associated with a positive eigenvalue $\mu > 0$. We note that (3.1) holds in this case.

4.1. φ_1. We set $\varphi_1 = P(x)g(y)$. Then it follows from (3.9), (3.10) and (3.11) that P satisfies

$$\begin{cases} P''(x) + (\lambda\kappa - \mu)P(x) = 0, & 0 < x < c, \\ P''(x) - (\lambda + \mu)P(x) = 0, & c < x < 1, \\ P'(0) = P'(1) = 0, \\ P'(c+) - P'(c-) = -\lambda(\kappa + 1)\varphi(c). \end{cases} \tag{4.1}$$

We solve (4.1) as follows.

Case I: $\mu \le \lambda\kappa$
We write P as

$$P(x) = \begin{cases} C\cos\sqrt{\lambda\kappa - \mu}\,x, & 0 < x < c, \\ D\cosh\sqrt{\lambda + \mu}\,(x - 1), & c < x < 1. \end{cases}$$

From the matching condition at $x = c$, we have

$$C\cos\sqrt{\lambda\kappa - \mu}\,c = D\cosh\sqrt{\lambda + \mu}\,(1 - c),$$
$$-C\sqrt{\lambda\kappa - \mu}\sin\sqrt{\lambda\kappa - \mu}\,c = -D\sqrt{\lambda + \mu}\sinh\sqrt{\lambda + \mu}\,(1 - c) + \lambda(\kappa + 1)\varphi(c).$$

By the first equality, we have

$$D = \frac{C\cos\sqrt{\lambda\kappa - \mu}\,c}{\cosh\sqrt{\lambda + \mu}\,(1 - c)}.$$

Using this, we obtain

$$C = \frac{\lambda(\kappa + 1)\varphi(c)}{\cos\sqrt{\lambda\kappa - \mu}\,c\Big\{\sqrt{\lambda + \mu}\tanh\sqrt{\lambda + \mu}\,(1 - c) - \sqrt{\lambda\kappa - \mu}\tan\sqrt{\lambda\kappa - \mu}\,c\Big\}}.$$

Case II: $\mu \ge \lambda\kappa$
We write P as

$$P(x) = \begin{cases} C\cosh\sqrt{-\lambda\kappa + \mu}\,x, & 0 < x < c, \\ D\cosh\sqrt{\lambda + \mu}\,(x - 1), & c < x < 1. \end{cases}$$

From the matching condition at $x = c$, we have

$$C\cosh\sqrt{-\lambda\kappa + \mu}\,c = D\cosh\sqrt{\lambda + \mu}\,(1 - c),$$
$$C\sqrt{-\lambda\kappa + \mu}\sinh\sqrt{-\lambda\kappa + \mu}\,c = -D\sqrt{\lambda + \mu}\sinh\sqrt{\lambda + \mu}\,(1 - c) + \lambda(\kappa + 1)\varphi(c).$$

By the first condition, we have

$$D = \frac{C\cosh\sqrt{-\lambda\kappa + \mu}\,c}{\cosh\sqrt{\lambda + \mu}\,(1 - c)}.$$

Using this, we obtain

$$C = \frac{\lambda(\kappa + 1)\varphi(c)}{\cosh\sqrt{-\lambda\kappa + \mu}\,c\Big\{\sqrt{-\lambda\kappa + \mu}\tanh\sqrt{-\lambda\kappa + \mu}\,c + \sqrt{\lambda + \mu}\tanh\sqrt{\lambda + \mu}\,(1 - c)\Big\}}.$$

4.2. $I[g]$. Let us compute $I[g]$. We note that $I[g]$ is written as

$$I[g] = \{P(c) + \varphi'(c)\} \int_D g^2(y) dy. \tag{4.2}$$

In Case I ($\mu \le \lambda\kappa$), we have

$$P(c) + \varphi'(c) = C \cos \sqrt{\lambda\kappa - \mu}\, c + \varphi'(c).$$

Here, from the above computation, we have

$$C \cos \sqrt{\lambda\kappa - \mu}\, c = \frac{\lambda(\kappa + 1)\varphi(c)}{-\sqrt{\lambda\kappa - \mu}\, \tan \sqrt{\lambda\kappa - \mu}\, c + \sqrt{\lambda + \mu}\, \tanh \sqrt{\lambda + \mu}\,(1 - c)},$$

and

$$\varphi(c) = A \cos \sqrt{\lambda\kappa}\, c, \qquad \varphi' = -A\sqrt{\lambda\kappa}\, \sin \sqrt{\lambda\kappa}\, c$$

by (2.2). Hence $P(c) + \varphi'(c) > 0$ (i.e., $I[g] > 0$) if

$$(\kappa + 1)\sqrt{\lambda/\kappa}\, \cot \sqrt{\lambda\kappa}\, c$$
$$> -\sqrt{\lambda\kappa - \mu}\, \tan \sqrt{\lambda\kappa - \mu}\, c + \sqrt{\lambda + \mu}\, \tanh \sqrt{\lambda + \mu}\,(1 - c). \tag{4.3}$$

Recalling the characteristic equation (1.6), we see that the left-hand side satisfies

$$(\kappa + 1)\sqrt{\lambda/\kappa}\, \cot \sqrt{\lambda\kappa}\, c = (\kappa + 1)\sqrt{\lambda}\, \coth \sqrt{\lambda}\,(1 - c) > (\kappa + 1)\sqrt{\lambda}.$$

On the other hand, since $\mu \le \lambda\kappa$, we have

$$\sqrt{\lambda + \mu}\, \tanh \sqrt{\lambda + \mu}\,(1 - c) < \sqrt{\lambda + \lambda\kappa} = \sqrt{\kappa + 1}\sqrt{\lambda}.$$

These inequalities show that (4.3) always holds in Case I. Thus we have shown that $I[g] > 0$ if $\mu \le \lambda\kappa$.

In Case II ($\mu \ge \lambda\kappa$), we have

$$P(c) + \varphi'(c) = C \cosh \sqrt{-\lambda\kappa + \mu}\, c + \varphi'(c).$$

Here

$$C \cosh \sqrt{-\lambda\kappa + \mu}\, c = \frac{\lambda(\kappa + 1)\varphi(c)}{\sqrt{-\lambda\kappa + \mu}\, \tanh \sqrt{-\lambda\kappa + \mu}\, c + \sqrt{\lambda + \mu}\, \tanh \sqrt{\lambda + \mu}\,(1 - c)}$$

and

$$\frac{\varphi'(c)}{\varphi(c)} = -\sqrt{\lambda\kappa}\, \tan \sqrt{\lambda\kappa}\, c.$$

Hence $P(c) + \varphi'(c) > 0$ (i.e., $I[g] > 0$) if

$$(\kappa + 1)\sqrt{\lambda/\kappa}\, \cot \sqrt{\lambda\kappa}\, c$$
$$> \sqrt{-\lambda\kappa + \mu}\, \tanh \sqrt{-\lambda\kappa + \mu}\, c + \sqrt{\lambda + \mu}\, \tanh \sqrt{\lambda + \mu}\,(1 - c).$$

4.3. λ_2. Let $\Phi(s)$ be a function defined by

$$\Phi(s) := \sqrt{-\lambda\kappa + s}\, \tanh \sqrt{-\lambda\kappa + s}\, c + \sqrt{\lambda + s}\, \tanh \sqrt{\lambda + s}\,(1 - c). \tag{4.4}$$

Then (1.8) can be written as

$$\Phi(s) = (\kappa + 1)\sqrt{\lambda/\kappa}\, \cot \sqrt{\lambda\kappa}\, c.$$

Since

$$\Phi(\lambda\kappa) < \sqrt{\lambda(\kappa + 1)} < (\kappa + 1)\sqrt{\lambda}$$

and

$$\cot \sqrt{\lambda\kappa}\, c = \frac{\sqrt{\kappa}}{\tanh \sqrt{\lambda}(1 - c)} > \sqrt{\kappa}$$

by (1.6), we obtain

$$\Phi(\lambda\kappa) < (\kappa+1)\sqrt{\lambda/\kappa} \cot\sqrt{\lambda\kappa}\, c.$$

Noting that Φ is monotone increasing in s and $\Phi(s) \to \infty$ as $s \to \infty$, we can define μ_c uniquely by (1.8). Thus we have shown that

$$I[g] \begin{cases} < 0 & \text{if } \mu > \mu_c, \\ = 0 & \text{if } \mu = \mu_c, \\ > 0 & \text{if } \lambda\kappa \le \mu < \mu_c. \end{cases}$$

Consequently, we obtain the following lemma.

Lemma 4.1. *Let μ be the smallest positive eigenvalue of* (1.7) *and μ_c be the positive number defined by* (1.8)*. If $\mu < \mu_c$, then the number λ_2 defined by* (3.13) *satisfies $\lambda_2 < 0$.*

The following lemma gives a lower bound of μ_c.

Lemma 4.2. *Let μ_l be a positive number defined by*

$$\sqrt{-\lambda\kappa + \mu_l} + \sqrt{\lambda + \mu_l} = (\kappa+1)\sqrt{\lambda/\kappa}.$$

Then $\mu_c > \mu_l$ for any $c \in (0, c^*)$.

Proof. If $s \ge \mu_l$, then

$$\Phi(s) < \sqrt{-\lambda\kappa + s} + \sqrt{\lambda + s} \le \sqrt{-\lambda\kappa + \mu_l} + \sqrt{\lambda + \mu_l} = (\kappa+1)\sqrt{\lambda/\kappa}.$$

This shows $\mu_c > \mu_l$. ∎

Remark. It seems that the optimal lower bound of μ_c is μ^*, defined by (1.9). To show this, it suffices to prove the monotonicity of μ_c with respect to c.

5. **Rigorous proof by using a variational characterization.** If $\lambda_2 < 0$, then we expect that λ is not locally minimal. To prove this rigorously, we use the following well known variational characterization of the positive principal eigenvalue [1, 3, 13, 19]:

Lemma 5.1. *The positive principal eigenvalue λ of* (1.1) *is given by*

$$\lambda = \inf_{U \in \mathcal{S}(m)} \frac{\displaystyle\int_\Omega |\nabla U|^2}{\displaystyle\int_\Omega m(x) U^2}, \tag{5.5}$$

where

$$\mathcal{S}(m) := \left\{ U \in H^1(\Omega) : \int_\Omega m(x) U^2 > 0 \right\}.$$

Moreover, λ is simple, and the infimum is attained only by associated eigenfunctions that do not change sign in $\overline{\Omega}$.

Proof of Theorem 1.1.

Let $g(y)$ be an eigenfunction of (1.7) associated with a positive eigenvalue $\mu > 0$, and λ_ε be an eigenvalue of (3.2) for such g.

Using Lemma 5.1, we compare λ and λ_ε. To this purpose, we define a functional

$$J_\varepsilon[U] = -\int_\Omega |\nabla U|^2 + \lambda \int_\Omega m_\varepsilon(x) U^2, \tag{5.6}$$

and will show that $J_\varepsilon[U] > 0$ for some $U \in \mathcal{S}(m_\varepsilon)$ and small ε. In the argument in Section 3, φ_2 does not play any role in determining λ_2. Hence we take $\varphi_2 = 0$ and choose $U = \varphi + \varepsilon\varphi_1$ as a test function, where φ_1 is the solution of (3.9), (3.10), and (3.11) constructed in Subsection 4.1. Since $\varphi \in \mathcal{S}(m)$, we have $U \in \mathcal{S}(m_\varepsilon)$ by continuity for sufficiently small ε.

We have

$$J_\varepsilon[\varphi + \varepsilon\varphi_1] = -\int_\Omega |\nabla(\varphi + \varepsilon\varphi_1)|^2 + \lambda \int_\Omega m_\varepsilon(x)(\varphi + \varepsilon\varphi_1)^2.$$

The first term is written as

$$\int_\Omega |\nabla(\varphi + \varepsilon\varphi_1)|^2 = \int_\Omega |\nabla\varphi|^2 + 2\varepsilon \int_\Omega \nabla\varphi \cdot \nabla\varphi_1 + \varepsilon^2 \int_\Omega |\nabla\varphi_1|^2.$$

Replacing φ_ε by $\varphi + \varepsilon\varphi_1$ and taking $\psi = \varphi + \varepsilon\varphi_1$ in (3.6), we have

$$\begin{aligned}
\int_\Omega m_\varepsilon(\varphi + \varepsilon\varphi_1)^2 &= \int_\Omega m\varphi(\varphi + \varepsilon\varphi_1) \\
&+ \varepsilon\left[\int_\Omega m\varphi_1(\varphi + \varepsilon\varphi_1) + (\kappa + 1)\int_D \varphi(c)(\varphi(c) + \varepsilon\varphi_1(c,y))g(y)dy \right] \\
&+ \varepsilon^2\left[(\kappa + 1)\int_D \left\{ \varphi_1(c,y)\varphi(c)g(y) + \frac{\partial}{\partial x}(\varphi^2)\Big|_{x=c} \cdot \frac{g^2(y)}{2} \right\}dy \right] \\
&+ O(\varepsilon^3) \\
&= \int_\Omega m\varphi^2 + \varepsilon\left[2\int_\Omega m\varphi\varphi_1 + (\kappa + 1)\varphi(c)^2 \int_D g(y)dy \right] \\
&+ \varepsilon^2\left[\int_\Omega m\varphi_1^2 + (\kappa + 1)\int_D \varphi(c)\varphi_1(c,y)g(y)dy \right. \\
&\left. + (\kappa + 1)\varphi(c)\int_D \left\{ \varphi_1(c,y)g(y) + \varphi'(c)g(y)^2 \right\} \right] \\
&+ O(\varepsilon^3).
\end{aligned}$$

Hence, by using (3.1) and (3.13), we obtain

$$\begin{aligned}
\int_\Omega m_\varepsilon(\varphi + \varepsilon\varphi_1)^2 &= \int_\Omega m\varphi^2 + 2\varepsilon \int_\Omega m\varphi\varphi_1 \\
&+ \varepsilon^2\left[\int_\Omega m\varphi_1^2 + (\kappa + 1)\int_D \varphi(c)\varphi_1(c,y)g(y)dy - \frac{\lambda_2}{\lambda} \int_\Omega m\varphi^2 \right] \\
&+ O(\varepsilon^3).
\end{aligned}$$

Thus it is shown

$$J_\varepsilon[\varphi + \varepsilon\varphi_1] = -\int_\Omega |\nabla\varphi|^2 + \lambda \int_\Omega m\varphi^2 + 2\varepsilon\left[-\int_\Omega \nabla\varphi \cdot \nabla\varphi_1 + \lambda \int_\Omega m\varphi\varphi_1\right]$$

$$+\varepsilon^2\left[-\int_\Omega |\nabla\varphi_1|^2 + \lambda \int_\Omega m\varphi_1^2 + (\kappa+1)\lambda \int_D \varphi(c)\varphi_1(c,y)g(y)dy\right.$$

$$\left. -\lambda_2 \int_\Omega m\varphi^2\right] + O(\varepsilon^3)$$

$$= \varepsilon^2\left[-\int_\Omega |\nabla\varphi_1|^2 + \lambda \int_\Omega m\varphi_1^2 + (\kappa+1)\lambda \int_D \varphi(c)\varphi_1(c,y)g(y)dy - \lambda_2 \int_\Omega m\varphi^2\right]$$

$$+O(\varepsilon^3).$$

Here, by (3.9), (3.10), and (3.11), we have

$$\int_\Omega |\nabla\varphi_1|^2 = \int_{\Omega_0^+} |\nabla\varphi_1|^2 + \int_{\Omega_0^-} |\nabla\varphi_1|^2$$

$$= -\int_{\Omega_0^+} \varphi_1\Delta\varphi_1 + \int_D \frac{\partial\varphi_1}{\partial x}\bigg|_{x=c-} \varphi_1(c,y)dy - \int_{\Omega_0^-} \varphi_1\Delta\varphi_1$$

$$-\int_D \frac{\partial\varphi_1}{\partial x}\bigg|_{x=c+} \varphi_1(c,y)dy$$

$$= \lambda \int_\Omega m\varphi_1^2 + (\kappa+1)\lambda \int_D \varphi(c)\varphi_1(c,y)g(y)dy.$$

Consequently, we obtain

$$J_\varepsilon[\varphi + \varepsilon\varphi_1] = -\varepsilon^2\lambda_2 \int_\Omega m\varphi^2 + O(\varepsilon^3). \tag{5.7}$$

This together with Lemma 4.1 completes the proof. ∎

6. **Formal analysis in the case of** $\mu > \mu_c$. In this section, we show formally that if $\mu > \mu_c$, then λ is locally minimal in the class of

$$\mathcal{G} := \{g \in L^2(D) : g \text{ satisfies } (3.1)\}.$$

Let $\{V_j\}$ be an orthonormal basis which consists of eigenfunctions of (1.7):

$$\begin{cases} \Delta V_j + \mu_j V_j = 0 & \text{in } D, \\ \dfrac{\partial}{\partial n_y} V_j = 0 & \text{on } \partial D. \end{cases} \tag{6.1}$$

In particular, we set $\mu_0 = 0$ and $V_0 = 1/|D|$. Since g is orthogonal to V_0 by (3.1), we expand g as

$$g = \sum_{j=1}^\infty d_j V_j.$$

Then φ_1 is computed as

$$\varphi_1(x,y) = \sum_{j=1}^\infty d_j P_j(x)V_j(y),$$

where P_j satisfies

$$\begin{cases} P_j''(x) + (\lambda\kappa - \mu_j)P_j(x) = 0, & 0 < x < c, \\ P_j''(x) - (\lambda + \mu_j)P_j(x) = 0, & c < x < 1, \\ P_j'(0) = P_j'(1) = 0, \\ P_j'(c+) - P_j'(c-) = -\lambda(\kappa + 1)\varphi(c). \end{cases}$$

We compute

$$\begin{aligned} I[g] &= \int_D \Big\{ \varphi_1(c, y)g(y) + \varphi'(c)g^2(y) \Big\} dy \\ &= \int_D \Big\{ \sum_{j=1}^{\infty} d_j P_j(c)V_j(y) \Big\}\Big\{ \sum_{j=1}^{\infty} d_j V_j(y) \Big\} dy + \int_D \varphi'(c)\Big\{ \sum_{j=1}^{\infty} d_j V_j(y) \Big\}^2 dy \\ &= \sum_{j=1}^{\infty} d_j^2 P_j(c) + \varphi'(c)\sum_{j=1}^{\infty} d_j^2 \\ &= \{P_j(c) + \varphi'(c)\}\sum_{j=1}^{\infty} d_j^2, \end{aligned}$$

where we used

$$\int_D V_j(y)V_k(y)dy = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Here $P_j(c) + \varphi'(c) < 0$ if $\mu_j > \mu_c$. Since $\mu_j \geq \mu > \mu_c$ for all j, we obtain $I[g] < 0$ and hence $\lambda_2 > 0$. Thus it is shown that $\lambda_\varepsilon > \lambda$ holds for any $g \neq 0$ and sufficiently small $\varepsilon \neq 0$. This strongly suggests that λ is locally minimal.

We believe that the above formal analysis can be verified rigorously. That notwithstanding, it does not immediately mean that λ is minimal in the class $\mathcal{M}$, because we assumed that the boundary of the favorable region Ω_ϵ^+ is a graph of some function. Some further mathematical analysis will be needed to show the minimality of λ in the larger class $\mathcal{M}$.

7. **Numerical Simulations.** In this section, we show the numerical approach to find the optimal configuration $m(x)$ which minimizes the principle eigenvalue λ of (1.1) and satisfies (A1)

$$M = \int_\Omega m < 0. \tag{7.1}$$

Previously both theoretical and numerical approaches have been investigated to minimize the first eigenvalue value for positive $m(x)$ [10, 11, 9, 18]. Here we focus on the indefinite weight $m(x)$. Our approach is based on the projection gradient method [18]. The idea is to start from an initial guess for $m(x)$, evolve it along the gradient direction until it reaches the optimal configuration. However, the gradient direction may result in the violation of the constraint. A projection approach is used to project $m(x)$ back to the feasible set. Furthermore, we propose a new binary update for $m(x)$ to preserve the bang-bang structure.

At each iteration, we need to compute the principal eigenvalue and its corresponding eigenfunction. This is done by expanding φ in the FEM (finite element method) [14] basis, multiplying with a basis element, and integrating on the domain Ω. It yields a generalized eigenvalue equation which can be solved by Arnoldi

algorithm [16]. This can be implemented easily by using Matlab Partial Differential Equation Toolbox.

7.1. **Gradient Descent Approach.** Consider a variation in $m(x)$ by an amount δm, which causes variations in φ and λ by $\delta\varphi$ and $\delta\lambda$ respectively. The formula for the gradient of λ with respect to $m(x)$ is

$$D_m\lambda \cdot \delta m = \delta\lambda = -\frac{\lambda \int_\Omega \delta m \varphi^2 dx}{\int_\Omega m\varphi^2 dx}.$$

The descent direction can be chosen as $\delta m = a\varphi^2$ with $a > 0$, because

$$\delta\lambda = -\frac{\lambda a \int_\Omega \varphi^4 dx}{\int_\Omega m\varphi^2 dx} < 0.$$

This implies that if $m(x)$ increases linearly w.r.t φ^2, the principle eigenvalue decreases. However, this descent direction increases $m(x)$ everywhere and results in the violation of the mass constraint. The way to resolve this is by using the Lagrange Multiplier Method. The descent direction is then modified to be

$$\delta m = a\varphi^2 + b,$$

where the constant b can be obtained by enforcing the constraint

$$\int_\Omega \delta m dx = \int_\Omega (a\varphi^2 + b)dx = 0;$$

that is,

$$b = -\frac{\int_\Omega a\varphi^2 dx}{\int_\Omega dx}.$$

This gives

$$\delta m = a(\varphi^2 - \frac{\int_\Omega \varphi^2 dx}{\int_\Omega dx}), \qquad (7.2)$$

which projects $m(x)$ back to the feasible set.

Since we know that the optimal configuration for $m(x)$ needs to be bang-bang type, we search for $m(x)$ satisfying

$$m(x) := \begin{cases} 1 & \text{on} \quad \Omega^+, \\ -1 & \text{on} \quad \Omega^-, \end{cases}$$

with $\kappa = 1$ in (1.2). No intermediate value is allowed. Instead of using (7.2) to evolve $m(x)$ exactly, we give a binary update for $m(x)$ which still follow the idea of the projection gradient method.

For ease of representation, we give a one-dimensional illustration in Figure 1. We start from an initial guess for $m(x)$ shown as the blue curve in Figure 1a and its corresponding eigenfunction $\varphi(x)$ shown as the blue curve in Figure 1b. First, we find the minimum of eigenfunction on Ω^+ :

$$\varphi_{\min} = \min\{\varphi(x)|x \in \Omega^+\}$$

and then look for the set
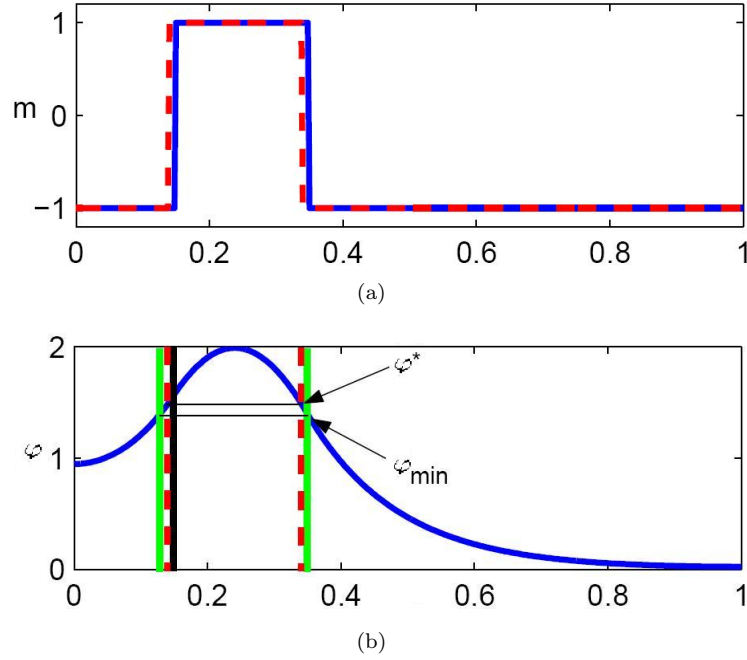
$$G = \{x|\varphi(x) \geq \varphi_{\min}\},$$

FIGURE 1. The update process for $m(x)$ in one dimension.

which is the interval shown between green lines in Figure 1b. The reason we look for this set is because the points on the boundary of G : $\partial G = \{x|\varphi(x) = \varphi_{\min}\}$ have same δm along the gradient direction. It is clear that

$$m^G(x) := \begin{cases} 1 & \text{on} & G \\ -1 & \text{on} & \Omega \backslash G \end{cases}$$

yields a smaller principle eigenvalue λ. Since $\Omega^+ \subset G$, $m_G \geq m$ a.e. and the inequality is strict if G/Ω^+ has positive measure. Then one can apply the comparison principle for the principal eigenvalue with indefinite weight [13] to conclude that $\lambda(m^G) < \lambda(m)$.

To satisfy the constraint, we need to project $m(x)$ back to feasible set. This can be done by looking for Ω_{new}^+ and φ^* such that

$$\Omega_{new}^+ = \{x|x \in R\} \text{ and } \int_{\Omega_{new}^+} dx = \int_{\Omega^+} dx,$$

where

$$R = \{x|\varphi(x) \geq \varphi^*\}$$

is the interval shown between red dash lines. This one-step process gives us Ω_{new}^+, which satisfies the constraint, and it turns out that the eigenvalue for

$$m^{new}(x) := \begin{cases} 1 & \text{on} & \Omega_{new}^+, \\ -1 & \text{on} & \Omega_{new}^- \end{cases}$$

is smaller than the original eigenvalue for

$$m(x) := \begin{cases} 1 & \text{on} & \Omega^+, \\ -1 & \text{on} & \Omega^-. \end{cases}$$

We then iterate this process until it reaches the optimal configuration.

The binary update algorithm we used yields fast evolution of $m(x)$. In our numerical simulations, we found that eigenvalue of $m^{new}(x)$ is smaller than that of $m(x)$ even though the project step may increase the eigenvalue. A similar construction was used in [4], and $\lambda(m^{new}) < \lambda(m)$ follows from $\int_\Omega m(x)U^2 < \int_\Omega m^{new}(x)U^2$, where U is the corresponding eigenfunction of $m(x)$.

In [17], the optimal configuration for $m(x)$ which minimizes the principle eigenvalue is distributed at one of the two ends in one dimension. For higher dimensions, the problem remains open.

In Figure 2, we show the numerical simulation on a square domain $\Omega = [-1,1] \times [-1,1]$ with the initial $m(x) = 1$ on $\Omega^+ = \{x|x + 0.05\sin(2\pi y) + 0.5 < 0\}$ (light gray) and $m(x) = -1$ on $\Omega^- = \{x|x + 0.05\sin(2\pi y) + 0.5 > 0\}$ (dark gray). In this case, $M = -2$. We show the distribution of $m(x)$ at the iteration $1, 2, 8, 9, 10$, and 24 in Figure 2 a through g, respectively. The domain Ω^+ tends to become a strip first and move to the upper left corner. As we iterate, the principle eigenvalue decreases until it reaches a stable value of around 1.856 after 15 iterations in Figure 2g. We have tried several different initial configurations with $M \leq -2$, the domains of Ω^+ always reach one of the four corners.

However, an interesting thing happens when M is increased. In Figure 3, we choose the initial distribution having $M = -1$. The optimal domain of Ω^+ tends to stay as a strip with eigenvalue $\lambda = 0.83$ without going to the corner. It seems that there exists a threshold M^* such that the optimal domain of Ω^+ is a strip if $M > M^*$ while the optimal domain is at the corner if $M < M^*$.

In Table (1), we list the eigenvalues of a quarter of an exact circle at one of the four corners and a strip at one end of the square domain $\Omega = [-1,1] \times [-1,1]$. Since the eigenvalues are computed numerically, we only list the first few digits. It can be seen that a quarter of a circle has a smaller principle eigenvalue when $M < M^* \approx -1.15$. So far we do not know the threshold value theoretically. We will study this in future work.

TABLE 1. Comparison of principles eigenvalues of a quarter of a circle at one of the four corners and a strip at the one end on square domain $\Omega = [-1,1] \times [-1,1]$.

M	quarter-circle at one of the four corners	strip at one end
-1.4	1.216	1.291
-1.2	1.033	1.043
-1.15	0.989	0.988
-1.1	0.945	0.933
-1.0	0.859	0.831

Furthermore, we show the numerical results on the rectangular domain in Figure 4 and Figure 5. The domains of Ω^+ both occupy a quarter of the full domain Ω as the first example in the square domain. However, Figure 4c shows that the optimal configuration stays as a strip at the left while Figure 5i shows that the optimal configuration stays as a strip at the top. Notice that both are at the one end with shorter edge. In Figure 5, Ω^+ evolves to the upper left corner first at the sixth iteration with eigenvalue 0.70 and keeps changing until it reaches the

(a) Initial m   (b) iteration 2   (c) iteration 8

(d) iteration 9   (e) iteration 10   (f) iteration 24
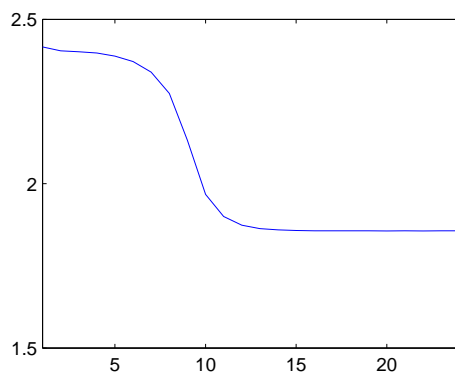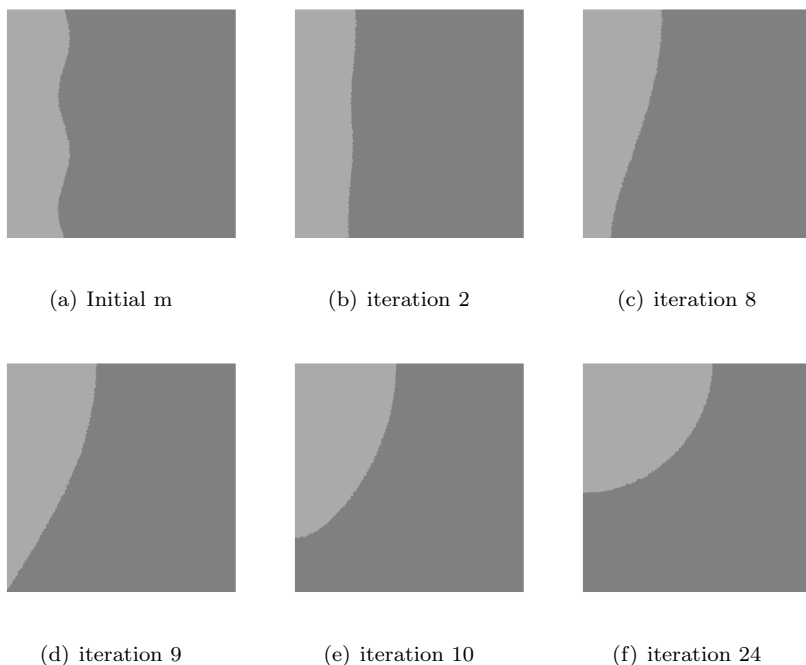
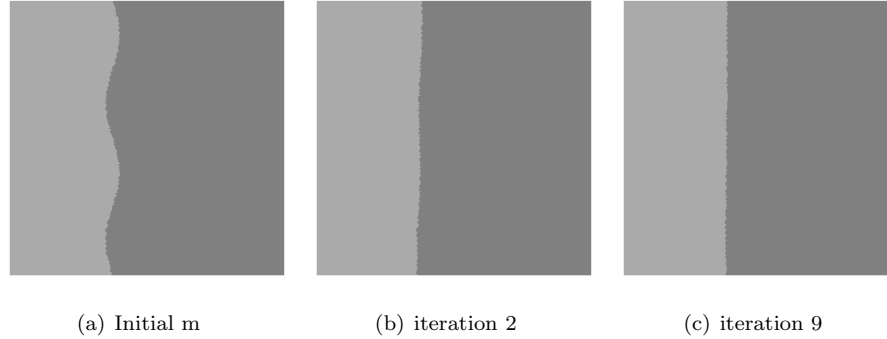(g) The principal eigenvalue v.s. the number
of iterations

FIGURE 2. Domain: $[-1, 1] \times [-1, 1]$; the initial configuration:
$m(x) = 1$ for $\{x | x + 0.05 \sin(2\pi y) + 0.5 < 0\}$ and $m(x) = -1$
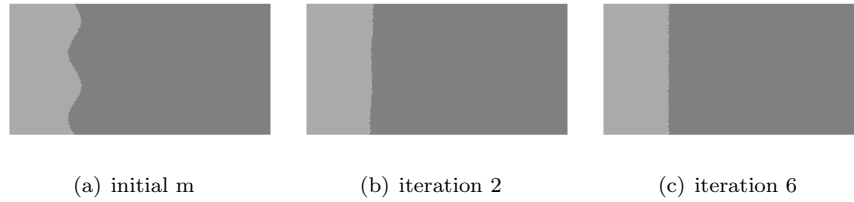otherwise. $M = -2$.(a)-(f) The configurations of $m(x)$ at iteration
$1, 2, 8, 9, 10$, and 24. (g) The corresponding principal eigenvalue at
different iterations.

optimal configuration at the top with eigenvalue 0.60. The corresponding principal
eigenvalue versus the number of iterations is shown in Figure 5j.

    The conclusion we have here is that the optimal configuration of Ω^+ on a rect-
angular domain is either a circular type domain at one of the four corners or a strip
at the one end with shorter edge.

| (a) Initial m | (b) iteration 2 | (c) iteration 9 |

FIGURE 3. Domain: $[-1,1] \times [-1,1]$; the initial configuration: $m(x) = 1$ for $\{x|x + 0.05\sin(2\pi y) + 0.25 < 0\}$ and $m(x) = -1$ otherwise. $M = -1$. (a)-(c) The configurations of $m(x)$ at iteration $1, 2,$ and 9.


| (a) initial m | (b) iteration 2 | (c) iteration 6 |

FIGURE 4. Domain: $[-1,1] \times [-\frac{1}{2}, \frac{1}{2}]$; the initial configuration: $m(x) = 1$ for $\{x|x + 0.05\sin(4\pi y) + 0.5 < 0\}$ and $m(x) = -1$ otherwise. $M = -1$. (a)-(c) The configurations of $m(x)$ at iteration $1, 2,$ and 6.

8. **Discussions.** We are interested in the minimization of the positive principal eigenvalue under the constraint that the weight is bounded by a positive and a negative constant and the total weight is a fixed negative constant. Biologically, this minimization problem is motivated by the question of determining the optimal spatial arrangement of favorable and unfavorable regions for a species to survive. This issue is important for public policy decisions on conservation of species with limited resources.

It was shown in [17] that the global minimizers of $\lambda(m)$ in $\mathcal{M}$ must be of the "bang-bang" type. When $N = 1$ and Ω is an interval, it is further shown in [17] that there are exactly two global minimizers of $\lambda(m)$ (up to change of a set of measure zero). The major open problem is the characterization of the optimal set E in $\mathcal{M}$ for higher-dimensional domains. We show that for cylindrical domains, a strip-type domain is not locally minimal if $\mu < \mu_c$, and numerical simulation suggests that the optimal favorable region is a circular type domain located at one of the four corners of the rectangles. Quite interestingly and in strong contrast, when $\mu > \mu_c$, both our analysis and numerical simulation strongly indicate that the strip located at the one end with shorter edge is locally minimal, at least in a sufficiently wide

(a) initial 1   (b) iteration 2   (c) iteration 3   (d) iteration 4   (e) iteration 6
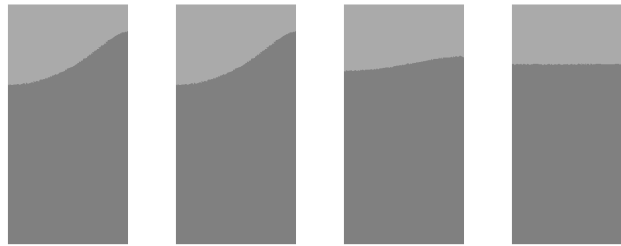
(f) iteration 7   (g) iteration 8   (h) iteration 9   (i) iteration 16
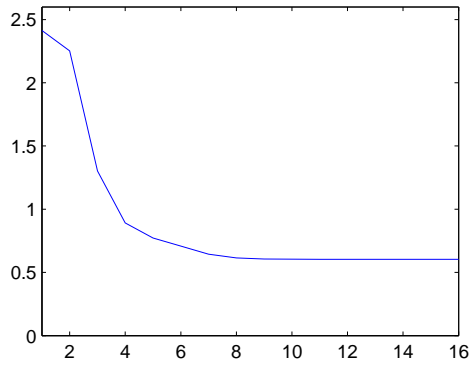
(j) The principal eigenvalue v.s. the number of
iterations

FIGURE 5. Domain: $[-1, 1] \times [-2, 2]$; the initial configuration: $m(x) = 1$ for $\{x|x + 0.05\sin(\pi y) + 0.5 < 0\}$ and $m(x) = -1$ otherwise. $M = -4$. (a)-(i) The configurations of $m(x)$ at iteration $1, 2, 3, 4, 6, 7, 8, 9$, and 16. (j)The corresponding principal eigenvalue at different iterations for numerical simulation.

class of perturbation. In particular, we conjecture that such strip is global minimal for a thin cylinder.

For general domains, more study is needed. At the end of this paper, we show the simulation on the ellipse domain $\Omega = \left\{x^2 + y^2/4 = 1\right\}$ with the initial guess for $\Omega^+ = \left\{(x, y)|(x - 0.3)^2 + (y - 0.1)^2 < (0.2)^2\right\}$ is a disk. In Figure 6a through d,

the domain of Ω^+ moves toward the upper boundary and the principle eigenvalue decreases dramatically. After it reaches the boundary, it moves slowly to the right until it reaches the optimal configuration shown in Figure 6f. The final eigenvalue becomes 17.09. So far we do not know what the optimal configuration is for general domain Ω; for example, does the optimal Ω^+ prefer to stay at the higher curvature regions if the boundary is smooth? We will do further study and report it in a future paper.
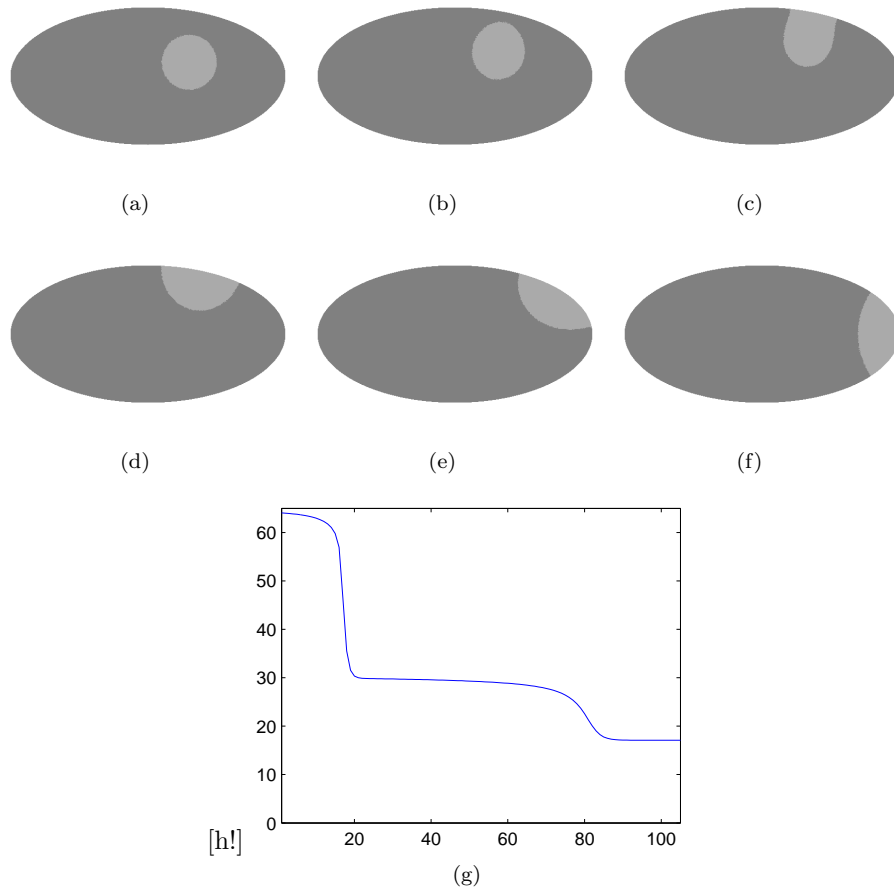

FIGURE 6. Ellipse domain: $x^2 + y^2/4 = 1$. (a)-(f) The configurations of $m(x)$ at iteration $1, 15, 18, 22, 78$, and 105. (g) The principal eigenvalue v.s. the iterations.

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