MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **3**, Number **1**, January **2006**

http://www.mbejournal.org/

pp. **67–77**

SOME BIFURCATION METHODS OF FINDING LIMIT CYCLES

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Abstract. In this paper we outline some methods of finding limit cycles for planar autonomous systems with small parameter perturbations. Three ways of studying Hopf bifurcations and the method of Melnikov functions in studying Poincaré bifurcations are introduced briefly. A new method of stability-changing in studying homoclinic bifurcation is described along with some interesting applications to polynomial systems.

In honor of Professor Zhien Ma's 70th birthday

1. Introduction.

In 1901, Hilbert [24] posed 23 mathematical problems, of which the second part of the sixteenth one is to find the maximal number and relative position of limit cycles of the polynomial system of degree n:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y).$$

Numerous researchers have studied the above problem, especially for quadratic and cubic systems. A detailed introduction and related literature can be found in Schlomiuk [38] and Li [33]. In the following section, we only introduce some methods of finding limit cycles which are used to study Hopf, Poincaré and homoclinic bifurcations.

2. Methods to find limit cycles.

As mentioned in Li [33], the fundamental problems in studying limit-cycle bifurcations for planar systems are

- multiple Hopf bifurcations near a center or a focus;
- homoclinic or heteroclinic bifurcations near a separatrix loop consisting of hyperbolic saddles and orbits connecting them;
- Poincaré bifurcations from a period annulus;
- limit-cycle bifurcations from a multiple limit cycle.

In this section we introduce some methods of studying limit-cycle bifurcations related to the first three types above.

2000 *Mathematics Subject Classification.* 34C07.
Key words and phrases. limit cycle, Hilbert's 16th problem, Melnikov function, stability-changing, Poincaré bifurcations, homoclinic bifurcation, Hopf bifurcations.

2.1. **Hopf bifurcation.** Consider the system

$$\begin{aligned} \dot{x} &= ax + by + f_2(x, y), \\ \dot{y} &= -bx + ay + g_2(x, y), \end{aligned} \tag{2.1}$$

where $f_2, g_2 = O(|x, y|^2), b \neq 0$. The Poincaré map of (2.1) near the origin has the form

$$P(r) = r + 2\pi \sum_{j\geq 1} v_j r^j, \tag{2.2}$$

where $v_1 = \frac{1}{2\pi}[e^{\frac{2\pi}{|b|}a} - 1], v_{2m} = O(|v_1, v_3, \ldots, v_{2m-1}|), m \geq 1$.

The origin is said to have order $k(\geq 0)$ if

$$v_j = 0, j = 1, \ldots, 2k, v_{2k+1} \neq 0.$$

In this case the sign of v_{2k+1} determines the stability of the focus at the origin. We remark that the stability and the order of a focus do not depend on the choice of cross sections in defining a Poincaré map. Further, it is easy to prove by using Rolle's theorem.

THEOREM 2.1. *A focus of order k generates at most k limit cycles under C^∞ perturbations, and k limit cycles can appear by suitable perturbations.*

A typical way to find limit cycles in a Hopf bifurcation is to change the stability of the focus. More precisely, if

$$0 < |v_1| \ll |v_3| \ll \ldots \ll |v_{2k+1}|, v_{2j-1}v_{2j+1} < 0, j = 1, 2, \ldots, k,$$

then (2.1) has k limit cycles near the origin.

For quadratic systems, we have the following theorem obtained by Bautin [3].

THEOREM 2.2. *A focus of a quadratic system has at most an order of three. Further, for this system a focus or center can generate at most three limit cycles under perturbations of its coefficients.*

Chen and Wang [4] (by the bifurcation method) and Shi [39] (by using the Poincaré-Bendixson theorem) separately found a quadratic system with four limit cycles. Increasingly, more mathematicians have suggested the following.

Conjecture 2.1. *Quadratic systems have at most four limit cycles.*

By normal form theory, Equation (2.1) has the following formal normal form in polar coordinates:

$$\begin{aligned} \dot{r} &= ar + a_1r^3 + a_2r^5 + \ldots, \\ \dot{\theta} &= -b - b_1r^2 - b_2r^4 - \ldots. \end{aligned} \tag{2.3}$$

By using the normal form method and stability analysis, Han, Lin and Yu [21] obtained sufficient conditions for a cubic system to have 10 limit cycles, and Yu and Han [40, 41] found some cubic systems having 12 limit cycles (all with small amplitude). Earlier, James and Lloyd [26] found a cubic system having 8 limit cycles. It seems that the maximal number of limit cycles for cubic systems is 12.

One can prove that there exist a formal series

$$V(x, y) = x^2 + y^2 + \sum_{i+j\geq 3} c_{ij}x^i y^j$$

and constants $L_2, L_3, \ldots,$ (called Lyapunov constants) such that

$$\frac{dV}{dt}\,|_{(2.1)} = \sum_{k\geq 2} L_k(x^2 + y^2)^k. \tag{2.4}$$

For a relationship of coefficients in (2.2)–(2.4) we have the following.

THEOREM 2.3. *The following three statements are equivalent to each other:*

i. $v_j = 0$ *for* $j \leq 2m, v_{2m+1} \neq 0$.

ii. $a_j = 0$ *for* $j \leq m-1, a_m \neq 0$.

iii. $L_j = 0$ *for* $j \leq m, L_{m+1} \neq 0$.

Moreover, if one of the above conditions holds, then

$$v_{2m+1} = \frac{a_m}{|b|} = \frac{L_{m+1}}{2|b|}.$$

For Liénard Hopf bifurcations and some cubic systems, see Han [17], Gasull and Torregrosa [10, 11], Christopher and Lloyd [6, 7], and Christopher and Lynch [8]. The following result was obtained by Han [15].

THEOREM 2.4. *The Liénard system*

$$\dot{x} = p(y) - \Sigma_{i=1}^n a_i x^i, \dot{y} = -x(1+x)$$

has Hopf cyclicity $[\frac{2n-1}{3}]$ *at the origin, where* p *is a* C^∞ *function satisfying* $p(0) = 0$ *and* $p'(0) > 0$. *Here, cyclicity* k *means that the system has at most* k *limit cycles near the origin and that* k *limit cycles can appear in an arbitrary neighborhood of the origin.*

2.2. **The method of Melnikov functions.** Consider a system of the form

$$\dot{x} = H_y + \varepsilon f(x,y), \qquad \dot{y} = -H_x + \varepsilon g(x,y). \tag{2.5}$$

The function $H(x,y)$ is called the Hamiltonian of (2.5) for $\varepsilon = 0$.

Suppose the equation $H(x,y) = h$ defines a smooth closed curve L_h for $h \in J \subset \mathbb{R}$. The Poincaré map of (2.5) in parameter h has the form

$$P(h,\varepsilon) = h + \varepsilon[M(h) + O(\varepsilon)],$$

where

$$M(h) = \oint_{L_h} g dx - f dy, \tag{2.6}$$

which is called the first-order Melnikov function (it is an Abelian integral).

For system (2.5) we have the so-called weakened Hilbert's sixteenth problem posed by Arnold [1, 2]: for given real polynomials H of degree n and f and g of degree m, find the total number of zeros of the Abelian integral (2.6) (taking into account multiplicity).

The above problem is very closely related to the number of limit cycles of system (2.5) for $\varepsilon \neq 0$ small. In fact, the implicit function theorem implies the following.

THEOREM 2.5. *According to the implicit function theorem, we have the following statements:*

i. *The system (2.5) has* k *limit cycles for* $\varepsilon \neq 0$ *and small if* $M(h)$ *has* k *simple zeros on the interval* J.

ii. *The system (2.5) has at most* k *limit cycles bifurcated from the period annulus associated with the interval* J *for* $\varepsilon \neq 0$ *and small if* $M(h)$ *has at most* k *zeros (taking into account multiplicity) on any compact set of the interval* J.

iii. *The system (2.5) has at least* k *limit cycles for* $\varepsilon \neq 0$ *and small if* $M(h)$ *has* k *zeros each with odd multiplicity on the interval* J.

For quadratic systems (i.e, $n = 3, m = 2$), it has been proved for different cases that $M(h)$ has at most two zeros if it is not zero identically. For details, we refer to the recent paper Chow, Li and Yi's recent paper [5], which dealt with the last case.

The Melnikov function $M(h)$ can also be used to study Hopf bifurcation for system (2.5). For the purpose, we suppose that the origin is an elementary singular point with index +1 and that L_h approaches the origin as h goes to zero. Also, suppose that the functions f and g in (2.5) depend on a vector parameter a in $\mathbb{R}^m$ so that $M(h)$=$M(h, a)$ also depends on a. Then we have the following (see Han [18]).

Theorem 2.6. *Under the above conditions, we have the following:*

i. *The Melnikov function $M(h, a)$ is of class C^∞ (resp., C^ω) in h at $h = 0$ if the functions H, f and g are of class C^∞ (resp., C^ω) in (x, y).*

ii. *If there exists a compact set D_0 in R^m and a function $B_k(a) \neq 0$ for a in D_0 such that $M(h, a) = B_k(a)h^{k+1} + O(h^{k+2})$ for $|h|$ small, then there exist $\epsilon_0 > 0$ and an open set $U(D_0)$ containing D_0 and a neighborhood V of the origin such that (2.5) has at most k limit cycles in V for $0 < \epsilon < \epsilon_0$ and $a \in U(D_0)$.*

iii. *Suppose that the functions f and g are linear in a and that*

$$M(h, a) = b_0(a)h + b_1(a)h^2 + \ldots + b_k(a)h^{k+1} + O(h^{k+2})$$

for $0 < h \ll 1$. If

$$rank \frac{\partial(b_0, \ldots, b_k)}{\partial(a_1, \ldots, a_m)} = k + 1,$$

and there exist functions $\phi_j(\epsilon) = O(\epsilon)$, $j = 0, \ldots, k$ such that (2.5) has a center at the origin for b_j=$\phi_j(\epsilon)$, $j = 0, \ldots, k$, then equation (2.5) has at most k limit cycles near the origin for all $a \in \mathbb{R}^m$ and ϵ suffciently small, and k limit cycles can appear for some (ϵ, a). In other words, (2.5) has Hopf cyclicity k at the origin.

In many cases the function M has the form

$$M(h) = I(h)[\lambda - P(h)], \tag{2.7}$$

where $I(h) \neq 0$ on J and λ is a real parameter. The function P is called a detection function corresponding to the periodic family L_h. The graph of $\lambda = P(h)$ in the plane (h, λ) is called a detection curve.

On the basis of the Poincaré-Pontrjagin-Andronov theorem on the global center bifurcation and Melnikov method (see Melnikov [36]), Li et al. [28, 32] obtained the following result for the bifurcation limit cycles.

Theorem 2.7. *Suppose that (2.7) holds on the interval J. For a given $\lambda = \lambda_0$ considering the set S of the intersection points of the straight line $\lambda = \lambda_0$ and the curve $\lambda = P(h)$ in the (h, λ)-plane with $h \in J$, we have that*

i. *if S consists of exactly one point (h_0, λ_0) and $P'(h_0) \neq 0$ then there exists a hyperbolic limit cycle of (2.5) near L_{h_0};*

ii. *if S consists of two points (h_{01}, λ_0) and (h_{02}, λ_0) having $h_{02} > h_{01}$ and $P'(h_{01})$ $P'(h_{02}) < 0$, then there exist two limit cycles near $L_{h_{01}}$ and $L_{h_{02}}$, respectively;*

iii. *if S contains a point (h_0, λ_0) and $P'(h_0)$= $P''(h_0)$ $= \cdots =$ $P^{(k-1)}(h_0)$= 0, but $P^{(k)}(h_0) \neq 0$, then (2.5) has at most k limit cycles near L_{h_0};*

iv. *if S is empty, then (2.5) has no limit cycle.*

Remark 1. *When we use the above Theorem 2.7 to study the number of limit cycles we can consider the values of P and the signs of P' at the endpoints of the interval*

J. Also, to get more limit cycles, we can take advantage of symmetry of (2.5) and consider the function M for different families L_h defined on different intervals J as well.

By using Theorem 2.7, Li and Huang [29] and Li and Liu [30, 31] gave different cubic systems having 11 limit cycles with the same distributions of limit cycles. Then Li [33] found a system of degree 5 having 24 limit cycles.

Christopher and Lloyd [6] introduced a method of quadruple transformation and studied the number of limit cycles for some polynomial systems of particular degrees by perturbing some families of closed orbits of a Hamiltonian system sequence in small neighborhoods of some center points. The method is interesting and was developed further in Li [33], where the results of Christopher and Lloyd [6] were improved.

Many results have also been also obtained for the cases of general perturbations of some special Hamiltonian systems. Higher-order Melnikov functions should be considered in general to find the maximal number of limit cycles that bifurcate from the periodic orbits of a period annulus in degenerate cases (see Li [33], Schlomiuk [38] and Ilyashenko Yu [25]).

2.3. **A new method to find limit cycles: Stability changing of a homoclinic loop.** Consider a polynomial system of the form

$$\dot{x} = \lambda_1 x + f(x, y), \quad \dot{y} = \lambda_2 y + g(x, y), \tag{2.8}$$

where $\lambda_1 > 0$, $\lambda_2 < 0$, f, g=$O(|x, y|^2)$. Equation (2.8) has a hyperbolic saddle at the origin. Let $\alpha_0 = \lambda_1 + \lambda_2$. The saddle is called rough (fine) if $\alpha_0 \neq 0$ ($\alpha_0 = 0$).

One can prove that if $\alpha_0 = 0$, then a formal transformation of the form

$$u = x + \textstyle\sum_{i+j\geq 2} a_{ij} x^i y^j,$$
$$v = y + \textstyle\sum_{i+j\geq 2} b_{ij} x^i y^j$$

exists which carries equation (2.8) into the normal form

$$\dot{u} = \lambda_1 u[1 + \textstyle\sum_{m\geq 1} a_m (uv)^m],$$
$$\dot{v} = -\lambda_1 v[1 + \textstyle\sum_{m\geq 1} b_m (uv)^m].$$

Set $\alpha_m = a_m - b_m$, which is called the mth order saddle value of the origin.

Now suppose equation (2.8) has a homoclinic loop L. The Poincaré map $P(x)$ of equation (2.8) near L has the following form:

(a) $\alpha_0 \neq 0$: $P(x) = cx^r(1 + o(1)))$, $c > 0$, $r = \frac{-\lambda_2}{\lambda_1}$;

(b) $\alpha_0 = 0$: $P(x) = x + \beta_k x^k + \alpha_k x^{k+1} \ln x+$ high-order terms, $k \geq 1$,

where β_k is called the kth-order separatrix value.

Set

$$c_{2k-1} = \alpha_k, \ \ k \geq 0,$$
$$c_{2k} = \beta_k, \ \ k \geq 1.$$

The sequence $c_1, c_2, c_3, c_4, \ldots$ is called a Dulac sequence and c_k is called the kth-order homoclinic constant. The homoclinic loop L has order k if

$$c_j = 0, j = 0, \ldots, k - 1, c_k \neq 0.$$

A general theorem on homoclinic bifurcation is as follows (Roussarie [37]).

THEOREM 2.8 (Leontovich-Roussarie). *A homoclinic loop of order k generates at most k limit cycles under perturbations. Moreover, k limit cycles can appear by suitable perturbations.*

Consider equation (2.5). Suppose for $\varepsilon = 0$ (2.5) has a homoclinic loop L_0 given by $H(x, y) = 0$. If the periodic orbits L_h near L_0 are given by $H(x, y) = h$, $0 < |h| \ll 1$, then the first-order Melnikov function $M(h)$ given by (2.6) has the following asymptotic expansion:

$$M(h) = m_0 + m_1 h \ln |h| + m_2 h + m_3 h^2 \ln |h| + m_4 h^2 + \dots . \tag{2.9}$$

THEOREM 2.9 (Roussarie [37]). *If the coefficients in (2.9) satisfy*

$$m_j = 0, j = 0, \dots, k-1, m_k \neq 0, k \geq 0,$$

then for $\varepsilon \neq 0$ small, equation (2.5) has at most k limit cycles in a neighborhood of L_0.

If the equation $H(x, y) = 0$ gives a heteroclinic loop, then the expansion (2.9) remains valid [27]. However, how Jiang and Han determine the maximal number of limit cycles near the loop is not known. In this aspect, Han and Zhang [17] gave a generic condition for a 2-polycycle to generate at most two limit cycles. Recently, Han, Wu and Bi [22] gave a condition for an n-polycycle to generate at least n limit cycles

Using Theorem 2.9 to study homoclinic bifurcations makes it difficult to compute the coefficients in the expansion (2.9), which we will discuss later.

We next study the stability of an isolated homoclinic loop. Suppose, as before, that L is a homoclinic loop of (2.8) passing through the origin. Let $c_1 = \lambda_1 + \lambda_2$. Then it is well known that L is stable (resp., unstable) if $c_1 < 0$ (resp., $c_1 > 0$). Ma and Wang [35] proved that if $c_1 = 0$, then Ma and Wang [35] proved that the integral $c_2 = \oint_L (f_x + g_y) dt$ is convergent, and then Feng and Qian [9] verified that L is stable (resp., unstable) if $c_2 < 0$ (resp., $c_2 > 0$). When $c_1 = c_2 = 0$, Joyal and Rousseau [34] gave a computing formula for the first saddle value c_3 of Eq.(2.8):

$$c_3 = \frac{1}{2\lambda_1}[f_{xxy} + g_{xyy} - (f_{xx} f_{xy} - g_{xy} g_{yy})/\lambda_1]|_{x=y=0}. \tag{2.10}$$

If, replacing equation (2.8), we have a system of the form

$$\dot{x} = \lambda y + f(x, y), \quad \dot{y} = \lambda x + g(x, y),$$

then, instead of (2.10), the first saddle value c_3 at the origin has the following computation formula:

$$c_3 = \frac{1}{2\lambda}[f_{xxx} - f_{xyy} + g_{xxy} - g_{yyy} + (f_{xy}(f_{yy} - f_{xx}) + g_{xy}(g_{yy} - g_{xx}) - f_{xx} g_{xx} + f_{yy} g_{yy})/\lambda], \tag{2.11}$$

where the right-hand side function is evaluated at the origin.

Han, Hu and Liu [20] found that if $c_1 = c_2 = 0, c_3 \neq 0$, then the stability of L depends on the sign of c_3, the orientation of L and the side on which the Poincaré map is well-defined.

The formula for c_4, which Han and Zhu recently obtained, is very complicated.

Han, Hu and Liu [20] also obtained similar conclusions on the stability of a double homoclinic loop.

We now describe how to find limit cycles near a homoclinic loop by the method of stability changing. For this purpose, consider a system of the form with parameters

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x, y, a), \\ \dot{y} &= -H_x + \varepsilon g(x, y, a), \end{aligned} \tag{2.12}$$

where ε is small, $a \in \mathbb{R}^n, n \geq 1$.

First, let us suppose that for $\varepsilon = 0$, (2.12) has a homoclinic loop L_0 passing through a hyperbolic saddle S_0. For $\varepsilon \neq 0$ and small, there exist a saddle point S_ε near S_0 and separatrices L_ε^u and L_ε^s near L_0.

The directed distance between L_ε^s and L_ε^u on a section l is given by

$$d(\varepsilon, a) = \varepsilon N[m_0(a) + O(\varepsilon)],$$

where N is a positive constant and

$$m_0(a) = \oint_{L_0} g dx - f dy.$$

Equation (2.12) has a homoclinic loop L_ε near L_0 for $\varepsilon \neq 0$ and small if and only if $d(\varepsilon, a) = 0$. When $d(\varepsilon, a) \neq 0$, its sign determines the relative position of L_ε^s and L_ε^u. For (2.12), we introduce the following functions:

$$\begin{array}{l} d_0(L_0, a) = m_0(a), \\ d_1(L_0, a) = (f_x + g_y)(S_0), \\ d_2(L_0, a) = \oint_{L_0} [f_x + g_y - d_1(L_0, a)] dt, \\ d_3(L_0, a) = \frac{\partial c_3(\varepsilon, a)}{\partial \varepsilon}|_{\varepsilon=0}, \end{array} \tag{2.13}$$

where $c_3(\varepsilon, a)$ is the first saddle value of equation (2.12) at the saddle S_ε which can be obtained by using formula (2.10) or (2.11). By Han [12] we have the following theorem.

THEOREM 2.10.

i. *If there exists $a_0 \in \mathbb{R}^n$ with $n \geq 2$ such that*

$$d_0(L_0, a_0) = d_1(L_0, a_0) = 0, d_2(L_0, a_0) \neq 0, \det \frac{\partial(d_0, d_1)}{\partial(a_1, a_2)}(a_0) \neq 0,$$

then for any $\varepsilon_0 > 0$ and neighborhood U of a_0 there exists an open subset $V_\varepsilon \in U$ for $0 < |\varepsilon| < \varepsilon_0$ such that equation (2.12) has 2 limit cycles near L_0 for $a \in V_\varepsilon$.

ii. *If there exists $a_0 \in \mathbb{R}^n$ with $n \geq 3$ such that*

$$d_j(L_0, a_0), j = 0, 1, 2, d_3(L_0, a_0) \neq 0, \det \frac{\partial(d_0, d_1, d_2)}{\partial(a_1, a_2, a_3)}(a_0) \neq 0,$$

then for any $\varepsilon_0 > 0$ and neighborhood U of a_0 there exists an open subset V_ε of U for $0 < |\varepsilon| < \varepsilon_0$ such that equation (2.12) has 3 limit cycles near L_0 for $a \in V_\varepsilon$.

We briefly outline the proof of the first conclusion. For definiteness, suppose L_0 is oriented clockwise and the Poincaré map is well-defined inside it. By the assumption, we can suppose $d'_{00} = \frac{\partial d_0}{\partial a_1}(a_0) \neq 0$. The implicit function theorem implies that a unique function $a_1 = \phi_1(\varepsilon, a_2, \ldots, a_n) = \phi_{10}(a_2, \ldots, a_n) + O(\varepsilon)$ exists such that for $\varepsilon > 0$ and $|a - a_0|$ small $d(\varepsilon, a) \geq 0 \Leftrightarrow d'_{00}[a_1 - \phi_1] \geq 0$. Hence, a homoclinic loop L_ε appears near L_0 if $a_1 = \phi_1$. Let $a_1 = \phi_1$ and define

$$c_1(\varepsilon, a_2, \ldots, a_n) = \varepsilon(f_x + g_y)(S_\varepsilon) = \varepsilon[c_{10}(a_2, \ldots, a_n) + O(\varepsilon)],$$

where $c_{10}(a_2, \ldots, a_n) = (f_x + g_y)(S_0)|_{a_1 = \phi_{10}}$. Let $a_0 = (a_{10}, \ldots, a_{n0})$. Then our assumption implies that

$$c_{10}(a_{20}, \ldots, a_{n0}) = 0, d'_{10} = \frac{\partial c_{10}}{\partial a_2}(a_{20}, \ldots, a_{n0}) \neq 0.$$

Hence, a unique function $a_2 = \phi_2(\varepsilon, a_3, \cdots, a_n) = \phi_{20}(a_3, \ldots, a_n) + O(\varepsilon)$ exists such that

$$c_1(\varepsilon, a_2, \ldots, a_n) \geq 0 \Leftrightarrow d'_{10}[a_2 - \phi_2] \geq 0.$$

Let $a_1 = \phi_1, a_2 = \phi_2$ and define

$$c_2(\varepsilon, a_3, \ldots, a_n) = \varepsilon \oint_{L_\varepsilon} (f_x + g_y)dt = \varepsilon[c_{20}(a_3, \ldots, a_n) + O(\varepsilon)],$$

where

$$c_{20}(a_3, \ldots, a_n) = \oint_{L_0} (f_x + g_y)|_{a_1=\phi_{10}, a_2=\phi_{20}} dt.$$

It is easy to see that $c_{20}(a_{30}, \ldots, a_{n0}) = d_2(L_0, a_0) \neq 0$, say, $c_{20}(a_{30}, \ldots, a_{n0}) > 0$. Then for $\varepsilon > 0$, $a_1 = \phi_1, a_2 = \phi_2$ and $|\varepsilon| + |a - a_0|$ small L_ε is unstable. Fix $\varepsilon > 0$ and a_j near a_{j0}, $j = 3, \ldots, n$ and change a_1 and a_2 such that

$$a_1 = \phi_1, 0 < |a_2 - \phi_2| \ll 1, c_1(\varepsilon, a_2, \ldots, a_n) < 0.$$

Then L_ε has changed its stability from unstable into stable and therefore an unstable limit cycle has appeared near it at the same time. Next, noting that we have assumed that L_0 is oriented clockwise and the Poincaré map is well defined inside it, we then change a_1 such that $0 < |a_1 - \phi_1| \ll |a_2 - \phi_2|, d(\varepsilon, a) < 0$. Clearly, L_ε has broken, and a stable limit cycle has appeared. Thus, two limit cycles can appear near L_0. In the same way, three limit cycles can be obtained under the conditions of the second conclusion in Theorem 2.10.

For a relationship between the coefficients of the expansion (2.9) of the function $M(h, a)$ and the functions d_j in (2.13), from Han and Ye [16] and Han, Hu and Liu [20] we have

THEOREM 2.11. *Assume that L_0 is oriented clockwise and the Poincaré map is well defined inside it. Then*

$$\begin{aligned} m_1(a) &= -\tfrac{1}{\lambda_1(0,a)} d_1(L_0, a) + O(|d_0|), \\ m_2(a) &= d_2(L_0, a) + O(|d_0| + |d_1|), \\ m_3(a) &= N d_3(L_0, a) + O(|d_0| + |d_1| + |d_2|), \end{aligned}$$

where $N < 0$ is a constant.

Now we consider the more interesting case that (2.12) has a double homoclinic loop $L = L_0 \bigcup L_1$ for $\varepsilon = 0$, which are both homoclinic to a hyperbolic saddle S_0. Applying the formulas in (2.13), we can obtain functions as follows:

$$d_{ji}(a) = d_j(L_i, a), i = 0, 1, j = 0, 1, 2, 3,$$

where $d_{10} = d_{11}$, $d_{30} = d_{31}$.

Then following Han and Chen [19] and Han, Hu and Liu [20], we can prove the following.

THEOREM 2.12. *Suppose that the functions in the right-hand side of (2.12) are odd in (x, y) so that the vector field defined by (2.12) is centrally symmetric. Then accordingly*

i. *If there exists $a_0 \in \mathbb{R}^n$ with $n \geq 2$ such that*

$$d_{00}(a_0) = d_{10}(a_0) = 0, d_{20}(a_0) \neq 0, \det \frac{\partial(d_{00}, d_{10})}{\partial(a_1, a_2)}(a_0) \neq 0,$$

then for any $\varepsilon_0 > 0$ and neighborhood U of a_0 there exists an open subset $V_\varepsilon \in U$ for $0 < |\varepsilon| < \varepsilon_0$ such that equation (2.12) has 5 limit cycles near L for $a \in V_\varepsilon$;

ii. *If there exists $a_0 \in \mathbb{R}^n$ with $n \geq 3$ such that*

$$d_{j0}(a_0), j = 0, 1, 2, d_{30}(a_0) \neq 0, \det \frac{\partial(d_{00}, d_{10}, d_{20})}{\partial(a_1, a_2, a_3)}(a_0) \neq 0,$$

then for any $\varepsilon_0 > 0$ *and neighborhood* U *of* a_0 *there exists an open subset* $V_\varepsilon \in U$ *for* $0 < |\varepsilon| < \varepsilon_0$ *such that equation (2.12) has 7 limit cycles near* L *for* $a \in V_\varepsilon$.

For the nonsymmetric case we have the following theorem.

THEOREM 2.13.
i. *If there exists* $a_0 \in \mathbb{R}^n$ *with* $n \geq 3$ *such that*

$$d_{ji}(a_0) = 0, j, i = 0, 1, d_{20}(a_0)d_{21}(a_0) > 0, \det \frac{\partial(d_{00}, d_{01}, d_{10})}{\partial(a_1, a_2, a_3)}(a_0) \neq 0,$$

then for any $\varepsilon_0 > 0$ *and neighborhood* U *of* a_0 *there exists an open subset* $V_\varepsilon \in U$ *for* $0 < |\varepsilon| < \varepsilon_0$ *such that equation (2.12) has 5 limit cycles near* L *for* $a \in V_\varepsilon$.
ii. *If there exists* $a_0 \in \mathbb{R}^n$ *with* $n \geq 5$ *such that*

$$d_{ji}(a_0), j = 0, 1, 2, i = 0, 1, d_{30}(a_0) \neq 0, \det \frac{\partial(d_{00}, d_{01}, d_{10}, d_{20}, d_{21})}{\partial(a_1, a_2, a_3, a_4, a_5)}(a_0) \neq 0,$$

then for any $\varepsilon_0 > 0$ *and neighborhood* U *of* a_0 *there exists an open subset* $V_\varepsilon \in U$ *for* $0 < |\varepsilon| < \varepsilon_0$ *such that equation (2.12) has 7 limit cycles near* L *for* $a \in V_\varepsilon$.

Recently, the authors and their colleagues have used the methods stated in the above two theorems to study the number of limit cycles of polynomial systems with degree 3, 4 and 5, and so on. For example, Zhang, Zang and Han [43] studied a cubic system and found out that it has 11 limit cycles with two different distributions, of which one is new. Zhang et al. [44] discussed a polynomial system of degree 4 and proved that it can have 15 limit cycles. Wu, Han and Chen [42] verified that the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(x^2 - 1) - (c_1 + c_2x^2 + c_3y^2 + c_4x^4)y \end{aligned}$$

can have 7 limit cycles. Just recently, we proved that the cubic system

$$\begin{aligned} \dot{x} &= y + \varepsilon \textstyle\sum_{i+j=3} a_{ij}x^iy^j \\ \dot{y} &= -x(x^2 - 1) + \varepsilon \textstyle\sum_{i+j=3} b_{ij}x^iy^j \end{aligned}$$

can have 7 limit cycles for $\varepsilon \neq 0$ small.

We finally remark that the method of studying homoclinic bifurcations introduced above can also be used to study heteroclinic bifurcations. For details, see Han [13], Han and Zhang [17], Han, Wu and Bi [22], and Han and Yang [23].

Acknowledgments. This work was partially supported by the National Natural Sciences Foundation of China (grant no. 10371072).

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Received on January 10, 2005. Revised on April 6, 2005.

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