MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **3**, Number **1**, January **2006**

http://www.mbejournal.org/

pp. **79–87**

WEAKLY COUPLED TRAVELING WAVES FOR A MODEL OF GROWTH AND COMPETITION IN A FLOW REACTOR

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Abstract. For a reaction-diffusion model of microbial flow reactor with two competing populations, we show the coexistence of weakly coupled traveling wave solutions in the sense that one organism undergoes a population growth while another organism remains in a very low population density in the first half interval of the space line; the population densities then exchange the position in the next half interval. This type of traveling wave can occur only if the input nutrient slightly exceeds the maximum carrying capacity for these two populations. This means, lacking an adequate nutrient, two competing organisms will manage to survive in a more economical way.

Dedicated to Professor Zhien Ma on the occasion of his 70th birthday

1. Introduction.

Recently, a model of microbial competition for a nutrient in a tubular reactor was introduced and studied in [2] to understand the circumstance under which a population of microorganisms can survive in a flow reactor and to understand the circumstances under which the coexistence of two populations is possible. This model is a modification of a model formulated earlier in [7]. Unlike most models of microbial growth in a limited nutrient where the nutrient and population are assumed to be homogeneously distributed, the diffusion effect has been accounted for in the model. Consider a reactor occupying the part of a long thin tube from $x = 0$ to $x = L$. Suppose a fresh flow containing an amount S^0 of nutrient enters at the end $x = 0$ with velocity α, and carries unutilized nutrient and organism out of the reactor at $x = L$. Let the nutrient density at time t and location x be denoted by $S(x,t)$, and let $P_i(x,t)$ denote the density of organisms i, for $i = 1,2$. Then the competition model is given by a system of reaction-diffusion equations

$$\begin{aligned} \frac{\partial S}{\partial t} &= \rho \frac{\partial^2 S}{\partial x^2} - \alpha \frac{\partial S}{\partial x} - f_1(S)P_1 - f_2(S)P_2, \\ \frac{\partial P_i}{\partial t} &= d_i \frac{\partial^2 P_i}{\partial x^2} - \alpha \frac{\partial P_i}{\partial x} + \big[f_i(S) - K_i\big]P_i, \end{aligned} \qquad i = 1,2. \tag{1.1}$$

with Danckwerts boundary conditions (see [2] for the detail), where ρ, d_1, and d_2 are diffusion coefficients, f_i, $i = 1,2$, is uptake function for the organism i.

2000 *Mathematics Subject Classification.* 34D20, 34D23, 92D30.

Key words and phrases. Microbial flow reactor model, reaction-diffusion equations, unstable manifold, weakly coupled traveling waves.

Although the main interest of [2] is to investigate effects of random motility on the competition and coexistence of populations in the reactor, our interest in this paper is to study the phenomenon of wave propagation. To best describe this phenomenon (see [8]), let us consider a long flow reactor that we treat it mathematically to be infinitely long. Suppose that the amount S^0 of nutrient is input at a constant velocity α at one end of the flow reactor, says at $x = -\infty$. If there is no bacteria population, then the concentration of nutrient remains constant and is washed out at the other end of reactor. On the other hand, suppose that the uptake function $f_i(S)$ of bacteria cell i is increasing with respect to S and $f_i(S^0) - K_i > 0$, and let a small quantity of bacteria i for $i = 1, 2$ be introduced, then the population P_i increases when growth rate $f_i(S) - K_i > 0$. The growth rate eventually becomes negative because of the reduction of the nutrient so that the bacteria population declines. Hence one may expect that a hump-shaped bacteria population density $P_i(x,t)$, as t increases, moves towards the other end of reactor. That is, we expect that there are constants c, S_0, with $f(S_0) < K$, and a nonnegative traveling wave solution

$$(S(x,t), P_1(x,t), P_2(x,t))) = (U(x+ct), V_1(x+ct), V_2(x+ct))$$

such that

$$\begin{aligned} \lim_{z\to-\infty} \quad & U(z) = S^0, \quad \lim_{z\to\infty} U(z) = S_0, \\ \lim_{z\to-\infty} \quad & V_i(z) = \lim_{z\to\infty} V_i(z) = 0, \quad i = 1, 2, \end{aligned} \tag{1.2}$$

where $z = x + ct$.

A particular case of Eq.(1.1) is the absence of one organism, that is $P_i(x,t) \equiv 0$ for $i = 1$ or $i = 2$. In this circumstance Eq.(1.1) is reduced to a single population model. For this case, the existence and uniqueness of traveling wave solution with boundary condition (1.2) has recently been completely solved (see [1,3,4,5,6,8]). However, the existence of traveling wave solutions for two competing populations has not been studied yet. In this paper we will investigate the coexistence of traveling wave solutions with both $V_1(z)$ and $V_2(z)$ are positive.

The paper is organized as follows. In Section 2 we provide some known result on the existence of traveling wave solutions for a single population. The coexistence of traveling wave solutions will be established in Section 3 by using a perturbation method with the utility of the results in Section 2. A short conclusion will be given in Section 4.

2. **Traveling wave solutions for a single model.** Let us begin with a single population model

$$\begin{aligned} \frac{\partial S}{\partial t} &= \rho\frac{\partial^2 S}{\partial x^2} - \alpha\frac{\partial S}{\partial x} - f(S)P \\ \frac{\partial P}{\partial t} &= d\frac{\partial^2 P}{\partial x^2} - \alpha\frac{\partial P}{\partial x} + \big[f(S) - K\big]P, \end{aligned} \tag{2.1}$$

where ρ, d, K are all positive constants. Upon a substitution of $S(x,t) = U(x+ct)$ and $P(x,t) = V(x+ct)$ we obtain the equations for the traveling wave U and V as

$$\begin{aligned} C\dot{U} &= \rho\ddot{U} - f(U)V \\ C\dot{V} &= d\ddot{U} + [f(U) - K]V \end{aligned} \tag{2.2}$$

with $C = c + \alpha$. For the convenience of discussion we further reverse the time by letting $u(t) = U(-t)$ and $v(t) = V(-t)$ for $t \in \mathbb{R}$. Then the equations for u and v are

$$\begin{aligned} \rho\ddot{u} &= -C\dot{u} + f(u)v \\ d\ddot{v} &= -C\dot{v} - [f(u) - K]v. \end{aligned} \tag{2.3}$$

We state two known results that can be found in the reference.

Lemma 2.1. *([5. Theorem 1.1]) Suppose that f is monotone increasing with $f(0) = 0$ and $f(S_K) = K$ for some positive number S_K. Then, given $u^0 > S_K$ and $C > 0$, there exists $u_0 < S_K$ such that Eq. (2.3) has a nonnegative traveling wave (or heteroclinic) solution $(u(t), v(t))$ satisfying the boundary condition*

$$\lim_{t\to-\infty} u(t) = u_0, \quad \lim_{t\to\infty} u(t) = u^0,$$

$$\lim_{t\to-\infty} v(t) = \lim_{t\to\infty} v(t) = 0,$$

if and only if

$$C^2 \geq 4d[f(u^0) - K]. \tag{2.4}$$

Moreover,
a. *The number u_0 is uniquely determined by u^0;*
b. *$u(t)$ is strictly monotone decreasing;*
c. *$v(t)$ is strictly positive for $t \in \mathbb{R}$ and there is a unique t_0 such that $v(t)$ is increasing in $(-\infty, t_0)$ and decreasing in (t_0, ∞).*

Lemma 2.2. *([5. Theorem 3.3]) Given $C > 0$ and $\tilde{u}^0 > S_K$ with*

$$C^2 \geq 4d[f(\tilde{u}^0) - K],$$

for each $u^0 \in (S_K, \tilde{u}^0]$ let $\xi(u^0)$ be the unique number such that Eq. (2.3) admits a positive traveling wave solution $(u(t), v(t))$ connecting $(\xi(u^0), 0)$ and $(u^0, 0)$. Then
(1) $\xi(u^0)$ is a decreasing function on u^0 for $u^0 \in (S_K, \tilde{u}^0]$.
(2) $\xi((S_K, \tilde{S}^0]) = [\xi(\tilde{u}^0), S_K)$, where $\xi((S_K, \tilde{S}^0])$ denotes the range of ξ.

The following lemma is also needed in the next section to establish the coexistence of traveling wave solutions.

Lemma 2.3. *For the second order differential equation*

$$d\ddot{v} = -C\dot{v} + g(t)v, \tag{2.5}$$

where d, C are positive number and $g(t)$ is continuous and $g(t) > 0$ for all $t \in (-\infty, t_0]$, if either $v(t_0) > 0$ and $\dot{v}(t_0)) = 0$ or $v(t_0) = 0$ and $\dot{v}(t_0) > 0$, then $(v(t), \dot{v}(t))$ can not converge to $(0, 0)$ as $t \to -\infty$.

Proof. We shall prove the lemma only for the case $v(t_0) > 0$ and $\dot{v}(t_0)) = 0$. The proof for another case is analogous. First we have

$$d\ddot{v}(t_0) = g(t_0)v(t_0) > 0.$$

This implies that for a small $\epsilon > 0$, $\dot{v}(t) < 0$ for all $t \in [t_0 - \epsilon, t_0)$. It follows that $v(t)$ is decreasing in $[t_0 - \epsilon, t_0]$ and hence $v(t) > v(t_0) > 0$ for $t \in [t_0 - \epsilon, t_0]$. Observe that

$$d\ddot{v}(t) = -C\dot{v}(t) + g(t)v(t) > 0$$

whenever $\dot{v}(t) < 0$ and $v(t) > 0$, one therefore easily concludes that $\dot{v}(t) < 0$ for all $t \leq 0$. Hence $v(t)$ is increasing as t decreases. Thus $(v(t), \dot{v}(t))$ can not converge to $(0, 0)$ as $t \to -\infty$. □

3. **Coexistence of traveling wave solutions.** Now let us return to the two-population model. Let

$$S(x,t) = U(x+ct), \quad P_i(x,t) = V_i(x+ct), \quad i = 1,2,$$

be a traveling wave solution, then $(U(t), V_1(t), V_2(t))$ satisfies the system

$$\begin{aligned} C\dot{U} &= \rho\ddot{S} - f_1(U)V_1 - f_2(U)V_2 \\ C\dot{P}_i &= d_i\ddot{P}_i - +\big[f_i(U) - K_i\big]V_i, \\ &\quad i = 1,2. \end{aligned} \tag{3.1}$$

As in the single population model, we let $u(t) = U(-t)$, $v_i(t) = V_i(-t)$, $t \in \mathbb{R}$ and $i = 1, 2$. Then

$$\begin{aligned} \rho\ddot{u} &= -C\dot{u} + f_1(u)v_1 + f_2(u)v_2 \\ d_i\ddot{v}_i &= -C\dot{v}_i - [f_i(u) - K_i]v_i, \\ &\quad i = 1,2. \end{aligned} \tag{3.2}$$

Throughout the paper we suppose that, for $i = 1, 2$,

A1 $f_i(u)$ is strictly increasing and is Lipschitz continuous.

A2 $f_i(0) = 0$, and there is a $S_{K_i} > 0$ such that $f_i(S_{K_i}) = K_i$ (S_{K_i} serves as the carrying capacity for the population i.)

In addition, we suppose $S_{K_1} \neq S_{K_2}$. To be specific we suppose

$$S_{K_1} < S_{K_2}. \tag{3.3}$$

Notice that S_{K_i} is the carrying capacity for population i. So to support a coexistence traveling wave the input nutrient u^0 must be larger than S_{K_2}. Choose $u^* > S_{K_2}$. Let $C > 0$ such that

$$C^2 > \max\{4d_i[f_i(u^*) - K_i],\ i = 1, 2\}. \tag{3.4}$$

Then $C^2 > 4d_2[f_i(u^*) - K_2]$. By Theorem 2.2 there is a $S_{K_2} < \bar{u}^0 \leq u^*$ such that for each u^0 with $S_{K_2} < u^0 < \bar{u}^0$, there is a unique number $\xi(u^0)$ with

$$S_{K_1} < \xi(u^0) < S_{K_2}$$

such that Eq.(3.2) has a traveling wave solution $(u_2(t), 0, v_{22}(t))$ joining the equilibrium points $(\xi(u^0), 0, 0)$ and $(u^0, 0, 0)$, where $u_2(t)$ is increasing and $v_{22}(t) > 0$ for all $t \in \mathbb{R}$. Since $S_{K_1} < \xi(u^0) < S_{K_2}$ implies that

$$C^2 > 4d_1[f_1(\xi(u^0) - K_1],$$

again by Theorem 2.2 there is a unique $u_0(u^0) < S_{K_1}$ such that Eq.(3.2) has a traveling wave solution $(u_1(t), v_{11}(t), 0)$ connecting $(u_0(u^0), 0, 0)$ and $(\xi(u^0), 0, 0)$. In what follows we show that, under an appropriate perturbation on the traveling wave $(u_1(t), v_{11}(t), 0)$, we can obtain a positive coexistence traveling wave solution that is close to $(u_1(t), v_{11}(t), 0)$ for sufficiently negative t and close to $(u_2(t), 0, v_{22}(t))$ for sufficiently large t. To this end let us first study the unstable manifold associated with the equilibrium

$$E(u_0) = (u, \dot{u}, v_1, \dot{v}_1, v_2, \dot{v}_2) = (u_0, 0, 0, 0, 0, 0), \quad u_0 = u_0(u^0).$$

Here we consider Eq.(3.2) as a six-dimensional system. A straightforward computation shows that the linearization of (3.2) at the equilibrium $E(u_0)$ is

$$\begin{aligned} \rho \ddot{u} \quad &= -C\dot{u} + f_1(u_0)v_1 + f_2(u_0)v_2 \\ d_i \ddot{v}_i \quad &= -C\dot{v}_i - [f_i(u_0) - K_i]v_i, \quad i = 1, 2 \end{aligned} \tag{3.5}$$

and the corresponding characteristic equation is

$$\begin{aligned} \Delta(\lambda) \quad &= \left[\rho\lambda^2 + C\lambda\right]\left[d_1\lambda^2 + C\lambda + f_1(u_0) - K_1\right]\left[d_2\lambda^2 + C\lambda + f_2(u_0) - K_2\right] \\ &= 0. \end{aligned} \tag{3.6}$$

Since $u_0 = u_0(u^0) < S_{K_1}$ implies that $K_1 > f_1(u_0)$and $K_2 > f_2(u_0)$, form (3.6) it follows that the equilibrium $E(u_0)$ has one zero eigenvalue, three negative eigenvalues, and two positive eigenvalues

$$\lambda_i = \frac{-C + \sqrt{C^2 + d_i[K_i - f_i(u_0)]}}{2d_i}, \quad i = 1, 2.$$

Moreover, following a straightforward computation we obtain the eigenvector h_i associated to the positive eigenvalue λ_i as

$$\begin{aligned} h_1 \quad &= \left(\frac{f_1(u_0)}{\rho\lambda_1^2 + C\lambda_1}, \frac{\lambda_1 f_1(u_0)}{\rho\lambda_1^2 + C\lambda_1}, 1, \lambda_1, 0, 0\right), \\ h_2 \quad &= \left(\frac{f_2(u_0)}{\rho\lambda_2^2 + C\lambda_2}, \frac{\lambda_2 f_2(u_0)}{\rho\lambda_2^2 + C\lambda_2}, 0, 0, 1, \lambda_2\right). \end{aligned} \tag{3.7}$$

By the local unstable manifold theorem, Eq.(3.2) at equilibrium $E(u_0)$ has a two-dimensional local smooth unstable manifold $\mathcal{M} = \mathcal{M}(u_0)$ such that any solution starting in $\mathcal{M}$ stays in $\mathcal{M}$ and converges to $E(u_0)$ as $t \to -\infty$. Moreover, since $0 < u_0 < S_{K_1}$, without loss of generality, we can choose $\mathcal{M}$ small enough such that $(u, \dot{u}, v_1, \dot{v}_1, v_2, \dot{v}_2) \in \mathcal{M}$ implies that $0 < u < S_{K_1}$. (We remark that the center manifold corresponding to the zero eigenvalue is $\{(u_1, 0, 0, 0, 0, 0) : \ u_1 \in \mathbb{R}\}$, which is a line consisting of all equilibrium points. So any solution of Eq.(3.2) can not approach the $E(u_0)$ along the center manifold unless it is the equilibrium $E(u_0)$.)

First we observe that the subspace $X = \{(u, \dot{u}, v, \dot{v}, 0, 0)\} \subset \mathbb{R}^6$ is invariant to Eq. (3.2) and the eigenvector $h_1 \in X$. It follows that $\mathcal{M} \cap X$ is one-dimensional that is tangent to the line $\{E(u_0) + \alpha h_1 : \alpha \in \mathbb{R}\}$ at $E(u_0)$. Hence by unstable manifold theory there is a smooth function $\phi : (-\alpha_1, \alpha_1) : \mathbb{R}^6$ such that

$$\mathcal{M} \cap X = \{\phi(\alpha) : \ \alpha \in (-\alpha_1, \alpha_1)\},$$

where α_1 is a small positive number and

$$\phi(\alpha) = E(u_0) + \alpha h_1 + 0(\alpha^2), \quad \alpha \in (-\alpha_1, \alpha_1).$$

Now $\mathcal{M}$ is tangent to the two-dimensional plane $\mathcal{P}$ spanned by the eigenvectors h_1 and h_2 at $E(u_0)$, where

$$\mathcal{P} = \{\alpha E(u_0) + h_1 + \beta h_2 : \alpha, \beta \in \mathbb{R}\}.$$

Hence the local unstable manifold $\mathcal{M}$ can be parametrized by a smooth function $\Phi : \mathcal{V} \to \mathbb{R}^6$, which is an extension of ϕ to a small neighborhood $\mathcal{V}$ of origin in $\mathbb{R}^2$ such that

$$\mathcal{M} = \{\Phi(\alpha, \beta) : (\alpha, \beta) \in \mathcal{V}\}, \quad \Phi(\alpha, 0) = \phi(\alpha),$$

and

$$\Phi(\alpha, \beta) = E(u_0) + \alpha h_1 + \beta h_2 + O(\alpha^2 + \beta^2), \quad (\alpha, \beta) \in \mathcal{V}. \tag{3.8}$$

Lemma 3.1. *Let $\alpha_0 > 0$ be fixed such that $(\alpha_0, 0) \in \mathcal{V}$. Then for each small $\beta > 0$ with $(\alpha_0, \beta) \in \mathcal{V}$, if $(u(t), v_1(t), v_2(t))$ is a solution of Eq. (3.2) satisfying the initial condition*

$$(u(0), \dot{u}(0), v_1(0), \dot{v}_1(0), v_2(0), \dot{v}_2(0)) = \Phi(\alpha_0, \beta),$$

then $v_1(t)$ and $v_2(t)$ are all positive for all $t \in (-\infty, 0)$.

Proof. First it is clear that for $i = 1, 2$, $(v_i(t), \dot{v}_i(t)) \to (0, 0)$ as $t \to -\infty$ because $\Phi(\alpha_0, \beta) \in \mathcal{M}$. Moreover, it is clear that for small positive $\beta > 0$, $\Phi(\alpha_0, \beta)$ is strictly positive (each component of $\Phi(\alpha_0, \beta)$ is positive) by the expressions of Φ in (3.8), vectors h_1 and h_2 in (3.7). Hence

$$(u(t), \dot{u}(t), v_1(t), \dot{v}_1(t), v_2(t), \dot{v}_2(t)) \notin \mathcal{M} \cap X, \quad t \leq 0,$$

for, $\mathcal{M} \cap X$ is invariant. It follows that $(v_2(t), \dot{v}_2(t)) \neq (0, 0)$ for all $t \leq 0$. We claim that $v_2(t)$ and $\dot{v}_2(t)$ remain positive for all $t < 0$. For if this is false, then there must be a time $t_0 < 0$ such that either $v_2(t_0) > 0$ and $\dot{v}_2(t_0) = 0$ or $v_2(t_0) = 0$ and $\dot{v}_2(t_0) > 0$. Note that $v_2(t)$ satisfies Eq. (2.5) with $d = d_2$ and $g(t) = K_2 - f_2(u(t) > 0$ for all $t \in (-\infty, 0)$ because $u(t) < S_{K_1} < S_{K_2}$. Therefore Lemma 2.3 yields that $(v_2(t), \dot{v}_2(t))$ does not go to $(0, 0)$ as $t \to -\infty$. This leads to a contradiction. Observing that the subspace

$$Y = \{(u, \dot{u}, 0, 0, v_2, \dot{v}_2)\} \subset \mathbb{R}^6$$

is also invariant to Eq. (3.2). A similar argument shows that $v_1(t) > 0$ for $t \leq 0$. □

Lemma 3.2. *Let $u^0 > S_{K_2}$ be fixed. Then for each $\sigma > 0$, there is a $\gamma > 0$ such that if $W(t) = (u(t), \dot{u}(t), v_1(t), \dot{v}_1(t), v_2(t), \dot{v}_2(t))$ is a solution of Eq.(3.2) with*

$$\|W(t_1) - E(u^0)\| < \gamma, \tag{3.9}$$

where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^6$, then $W(t)$ exists for all $t \geq t_1$ and

$$|u^0 - u(t)| < \sigma, \quad t \geq t_1, \qquad \lim_{t\to\infty} v_i(t) = 0, \quad i = 1, 2.$$

For the proof of this lemma, see [4, Lemma 3.2, p.758].

Let $\Phi(\alpha, \beta)$ be defined as above and let $\alpha_0 >$ be a small fixed number. For any small $\beta > 0$ with $(\alpha_0, \beta) \in \mathcal{V}$, we let $(u(t, \beta), v_1(t, \beta), v_2(t, \beta))$ denote a solution of Eq. (3.2) that satisfies the initial condition

$$(u(0, \beta), \dot{u}(0, \beta), v_1(t, \beta), \dot{v}(0, \beta), v_2(t, \beta), \dot{v}_2(0, \beta)) = \Phi(\alpha_0, \beta).$$

Then by Lemma 3.1 $v_1(t, \beta)$ and $v_2(t, \beta)$ are positive for all $t \leq 0$. We let $T_M(\beta) \leq +\infty$ be such that $(-\infty, T_M(\beta))$ is a maximum interval of existence for the solution $(u(t, \beta), v_1(t, u\beta), v_2(t, \beta))$ and define

$$t_M(\beta) = \sup\{t < T_M(\beta) :\ v_i(s, \beta) > 0, s \in (-\infty, t],\ i = 1, 2\}.$$

Let $u(t) = u(t, \beta)$, $v_i(t) = v_i(t, \beta)$. By applying the variation-of-constant formula to the first equation in (3.2) we obtain

$$\dot{u}(t) = \frac{1}{\rho} \int_{-\infty}^{t} e^{-C(t-s)/\rho} [f_1(u(s))v_1(s) + f_2(u(s))v_2(s)] ds.$$

From the above expression one immediately concludes that $u(t)$ is strictly monotone increasing on $(-\infty, t_M(\beta))$, and hence

$$u^+(\beta) = \lim_{t \to t_M(\beta)} u(t, \beta)$$

is well defined.

Lemma 3.3. *If*

$$4d_i[f_i(u^+(\beta) - K_i] < C^2, \quad i = 1, 2,$$

then
1. $t_M(\beta) = +\infty$*;*
2. For $i = 1, 2$*,* $v_i(t, \beta) \to 0$ *as* $t \to \infty$*.*

Hence $(u(t, \beta), v_1(t, \beta), v_2(t, \beta))$ *is a coexistence traveling wave solution connecting* $(u_0, 0, 0)$ *and* $(u^+(\beta), 0, 0))$ *such that* $u(t, \beta)$ *is monotone increasing and* $v_i(t, u_0) > 0$*,* $i = 1, 2$*, for all* $t \in \mathbb{R}$*.*

Proof. Applying [4, Lemma 2.5] to the equation

$$d_i \ddot{v}_i =_C \dot{v}_i - [f_i(u(t, \beta)) - K_i]v_i$$

respectively for $i = 1, 2$ we obtain this lemma. □

Lemma 3.4. *Let*

$$W_2(t) = (u_2(t), \dot{u}_2(t), 0, 0, v_{22}(t), \dot{v}_{22}(t)),$$

where $(u_2(t), 0, v_{22}(t))$ *is a traveling wave solution connecting* $(\xi(u^0), 0, 0)$ *and* $(u^0, 0, 0)$ *as defined at the beginning of this section. Let* t_0 *be any fixed number. Then for any* $\sigma > 0$*, there is a* $\delta > 0$ *such that if* $0 < \beta < \delta$*, then there exists* $t_1 < t_M(\beta)$ *such that* $\|W(t_1, \beta) - W_2(t_0)\| < \sigma$*, where* $W(t, \beta)$ *is the solution of Eq. (3.2) with* $W(0, \beta) = \Phi(\alpha_0, \beta)$*.*

Recall that $u_0 = u_0(u^0)$ and the traveling wave solution $(u_1(t), v_{11}(t), 0)$ connecting the equilibrium points $(u_0(u^0), 0, 0)$ and $(\xi(u^0), 0, 0)$. Let

$$W_1(t) = (u_1(t), \dot{u}_1(t), v_{11}(t), \dot{v}_{11}(t), 0, 0).$$

Then there is a t_0 such that $W_1(t_0) \in \mathcal{M} \cap X$. So

$$W_1(t_0) = \Phi(\alpha_0, 0)$$

for some small positive α_0. Without loss of generality we can suppose $t_0 = 0$, for otherwise we can make a time translation because Eq.(3.2) is an autonomous system. Thus for small $\beta > 0$, $W(0, \beta)$ is a small perturbation of the traveling wave $W_1(t)$. The proof of Lemma 3.4 uses the fact that as long as $\beta > 0$, the component $v_2(t)$ of $W(t)$ is increasing when $u(t) \le S_{K_2}$. Hence the component $u(t)$ must passes $\xi(u^0)$ as t increases. So that $W(t)$ does not belong to the stable manifold of the equilibrium point $E(\xi(u^0)) = (\xi(u^0), 0, 0, 0, 0, 0)$. On the other hand, we can show that when $u(t) > \xi(u^0)$, $W(t, \beta)$ can get as close to the orbit of $W_2(t)$ as we want by letting $\beta \to 0$. We shall omit the detailed proof because it is little too long for this paper. A complete proof will be given in a separate paper.

We are now ready to state and prove the coexistence of the traveling wave solution.

Theorem 3.5. *Let* $C > 0$ *and* $u^* > S_{K_2}$ *such that (3.4) is satisfied. For each* $u^0 \in (S_{K_2}, \bar{u}^0)$*, let* $u_0(u^0)$ *be defined as in the beginning of this section. Then there is a* $\delta > 0$ *such that for each* $\beta \in (0, \delta)$*, the solution* $W(t, \beta)$ *of Eq.(3.2) with* $W(0, \beta) = \phi(\alpha_0, \beta)$ *is a strictly positive coexistence traveling wave joining the equilibrium* $(u_0(u^0), 0, 0, 0, 0, 0)$ *and* $(\tilde{u}^0, 0, 0, 0, 0)$ *for some* $\tilde{u}^0 \in (S_{K_2}, u^*)$*.*

Proof. Note that $u^0 < u^*$, we can pick a small positive number σ such that $u^0+\sigma < u^*$. Let $\gamma > 0$ be the number in Lemma 3.2 corresponding to the number σ. Since $W_2(t) \to E(u^0)$ as $t \to \infty$, there is a sufficiently large t_0 such that

$$\|W_2(t_0) - E(u^0)\| < \frac{\gamma}{2}.$$

By Lemma 3.4, there is a $\delta > 0$ such that for each $\beta \in (0, \delta)$, there is a $t_1 = t_1(\beta) < t_M(\beta)$ such that

$$\|W(t_1, \beta) - W_2(t_0)\| < \frac{\gamma}{2}.$$

This yields that

$$\|W(t_1, \beta) - E(u^0)\| \leq \|W(t_1, \beta) - W_2(t_0)\| + \|W_2(t_0) - E(u^0)\| < \gamma.$$

One therefore deduces by Lemma 3.2 that

$$u(t, \beta) \leq |u(t, \beta) - u^0| + u^0 < \sigma + u^0 < u^*, \quad t \geq t_1(\beta).$$

It follows that

$$u^+(\beta) < u^*.$$

So that

$$C^2 > 4d_i[f_i(u^+(\beta)) - K_i], \quad i = 1, 2.$$

Thus, from Lemma 3.3 it follows that $(u(t, \beta), v_1(t, \beta), v_2(t, \beta))$ is a positive coexistence traveling wave solution of Eq.(3.2) connecting the equilibrium points $(u_0(u^0), 0, 0)$ and $(u^+(\beta), 0, 0)$. □

4. **A short discussion.** The positive coexistence traveling wave

$$w(t, \beta) = (u(t, \beta), v_1(t, \beta), v_2(t, \beta))$$

obtained actually bifurcates from boundary traveling waves $w_1(t) = (u_1(t), v_{11}(t), 0)$ and $w_2(t) = (u_2(t), 0, v_{22}(t))$. In fact, we can further show that there are $T_1 < T_2$ and a small $\epsilon > 0$ such that

$$\|w(t, \beta) - w_1(t)\| \leq \epsilon, \quad t \in (-\infty, T_1],$$

$$\|w(t, \beta) - w_2(t)\| \leq \epsilon, \quad t \in [T_2, +\infty),$$

and $w(t, \beta)$ stays in a small neighborhood of the equilibrium point $(\xi(u^0), 0, 0)$ for $t \in [T_1, T_2]$. Hence we can call $w(t, \beta)$ a weakly coupled positive coexistence traveling wave in the sense that $v_2(t, \beta)$ is small for $t \in (-\infty, T_1]$ while $v_1(t, \beta)$ is small when $t \in [T_2, +\infty)$. This can occur because $\xi(u^0)$ is between the two carrying capacities S_{K_1} and S_{K_2}. This requires that $u^0 > S_{K_2}$ but is close to S_{K_2}. That is, the weakly coupled coexistence traveling wave can occur only if the input nutrient u^0 slightly exceeds the maximum of the two carrying capacities. This means, in the absence of adequate nutrient, the two competing organisms will manage to survive in a more economical way. We will show in a separate paper that this weakly coupled traveling wave no longer exists when the input quantity of nutrient u^0 is large.

5. **Acknowledgment.** This research was supported in part by NSF grant DMS-0204676.

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Received on March 2, 2005. Revised on April 4, 2005.

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