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# A NONLINEAR $L^2$-STABILITY ANALYSIS FOR TWO-SPECIES POPULATION DYNAMICS WITH DISPERSAL

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Abstract. The nonlinear $L^2$-stability (instability) of the equilibrium states of two-species population dynamics with dispersal is studied. The obtained results are based on (i) the rigorous reduction of the $L^2$-nonlinear stability to the stability of the zero solution of a linear binary system of ODEs and (ii) the introduction of a particular Liapunov functional $V$ such that the sign of $\frac{dV}{dt}$ along the solutions is linked directly to the eigenvalues of the linear problem.

1. **Introduction.** Let $\Omega \subset I\!R^3$ be a bounded smooth domain. The nonlinear stability analysis of an equilibrium state in $\Omega$ of two-species population dynamics with dispersal, very often can be traced back to the nonlinear stability analysis of the zero solution of a dimensionless nonlinear binary system of PDEs like (see [1], [2], [3] and the references quoted therein and section 6)

$$\begin{cases} \dfrac{\partial C_1}{\partial t} = a_1\, C_1 - b_2\, C_2 + \gamma_1 \Delta C_1 + f(C_1, C_2) \\[2ex] \dfrac{\partial C_2}{\partial t} = b_3\, C_1 + a_4\, C_2 + \gamma_2 \Delta C_2 + g(C_1, C_2) \end{cases} \tag{1}$$

with $a_i\,(i = 1, 4)$, $b_i > 0\,(i = 2, 3)$, $\gamma_i > 0\,(i = 1, 2)$ constants; $C_i$ perturbations (of finite amplitude) to the equilibrium concentrations of species; and $f$, $g$ nonlinear smooth functions of $C_1$, $C_2$ verifying the conditions

$$f(0, 0) = g(0, 0) = 0\,. \tag{2}$$

In the present paper we consider the Dirichlet boundary conditions

$$C_1 = C_2 = 0 \qquad \text{on } \partial\Omega,\ \forall\, t \geq 0 \tag{3}$$

and denote by $< \cdot, \cdot >$ the scalar product in $L^2(\Omega)$; $\| \cdot \|$ the $L^2(\Omega)$-norm; $H_0^1(\Omega)$ the Sobolev space such that

$$\varphi \in H_0^1(\Omega) \to \{\varphi^2 + (\nabla\varphi)^2 \in L^2(\Omega),\, \varphi = 0 \text{ on } \partial\Omega\}\,.$$

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Our aim is to study the stability of the null solution ($C_1^* = C_2^* = 0$) in the $L^2(\Omega)$-norm with respect to the perturbations $(C_1, C_2) \in \left[H_0^1(\Omega)\right]^2$. Precisely assuming

$$\|f\| + \|g\| = o[(\|C_1\|^2 + \|C_2\|^2)^{1/2}] \tag{4}$$

with $\varepsilon, k$ positive constants and setting

$$\begin{cases} b_1 = a_1 - \bar{\alpha}\,\gamma_1 \\ \\ b_4 = a_4 - \bar{\alpha}\,\gamma_2 \end{cases} \tag{5}$$

with $\bar{\alpha}$ positive constant appearing in the embedding [1] Poincaré inequality

$$\|\nabla\phi\|^2 \geq \bar{\alpha}\,\|\phi\|^2, \qquad \forall\phi \in H_0^1(\Omega)\,, \tag{6}$$

our aim is to link the stability (instability) of ($C_1^* = C_2^* = 0$) to the stability (instability) of the solution ($\xi_* = \eta_* = 0$) to the linear system of ODEs:

$$\begin{cases} \dfrac{d\xi}{dt} = b_1\,\xi - b_2\,\eta \\ \\ \dfrac{d\eta}{dt} = b_3\,\xi + b_4\,\eta\,. \end{cases} \tag{7}$$

The eigenvalues of (7) are

$$\lambda = \frac{I \pm \sqrt{I^2 - 4A}}{2} \tag{8}$$

with

$$\begin{cases} I = b_1 + b_4 \\ \\ A = b_1 b_4 + b_2 b_3\,; \end{cases} \tag{9}$$

hence

$$\begin{cases} I < 0 \\ \\ A > 0 \end{cases} \tag{10}$$

guarantee the stability of ($\xi_* = \eta_* = 0$), while the instability is guaranteed by

$$\begin{cases} I > 0 \\ \\ A > 0 \end{cases} \tag{11}$$

or by

$$A < 0\,. \tag{12}$$

Our goal is to prove the followings theorems.

**Theorem 1.** *Let (4) and (10) hold. Then* ($C_1^* = C_2^* = 0$) *is nonlinearly asymptotically stable with respect to the $L^2(\Omega)$-norm.*

**Theorem 2.** *Let (4) and (11) or (4) and (12) hold. Then* ($C_1^* = C_2^* = 0$) *is unstable with respect to the $L^2(\Omega)$-norm.*

The plan of the paper is the following. Section 2 is dedicated to a particular Liapunov functional such that the sign of its derivative along the solutions of (1) essentially depends on the eigenvalues (8) through the product $AI$. In section 3 we obtain Theorem 1, while section 4 is dedicated to the instability Theorem 2. In

[1] As it is well known, $\bar{\alpha} = \bar{\alpha}(\Omega)$ is the lowest eigenvalue $\lambda$ of $\Delta\Phi + \lambda\Phi = 0$ with $\Phi \in H_0^1(\Omega)$.

section 5 the stabilizing-destabilizing effect of diffusivity is studied, while in section 6, we apply the obtained results to some classical model for two-species population dynamics with dispersal. In section 7, we conclude with some remarks concerning possible generalizations of the obtained results.

## 2. **Two particular Liapunov functionals.**

Setting

$$C_1 = \alpha\, u, \qquad C_2 = \beta\, v \tag{13}$$

with $\alpha$ and $\beta$ suitable constants to be chosen appropriately later, in view of (1), we obtain

$$\begin{cases} u_t = a_1 u - \dfrac{\beta}{\alpha} b_2 v + \gamma_1 \Delta u + \bar{f} \\ \\ v_t = \dfrac{\alpha}{\beta} b_3 u + a_4 v + \gamma_2 \Delta v + \bar{g} \end{cases} \tag{14}$$

with

$$\bar{f} = \frac{1}{\alpha} f(\alpha u, \beta v), \qquad \bar{g} = \frac{1}{\beta} g(\alpha u, \beta v)\,. \tag{15}$$

Setting

$$f^* = \gamma_1(\Delta u + \bar{\alpha} u)\,, \qquad g^* = \gamma_2(\Delta v + \bar{\alpha} v) \tag{16}$$

by virtue of (5) and (13)–(16), it follows that

$$\begin{cases} u_t = b_1 u - \dfrac{\beta}{\alpha} b_2 v + f^* + \bar{f} \\ \\ v_t = \dfrac{\alpha}{\beta} b_3 u + b_4 v + g^* + \bar{g} \end{cases} \tag{17}$$

under the boundary conditions

$$u = v = 0 \qquad \text{on } \partial\Omega, \quad \forall\, t \geq 0\,. \tag{18}$$

The analysis of the present paper is based essentially on the Liapunov funtional

$$V = \frac{1}{2}\left[ A(\|u\|^2 + \|v\|^2) + \|b_1 v - \frac{\alpha}{\beta} b_3 u\|^2 + \|\frac{\beta}{\alpha} b_2 v + b_4 u\|^2 \right], \tag{19}$$

which is very particular. In fact, along the solutions of (17) it turns out that

$$\frac{dV}{dt} = AI(\|u\|^2 + \|v\|^2) + \Psi^* + \Psi \tag{20}$$

with

$$\begin{cases} \Psi^* = < \alpha_1 u - \alpha_3 v, f^* > + < \alpha_2 v - \alpha_3 u, g^* > \\ \\ \Psi = < \alpha_1 u - \alpha_3 v, \bar{f} > + < \alpha_2 v - \alpha_3 u, \bar{g} > \\ \\ \alpha_1 = A + \dfrac{\alpha^2}{\beta^2} b_3^2 + b_4^2, \quad \alpha_2 = A + b_1^2 + \dfrac{\beta^2}{\alpha^2} b_2^2, \quad \alpha_3 = \dfrac{\alpha}{\beta} b_1 b_3 - \dfrac{\beta}{\alpha} b_2 b_4\,. \end{cases} \tag{21}$$

Hence, the eigenvalues (8) influence $\dfrac{dV}{dt}$ in a simple direct way through the product $AI$.

**Remark 1.** *We notice the following:*

i. *Setting*

$$f_1^* = -\frac{\beta}{\alpha} b_2 v + f^* \,, \quad g_1^* = \frac{\alpha}{\beta} b_3 u + g^* \,, \tag{22}$$

*(17) becomes*

$$\begin{cases} u_t = b_1 u + f_1^* + \bar{f} \\ v_t = b_4 v + g_1^* + \bar{g} \,. \end{cases} \tag{23}$$

*The functional* $V$ *for the system (23) becomes*

$$\hat{V} = \frac{1}{2}\left[b_1 b_4(\|u\|^2 + \|v\|^2) + b_1^2\|v\|^2 + b_4^2\|u\|^2\right] \,, \tag{24}$$

*then, along (23), one obtains*

$$\frac{d\hat{V}}{dt} = b_1 b_4 (b_1 + b_4)(\|u\|^2 + \|v\|^2) + \hat{\Psi}^* + \hat{\Psi} \tag{25}$$

*with*

$$\begin{cases} \hat{\Psi}^* = < \hat{\alpha}_1 u, f_1^* > + < \hat{\alpha}_2 v, g_1^* > \,, \quad \hat{\Psi} = < \hat{\alpha}_1 u, \bar{f} > + < \hat{\alpha}_2 v, \bar{g} > \\ \hat{\alpha}_1 = b_4(b_1 + b_4), \quad \hat{\alpha}_2 = b_1(b_1 + b_4), \quad \hat{\alpha}_3 = 0 \,; \end{cases} \tag{26}$$

*that is, (in view of (23)),*

$$\frac{d\hat{V}}{dt} = (b_1 + b_4)\left[b_1 b_4(\|u\|^2 + \|v\|^2) + \left(\frac{\alpha}{\beta} b_1 b_3 - \frac{\beta}{\alpha} b_2 b_4\right) < u, v >\right] + \\ + \hat{\Psi}_1^* + \hat{\Psi}_1 \tag{27}$$

*with*

$$\hat{\Psi}_1^* = \hat{\alpha}_1 < u, f^* > + \hat{\alpha}_2 < v, g^* >, \quad \hat{\Psi}_1 = \hat{\alpha}_1 < u, \bar{f} > + \hat{\alpha}_2 < v, \bar{g} > \,. \tag{28}$$

ii. *By virtue of (19),* $A > 0$ *implies that* $V$ *is a positive definite functional of* $(u, v)$*. Further,* $V$ *denotes a norm equivalent to the* $L^2(\Omega)$*-norm so that there exist two positive constants* $k_1, k_2$*, such that*

$$k_1(\|u\|^2 + \|v\|^2) \le V \le k_2(\|u\|^2 + \|v\|^2) \,. \tag{29}$$

*In fact, on choosing*

$$k_1 = \frac{A}{2} \,, \qquad k_2 = \frac{3}{2} \max\left\{A, \, 2\left(b_1^2 + \frac{\alpha^2}{\beta^2} b_3^2\right), \, 2\left(\frac{\beta^2}{\alpha^2} b_2^2 + b_4^2\right)\right\} \tag{30}$$

*by virtue of (19), (29) immediately follows.*

iii. *By virtue of (24),* $b_1 b_4 > 0$ *implies that* $\hat{V}$ *is a positive definite functional of* $(u, v)$*. Further, it develops that*

$$k_3(\|u\|^2 + \|v\|^2) \le \hat{V} \le k_4(\|u\|^2 + \|v\|^2) \tag{31}$$

*with*

$$k_3 = \frac{1}{2} b_1 b_4, \qquad k_4 = (b_1 + b_4)^2 \,. \tag{32}$$

3. **Nonlinear stability: Proof of Theorem 1.** For any constant $\bar\varepsilon$ such that

$$0 < \bar\varepsilon < \inf\left(\frac{|I|}{2\bar\alpha}, \frac{A}{\bar\alpha\,|I|}, \gamma_1, \gamma_2\right), \tag{33}$$

setting

$$\begin{cases} \bar b_i = b_i + \bar\alpha\bar\varepsilon\,, & (i = 1, 4) \\ \bar\gamma_i = \gamma_i - \bar\varepsilon\,, & (i = 1, 2) \end{cases} \tag{34}$$

it easily turns out that [2]

$$\begin{cases} \bar I = \bar b_1 + \bar b_4 < 0 \\ \bar A = \bar b_1 \bar b_4 + b_2 b_3 > 0\,. \end{cases} \tag{35}$$

By virtue of (14) and (34), we obtain

$$\begin{cases} u_t = \bar b_1 u - \dfrac{\beta}{\alpha} b_2 v + \bar f^* + \bar f \\ v_t = \dfrac{\alpha}{\beta} b_3 u + \bar b_4 v + \bar g^* + \bar g \end{cases} \tag{36}$$

with $\bar f$, $\bar g$ given by (15) and

$$\begin{cases} \bar f^* = \bar\gamma_1(\Delta u + \bar\alpha u) + \bar\varepsilon\Delta u \\ \bar g^* = \bar\gamma_2(\Delta v + \bar\alpha v) + \bar\varepsilon\Delta v\,. \end{cases} \tag{37}$$

Then, using the substitution

$$\begin{pmatrix} \bar b_1 & b_2 & b_3 & \bar b_4 & \bar f^* & \bar g^* & \bar f & \bar g \\ b_1 & b_2 & b_3 & b_4 & f^* & g^* & \bar f & \bar g \end{pmatrix} \tag{38}$$

from (19)–(21) we obtain that along the solutions of (36), it turns out that

$$\frac{d\bar V}{dt} = \bar A\,\bar I(\|u\|^2 + \|v\|^2) + \bar\Psi^* + \bar\Psi \tag{39}$$

with

$$\bar V = \frac{1}{2}\left[\bar A(\|u\|^2 + \|v\|^2) + \|\bar b_1 v - \frac{\alpha}{\beta}\, b_3 u\|^2 + \|\frac{\beta}{\alpha} b_2 v - \bar b_4 u\|^2\right] \tag{40}$$

and

$$\begin{cases} \bar\Psi^* =< \bar\alpha_1 u - \bar\alpha_3 v, \bar f^* > + < \bar\alpha_2 v - \bar\alpha_3 u, \bar g^* > \\ \bar\Psi =< \bar\alpha_1 u - \bar\alpha_3 v, \bar f > + < \bar\alpha_2 v - \bar\alpha_3 u, \bar g > \\ \bar\alpha_1 = \bar A + \dfrac{\alpha^2}{\beta^2}\, b_3^2 + \bar b_4^2, \quad \bar\alpha_2 = \bar A + \bar b_1^2 + \dfrac{\beta^2}{\alpha^2}\, b_2^2, \quad \bar\alpha_3 = \dfrac{\alpha}{\beta}\bar b_1 b_3 - \dfrac{\beta}{\alpha} b_2 \bar b_4\,. \end{cases} \tag{41}$$

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[2] In fact, in view of $I < 0$, it follows that

$$\begin{cases} 0 < \bar\varepsilon < \dfrac{|I|}{2\bar\alpha} \Rightarrow \bar I = I + 2\bar\alpha\bar\varepsilon < 0 \\ 0 < \bar\varepsilon < \dfrac{A}{\bar\alpha\,|I|} \Rightarrow \bar A = A + (\bar\alpha\bar\varepsilon)^2 + \bar\alpha\bar\varepsilon I > 0\,. \end{cases}$$

Choosing

$$\alpha = \sqrt{\frac{b_2\bar{b}_4}{\bar{b}_1 b_3}}, \qquad \beta = 1\,, \tag{42}$$

it follows that

$$\alpha_3 = 0 \tag{43}$$

and, by virtue of (37) and (41), we obtain

$$\bar{\Psi}^* = \bar{\alpha}_1 < u, \bar{f}^* > + \bar{\alpha}_2 < \bar{v}, \bar{g}^* > = \bar{\alpha}_1\bar{\gamma}_1(-\|\nabla u\|^2 + \bar{\alpha}\|u\|^2) +$$

$$\bar{\alpha}_2\bar{\gamma}_2(-\|\nabla v\|^2 + \bar{\alpha}\|v\|^2) - \bar{\varepsilon}(\bar{\alpha}_1\|\nabla u\|^2 + \bar{\alpha}_2\|\nabla v\|^2)\,;$$

that is,

$$\bar{\Psi}^* \le -k_*(\|\nabla u\|^2 + \|\nabla v\|^2) \tag{44}$$

with

$$0 < k_* = \bar{\varepsilon}\ \inf(\bar{\alpha}_1,\, \bar{\alpha}_2)\,. \tag{45}$$

On the other hand, (4), (15), (41) and (42) imply

$$\bar{\Psi} \le \frac{\bar{\alpha}_1}{\alpha^2} < C_1,\, f(C_1, C_2) > + \bar{\alpha}_2 < C_2,\, g(C_1, C_2) > \le$$

$$k\left(\frac{\bar{\alpha}_1}{\alpha^2} + \bar{\alpha}_2\right)(\|C_1\|^2 + \|C_2\|^2)^{\varepsilon}\,[\|\nabla C_1\|^2 + \|\nabla C_2\|^2] \le$$

$$k\left(\frac{\bar{\alpha}_1}{\alpha^2} + \bar{\alpha}_2\right)(\alpha^2 + 1)^{(1+\varepsilon)}(\|u\|^2 + \|v\|^2)^{\varepsilon}\,[\|\nabla u\|^2 + \|\nabla v\|^2]\,;$$

that is,

$$\bar{\Psi} \le \tilde{k}(\|u\|^2 + \|v\|^2)^{(1+\varepsilon)}\,[\|\nabla u\|^2 + \|\nabla v\|^2] \tag{46}$$

with

$$\tilde{k} = k\left(\frac{\bar{\alpha}_1}{\alpha^2} + \bar{\alpha}_2\right)(1 + \alpha^2)^{(1+\varepsilon)}\,. \tag{47}$$

Therefore, by virtue of (39), (44) and (46), we obtain

$$\frac{d\bar{V}}{dt} \le -\bar{A}\,|\bar{I}|(\|u\|^2 + \|v\|^2) - [k_* - \tilde{k}(\|u\|^2 + \|v\|^2)^{\varepsilon}][\|\nabla u\|^2 + \|\nabla v\|^2]\,,$$

and hence, in view of (29), it turns out that

$$\frac{d\bar{V}}{dt} \le -\frac{\bar{A}\,|\bar{I}|}{\bar{k}_2}\,\bar{V} - \left(k_* - \frac{\tilde{k}}{\bar{k}_1^{\varepsilon}}\,\bar{V}^{\varepsilon}\right)[\|\nabla u\|^2 + \|\nabla v\|^2] \tag{48}$$

with

$$\bar{k}_1 = \frac{\bar{A}}{2},\ \bar{k}_2 = \max\left\{\bar{A},\, 2\left(\bar{b}_1^2 + \alpha^2\, b_3^2\right),\, 2\left(\frac{1}{\alpha^2}\, b_2^2 + \bar{b}_4^2\right)\right\}. \tag{49}$$

By recursive arguments, one obtains that

$$\bar{V}_0^{\varepsilon} < \frac{k_*\,\bar{k}_1^{\varepsilon}}{\tilde{k}} \tag{50}$$

implies

$$\frac{d\bar{V}}{dt} \le 0 \qquad \forall\, t \ge 0\,, \tag{51}$$

and in view of (6), setting

$$0 < \delta = \frac{1}{\bar{k}_2}\left[\bar{A}\,|\bar{I}| + \bar{\alpha}\left(k_* - \frac{\tilde{k}}{\bar{k}_1^{\varepsilon}}\, V_0^{\varepsilon}\right)\right], \tag{52}$$

it easily follows

$$\frac{d\bar{V}}{dt} \leq -\delta\bar{V}\,;$$

that is,

$$\bar{V} \leq \bar{V}_0\, e^{-\delta\, t}\,. \tag{53}$$

**Remark 2.** *We observe the following:*

i. *The proof of Theorem 1 is not based on the methodology of the usual energy method ([4] pp. 30-39). In particular the stability condition is not linked to the solution of a variational problem embedded in $H_0^1(\Omega)$, but is simply linked to the (algebraic) inequalities (10).*

ii. *The inequalities in (10) do not imply*

$$b_i < 0 \qquad \forall\, i \in \{1, 2\} \tag{54}$$

*In fact, for instance, the inequalities (10) are verified by*

$$0 < b_1 < \sqrt{b_2 b_3}\,, \quad b_4 = -b_1 - \varepsilon\,, \quad 0 < \varepsilon < \frac{b_2 b_3}{b_1} - b_1\,. \tag{55}$$

iii. *When (54) hold, Theorem 1 can be obtained, through an analogous but simpler procedure, from (27).*

4. **Instability: Proof of Theorem 2.** By definition, the instability is guaranteed by the existence of at least one destabilizing admissible perturbation. The optimum occurs when the destabilizing perturbations are dynamically admissible.

In view of (17) with $\alpha = \beta = 1$, the $L^2$-energy system

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|u\|^2 = < u,\, b_1 u - b_2 v > + < u,\, f^* + f > \\[2ex] \dfrac{1}{2}\dfrac{d}{dt}\|v\|^2 = < v,\, b_3 u + b_4 v > + < v,\, g^* + g > \end{cases} \tag{56}$$

easily follows. Let us look for solutions of (56) having the multiplicative form

$$u = p = X(t)\varphi\,, \quad v = q = Y(t)\varphi \tag{57}$$

with $\varphi$ principal eigenfunction of $-\Delta$ in $H_0^1(\Omega)$; that is,

$$\Delta\varphi + \bar{\alpha}\varphi = 0\,, \qquad \varphi \in H_0^1(\Omega)\,. \tag{58}$$

Then (57)–(58) imply

$$\begin{cases} \Delta p + \bar{\alpha} p = \Delta q + \bar{\alpha} q = f^*(p) = g^*(q) = 0 \\[2ex] \|\nabla p\|^2 = \bar{\alpha}\|p\|^2, \;\; \|\nabla q\|^2 = \bar{\alpha}\|q\|^2\,, \end{cases} \tag{59}$$

and any nonzero solution of

$$\begin{cases} \dfrac{dX}{dt} = b_1 X - b_2 Y + F(X,\, Y) \\[2ex] \dfrac{dY}{dt} = b_3 X + b_4 Y + G(X,\, Y)\,, \end{cases} \tag{60}$$

with

$$\begin{cases} F(X,Y) = \dfrac{1}{\|\varphi\|^2} < \varphi, f(\varphi X, \varphi Y) > \\ \\ G(X,Y) = \dfrac{1}{\|\varphi\|^2} < \varphi, g(\varphi X, \varphi Y) > \end{cases} \tag{61}$$

nonlinear smooth functions of $X$, $Y$ such that

$$F(0,0) = G(0,0) = 0 \tag{62}$$

is a solution of (56). The global existence of the multiplicative solutions (57) of (56) is guaranteed by the global existence of the solutions of the binary system of ODEs (60), and the instability of the null solution ($X_* = Y_* = 0$) of (60) implies the instability of the null solution ($C_1^* = C_2^* = 0$) of (1). The linear version of (60) coincides with (7); hence, its eigenvalues are given by (8). In the cases (11)–(12), at least one of the eigenvalues is real positive or complex with positive real part. Although it is well known that in the case at hand, the null solution of (60) is nonlinearly unstable, for the sake of completeness, we present here a simple direct proof of it.

In case (11), the appropriate Liapunov functional for the instability is the analogous of (18)

$$W = \frac{1}{2}\left[A(X^2+Y^2) + (b_1Y - b_3X)^2 + (b_2Y + b_4X)^2\right] \tag{63}$$

Along (60) it follows that

$$\frac{dW}{dt} = AI(X^2+Y^2) + \Psi_2 \tag{64}$$

with

$$\begin{cases} \Psi_2 = FF_1 + GG_1 \\ \\ F_1 = (A + b_3^2 + b_4^2)X - (b_1b_3 - b_2b_4)Y \\ \\ G_1 = (A + b_1^2 + b_2^2)Y - (b_1b_3 - b_2b_4)X\,. \end{cases} \tag{65}$$

But it easily follows that there exists a positive constant $k_4$ such that

$$|\Psi_2| \le k_4(X^2+Y^2)^{1+\varepsilon}\,, \tag{66}$$

and hence

$$\frac{dW}{dt} \ge AI(X^2+Y^2) - k_4(X^2+Y^2)^{1+\varepsilon}\,. \tag{67}$$

Therefore in the sphere $S_r$ of radius $r \le \left(\dfrac{AI}{k_4}\right)^{1/\varepsilon}$ centered at ($X = Y = 0$), $W$ is positive definite and $\dfrac{dW}{dt} > 0$. Then the instability is guaranteed by the Liapunov instability theorem [5], [6]. [3]

[3]In case (11), the instability can be obtained directly as follows. There exist two positive constants $\delta_1$, $\delta_2$ such that $|\Psi_2| \le \delta_2 W^{1+\varepsilon}$, $X^2+Y^2 \le \dfrac{\delta_1}{AI}W$ and hence (64) implies

$$\frac{dW}{dt} \ge \delta_1 W - \delta_2 W^{1+\varepsilon}\,.$$

Integrating, one obtains

$$W^\varepsilon \ge \frac{\delta_1 W_0^\varepsilon e^{\varepsilon\delta_1 t}}{\delta_1 + \delta_2\, W_0^\varepsilon e^{\varepsilon\delta_1 t}}\,,$$

In case (12), $(X_* = Y_* = 0)$ is a saddle point, and via the transformation

$$X_1 = -b_4 X + (b_1 - \lambda_1)Y\,, \quad Y_1 = -b_4 X + (b_1 - \lambda_2)Y\,, \tag{68}$$

(60) can be reduced to

$$\begin{cases} \dfrac{dX_1}{dt} = \lambda_1 X_1 + F_1 \\[2ex] \dfrac{dY_1}{dt} = \lambda_2 Y_1 + G_1\,, \end{cases} \tag{69}$$

with $(p, q, r, s$ being suitable constants)

$$\begin{cases} F_1 = pF[X(X_1, Y_1),\, Y(X_1, Y_1)] + qG[X(X_1, Y_1),\, Y(X_1, Y_1)] \\[2ex] G_1 = rF[X(X_1, Y_1),\, Y(X_1, Y_1)] + sG[X(X_1, Y_1),\, Y(X_1, Y_1)] \end{cases} \tag{70}$$

and

$$\begin{cases} X(X_1,\, Y_1) = \dfrac{1}{b_4(\lambda_1 - \lambda_2)}[(b_1 - \lambda_1)Y_1 + (\lambda_2 - b_1)X_1] \\[2ex] Y(X_1,\, Y_1) = \dfrac{Y_1 - X_1}{\lambda_1 - \lambda_2}\,. \end{cases} \tag{71}$$

In view of $\lambda_1\lambda_2 < 0$, without loss of generality, one can assume $\lambda_1 > 0$. The appropriate Liapunov functional in this case is

$$E = \frac{1}{2}(X_1^2 - Y_1^2)\,, \tag{72}$$

and along the solutions of (69), it follows that

$$\frac{dE}{dt} = \lambda_1 X_1^2 + |\lambda_2| Y_1^2 + X_1 F + Y_1 G\,. \tag{73}$$

Setting

$$\delta = \min(\lambda_1,\, |\lambda_2|)$$

and recalling that (4), (57)–(59) and (71) imply

$$|X_1 F_1 + Y_1 G_1| \le a(X_1^2 + Y_1^2)^{1+\varepsilon} \tag{74}$$

with $a$ positive constant, it turns out that

$$\frac{dE}{dt} > \delta(X_1^2 + Y_1^2) - a(X_1^2 + Y_1^2)^{1+\varepsilon}\,. \tag{75}$$

Therefore in the sphere $S_r$ of radius $r \le \left(\dfrac{\delta}{a}\right)^{1/\varepsilon}$ centered at $(X_1 = Y_1 = 0)$, it turns out that $\dfrac{dE}{dt} > 0$. By virtue of

$$Y_1 = 0 \Rightarrow E > 0, \tag{76}$$

also in case (12), the instability is guaranteed by the Liapunov instability theorem.

**Remark 3.** *We observe that*

i. *The classical energy method of nonlinear $L^2$-stability generally does not allow to obtain conditions guaranteeing instability* [4]*;*

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and the instability is implied by

$$\lim_{t\to\infty} W^\varepsilon \ge \frac{\delta_1}{\delta_2}, \quad \forall W_0\,.$$

ii. *If (11) and (12) are respectively equivalent to* $b_1 > R_C^{(1)}(b_2, b_3, b_4)$ *and* $b_1 > R_C^{(2)}(b_2, b_3, b_4)$*, then the effective instability critical value for* $b_1$ *is* $R_C = \inf(R_C^{(1)}, R_C^{(2)})$.

5. **Stabilizing-destabilizing effect of diffusivity.** Immediate consequences of Theorems 1–2 are as follows:

**Theorem 3.** *Let (4), (10) and*

$$\begin{cases} I_0 = a_1 + a_4 > 0 \\ A_0 = a_1 a_4 + b_2 b_3 > 0 \end{cases} \tag{77}$$

*or*

$$A_0 = a_1 a_4 + b_2 b_3 < 0 \tag{78}$$

*hold. Then* $(C_1^* = C_2^* = 0)$ *- - unstable in the absence of diffusivity - - is stabilized by diffusivity.*

**Theorem 4.** *Let (4), (12) and*

$$\begin{cases} I_0 = a_1 + a_4 < 0 \\ A_0 = a_1 a_4 + b_2 b_3 > 0 \end{cases} \tag{79}$$

*hold. Then* $(C_1^* = C_2^* = 0)$ *- - stable in the absence of diffusivity - - is destabilized by diffusivity.*

It remains only to show the consistency of the assumptions (79). From

$$A = \gamma_1 \gamma_2 \bar{\alpha}^2 - (\gamma_1 a_4 + \gamma_2 a_1)\bar{\alpha} + A_0 < 0\,, \tag{80}$$

it follows that the consistency of (79) requires

$$\begin{cases} \gamma_1 \neq \gamma_2 \\ a_1 a_4 < 0\,. \end{cases} \tag{81}$$

Let

$$a_1 < 0\,; \tag{82}$$

then (80) becomes

$$\gamma_1 > \frac{1}{a_4}(|a_1| + \gamma_1 \bar{\alpha})\gamma_2 + \frac{A_0}{a_4 \bar{\alpha}}\,, \tag{83}$$

and the consistency of (79) is guaranteed by (81)–(82) and

$$\begin{cases} \gamma_1 > \dfrac{(1+\delta)A_0}{a_4 \bar{\alpha}} \\ \gamma_2 < \dfrac{\delta A_0}{(|a_1| + \gamma_1 \bar{\alpha})\bar{\alpha}}\,. \end{cases} \qquad \delta = \text{const.} > 0 \tag{84}$$

Analogously if

$$a_4 < 0\,, \tag{85}$$

the consistency is guaranteed by (81), (85) and

$$\begin{cases} \gamma_2 \geq \dfrac{(1+\delta)A_0}{a_1\bar{\alpha}} \\ \\ \gamma_1 < \dfrac{\delta A_0}{(|a_4| + \gamma_2\bar{\alpha})\bar{\alpha}}\,. \end{cases} \qquad \delta = \text{const.} > 0 \tag{86}$$

**Remark 4.** *The stabilizing-destabilizing effect of diffusivity on the linear stability is well known (*[1] *pag. 351,* [7]*). When* $\Omega$ *is a torus and the perturbations verify the plan-form equations, the nonlinear stabilizing-destabilizing effect of diffusivity has been considered in* [8]*.*

## 6. Applications.

Let $a_{ij}$ $(i, j = 1, 2)$ be real constants and consider the case

$$\begin{cases} f = a_{11}C_1^2 + a_{12}C_1C_2 + a_{13}C_2^2 \\ \\ g = a_{21}C_1^2 + a_{22}C_1C_2 + a_{23}C_2^2\,, \end{cases} \tag{87}$$

which is encountered, for instance, in the stability of the equilibrium states of the Lotka-Volterra prey-predator model

$$\begin{cases} \dfrac{\partial\Gamma_1}{\partial t} = h_1\Gamma_1 + \gamma_1\Delta\Gamma_1 - d_1\Gamma_1\Gamma_2 \\ \\ \dfrac{\partial\Gamma_2}{\partial t} = -h_4\Gamma_2 + \gamma_2\Delta\Gamma_2 + d_2\Gamma_1\Gamma_2 \end{cases} \tag{88}$$

and in the stability of the equilibrium states of its modified version of Segel and Jackson [1]

$$\begin{cases} \dfrac{\partial\Gamma_1}{\partial t} = h_1\Gamma_1 + \gamma_1\Delta\Gamma_1 - d_1\Gamma_1\Gamma_2 + e_1\Gamma_1^2 \\ \\ \dfrac{\partial\Gamma_2}{\partial t} = -h_4\Gamma_2 + \gamma_2\Delta\Gamma_2 + d_2\Gamma_1\Gamma_2 - e_2\Gamma_2^2 \end{cases} \tag{89}$$

with $\Gamma_1$, $\Gamma_2$ biomass of prey and predator, respectively, and $h_i \geq 0\,(i = 1, 4); \gamma_i > 0$, $d_i > 0$, $e_i > 0\,(i = 1, 2)$ constants. In fact the stability of an equilibrium state $(\Gamma_1^*, \Gamma_2^*)$ of (88) or (89), setting

$$\begin{cases} \Gamma_1 = \Gamma_1^* + C_1 \\ \\ \Gamma_2 = \Gamma_2^* + C_2\,, \end{cases} \tag{90}$$

is easily traced back to the stability of the null solution of (1) with $f$ and $g$ given by (87) with

$$a_{11} = a_{13} = a_{21} = a_{23} = 0 \tag{91}$$

in the case (88) and by (87) with

$$a_{13} = a_{21} = 0 \tag{92}$$

in the case (89).

By using the Cauchy-Schwarz inequality and the Sobolev embedding inequality

$$\int_\Omega \Psi^4\, d\Omega \leq \delta_1^2 \|\nabla\Psi\|^4, \qquad \Psi \in H_0^1(\Omega)\,, \tag{93}$$

one easily obtains that $\Psi_1, \Psi_2 \in H_0^1(\Omega)$ imply

$$\begin{cases} \displaystyle\int_\Omega \Psi_i^3 \, d\Omega \leq \delta_1 \|\Psi_i\| \cdot \|\nabla\Psi_i\|^2 & i = 1,2 \\ \displaystyle\int_\Omega \Psi_i^2 \, \Psi_j \, d\Omega \leq \delta_1 \|\Psi_j\| \cdot \|\nabla\Psi_i\|^2 & i \neq j, \, i,j = 1,2 \end{cases} \tag{94}$$

with $\delta_1 =$ positive constant, and hence (4) immediately follows. Then all the results of the previous section can be applied. In particular the results below hold true:

i. By virtue of Theorem 3 the condition

$$a_1 < \bar{\alpha}\gamma_1 \tag{95}$$

implies that the null solution of (88), which is unstable in the absence of diffusivity, is stabilized by the diffusivity.

ii. Model (88) admits the nonzero spatially uniform equilibrium

$$\Gamma_1^* = \frac{h_4}{d_2}\,, \qquad \Gamma_2^* = \frac{h_1}{d_1}\,. \tag{96}$$

In view of (90), one obtains (1) with

$$\begin{cases} a_1 = h_1 - d_1\Gamma_2^* = 0\,, & b_2 = d_1\Gamma_1^*\,, & b_3 = d_2\Gamma_2^* \\ a_4 = d_2\Gamma_1^* - h_4 = 0\,, & f = -d_1C_1C_2\,, & g = d_2C_1C_2\,, \end{cases} \tag{97}$$

and hence because

$$\begin{cases} I = b_1 + b_4 = -(\gamma_1 + \gamma_2)\bar{\alpha} < 0 \\ A = b_1b_4 + b_2b_3 = \gamma_1\gamma_2 + b_2b_3 > 0\,, \end{cases} \tag{98}$$

Theorem 1 implies the asymptotic exponential $L^2$-stability of (96).

iii. Provided $\{d_1d_2 > e_1e_2,\ h_4 = 0,\ h_1 > 0\}$, (89) admits the spatially uniform equilibrium state

$$\Gamma_1^* = \frac{h_1e_2}{d_1d_2 - e_1e_2}\,, \qquad \Gamma_2^* = \frac{h_1d_2}{d_1d_2 - e_1e_2}\,. \tag{99}$$

Substituting (99) in (89), setting

$$\begin{cases} \bar{t} = h_1t & \bar{\gamma}_i = \dfrac{\gamma_i}{h_1}\,, (i = 1,2) & \\ \dfrac{e_1}{d_2} = a^2 - b^2\,, & \dfrac{d_1}{e_2} = a^2\,, & b^2 = \dfrac{h_1}{d_2\Gamma_1^*} = \dfrac{h_1}{e_2\Gamma_2^*} \\ \bar{C}_1 = \dfrac{d_2}{h_1}\,C_1\,, & \bar{C}_2 = \dfrac{e_2}{h_1}\,C_2 & \end{cases} \tag{100}$$

and omitting the bars, one obtains the system (1) with

$$\begin{cases} a_1 = \dfrac{a^2}{b^2} - 1, \quad b_2 = \dfrac{a^2}{b^2}, \quad b_3 = \dfrac{1}{b^2}, \quad a_4 = -\dfrac{1}{b^2} \\ \gamma_1 = 1, \quad \gamma_2 = \theta^2, \quad f = (a^2 - b^2)C_1^2 - a^2C_1C_2, \quad g = C_1C_2 - C_2^2\,. \end{cases} \tag{101}$$

In view of

$$\begin{cases} I = \dfrac{a^2-1}{b^2} - [1+(1+\theta^2)\bar{\alpha}] \\ \\ A = -\dfrac{a^2}{b^2}\,\theta^2\bar{\alpha} + (1+\bar{\alpha})\left(\dfrac{1}{b^2}+\theta^2\bar{\alpha}\right), \end{cases} \tag{102}$$

it follows that

1. By virtue of Theorem 1,

$$a^2 < \inf\left\{[1+(1+\theta^2)\bar{\alpha}]b^2,\ \frac{1+\bar{\alpha}}{\bar{\alpha}\theta^2}+(1+\bar{\alpha})b^2\right\} \tag{103}$$

guarantees the asymptotic exponential stability in the $L^2(\Omega)$-norm of (99). Setting

$$\theta_c^2 = \frac{-1+\sqrt{1+4(1+\bar{\alpha})b^2}}{2\bar{\alpha}b^2}, \tag{104}$$

one obtains

$$\theta^2 \ \substack{\leq \\ >}\ \theta_c^2 \Rightarrow 1+[1+(1+\theta^2)\bar{\alpha}]b^2 \ \substack{\leq \\ >}\ \frac{1+\bar{\alpha}}{\bar{\alpha}\theta^2}+(1+\bar{\alpha})b^2, \tag{105}$$

and the stability of (99) is guaranteed by

$$\begin{cases} \theta^2 \leq \theta_c^2 \\ \\ a^2 \leq 1+[1+(1+\theta^2)\bar{\alpha}]b^2 \end{cases} \tag{106}$$

and by

$$\begin{cases} \theta^2 \geq \theta_c^2 \\ \\ a^2 \leq \dfrac{1+\bar{\alpha}}{\bar{\alpha}\theta^2}+(1+\bar{\alpha})b^2. \end{cases} \tag{107}$$

2. In view of Theorem 2, the instability of (99) is guaranteed by

$$1+[1+(1+\theta^2)\bar{\alpha}]b^2 < a^2 < \frac{1+\bar{\alpha}}{\bar{\alpha}\theta^2}+(1+\bar{\alpha})b^2 \tag{108}$$

and by

$$a^2 > \frac{1+\bar{\alpha}}{\bar{\alpha}\theta^2}+(1+\bar{\alpha})b^2. \tag{109}$$

By virtue of (105) the consistency of (108) is guaranteed only for $\theta^2 < \theta_c^2$; hence

$$\begin{cases} \theta^2 \leq \theta_c^2 \\ \\ a^2 > 1+[1+(1+\theta^2)\bar{\alpha}]b^2 \end{cases} \tag{110}$$

and

$$\begin{cases} \theta^2 \geq \theta_c^2 \\ \\ a^2 > \dfrac{1+\bar{\alpha}}{\bar{\alpha}\theta^2}+(1+\bar{\alpha})b^2 \end{cases} \tag{111}$$

imply instability. In other words (105)–(106) are also necessary for the $L^2(\Omega)$-stability.

3. In the absence of diffusivity, the stability is guaranteed by

$$a^2 < 1 + b^2 \,. \tag{112}$$

Looking for the destabilizing effect of diffusivity, by virtue of (84) it is requested that

$$a^2 > b^2 \,. \tag{113}$$

In view of (112)–(113) and $A < 0$, it turns out that the destabilizing effect of diffusivity is implied by

$$\begin{cases} b^2 < a^2 < 1 + b \\ \theta^2 > \dfrac{1 + \bar{\alpha}}{a^2 - b^2} \,. \end{cases} \tag{114}$$

## 7. Final remarks.

i. Theorems 1–2 allow one to obtain the coincidence between the conditions of linear stability (via normal modes) and the conditions of nonlinear stability with respect to the $L^2(\Omega)$-norm.

ii. By virtue of the Holder inequality [9] and of the Sobolev inequality below [4]

$$\int_\Omega \Phi^6 \, d\Omega \le \delta^3 \, \|\nabla \Phi\|^6 \qquad (\delta = \text{const.} > 0) \,, \tag{115}$$

it turns out that

$$\int_\Omega \Phi_1^{4/3} \Phi_2^2 \, d\Omega \le \delta \|\Phi_1\|^{2/3} \cdot \|\nabla \Phi_2\|^2 \,. \tag{116}$$

Therefore (4) is fulfilled also in the more general case

$$\begin{cases} f = f_1 \cdot (a_{11} C_1^2 + a_{12} C_1 C_2 + a_{13} C_2^2 + a_{14} C_1 C_2^{4/3} + a_{15} C_1^{1/3} C_2^2 + a_{16} C_1^{7/3}) \\ g = g_1 \cdot (a_{21} C_1^2 + a_{22} C_1 C_2 + a_{23} C_2^2 + a_{24} C_2 C_1^{4/3} + a_{25} C_2^{1/3} C_1^2 + a_{26} C_2^{7/3}) \end{cases} \tag{117}$$

with

$$a_{ij} = \text{const.} \,, \quad |f_1(C_1,\, C_2)| + |g_1(C_1,\, C_2)| \le \delta_1 = \text{const.} \tag{118}$$

iii. In the one-dimensional case $(i = 1, 2)$

$$\begin{cases} C_i = C_i(x, t) & x \in [0, 1] \\ C_i = 0 & x = 0, 1 \,, \end{cases} \tag{119}$$

it turns out that [12]

$$C_i^2 \le \frac{1}{\pi} \|\nabla C_i\|^2 \qquad (i = 1, 2) \,; \tag{120}$$

hence, $(i, j = 1, 2)$

$$\begin{cases} \displaystyle\int_0^1 C_i^2 C_j^2 dx \le \frac{1}{\pi} \|\nabla C_i\|^2 \cdot \|C_j\|^2 \\ \displaystyle\int_0^1 |C_i C_j^3| dx \le \frac{1}{\pi} \|\nabla C_j\|^2 \cdot \int_0^1 |C_i C_j| dx \le \\ \qquad \le \dfrac{1}{2\pi} \|\nabla C_j\|^2 \cdot (\|C_i\|^2 + \|C_j\|^2) \,. \end{cases} \tag{121}$$

In view of (121) it follows that - - at least in the one dimensional case - - (4) also continues to be fulfilled when one adds to the right hand side of (117) the functions $f_2$ and $g_2$, respectively, given by ($b_{ij}$ = constants)

$$\begin{cases} f_2 = f_1(C_1, C_2)(b_{11}C_1^3 + b_{12}C_1^2C_2 + b_{13}C_1C_2^2 + b_{14}C_2^3) \\ \\ g_2 = g_1(C_1, C_2)(b_{21}C_1^3 + b_{22}C_1^2C_2 + b_{23}C_1C_2^2 + b_{24}C_2^3)\,. \end{cases} \tag{122}$$

Then, in the one-dimensional case, the stability-instability results obtained in the present paper can be applied to many other models. In particular, they can be applied to the second model presented in [7] by Segel and Jackson in which the source terms $\pm C_1^2C_2$ appear.

iv. When a maximum principle implies

$$|C_i| \leq M = \text{const.} \qquad i = 1, 2\,,$$

then (4) is fulfilled when $f$ and $g$ are polynomials of any degree $> 1$ of $C_i$, ($i = 1, 2$).

v. Theorem 2 holds for any nonlinearity such that

$$|F| + |G| \leq o(\sqrt{X^2 + Y^2})\,.$$

vi. When the coefficients $a_i$, $b_i$ contained in (1) are not constant in $\Omega$, subregions $\Omega^* \subset \Omega$ can exist in which the instability begins [10], [11] . This problem will be considered in a future paper.

vii. Theorems 1–2 continue to hold also in the case of Neumann boundary conditions

$$\frac{du}{d\mathbf{n}} = \frac{dv}{d\mathbf{n}} = 0 \tag{123}$$

($\mathbf{n}$ being the unit outward normal to $\partial\Omega$) in the class of the perturbations such that

$$\int_\Omega u\, d\Omega = \int_\Omega v\, d\Omega = 0\,. \tag{124}$$

In fact, when (123)–(124) hold, the inequality (6) continues to hold (generally with a different value for the constant $\bar{\alpha}$) [12].

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