MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **3**, Number **2**, April **2006**

http://www.mbejournal.org/

pp. **297–312**

THE EFFECTS OF VERTICAL TRANSMISSION ON THE SPREAD OF HIV/AIDS IN THE PRESENCE OF TREATMENT

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(Communicated by Zhilan Feng)

Abstract. In this study, we develop a model that incorporates treatment of both juveniles who were infected with HIV/AIDS through vertical transmission and HIV/AIDS-infected adults. We derive conditions under which the burden of HIV/AIDS can be reduced in the population both in the absence of and in the presence of vertical transmission. We have determined the critical threshold parameter (R_v^*), which represents the demographic replacement of infectives through vertical transmission, below which treated infected juveniles can reach adulthood without causing an epidemic. Five countries in sub-Saharan Africa are used to illustrate our results. We have concluded that R_v^* is dependent on the current prevalence rate but that a significant proportion of infected juveniles receiving treatment can reach adulthood without causing an epidemic.

1. **INTRODUCTION.** Diseases can be transmitted many ways, some of which can be classified as either horizontal or vertical. In the case of HIV/AIDS, horizontal transmission can result from direct physical contact between an infected individual and a susceptible individual. Vertical transmission, on the other hand, can result from direct transfer of a disease from an infected mother to an unborn or newborn offspring. Diseases that can be transmitted vertically include chagas, dengue fever, hepatitis B and HIV/AIDS just to name a few. Vertical transmission of HIV/AIDS can occur during pregnancy, delivery or breastfeeding and is influenced by many factors, including maternal viral load and the type of delivery [16]. According to [3] and [19], about 20% of the children infected with HIV develop AIDS in the first year of their lives, and most of them die by the age of 4 years. The others, up to 80% of infected children, develop symptoms of HIV/AIDS at school entry age (7–9 years) or even during adolescence. In the study conducted in Kenya [7], it was found that at least 89% of children infected with HIV-1, die before their third birthday, and long term non-progressors are rare. It has been noted in [19] that in industrialized countries where antiretroviral therapy is widely available, HIV-infected children can survive into their twenties and produce children of their

2000 *Mathematics Subject Classification.* 37N25.

Key words and phrases. vertical transmission, HIV/AIDS, demographic replacement, prevalence, treatment rate.

own. Awareness, early diagnosis and correct management of the common childhood diseases are key factors for the success of HIV-infected children to live beyond the adolescent stage.

According to the UNAIDS/WHO [20], at the end of 2004 an estimated 39.4 million people globally were living with HIV/AIDS. Of this figure, 17.6 million were children under the age of 15 years. New infections were estimated at 4.94 million, comprising of 4.3 million adults (15–49 years) and 0.64 million children (under 15 years), while deaths were estimated at 2.6 million for adults and 0.51 million for children. Sub-Saharan Africa, the hardest hit region in the world, recorded 25.4 million adults and children living with HIV/AIDS, 3.1 million new infections (adults and children), and 2.3 million deaths, and the infection rate was estimated at 7.4%.

The impact of vertical transmission of HIV/AIDS has been felt mainly in Africa [16], where the level of literacy is low, the poverty level is very high and the quality of health services is generally very poor. A number of studies involving treatment of HIV-infected pregnant mothers (in Abidjan [17], and Thailand [15]) have shown that when HIV-infected mothers are treated with AZT (Zidovudine) and Nevirapine, the number of babies born infected is reduced significantly. Other studies [6, 9, 19] involving treatment of HIV-infected children have demonstrated further that effective treatment for these children can prolong their survival and significantly improve the quality of their lives. The current antiretroviral drugs are known [6] to be effective in lowering viral loads, and the infected children may as a result reach adulthood and become sexually active. Consequently, we may ask this question: what impact will this have on the spread of HIV/AIDS in the long term?

In this study, we extend the model by Busenburg et al. [4], who considered the effects of vertical transmission without treatment. Busenburg et al. [4] assumed that no juveniles born infected with HIV/AIDS lived long enough to reach the adolescent stage. This assumption was justified in 1991, since antiretroviral drugs capable of prolonging lives up to adulthood were unknown or not widely available. Using models which assume constant total population in arriving at or in assessing the effectiveness of public health policies aimed at the control of such epidemics may fail to capture the severity of the epidemic in populations which are undergoing demographic changes. Consequently, we formulate a non-linear system of differential equations that models the dynamics of transmission in a varying population. Models of HIV/AIDS dynamics that ignore the effects of vertical transmission, particularly during the current high usage of antiretroviral drugs, may fail to capture the actual impact of HIV/AIDS in a population.

2. **MODEL FORMULATION.**

We consider a model that describes the dynamics of HIV infection among four (4) sub-populations, namely, the susceptible population (S), the untreated infected population (U), the treated infected population (I) and the AIDS population (A). We assume that the susceptible population is generated from two sources: the normal susceptible population at a constant recruitment rate bS^0 per unit time, where S^0 is the initial susceptible population, and by births from HIV-infected mothers at a rate $be^{-\mu_0\tau}(1-\epsilon)(U+I)$, where b is the natural birth rate and ϵ is the fraction of offsprings born with HIV virus. In the presence of a treatment strategy, the susceptible population may be infected upon contact with either untreated infectives with a probability β_0 or treated infectives with probability β_1.

Let c_0 and c_1 denote the average number of sexual partners per unit time for the untreated and treated infectives, respectively. Then the products $c_0\beta_0$ and $c_1\beta_1$ give the net disease transmission rates for the untreated and treated infectives, respectively. HIV/AIDS treatment may affect transmission of the disease either positively by making infected individuals less infective or negatively by promoting high-risk behavior. It has also been argued that the longer life span of HIV patients under treatment may actually increase transmission [8]. It has been observed in KwaZulu-Natal, South Africa that the prevalence of HIV/AIDS among pregnant mothers increased from 0.76% in the 1980s to 23% in 1998 [18] despite the expansion of intervention programs. Against this background, we shall ignore the positive aspects of treatment and assume that treatment induces risk behavior so that $c_0\beta_0 < c_1\beta_1$. This condition is realistic for sub-Saharan Africa, where the literacy level among women is very low and treatment with antiretroviral drugs may be taken as a cure. We note that because treatment lowers the infectivity of an individual, if it is accompanied by risk-free behavior, then the appropriate assumption would be $c_0\beta_0 > c_1\beta_1$ [10]. We neglect transmission by the AIDS population under the assumption that individuals in this group have significantly reduced their sexual activity or infectivity to the extent that the transmission caused by them is negligible compared to the rest of the infected population [12]. The proportion p of infected offsprings is subjected to treatment, and those that survive the maturation age τ (15 years) progress to the treated infectives class. The probability of surviving the juvenile stage (0–15 years) is given by $e^{-\mu_0\tau}$, where μ_0 is the natural death rate of juveniles. It is assumed that untreated offsprings progress to full-blown AIDS and never live long to become adult infectives. Individuals in the AIDS class are assumed to have a higher removal rate $\delta \geq \mu$, where δ accounts for both natural death in the absence of HIV infection (μ) and disease related deaths. Parameters ν_1 and ν_2 are progression rates to AIDS for untreated and treated infectives, respectively, while the parameter σ denotes the treatment rate. The dynamics of the model are governed by the following system of differential equations:

$$\begin{cases} \dot{S} = bS^0 + be^{-\mu_0\tau}(1-\epsilon)(U+I) - BS - \mu S, \\ \dot{U} = BS - (\sigma + \mu + \nu_1)U, \\ \dot{I} = \sigma U + bp\epsilon e^{-\mu_0\tau}(U+I) - (\mu+\nu_2)I, \\ \dot{A} = b(1-p)\epsilon e^{-\mu_0\tau}(U+I) + \nu_1 U + \nu_2 I - \delta A, \end{cases} \tag{1}$$

where

$$N = S + U + I, \quad p + q = 1, \quad \text{and} \quad B = \frac{c_0\beta_0 U + c_1\beta_1 I}{N}.$$

3. **MODEL ANALYSIS.**

3.1. **Steady-state solutions.** It is more convenient to rearrange system (1) to the following form:

$$\begin{cases} \dot{S} = bS^0 + be^{-\mu_0\tau}(1-\epsilon)(U+I) - BS - \mu S, \\ \dot{U} = BS - (\sigma+\mu+\nu_1)U, \\ \dot{I} = (\sigma + bp\epsilon e^{-\mu_0\tau})U - (\mu+\nu_2)(1-R_v)I, \\ \dot{A} = \alpha_1 U + \alpha_2 I - \delta A, \end{cases} \tag{2}$$

where

$$R_v = \frac{bp\epsilon e^{-\mu_0\tau}}{\mu+\nu_2}, \quad \alpha_1 = \nu_1 + b[1-\epsilon(1-p)]e^{-\mu_0\tau} \text{ and } \alpha_2 = \nu_2 + b[1-\epsilon(1-p)]e^{-\mu_0\tau}.$$

Adding the equations in (2) yields

$$\dot{N} = bS^0 - \mu N - \alpha_1 U - \alpha_2 I. \tag{3}$$

Using the relation $S = N - U - I$ and omitting the equation for the AIDS class, system (2) can be reduced to an equivalent system:

$$\begin{cases} \dot{N} = bS^0 - \mu N - \alpha_1 U - \alpha_2 I, \\ \dot{U} = B(N-U-I) - (\sigma+\mu+\nu_1)U, \\ \dot{I} = (\sigma + bp\epsilon e^{-\mu_0\tau})U - (\mu+\nu_2)(1-R_v)I. \end{cases} \tag{4}$$

System (4) has two equilibrium points, namely, the disease-free equilibrium point (DFE) given by

$$(N^*, U^*, I^*) = \left(\frac{bS^0}{\mu}, 0, 0\right) \tag{5}$$

and the endemic equilibrium point (EEP) given by

$$\begin{aligned} \hat{E} &= (N^*, U^*, I^*) \\ &= \left(\frac{bS^0}{D}, \frac{bS^0\psi(\Re_0-1)}{D(1+k)\Re_0}, \frac{bS^0 k\psi(\Re_0-1)}{D(1+k)\Re_0}\right), \end{aligned} \tag{6}$$

where $\psi = \sigma+\mu+\nu_1$, $D = \mu + \dfrac{(\alpha_1+k\alpha_2)(\Re_0-1)}{(1+k)\Re_0}$ and $k = \dfrac{\sigma + bp\epsilon e^{-\mu_0\tau}}{\mu+\nu_2 - bp\epsilon e^{-\mu_0\tau}}$.
Since the state solutions are positive, it follows then that $k > 0$, which consequently implies that

$$\mu + \nu_2 - bp\epsilon e^{-\mu_0\tau} > 0. \tag{7}$$

The parameter given by

$$\Re_0 = \frac{\mu+\nu_1}{\sigma+\mu+\nu_1}R_0 + \frac{\sigma+(\mu+\nu_2)R_v}{(\sigma+\mu+\nu_1)(1-R_v)}R_{0t} \tag{8}$$

is the effective reproduction number, while R_0 and R_{0t} given by

$$R_0 = \frac{c_0\beta_0}{\mu+\nu_1} \text{ and } R_{0t} = \frac{c_1\beta_1}{\mu+\nu_2}$$

are the basic reproduction numbers in the absence of any intervention, and the basic reproduction number for a population in which all infectives are treated, respectively. The threshold parameter

$$R_v = \frac{bp\epsilon e^{-\mu_0 \tau}}{\mu + \nu_2} \tag{9}$$

is the reproduction number for the demographic replacement of infectives through vertical transmission.

From condition (7), we can see clearly that the threshold parameter R_v is always less than one ($R_v < 1$). The implication of this is that the demographic replacement of infectives through vertical transmission alone cannot lead to an epidemic. The question we ask is this: how does vertical transmission, even though insignificant in causing an epidemic on its own, affect the overall spread of the disease when other factors such as horizontal transmission are taken into account?

From (5) and (6) we can draw the following conclusions summarized in the theorem below regarding the existence of equilibria.

THEOREM 3.1. *The disease-free equilibrium point E_0 exists for all values of $\Re_0$. On the other hand, a unique endemic equilibrium point $\hat{E}$ exists if and only if $\Re_0 > 1$ and $R_v < 1$.*

4. **MODEL REPRODUCTION NUMBERS.** The basic reproduction number $\Re_0$ represents the effect of control strategies, which involve treatment of the juvenile population and treatment of the adult population. The impact of these strategies on controlling the disease can be assessed by monitoring the effects these have in reducing or increasing the basic reproduction number $\Re_0$. In this section we carry out the analysis of $\Re_0$ and investigate the effects of treating the juvenile and adult populations in controlling the disease.

4.1. **Horizontal transmission.** In the absence of vertical transmission (i.e., $\epsilon = 0$), the reproduction number, given by (8) reduces to

$$\Re_0(\sigma) = \frac{\mu + \nu_1}{\mu + \nu_1 + \sigma} R_0 + \frac{\sigma}{\mu + \nu_1 + \sigma} R_{0t}. \tag{10}$$

We note here that

$$\lim_{\sigma \to 0} \Re_0 = R_0 \text{ and } \lim_{\sigma \to \infty} \Re_0 = R_{0t}. \tag{11}$$

Moreover, evaluating the difference between R_0 and $\Re_0$, we obtain

$$\triangle \Re_0 = \frac{\sigma}{\sigma + \mu + \nu_1}(R_0 - R_{0t}) > 0, \text{ if and only if } R_0 > R_{0t}. \tag{12}$$

We note further that

$$\frac{d\Re_0}{d\sigma} = -\frac{\mu + \nu_1}{(\sigma + \mu + \nu_1)^2}(R_0 - R_{0t}). \tag{13}$$

From this result, we can see that $\Re_0(\sigma)$ is a decreasing function of σ if and only if $R_0 > R_{0t}$. Hence the necessary condition for slowing the spread of the disease is that $R_{0t} < \Re_0 < R_0$. Last, we can determine the critical treatment value σ^* required to reduce $\Re_0$ below the threshold of one. It is easy to show that

$$\sigma^* = (\mu + \nu_1)\left(\frac{R_0 - 1}{1 - R_{0t}}\right) \tag{14}$$

is the critical choice of σ for which treatment succeeds in bringing $\Re_0$ below a threshold value of one and that σ^* exists for $R_0 > 1 > R_{0t}$. In other words, even if $R_0 > 1$ initially so long as $R_0 > R_{0t}$ and σ is chosen such that $\sigma > \sigma^*$, the strategy of treating the infected population can reduce the burden of the disease.

4.2. **Vertical transmission.** The effect of vertical transmission on the disease transmission can be found by investigating the dependence of the reproduction number $\Re_0$ on R_v. We note here that the equation (8) can be written as

$$\Re_0(R_v) = \frac{\mu+\nu_1}{\sigma+\mu+\nu_1}R_0 + \frac{\sigma+(\mu+\nu_2)R_v}{(\sigma+\mu+\nu_1)(1-R_v)}R_{0t}. \tag{15}$$

Differentiating $\Re_0(R_v)$, given by (15), with respect to R_v, we obtain

$$\frac{d\Re_0(R_v)}{dR_v} = \frac{\sigma+\mu+\nu_2}{(\sigma+\mu+\nu_1)(1-R_v)^2}R_{0t},$$

which shows that $\Re_0(R_v)$ is an increasing function of the parameter R_v (since $R_v < 1$). It is therefore important to determine R_v^* which must not be exceeded if the disease is to be kept under control. This is given by

$$R_v^* = \frac{\psi(1-\Re_0(\sigma))}{\psi+(c_1\beta_1-c_0\beta_0)}. \tag{16}$$

Since $c_1\beta_1 - c_0\beta_0 > 0$ by assumption, it follows from (16) that there exists R_v^* for $\Re_0(\sigma) < 1$ such that for values of $R_v < R_v^*$ the effects of vertical transmission would be insignificant. The parameter R_v^* is determined numerically in section 6.

5. STABILITY OF EQUILIBRIUM POINTS.

5.1. **Disease-free equilibrium point.** Consider system (4). The Jacobian matrix evaluated at the disease-free equilibrium is given by

$$J(E_0) = \begin{pmatrix} -\mu & b(1-\epsilon)e^{-\mu_0\tau} - c_0\beta_0 & b(1-\epsilon)e^{-\mu_0\tau} - c_1\beta_1 \\ 0 & c_0\beta_0 - \psi & c_1\beta_1 \\ 0 & \sigma + bp\epsilon e^{-\mu_0\tau} & bp\epsilon e^{-\mu_0\tau} - \mu - \nu_2 \end{pmatrix}.$$

The eigenvalues of the Jacobian are $\lambda_1 = -\mu$, and the other two are given by the characteristic equation

$$\lambda^2 + a\lambda + b = 0, \tag{17}$$

where

$$\begin{aligned} a &= (\mu+\nu_2)(1-R_v) + \psi(1-\Re_0) + kc_1\beta_1, \\ b &= \psi(\mu+\nu_2)(1-R_v)(1-\Re_0). \end{aligned}$$

The roots of the characteristic equation are

$$\lambda_{2,3} = \frac{-a \pm \sqrt{a^2-4b}}{2}. \tag{18}$$

We summarize the local stability of the disease-free equilibrium point in the following theorem:

THEOREM 5.1. *Let $R_v < 1$; then the disease-free equilibrium point, E_0, is locally asymptotically stable if $\Re_0 < 1$ and unstable if $\Re_0 > 1$.*

We want to show that subcritical bifurcation does not exist for our model; that is, the disease-free equilibrium point is globally stable.

5.2. **Global stability of the disease-free equilibrium point.**

THEOREM 5.2. *The disease-free equilibrium point is globally asymptotically stable if* $\Re_0 < 1$.

To prove theorem 5.2, we make use of the lemma stated in [5], which we restate here for easy of reference:

LEMMA 5.1. [Thieme (1993)]. *Let* $f : [0, \infty) \to \Re$ *be a bounded and twice differentiable with bounded second derivative. Let* t_n *tend to infinity, and* $f(t_n)$ *converge to* f^∞ *or* f_∞ *for* $n \to \infty$. *Then*

$$f'(t_n) \to 0, \ n \to \infty.$$

Consider system (2). Let $\Re_0 < 1$, and choose a sequence $s_n \to \infty$ such that

$$U(s_n) \ \to \ U^\infty, \ \dot{U}(s_n) \ \to \ 0. \tag{19}$$

Using the equation for $\dot{I}(t)$ in system (2) and applying Lemma 5.1, we obtain

$$(\sigma + bp\epsilon e^{-\mu_0 \tau})U^\infty - (\mu + \nu_2)(1 - R_v)I^\infty \geq 0 \text{ or } I^\infty \leq kU^\infty. \tag{20}$$

Similarly, choosing a sequence $\tau_n \to \infty$ such that

$$I(\tau_n) \ \to \ I^\infty, \ \dot{I}(\tau_n) \ \to \ 0, \tag{21}$$

and using the equation for $\dot{U}(t)$ in system (2) and Lemma 5.1, we obtain

$$[(c_0\beta_0 + kc_1\beta_1) - (\sigma + \mu + \nu_1)]U^\infty \geq 0 \text{ or } (1 - \Re_0)U^\infty \leq 0. \tag{22}$$

This implies that $U^\infty \leq 0$, since $\Re_0 < 1$, but $U^\infty \geq 0$, a contradiction. It follows then that $U^\infty = U_\infty = 0$, and $U(t) \to 0$ as $t \to \infty$. By the result (20), we also have $I(t) \to 0$ as $t \to \infty$.

From the equation governing the changes in the total population,

$$\dot{N} \ = \ bS^0 - \mu N - \alpha_1 U - \alpha_2 I, \tag{23}$$

applying Lemma 5.1 yields

$$bS^0 - \mu N^\infty - \alpha_1 U^\infty - \alpha_2 I^\infty \leq 0, \tag{24}$$

which together with $I^\infty = U^\infty = 0$ give

$$N^\infty \geq \frac{bS^0 - \alpha_1 U^\infty - \alpha_2 I^\infty}{\mu} = \frac{bS^0}{\mu}. \tag{25}$$

Noting also that $\dot{N} < 0$ for $N(t) > \dfrac{bS^0}{\mu}$, we can consider the solutions for model (1) in the feasible region with $N(t) \leq \dfrac{bS^0}{\mu}$, so that $N^\infty \leq \dfrac{bS^0}{\mu}$. It follows that

$$N^\infty = N_\infty = \frac{bS^0}{\mu}.$$

Hence, the disease-free equilibrium point E_0 is globally asymptotically stable if $\Re_0 < 1$. The theoretical implications of Theorem 5.2 are that disease would clear from the population irrespective of the order of the initial sub-population sizes.

5.3. **Endemic equilibrium point.** We now investigate the stability of the endemic equilibrium point given (6). The Jacobian matrix of this system (4) evaluated at $\hat{E}$ is given by

$$J(\hat{E}) = \begin{pmatrix} -\mu & -\alpha_1 & -\alpha_2 \\ \frac{B^*(U^*+I^*)}{N^*} & c_0\beta_0\Lambda - B^* - \psi & c_1\beta_1\Lambda - B^* \\ 0 & \sigma + bp\epsilon e^{-\mu_0\tau} & -(\mu+\nu_2)(1-R_v) \end{pmatrix}$$

$$= \begin{pmatrix} -\mu & -\alpha_1 & -\alpha_2 \\ \frac{\eta(\Re_0-1)^2}{(1+k)\Re_0} & \frac{c_0\beta_0}{\Re_0} - \psi - \frac{\eta(\Re_0-1)}{(1+k)\Re_0} & \frac{c_1\beta_1}{\Re_0} - \frac{\eta(\Re_0-1)}{(1+k)\Re_0} \\ 0 & \sigma + bp\epsilon e^{-\mu_0\tau} & -(\mu+\nu_2)(1-R_v) \end{pmatrix},$$

where $\Lambda = \dfrac{N^* - U^* - I^*}{N^*}$ and $\eta = c_0\beta_0 + kc_1\beta_1$. The characteristic equation arising from $J(\hat{E})$ is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{26}$$

where

$$a_0 = \psi(\mu+\nu_2)(1-R_v)\left[\mu + \tfrac{(\alpha_1+k\alpha_2)(\Re_0-1)}{1+k}\right](\Re_0-1) > 0, \text{ if } R_v < 1 \text{ and } \Re_0 > 1$$

$$a_1 = \psi(\mu+\nu_2)(1-R_v)(\Re_0-1) + \mu\left((\mu+\nu_2)(1-R_v) + \tfrac{\psi(\Re_0-1)}{1+k} + \tfrac{kc_1\beta_1}{\Re_0}\right) + \tfrac{\psi\alpha_1(\Re_0-1)^2}{1+k} > 0, \text{ if } R_v < 1 \text{ and } \Re_0 > 1,$$

$$a_2 = (\mu+\nu_2)(1-R_v) + \tfrac{\psi(\Re_0-1)}{1+k} + \mu + \tfrac{kc_1\beta_1}{\Re_0} > 0, \text{ if } R_v < 1 \text{ and } \Re_0 > 1.$$

Similarly, it can be shown that $a_1a_2 - a_0 > 0$. Hence, according to the Routh-Hurwitz stability criterion, we establish the following theorem:

THEOREM 5.3. *The endemic equilibrium point of system (4) exists, if and only if* $\Re_0 > 1$ *(equation 6) and is locally asymptotically stable if* $\Re_0 > 1$ *and* $R_v < 1$ *by the Routh-Hurwitz stability criterion above.*

6. **NUMERICAL RESULTS.** In this section, we illustrate the theoretical results of model (1) using five countries in sub-Saharan Africa. The five countries, Botswana, Senegal, Swaziland, Uganda and Zambia, were chosen for the following reasons. Swaziland was chosen because the demographic situation is such that birth and death rates are equal ($b = \mu = 0.02$), but the country has the highest HIV prevalence rate in the world. Botswana was chosen because the death rate ($\mu = 0.03$) is currently greater than the birth rate ($b = 0.02$), and the country has the second-highest HIV prevalence rate in the world. Zambia was chosen because, like Botswana, the death rate $\mu = 0.03$ is greater than the birth rate ($b = 0.02$), but the country has the lowest life expectancy rate in the world. Senegal was chosen because, although birth and death rates are equal ($b = \mu = 0.02$), the country has the highest life expectancy in sub-Saharan Africa and the lowest HIV prevalence in

this region. Finally, Uganda was chosen because the birth rate ($b = 0.03$) is greater than the death rate ($\mu = 0.02$) and the country is a success story in sub-Saharan Africa, in that a few years ago it had the highest HIV prevalence rate in the world but has managed to bring it under control, it now has one of the lowest prevalence rates. In this study, we use the demographic data on births, deaths, progression, disease prevalence rates, and HIV-related death rate extracted from [2], [14] and [21].

Prevalence rate is a parameter affected by many factors, including sexual behavior pattern, rapid urbanization leading to break down of traditional mechanisms for control of social behavior [14], poverty and the quality of education on prevention of HIV/AIDS. Public health authorities find it easier to determine the prevalence rate. In reality, the parameters that give insight into sexual behavior of individuals are the probability, β_0, that an infected individual will infect another during their sexual partnership and the reproduction number. The authorities also need to know the rate, σ, at which individuals should start treatment in order to reduce the burden of HIV in the population. These three parameters are very difficult to estimate in practice. In this study, we consider two hypothetical scenarios. The first is a situation in which the population consists of adults only. Using a simple numerical procedure, (i) we determine parameters values for the probability of infection, $\hat{\beta}_0$, at the current prevalence level and the corresponding reproduction number in the absence of any intervention; (ii) we introduce treatment in the absence of vertical transmission and determine the rate, σ^*, at which infected individuals should start treatment in order to reduce the reproduction number below one; and (iii) finally, we introduce vertical transmission and determine the appropriate rate, σ^*, at which the infected population should begin treatment in order to keep the disease under control. The second scenario considers this question: for the values of $\hat{\beta}_0$, and σ^*, for which the reproduction number in the adult population is reduced below one in (a), what is the maximum fraction, R_v^*, of infected juveniles receiving treatment that can join the adult population and still keep the disease under control?

Based on the current prevalence rates, the situation in the five countries considered in our study is summarized in Table 1. From Table 1, we conclude that the treatment programs are not necessary in Senegal and Uganda to reduce the spread of HIV/AIDS, because the situation is under control with $R_0 << 1$. However, for humanitarian reasons treatment should be available to infected individuals in these countries. The low values of R_0 in these two countries may be an indication that cultural practices are vital for controlling risky behavior, religious beliefs with regard to maintaining one partner or the same number of partners (where polygamy is allowed) for life, and HIV/AIDS educational and counselling programs are very effective in these two countries. Despite this conclusion, in the case of Senegal and Uganda we ask the following question: Can vertical transmission of the disease destabilize the situation and lead to an epidemic? In the case of Zambia, Botswana and Swaziland, where the disease is currently at epidemic levels, we ask this question: At what levels of the prevalence rate in each country can we reduce the HIV/AIDS burden with a sustainable number of individuals receiving treatment and an optimal number of juveniles receiving treatment reaching adulthood? The effects of vertical transmission on the spread of HIV/AIDS in Senegal and Uganda are illustrated in Figures 1(a) and 1(b). It is evident from these results that vertical transmission cannot cause an epidemic even if 90% of the infected juveniles receiving treatment reached adulthood.

TABLE 1. Disease indicators by country.

Country	Current Situation			Model Predictions	
				No Vert. Trans.	Vert. Trans.
	CPR	$\hat{\beta}_0$	$\Re_0$	σ^*	σ^*
Senegal	0.0080	$\ll 0.1000$	0.0055	-	-
Uganda	0.0410	$\ll 0.1000$	0.5655	-	-
Zambia	0.1650	0.1250	1.7241	0.1153	0.1154
Swaziland	0.3880	0.8000	11.0345	3.3950	3.4248
Botswana [13]	0.1700	0.5000	6.8966	1.3300	1.3343
Botswana [14]	0.2900	0.3000	4.1385	0.5791	0.5801
Botswana [20]	0.3730	0.7000	9.6552	2.5100	2.5243

CPR denotes current prevalence rates. $\hat{\beta}_0$ is the estimate of β_0 at current prevalence levels. Vert. Trans. denotes vertical transmission.

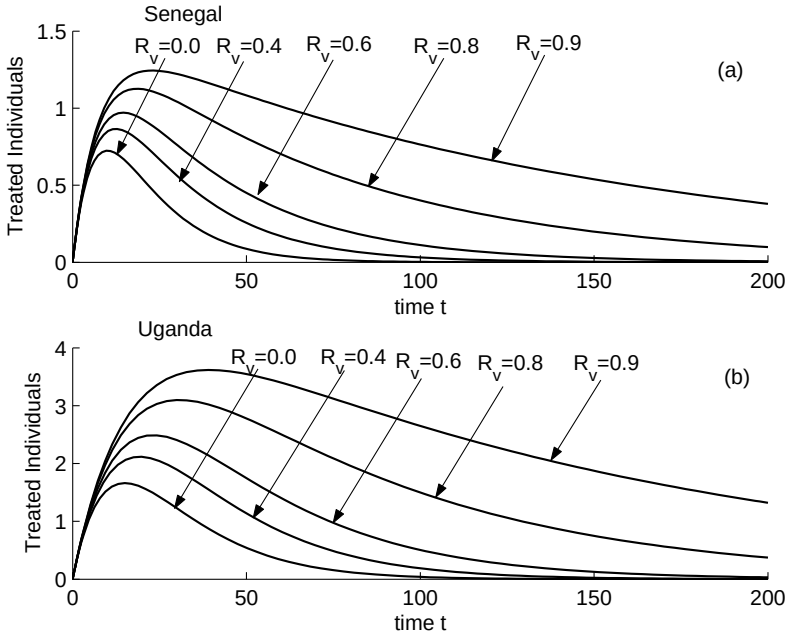

FIGURE 1. Senegal and Uganda HIV/AIDS epidemic in the presence of vertical transmission. Varying the parameter R_v for fixed parameters: $S^0 = 190000,\ b = 0.02,\ \mu_0 = 0.02,\ \mu = 0.02,\ \nu_1 = 0.125,\ \nu_2 = 0.05,\ \beta_0 = 0.008,\ \beta_1 = 0.05\beta_0,\ c_0 = 1,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 0.20$ for Senegal and $S^0 = 710000,\ b = 0.03,\ \mu_0 = 0.02,\ \mu = 0.02,\ \nu_1 = 0.125,\ \nu_2 = 0.05,\ \beta_0 = 0.041,\ \beta_1 = 0.05\beta_0,\ c_0 = 2,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 0.20$ for Uganda.

The rate at which individuals are starting treatment in the southern African countries is $\sigma = 0.17$ per year [20] for all countries. This rate is probably chosen so that countries can reach a certain number of infected individuals receiving

treatment and to ensure that provision of treatment is sustainable. Table 1 shows the calculated values of σ^* for which the reproduction number is equal to one and above which the reproduction number is less than one. Comparing the current rate $\sigma = 0.17$ per year with the calculated values of σ^* in Table 1, it is obvious that a lot remains to be done to help HIV/AIDS sufferers in Botswana and Swaziland. In sub-Saharan Africa, where the disease is currently at epidemic levels in most countries, at what level of the parameter σ can we reduce the model reproduction number below one and maximize the number of treated juveniles reaching adulthood in each country?

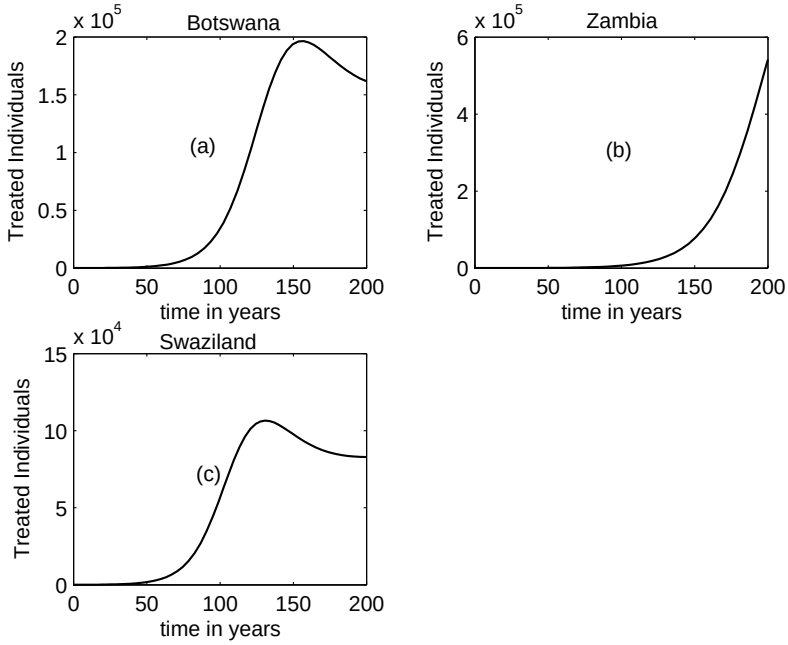

FIGURE 2. Botswana, Zambia and Swaziland HIV/AIDS epidemic in the absence of vertical transmission. Data set for Botswana: $S^0 = 40000,\ b = 0.02,\ \mu_0 = 0.03,\ \mu = 0.03,\ \nu_1 = 0.125,\ \mu_2 = 0.05,\ \beta_0 = 0.7,\ \beta_1 = 0.05\beta_0,\ c_0 = 2,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 0.20$. Swaziland: $S^0 = 22000,\ b = 0.02,\ \mu_0 = 0.03,\ \mu = 0.02,\ \nu_1 = 0.125,\ \mu_2 = 0.05,\ \beta_0 = 0.8,\ \beta_1 = 0.05\beta_0,\ c_0 = 2,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 0.20$. Zambia: $S^0 = 208000,\ b = 0.02,\ \mu_0 = 0.03,\ \mu = 0.03,\ \nu_1 = 0.125,\ \mu_2 = 0.05,\ \beta_0 = 0.125,\ \beta_1 = 0.05\beta_0,\ c_0 = 2,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 0.10$ [2, 20].

The estimates for the prevalence rate for Botswana vary from 29% [14] to 37.3% [20] for the age group 15 to 49 years (sexually active population). The estimate for the prevalence rate for the whole population (age group 18 months to 64 years) is 17% [13]. In the absence of any intervention and vertical transmission at the current prevalence level, the probability of infection for Botswana for the sexually active population varies from $\hat{\beta}_0 = 0.3$ to $\hat{\beta}_0 = 0.7$, corresponding to prevalence rates

29% and 37.3%, respectively. The model reproduction number, on the other hand, varies from $\Re_0 = 4.1385$ (for $\hat{\beta}_0 = 0.3$) to $\Re_0 = 9.6552$ (for $\hat{\beta}_0 = 0.7$). At the current prevalence rate, for the total population, the probability of infection is estimated to be $\hat{\beta}_0 = 0.5$ and the corresponding reproduction number is $\Re_0 = 6.8966$. When treatment of the sexually active adult infected population is applied, we find that the rate σ^* at which the adult population should begin treatment varies from $\sigma^* = 0.5791$ to $\sigma^* = 2.5100$, corresponding to prevalence rates 29% to 37.3%, respectively. On the other hand, at the prevalence rate of 17% for the total population, the rate at which the population should begin treatment is found to be $\sigma^* = 1.3300$. In the presence of vertical transmission at current estimates of prevalence rates for the sexually active population, assuming the proportion of children born with HIV is $\epsilon = 0.3$ [20] and the proportion of infected juveniles reaching adulthood is $p = 0.1$, we find that the rate σ^* at which the adult population should begin treatment varies from $\sigma^* = 0.5801$ to $\sigma^* = 2.5243$. The estimates in the presence of vertical transmission are slightly higher than those in the absence of vertical transmission (Table 1). For $\hat{\beta}_0$ and σ^* (found above for the sexually active group at a prevalence rate of 37.3%) the maximum fraction of infectives through vertical transmission who can reach adulthood without causing an epidemic is estimated to be 0.67 (Figure 4(a)).

In the absence of any intervention and vertical transmission at the current prevalence level, the probability of infection for Zambia is $\hat{\beta}_0 = 0.125$ and $\Re_0 = 1.7241$. When treatment of the adult infected population is applied at the current level of infection, our model suggests that $\sigma > \sigma^* = 0.1153$ leads to a reduction in the reproduction number below one (Table 1), while in the presence of vertical transmission (Figures 2(b) and 3(b)), taking $p = 0.1$ and $\epsilon = 0.3$, the model gives $\sigma^* = 0.1154$ as the rate for which the burden of HIV can be reduced in the population. In the presence of vertical transmission, the value of R_v^* is found to be 0.752 (Figure 4(b)). For Zambia the current rate at which individuals are starting treatment seems to achieve the desired results, as the prevalence rate has come down from 24% in 2000 [20] to 16.5% by the end of 2004. This may also be an indication that the intervention programs which started in the early 1990s may be bearing fruit.

In the absence of any intervention and vertical transmission at the current prevalence level, the probability of infection for Swaziland is found to be $\hat{\beta}_0 = 0.8$ and $\Re_0 = 11.035$. We note that the value of $\Re_0$ for Swaziland is very high. This can be explained by marriage customs which permit polygamy and the fact that a number of men in Swaziland are migrant workers in South Africa and visit their wives only when they are on vacation. The statistics, according to [20], show that the number of HIV/AIDS cases in Swaziland has risen rapidly from 50,000 in 1992 to over 300,000 in 2005. In the presence of a treatment strategy and vertical transmission (see Figures 2(c) and 3(c)), assuming $p = 0.1$ and $\epsilon = 0.3$, $\sigma^* = 3.4248$ leads to a reduction in the burden of HIV. In this case $R_v^* = 0.62$ (Figure 4(c)).

7. **DISCUSSION AND CONCLUSIONS.** In this study, we formulated and analyzed a deterministic HIV model which incorporates treatment of juveniles infected with HIV/AIDS through vertical transmission, and infected adults. We have established the local and global stability of the disease-free equilibrium point. From the stability analysis of this point, we have determined the model reproduction number $\Re_0$, which is found to be a decreasing function of the treatment rate

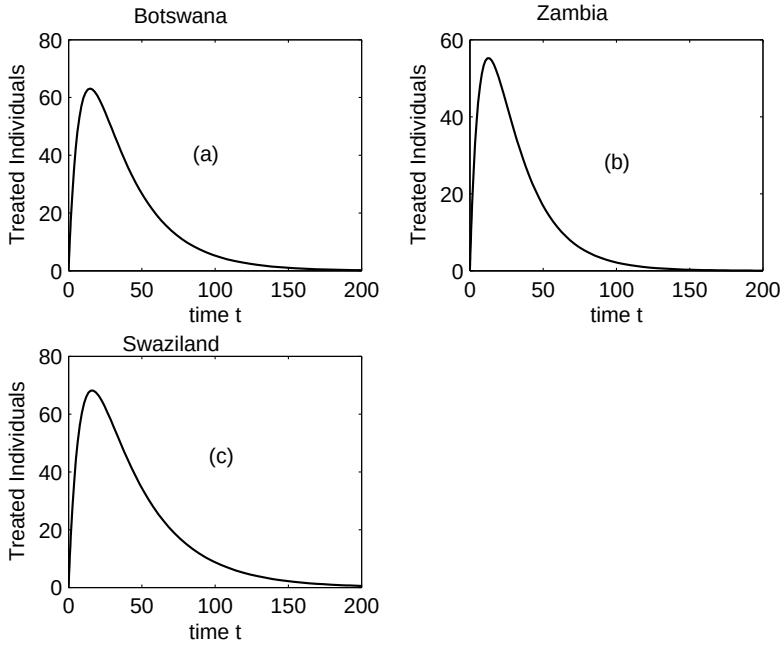

FIGURE 3. Botswana, Zambia and Swaziland HIV/AIDS epidemic in the absence of vertical transmission. Data set for Botswana: $S^0 = 40000,\ b = 0.02,\ \mu_0 = 0.03,\ \mu = 0.03,\ \nu_1 = 0.125,\ \mu_2 = 0.05,\ \beta_0 = 0.7,\ \beta_1 = 0.05\beta_0,\ c_0 = 2,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 2.60$. Swaziland: $S^0 = 22000,\ b = 0.02,\ \mu_0 = 0.03,\ \mu = 0.02,\ \nu_1 = 0.125,\ \mu_2 = 0.05,\ \beta_0 = 0.8,\ \beta_1 = 0.05\beta_0,\ c_0 = 2,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 3.50$. Zambia: $S^0 = 208000,\ b = 0.02,\ \mu_0 = 0.03,\ \mu = 0.03,\ \nu_1 = 0.125,\ \mu_2 = 0.05,\ \beta_0 = 0.125,\ \beta_1 = 0.05\beta_0,\ c_0 = 2,\ c_1 = 1,\ p = 0.1,\ \epsilon = 0.3,\ \delta = 0.01,\ \tau = 15,\ \sigma = 0.20$ [2, 20].

σ if $R_0 > R_{0t}$ but an increasing function of R_v, a parameter which measures the demographic replacement of infectives due to vertical transmission of HIV/AIDS.

Using current prevalence rates for five sub-Saharan countries, we have determined the corresponding basic reproduction numbers R_0 (Table 1) which suggest that well designed intervention programs aimed at reducing current prevalence levels must be embarked upon in Botswana, Swaziland and Zambia, where, according to our model, an infected individual is able to infect on average 9.6552, 11.0345 and 1.7241 individuals, respectively, during the infected individual's period of infectiousness. The values of R_0 for Botswana and Swaziland reported here are higher than those reported before (3–5 for Botswana [22]). However, considering that the numbers of HIV/AIDS individuals in Botswana and Swaziland have risen during the period from 1992 to 2005 from 66,000 and 50,000 to 330,000 and 300,000, respectively [20], the values of R_0 reported ([14] for Botswana) for these two countries seem to be on the low side. In the case of Swaziland where polygamy is practiced, the high values of R_0 found in this study can be explained in terms of marriage

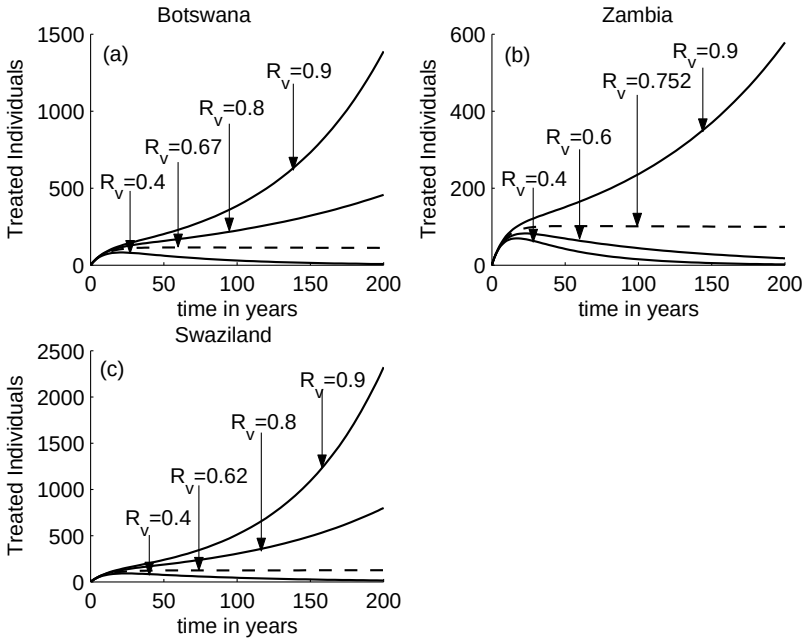

FIGURE 4. The effects of vertical transmission: The case of Botswana, Zambia and Swaziland. Data set for Botswana: $S^0 = 40000$, $b = 0.02$, $\mu_0 = 0.03$, $\mu = 0.03$, $\nu_1 = 0.125$, $\mu_2 = 0.05$, $\beta_0 = 0.7$, $\beta_1 = 0.05\beta_0$, $c_0 = 2$, $c_1 = 1$, $p = 0.1$, $\epsilon = 0.3$, $\delta = 0.01$, $\tau = 15$, $\sigma = 0.20$. Swaziland: $S^0 = 22000$, $b = 0.02$, $\mu_0 = 0.03$, $\mu = 0.02$, $\nu_1 = 0.125$, $\mu_2 = 0.05$, $\beta_0 = 0.8$, $\beta_1 = 0.05\beta_0$, $c_0 = 2$, $c_1 = 1$, $p = 0.1$, $\epsilon = 0.3$, $\delta = 0.01$, $\tau = 15$, $\sigma = 0.60$. Zambia: $S^0 = 208000$, $b = 0.02$, $\mu_0 = 0.03$, $\mu = 0.03$, $\nu_1 = 0.125$, $\mu_2 = 0.05$, $\beta_0 = 0.125$, $\beta_1 = 0.05\beta_0$, $c_0 = 2$, $c_1 = 1$, $p = 0.1$, $\epsilon = 0.3$, $\delta = 0.01$, $\tau = 15$, $\sigma = 0.10$. [2, 20]

patterns. Because of high unemployment rates in most southern African countries, especially Botswana, Lesotho and Swaziland, a number of men migrate to South Africa without their partners. In South Africa they may become clients of sex workers. Upon return home these men have a higher chance of infecting their partners. The partners who remained home may also have developed sexual relationships and may also infect the migrant men upon return home. The value of R_0 for Zambia may be compared with values of the number of sexual partners given as 2.8-3.1 [1].

The model reproduction number $\Re_0$ is difficult to measure in practice (see [11] for a detailed discussion of the reproduction number), because some of the parameters required to determine these numbers, such as progression-to-AIDS rates and the probability of infection, are statistically and medically difficult to measure. For this reason, the statistical parameter used to measure the severity of the disease in practice is the prevalence rate [2].

We have used the current prevalence rates for each of the three southern African countries, where intervention strategies are necessary, to estimate the probability of infection, which in turn is used to determine σ^* for which the prevalence rate

can be reduced to zero. Because the prevalence rate is easier to determine, good estimates of the parameters $\hat{\beta}_0$ and σ^* can provide significant insight into the transmission dynamics of the disease and can lead to effective strategies of controlling and managing the disease. Comparing the treatment rates in the absence of vertical transmission and those in the presence of vertical transmission, it is clear (Table 1) that treatment cost may not increase significantly as a result of treating infected juveniles, provided R_v^* is not exceeded.

In this study, we considered a very simple model which omitted a number of factors known to be relevant to HIV epidemiology, such as heterogeneous sexual behavior and realistic time-evolution of HIV infection. This work was constrained by the lack of reliable data or the unavailability of data. As a result, we resorted to iterative techniques to obtain estimates of certain parameters. However, some of the constraints raised above will be addressed in future work.

ACKNOWLEDGMENTS. We would like to thank the Botswana College of Agriculture and the University of Botswana for their support. We would also like to thank the reviewers for their constructive criticisms and suggestions which improved this paper.

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Received on September 5, 2005. Accepted on January 23, 2006.

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