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# TRAVELLING WAVE SOLUTIONS FOR HIGHER-ORDER WAVE EQUATIONS OF KDV TYPE (III)

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Abstract. By using the theory of planar dynamical systems to the travelling wave equation of a higher order nonlinear wave equations of KdV type, the existence of smooth solitary wave, kink wave and anti-kink wave solutions and uncountably infinite many smooth and non-smooth periodic wave solutions are proved. In different regions of the parametric space, the sufficient conditions to guarantee the existence of the above solutions are given. In some conditions, exact explicit parametric representations of these waves are obtain.

1. **Introduction.** This paper is a sister article of our paper(I) and (II)(see [5,10]). In 2002, E.Tzirtzilakis,etc [8] suggested studying a "more physically realistic form of water wave equations of the KdV type" as follows (also see Fokas [1]):

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x + \alpha\beta(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}) = 0, \tag{1.1}$$

where $\rho_i$ $(i = 1, 2, 3)$ are free parameters and $\alpha, \beta$ are positive real constants, which characterize, respectively, the long wavelength and short amplitude of the waves. For the equation (1.1), J. Li, Y. Long et al.[4] have studied the dynamics of solitary waves, kink waves and breaking waves for the case $\rho_3 = (p + 1)\rho_2,\ p < -1$, and obtained some exact explicit parametric representations of solitary wave, kink and anti-kink wave solutions. In our second paper [10], we have considered the dynamics of reduced travelling wave equation of (1.1) for the case $\rho_3 = (p+1)\rho_2,\ p > -1,\ \rho_2 > 0$. In this paper, we shall consider the case $\rho_2 < 0,\ p \geq 1$. As in [10], we consider the travelling wave equation of (1.1),

$$\beta(1 + \alpha\rho_2\phi)\phi'' + \frac{1}{2}\alpha\beta p\rho_2(\phi')^2 + \frac{1}{3}\alpha^2\rho_1\phi^3 + \frac{1}{2}\alpha\phi^2 + (1 - c)\phi = 0, \tag{1.2}$$

and its equivalent two-dimensional system,

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{3\alpha\beta p\rho_2 y^2 + 2\alpha^2\rho_1\phi^3 + 3\alpha\phi^2 + 6(1 - c)\phi}{6\beta(1 + \alpha\rho_2\phi)}. \tag{1.3}$$

Now, system (1.3) has the following first integral,

$$y^2 = h(1 + \alpha\rho_2\phi)^{-p} - \frac{1}{\Omega}\left(A_0 - \alpha\rho_2 p A_0\phi + C_0\phi^2 + D_0\phi^3\right), \tag{1.4}$$

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where $h$ is arbitrary constant and
$A_0 = 6[\rho_2^2(p+2)(p+3)(c-1)+\rho_2(p+3)-2\rho_1]$, $C_0 = 3p(p+1)\alpha^2\rho_2^2[(p+3)\rho_2-2\rho_1]$, $D_0 = 2p(p+1)(p+2)\alpha^3\rho_1\rho_2^3$, $\Omega = 3p(p+1)(p+2)(p+3)\alpha^2\beta\rho_2^4$. Suppose that $\phi(x-ct) = \phi(\xi)$ is a continuous solution of system (1.3) for $\xi \in (-\infty,\infty)$ and $\lim\limits_{\xi\to\infty}\phi(\xi) = a$, $\lim\limits_{\xi\to-\infty}\phi(\xi) = b$. Recall that (i) $\phi(x,t)$ is called a solitary wave solution if $a=b$; (ii) $\phi(x,t)$ is called a kink or anti-kink solution if $a \neq b$. Usually, a solitary wave solution of equation (1.1) corresponds to a homoclinic orbit of system (1.3); a kink (or anti-kink) wave solution equation (1.1) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of system (1.3). Similarly, a periodic orbit of system (1.3) corresponds to a periodically travelling wave solution of equation (1.1). Thus, to investigate all possible bifurcations of solitary waves and periodic waves of equation (1.1), we need to find all periodic annuli and homoclinic orbits of system (1.3), which depend on the system parameters. We notice that the right-hand side of the second equation in system (1.3) is generally not continuous when $\phi = -\frac{1}{\alpha\rho_2}$. In other words, on such straight lines in the phase plane $(\phi, y)$, the function $\phi''_\xi$ is not well-defined. It implies that the smooth system (1.1) sometimes has non-smooth travelling wave solutions(see [2, 3, 5,6]).

2. **Bifurcations of phase portraits of system (1.3).** System (1.3) has the same phase orbits as the following system,

$$\frac{d\phi}{d\tau} = 6\beta(1+\alpha\rho_2\phi)y, \quad \frac{dy}{d\tau} = -(3\alpha\beta p\rho_2 y^2 + 2\alpha^2\rho_1\phi^3 + 3\alpha\phi^2 + 6(1-c)\phi), \tag{2.1}$$

except for the straight line $\phi = \phi_s = -\frac{1}{\alpha\rho_2}$, where $d\tau = 6\beta(1+\alpha\rho_2\phi)d\xi$ for $\phi \neq -\frac{1}{\alpha\rho_2}$. Throughout in this paper, we assume that $\rho_2 < 0$, $p \geq 1$. (1.4) can be written as

$$H(\phi,y) = (1+\alpha\rho_2\phi)^p\left[y^2 + \frac{1}{\Omega}\left(A_0 - \alpha\rho_2 pA_0\phi + C_0\phi^2 + D_0\phi^3\right)\right] = h. \tag{2.2}$$

Write

$$f(\phi) = 2\alpha^2\rho_1\phi^3 + 3\alpha\phi^2 + 6(1-c)\phi.$$

Denote that

$$\phi_{1,2} = \frac{-3\pm\sqrt{\Delta_1}}{4\alpha\rho_1}, \quad Y_\pm = \pm\frac{1}{\alpha\rho_2^2}\sqrt{\frac{\Delta_2}{3p\beta}}, \tag{2.3}$$

where $\Delta_1 = 9+48\rho_1(c-1)$, $\Delta_2 = 6\rho_2^2(1-c)+2\rho_1-3\rho_2$. Then, when $\rho_1 \neq 0$, $\Delta_1 > 0$, system (2.1) has three equilibrium points at $O(0,0)$, $A_{1,2}(\phi_{1,2},0)$. When $\Delta_1 = 0$, system (2.1) has two equilibrium points at $O(0,0)$ and $A_{12}(\phi_{12},0), \phi_{12} = -\frac{3}{4\alpha\rho_1}$. When $\Delta_2 > 0$, there exist two equilibrium points of (2.1) in L: $\phi = -\frac{1}{\alpha\rho_2}$, at $S_\pm(\phi_s, Y_\pm)$. It is easy to see that for $\rho_1 > 0$, then when $0 < 1 - \frac{3}{16\rho_1} < c < 1$, $\phi_2 < \phi_1 < 0$. When $c = 1$, $\phi_2 = -\frac{3}{2\alpha\rho_1} < 0 = \phi_1$. When $c > 1$, $\phi_2 < 0 < \phi_1$. For $\rho_1 < 0$, then when $c = 1$, $\phi_1 = 0 < -\frac{3}{2\alpha\rho_1} = \phi_2$. when $1 < c < 1 - \frac{3}{16\alpha\rho_1}$, $0 < \phi_1 < \phi_2$. When $c > 1$, $\phi_1 < 0 < \phi_2$. For $\rho_1 = 0$, (2.1) has two equilibrium points at $O(0,0)$ and $A_0(\phi_0, 0)$, where $\phi_0 = \frac{2(c-1)}{\alpha}$. When $0 < c < 1$, $\phi_0 < 0$. When $c > 1$, $\phi_0 > 0$. Let $M(\phi_i, y_e)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point $(\phi_i, y_e)$. Then we have $Trace\ (M(\phi_{1,2},0)) = 0$ and

$$J(0,0) = detM(0,0) = 36\beta(1-c),\ J(\phi_{1,2},0) = 6\beta(1+\alpha\rho_2\phi_{1,2})f'(\phi_{1,2}),$$

$$J(\phi_s, Y_\pm) = detM(\phi_s, Y_\pm) = -36p\alpha^2\beta^2\rho_2^2 Y_\pm^2 < 0.$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point. If $J > 0$ and $Trace\ (M(\phi_i, 0)) = 0$, then it is a center point. If $J > 0$ and $(Trace\ (M(\phi_i, 0)))^2 - 4J(\phi_i, 0) > 0$, then it is a node. If $J = 0$ and the index of the equilibrium point is 0, then it is a cusp.

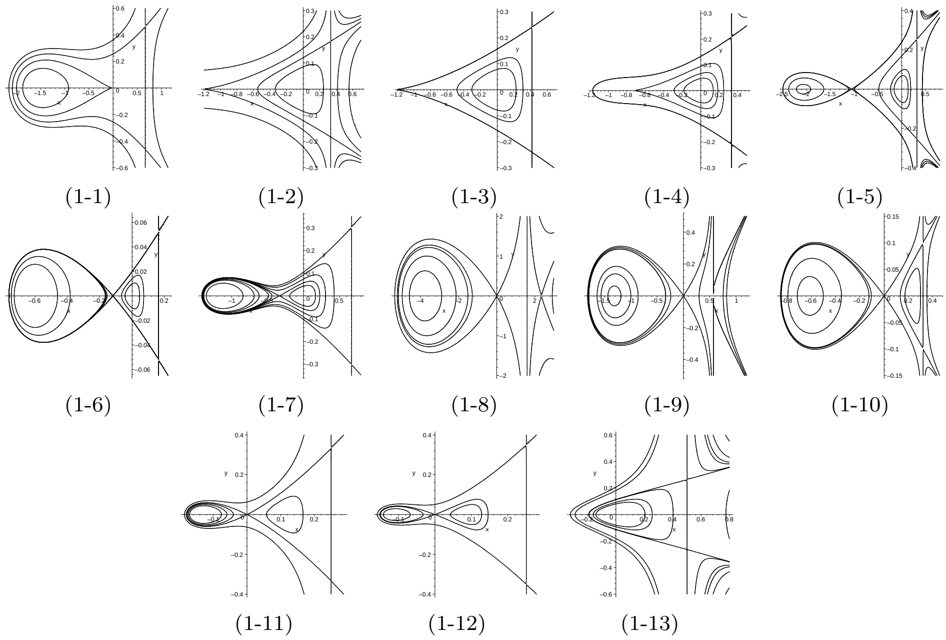



Figure 1. The phase portraits of system (2.1) for $\rho_1 > 0$, $\rho_2 < 0$. (1-1) $c = 1$, $\phi_2 < \phi_1 = 0 < \phi_s$, $\Delta_2 > 0$. (1-2) $c = 1 - \frac{3}{16\rho_1}$, $\phi_1 = \phi_2 < 0 < \phi_s$, $h_1 > 0$. (1-3) $c = 1 - \frac{3}{16\rho_1}$, $\phi_1 = \phi_2 < 0 < \phi_s$, $h_1 = 0$. (1-4) $c = 1 - \frac{3}{16\rho_1}$, $\phi_1 = \phi_2 < 0 < \phi_s$, $h_1 < 0$. (1-5) $1 - \frac{3}{16\rho_1} < c < 1$, $\phi_2 < \phi_1 < 0 < \phi_s$, $h_1 > 0, \Delta_2$ $\Delta_2 > 0$. (1-6) $1 - \frac{3}{16\rho_1} < c < 1$, $\phi_2 < \phi_1 < 0 < \phi_s$, $h_1 = 0$, $\Delta_2 > 0$. (1-7) $1 - \frac{3}{16\rho_1} < c < 1$, $\phi_2 < \phi_1 < 0 < \phi_s$, $h_1 < 0$, $\Delta_2 > 0$. (1-8) $c > 1$, $\phi_2 < 0 < \phi_s < \phi_1$, $\Delta_2 < 0$. (1-9) $c > 1$, $\phi_2 < 0 < \phi_1 = \phi_s$, $\Delta_2 = 0$. (1-10) $c > 1$, $\phi_2 < 0 < \phi_1 < \phi_s$, $A_0 > 0$, $\Delta_2 > 0$. (1-11) $c > 1, \phi_2 < 0 < \phi_1 < \phi_s$, $A_0 = 0, \Delta_2 > 0$. (1-12) $c > 1$, $\phi_2 < 0 < \phi_1 < \phi_s$, $A_0 < 0$, $\Delta_2 > 0$. (1-13) $0 < c < 1 - \frac{3}{16\rho_1}$, $0 < \phi_s$, $\Delta_2 > 0$.

For $H(\phi, y)$ defined by (2.2), we write

$$h_0 = H(0,0) = \frac{A_0}{\Omega}, \qquad h_s = H(\phi_s, Y_\pm) = 0,$$

$$h_{1,2} = H(\phi_{1,2}, 0) = \frac{1}{\Omega}\left(A_0 - \alpha\rho_2 p A_0 \phi_{1,2} + C_0\phi_{1,2}^2 + D_0\phi_{1,2}^3\right)(1 + \alpha\rho_2\phi_{1,2})^p,$$

$$h_* = H(\phi_0, 0) = \frac{1}{\Omega}\left(A_0 - \alpha\rho_2 p A_0 \phi_0 + C_0\phi_0^2 + D0\phi_0^3\right)(1 + \alpha\rho_2\phi_0)^p.$$

For a fixed $h$, the level curve $H(\phi, y) = h$ defined by (2.2) determines a set of (2.1), which contains different branches of curves. By using the above fact, we have the phase portraits of (2.1) shown in Figures 1-3 for $p \geq 1,\ \rho_2 < 0$.

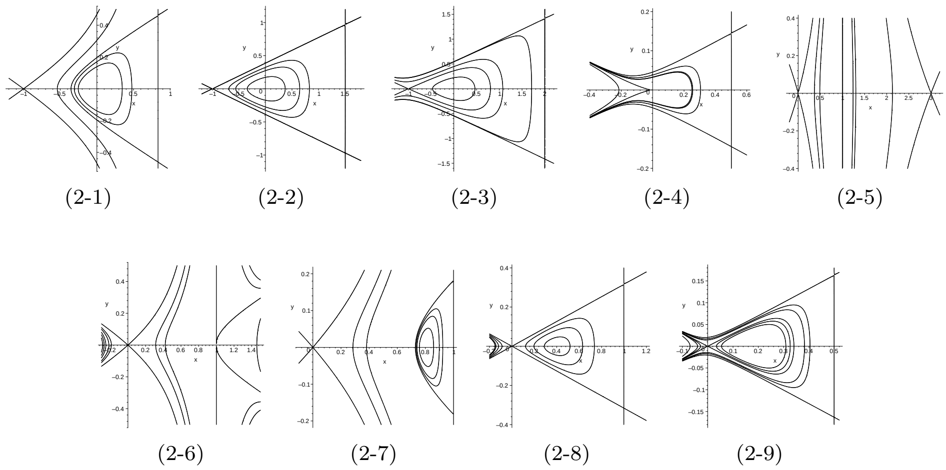
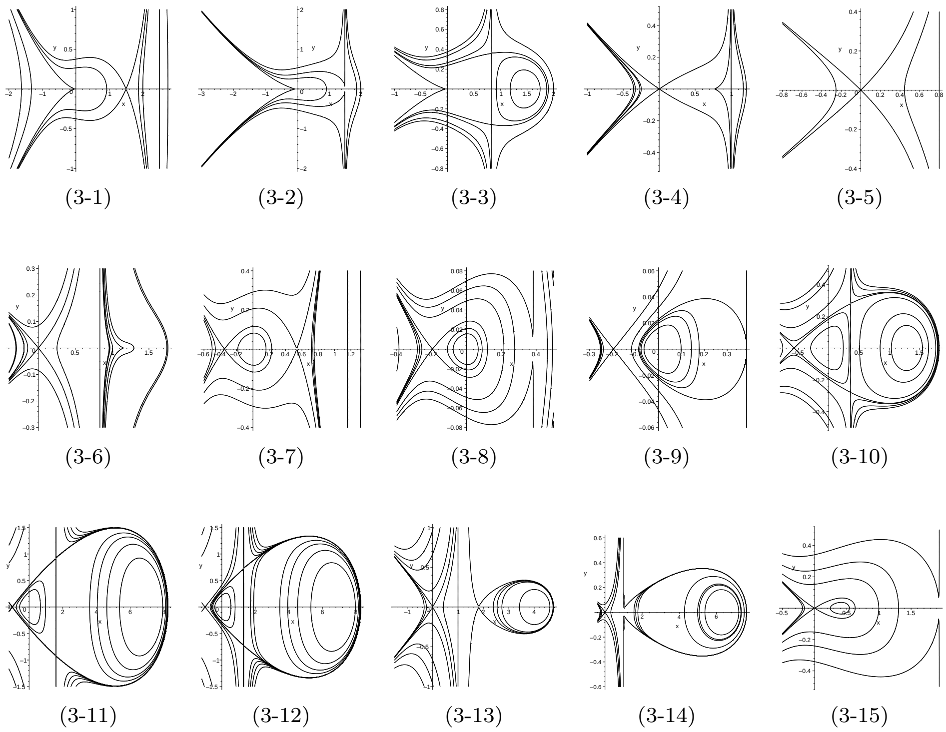
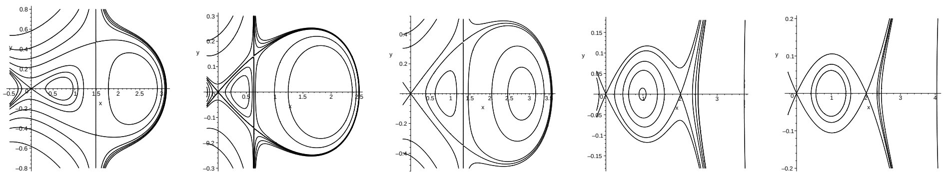
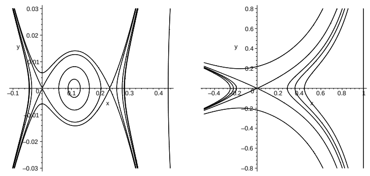



Figure 2. The phase portraits of system (2.1) for $\rho_1 = 0,\ \rho_2 < 0$.(2-1) $0 < c < 1,\ \phi_0 < 0 < \phi_s,\ h_* > 0,\ \Delta_2 > 0$. (2-2) $0 < c < 1,\ \phi_0 < 0 < \phi_s,\ h_* = 0,\ \Delta_2 > 0$. (2-3) $0 < c < 1,\ \phi_0 < 0 < \phi_s,\ h_* < 0,\ \Delta_2 > 0$. (2-4) $c = 1,\ \phi_0 = 0 < \phi_s$. (2-5) $c > 1,\ 0 < \phi_s < \phi_0,\ \Delta_2 < 0$. (2-6) $c > 1,\ 0 < \phi_s = \phi_0,\ \Delta_2 = 0$. (2-7) $c > 1,\ 0 < \phi_0 < \phi_s,\ h_0 > 0,\ \Delta_2 > 0$. (2-8) $c > 1,\ 0 < \phi_0 < \phi_s,\ h_0 = 0, \Delta_2 > 0$. (2-9) $c > 1,\ 0 < \phi_0 < \phi_s,\ h_0 < 0,\ \Delta_2 > 0$.





(3-16) (3-17) (3-18) (3-19) (3-20)



(3-21) (3-22)

FIGURE 3. The phase portraits of (2.1) for $\rho_1 < 0,\ \rho_2 < 0$. (3-1) $c = 1,\ 0 = \phi_1 < \phi_2 < \phi_s, < 0$. (3-2) $c = 1,\ 0 = \phi_1 < \phi_2 = \phi_s,\ \Delta_2 = 0$. (3-3) $c = 1, 0 = \phi_1 < \phi_s < \phi_2,\ \Delta_2 > 0$. (3-4) $c = 1 - \frac{3}{16\rho_1},\ 0 < \phi_{12} < \phi_s,\ \Delta_2 < 0$. (3-5) $c = 1 - \frac{3}{16\rho_1},\ 0 < \phi_{12} = \phi_s,\ \Delta_2 = 0$. (3-6) $c = 1 - \frac{3}{16\rho_1},\ 0 < \phi_s < \phi_{12},\ \Delta_2 < 0$. (3-7) $0 < c < 1,\ \phi_1 < 0 < \phi_2 < \phi_s,\ \Delta_2 < 0$. (3-8) $0 < c < 1,\ \phi_1 < 0 < \phi_2 = \phi_s,\ h_1 < 0,\ \Delta_2 < 0$. (3-9) $0 < c < 1,\ \phi_1 < 0 < \phi_2 = \phi_s,\ h_1 > 0,\ h_2 = 0,\ \Delta_2 = 0$. (3-10) $0 < c < 1,\ \phi_1 < 0 < \phi_s < \phi_2,\ h_1 < 0,\ \Delta_2 > 0$. (3-11) $0 < c < 1,\ \phi_1 < 0 < \phi_s < \phi_2,\ h_1 = 0,\ \Delta_2 > 0$. (3-12) $0 < c < 1,\ \phi_1 < 0 < \phi_s < \phi_2,\ h_1 > 0,\ \Delta_2 > 0$. (3-13) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_s < \phi_1 < \phi_2,\ \Delta_2 < 0$. (3-14) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_s = \phi_1 < \phi_2,\ h_1 = 0,\ \Delta_2 = 0$. (3-15) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_s = \phi_2,\ \Delta_2 = 0$. (3-16) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_s < \phi_2,\ A_0 < 0,\ \Delta_2 > 0$. (3-17) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_s < \phi_2,\ A_0 = 0,\ \Delta_2 > 0$. (3-18) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_s < \phi_2,\ A_0 > 0,\ \Delta_2 > 0$. (3-19) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_2 < \phi_s,\ h_0 > h_2,\ \Delta_2 < 0$. (3-20) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_2 < \phi_s,\ h_0 = h_2,\ \Delta_2 < 0$. (3-21) $1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_2 < \phi_s,\ h_0 < h_2,\ \Delta_2 < 0$. (3-22) $c > 1 - \frac{3}{16\rho_1},\ 0 < \phi_s,\ \Delta_2 < 0$.

We shall apply these phase portraits to discuss the travelling wave solutions of Equation (1.1) in section 3.

3. **Existence of smooth solitary wave solutions, periodic wave, kink wave and anti-kink wave solutions of equation (1.1).** In this section, we consider the existence of smooth solitary wave and smooth and non-smooth periodic wave solutions of equation (1.1). First, we discuss the existence of solitary wave solutions. We see from Figures 1-3 that the following conclusion holds. *Theorem 3.1.* (i.) Equation (1.1) has a smooth solitary wave solution of the valley type, which corresponds to $H(\phi, y) = \frac{A_0}{\Omega}$ under the following two conditions: (1) $\rho_1 > 0,\ c = 1,\ \ A_0 < 0,\ \Delta_2 > 0$ (see Fig. 1 (1-1)). (2) $\rho_1 > 0,\ c > 1$ (see Fig. 1 (1-8)-(1-12)). In addition, if $A_0 < 0$, equation (1.1) also has a smooth solitary wave solution of peak type (see Fig. 1 (1-12)). (ii.) Equation (1.1) has a smooth solitary wave solution of valley type, under the following conditions: (1) $\rho_1 > 0,\ 1 - \frac{3}{16\rho_1} < c < 1,\ \Delta_2 > 0$ (see Fig. 1 (1-5)-(1-7)), which corresponds to $H(\phi, y) = h_1$. In addition, if $h_1 < 0$, equation (1.1) also has a smooth solitary wave

solution of peak type (see Fig. 1 (1-7)). (2) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_2 < \phi_s,\ h_0 > h_2,\ \Delta_2 < 0$ (see Fig. 3 (3-19)), which corresponds to $H(\phi, y) = h_2$. (iii.) Equation (1.1) has a smooth solitary wave solution of peak type, which corresponds to $H(\phi, y) = \frac{A_0}{\Omega}$ under the following conditions: (1) $\rho_1 = 0,\ c > 1,\ A_0 < 0,\ \Delta_2 > 0$ (see Fig. 2 (2-9)). (2) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ A_0 < 0,\ \Delta_2 \geq 0$ (see Fig. 3 (3-15) and (3-16)). (3) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ h_0 < h_2,\ \Delta_2 < 0$ (see Fig. 3 (3-21). (iv.) Equation (1.1) has a smooth solitary wave solution of peak type, which corresponds to $H(\phi, y) = h_1$ under the the following conditions: (1) $\rho_1 > 0,\ c = 1 - \frac{3}{16\rho_1},\ h_1 < 0,\ \Delta_2 > 0$ (see Fig. 1 (1-4)). (2) $\rho_1 = 0,\ 0 < c < 1,\ h_* < 0,\ \Delta_2 > 0$ (see Fig. 2 (2-3)). (3) $\rho_1 < 0,\ 0 < c < 1,\ h_1 < 0$ (see Fig. 3 (3-7), (3-8) and (3-10)). (4) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ \Delta_2 < 0$ (see Fig. 3 (3-13)). Second, we consider the smooth periodic solutions of equation (1.1). *Theorem 3.2.* (i.) Equation (1.1) has a family of smooth periodic wave solutions, which corresponds to $H(\phi, y) = h,\ h \in (h_2, \frac{A_0}{\Omega})$ under the following conditions: (1) $\rho_1 > 0,\ c = 1,\ \phi_2 < \phi_1 = 0 < \phi_s,\ \Delta_2 > 0$ (see Fig. 1 (1-1)). (2) $\rho_1 > 0,\ c > 1,\ \phi_2 < 0$ (see Fig. 1 (1-8)-(1-12)). In addition, if $A_0 < 0, \Delta_2 > 0$, (1.1) also has a family of smooth periodic wave solutions, which corresponds to $H(\phi, y) = h,\ h \in (h_1, \frac{A_0}{\Omega})$ (see Fig. 1 (1-12)). (ii.) Equation (1.1) has a family of smooth periodic wave solutions, which corresponds to $H(\phi, y) = h,\ h \in (\frac{A_0}{\Omega}, h_1)$, under the following conditions: (1) $\rho_1 > 0,\ c = 1 - \frac{3}{16\rho_1},\ h_1 < 0,\ \Delta_2 > 0$ (see Fig. 1 (1-4)). (2) $\rho_1 < 0,\ 0 < c < 1,\ \phi_1 < 0 < \phi_2 \leq \phi_s,\ h_1 < 0,\ \Delta_2 \leq 0$ (see Fig. 3 (3-7) and (3-8)). (3) $\rho_1 < 0,\ 0 < c < 1,\ \phi_1 < 0 < \phi_s < \phi_2,\ h_1 < 0,\ \Delta_2 > 0$ (see Fig. 3 (3-10)).

(iii.) Equation (1.1) has a family of smooth periodic wave solutions, which corresponds to $H(\phi, y) = h,\ h \in (h_1, \frac{A_0}{\Omega})$ under the following conditions: (1) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_s \leq \phi_2,\ A_0 < 0,\ \Delta_2 \geq 0$ (see Fig. 3 (3-15) and (3-16)). (2) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_1 < \phi_2 < \phi_s,\ h_0 \leq h_2,\ \Delta_2 < 0$ (see Fig. 3 (3-20) and (3-21)). (iv.) Equation (1.1) has a family of smooth periodic wave solutions, under the following conditions: (1) $\rho_1 = 0,\ 0 < c < 1,\ \phi_0 < 0 < \phi_s,\ h_* < 0,\ \Delta_2 > 0$ (see Fig. 2 (2-3)), which corresponds to $H(\phi, y) = h,\ h \in (\frac{A_0}{\Omega}, h_*)$. (2) $\rho_1 = 0,\ c > 1,\ 0 < \phi_0 < \phi_s,\ h_0 < 0,\ \Delta_2 > 0$ (see Fig. 2 (2-9)), which corresponds to $H(\phi, y) = h,\ h \in (h_*, \frac{A_0}{\Omega})$. (v.) Equation (1.1) has a family of periodic wave solutions, which corresponds to $H(\phi, y) = h,\ h \in (h_2, h_1)$ under the following conditions: (1) $\rho_1 > 0,\ 1 - \frac{3}{16\rho_1} < c < 1,\ \phi_2 < \phi_1 < 0 < \phi_s,\ \Delta_2 > 0$ (see Fig. 1 (1-5)-(1-7)). In addition, if $h_1 < 0,\ \Delta_2 > 0$, equation (1.1) also has a family of smooth periodic wave solutions which corresponds to $H(\phi, y) = h,\ h \in (\frac{A_0}{\Omega}, h_1)$. (2) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_s < \phi_1 < \phi_2,\ \Delta_2 < 0$ (see Fig. 3 (3-13)). (vi.) Equation (1.1) has a family of periodic wave solutions under the following conditions: (1) $\rho_1 < 0,\ 0 < c < 1,\ \phi_1 < 0 < \phi_2 = \phi_s,\ h_1 > 0,\ \Delta_2 = 0,\ H(\phi, y) = h,\ h \in (h_0, 0]$ (see Fig. 3 (3-9)). (2) $\rho_1 < 0,\ 1 < c < 1 - \frac{3}{16\rho_1},\ 0 < \phi_s = \phi_1 < \phi_2,\ \Delta_2 = 0,\ H(\phi, y) = h,\ h \in (h_2, 0]$ (see Fig. 3 (3-14)). Finally, we notice that corresponding to the phase orbits which close to a segment of the straight line $\phi = \phi_s$, there exists a class of periodic cusp travelling wave solutions.

*Theorem 3.3.* It exists a class of uncountably infinite many-periodic travelling wave solutions of Equation (1.1), which will gradually lose smoothness of wave profiles when $h$ is varied in $H(\phi, y) = h$ under the following conditions: (i.) When $h$ varies from $h_0 = \frac{A_0}{\Omega}$ to 0, the profile of periodic wave evolve from smooth periodic wave to periodic cusp wave, under the following conditions: (1) $\rho_1 > 0$ $c = 1$, $A_0 < 0$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_0, 0)$ (see Fig. 1 (1-1)). (2) $\rho_1 > 0$ $c = 1 - \frac{3}{16\rho_1}$, $h_1 > 0$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_0, 0)$ (see Fig. 1 (1-2)). (3) $\rho_1 > 0$, $1 - \frac{3}{16\rho_1} < c < 1$, $h_1 > 0$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_0, 0)$ (see Fig. 1 (1-5)). (4) $\rho_1 > 0$, $0 < c < 1 - \frac{3}{16\rho_1}$, $0 < \phi_s$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_0, 0)$ (see Fig. 1 (1-13)). (5) $\rho_1 > 0$, $c > 1$, $\phi_2 < 0 < \phi_1 < \phi_s$, $A_0 < 0$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_0, 0)$ (see Fig. 1 (1-12)). (6) $\rho_1 = 0$, $0 < c < 1, h_* > 0$, $\Delta_2 > 0, H(\phi, y) = h$, $h \in (h_0, 0)$ (see Fig. 2 (2-1)). (ii.) When $h$ varies from $h_2$ to 0, the profile of periodic wave evolve from smooth periodic wave to periodic cusp waves, under the following conditions: (1) $\rho_1 < 0$, $c = 1$, $0 = \phi_1 < \phi_s < \phi_2$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_2, 0)$ (see Fig. 3 (3-3)). (2) $\rho_1 < 0$, $0 < c < 1$, $\phi_1 < 0 < \phi_s < \phi_2$, $h_1 \leq 0$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_2, 0)$ (see Fig. 3 (3-10)-(3-11)). (3) $\rho_1 < 0$, $1 < c < 1 - \frac{3}{16\rho_1}$, $0 < \phi_1 < \phi_s < \phi_2$, $A_0 \leq 0$, $\Delta_2 > 0$, $H(\phi, y) = h$, $h \in (h_2, 0)$ (see Fig. 3 (3-16)-(3-17)). (iii.) When $\rho_1 < 0$, $0 < c < 1$, $\phi_1 < 0 < \phi_s < \phi_2$, $h_1 > 0$, $\Delta_2 > 0$ corresponding to two families of periodic orbits of system (1.3) given by $H(\phi, y) = h$, $h \in (h_0, 0)$ and $h \in (h_2, 0)$, (see Fig. 3 (3-12)), there are two families of periodic wave solutions. As $h$ varies from $h_0$ to 0 and from $h_2$ to 0, respectively, the profile of periodic waves evolve from smooth periodic waves to periodic cusp waves. (iv.) When $\rho_1 < 0$, $1 < c < 1 - \frac{3}{16\rho_1}$, $0 < \phi_1 < \phi_s < \phi_2$, $A_0 > 0$, $\Delta_2 > 0$, corresponding to two families of periodic orbits of system (1.3) given by $H(\phi, y) = h$, $h \in (h_1, 0)$ and $h \in (h_2, 0)$ (see Fig. 3 (3-18)), there are two families of periodic wave solutions. When $h$ varies from $h_1$ to 0 and from $h_2$ to 0, respectively, the profiles of periodic waves evolve from smooth periodic waves to periodic cusp waves. *Theorem 3.4.* Suppose that $\rho_1 < 0$, $\rho_2 < 0$, $1 < c < 1 - \frac{3}{16\rho_1}$, $0 < \phi_1 < \phi_2 < \phi_s$, $h_0 = h_2$, $\Delta_2 < 0$. Then, corresponding to $h = h_0 = h_2 = \frac{A_0}{\Omega}$ in (2.2), equation (1.1) has a kink wave and an anti-kink wave solution. For $h \in (h_1, h_0)$, there exist uncountably infinite many-smooth periodic travelling wave solutions of equation (1.1) (See Fig. 3 (3-20)).

4. **Exact explicit parametric representations defined by** $H(\phi, y) = 0$ **in (2.2).** In this section, we shall describe the types of non-smooth solitary wave and periodic wave solutions that can appear for our system (1.1) which correspond to some homoclinic or heteroclinic orbits of (2.1) connecting to the straight line $\phi = -\frac{1}{\alpha\rho_2}$. To discuss the existence of cusp waves, we need to use the following lemma relating to the singular straight line (see [3], Theorem 3.1 and Theorem 3.2). *Lemma 4.1.* The boundary curves of a periodic annulus are the limit curves of closed orbits inside the annulus. If these boundary curves contain a segment of the singular straight line $\phi = \phi_s$ of system (1.3), then all along this segment and near this segment, in a very short time interval, $y = \phi_\xi$ rapidly jumps rapidly. By using Lemma 4.1, corresponding to the homoclinic and heteroclinic orbits connecting to $\phi = -\frac{1}{\alpha\rho_2}$ in section 2, we now consider the level curves of $H(\phi, y) = 0$ defined by (2.2). We have the following different types of solitary cusp wave and periodic cusp wave solutions.

4.1. **Solitary cusp wave**. 4.1.1. Suppose that $\rho_1 > 0$, $c = 1 - \frac{3}{16\rho_1}, \phi_1 = \phi_2 < 0 < \phi_s$, $h_1 = 0$ ,that is, $\rho_1 = -\frac{p\rho_2}{4}$, $\Delta_2 > 0$ (see Fig. 1 (1-3)). In this case, $H(\phi, y) = 0$ becomes $(1 + \alpha\rho_2\phi)^p = 0$ or $y^2 = \frac{(\alpha\rho_2 p\phi - 3)^3}{6\rho_2^3\beta\alpha^2 p^2(p+3)}$. Thus, we obtain the following parametric representation of solitary cusp wave solution of equation (1.1):

$$u(x,t)=\phi(x-ct)=\frac{3}{\alpha\rho_2 p}[1-\frac{p+3}{3(\Omega_1(x-ct)+1)^2}],\quad 2\sqrt{6\beta(-\rho_2)}<|x-ct|<\infty, \tag{4.1}$$

where $\Omega_1=\frac{1}{2\sqrt{6\beta(-\rho_2)}}$. 4.1.2. Suppose that $\rho_1=0,\ 0<c<1, \phi_0<0<\phi_s,\ h_*=0$ , that is, $c=1+\frac{1}{\rho_2 p},\ \Delta_2>0,$ (see Fig. 2 (2-2)). In this case, $H(\phi,y)=0$ becomes $(1+\alpha\rho_2\phi)^p=0$ or $y^2=-\frac{(2-\alpha\rho_2 p\phi)^2}{\rho_2\beta\alpha^2p^2(p+2)}$. Thus, we obtain the following parametric representation of solitary cusp wave solution of equation (1.1):

$$u(x,t)=\phi(x-ct)=\frac{1}{\alpha(-\rho_2)p}[(p+2)\exp(-\Omega_2|x-ct|)-2], \tag{4.2}$$

where $\Omega_2=\frac{1}{\sqrt{(-\rho_2)\beta(p+2)}},\ 0<|x-ct|<\infty$. 4.1.3. Suppose that $\rho_1=0,\ c>1,\ 0<\phi_0<\phi_s,\ h_0=0$ , that is, $c=1-\frac{1}{\rho_2(p+2)},\ \Delta_2>0$ (see Fig. 2 (2-8)). In this case, $H(\phi,y)=0$ becomes $(1+\alpha\rho_2\phi)^p=0$ or $y^2=-\frac{\phi^2}{\rho_2\beta(p+2)}$. Thus, we obtain the following parametric representation of solitary cusp wave solution of equation (1.1):

$$u(x,t)=\phi(x-ct)=\frac{1}{\alpha(-\rho_2)}\exp(-\Omega_3|x-ct|), \tag{4.3}$$

where $\Omega_3=\frac{1}{\sqrt{(-\rho_2)\beta(p+2)}},\ 0<|x-ct|<\infty$.

4.2. **Smooth periodic wave**. 4.2.1. Suppose that $\rho_1<0,\ 0<c<1,\ \phi_1<0<\phi_2=\phi_s,\ h_1>0,\ h_2=0,\ \Delta_2=0$ (see Fig. 3 (3-9)). In this case, we see that $\rho_1=3\rho_2^2(c-1)+1.5\rho_2$. Now, $H(\phi,y)=0$ becomes $(1+\alpha\rho_2\phi)^p=0$ or $y^2=-\,a_1(\phi^2+b_1\phi+c_1)(\phi+\frac{1}{\alpha\rho_2})=a_1(-\frac{1}{\alpha\rho_2}-\phi)(\phi-\phi_M)(\phi-\phi_m)$, where $a_1=\frac{\alpha(1+2\rho_2(c-1))}{\beta(p+3)},\ b_1=-\frac{2\beta(p+3)(1+\rho_2(p+5)(c-1))}{\rho_2(p+2)(1+2\rho_2(c-1))},\ c_1=\frac{2\beta(p+3)(1+\rho_2(p+5)\ (c-1))}{\alpha\rho_2^2(p+1)(p+2)(1+2\rho_2(c-1))}, \phi_M=\frac{1}{2}(\sqrt{b_1^2-4c_1}-b_1), \phi_m=-\frac{1}{2}(\sqrt{b_1^2-4c_1}+b_1)$. Hence, we have the following parametric representation of a smooth periodic wave solution of (1.1):

$$u(x,t)=\phi(x-ct)=\frac{\phi_M-\phi_m k^2sn^2(\Omega_4(x-ct),k)}{dn^2(\Omega_4(x-ct),k)}, \tag{4.4}$$

where $\Omega_4=\sqrt{\frac{\alpha(\phi_s-\phi_m)(1+2\rho_2(c-1))}{4\beta(p+3)\rho_2}},\quad k=\sqrt{\frac{\phi_s-\phi_M}{\phi_s-\phi_m}}$. 4.2.2. Suppose that $\rho_1<0,\ 1<c<1-\frac{3}{16\rho_1},\ 0<\phi_s=\phi_1<\phi_2,\ h_1=0,\ \Delta_2=0,$ (see Fig. 3 (3-14)). In this case, we see that $\rho_1=3\rho_2^2(c-1)+1.5\rho_2$. Now, $H(\phi,y)=0$ becomes $(1+\alpha\rho_2\phi)^p=0$ or $y^2=\frac{\alpha(1+2\rho_2(c-1))}{\beta(p+3)}(\phi^2+b_2\phi+c_2)=\frac{\alpha(1+2\rho_2(c-1))}{\beta(p+3)}(\phi_M-\phi)(\phi-\phi_s)(\phi-\phi_m)$, where $b_2=-\frac{2\beta(p+3)(1+\rho_2(p+5)(c-1))}{\rho_2(p+2)(1+2\rho_2(c-1))},\ c_2=\frac{2\beta(p+3)(1+\rho_2(p+5)(c-1))}{\alpha\rho_2^2(p+1)(p+2)(1+2\rho_2(c-1))}, \phi_M=\frac{1}{2}(\sqrt{b_2^2-4c_2}-b_2),\ \phi_m=-\frac{1}{2}(\sqrt{b_2^2-4c_2}+b_2)$. Hence, we have the following parametric representation of a smooth periodic wave solution of (1.1):

$$u(x,t)=\phi(x-ct)=\phi_M-(\phi_M-\phi_s)sn^2(\Omega_5(x-ct),k), \tag{4.5}$$

where $\Omega_5=\sqrt{\frac{\alpha(\phi_M-\phi_m)[1+2\rho_2(c-1)]}{4\beta(k+3)}},\quad k=\sqrt{\frac{\phi_M-\phi_s}{\phi_M-\phi_m}}$.

4.3. **Coexistence of a smooth solitary wave and a solitary cusp wave**. 4.3.1. Suppose that $\rho_1>0,\ 1-\frac{3}{16\rho_1}<c<1,\ \phi_2<\phi_1<0<\phi_s,\ h_1=0$, that is, $A_0-\alpha\rho_2kA_0\phi_1+C_0\phi_1^2+D_0\phi_1^3=0,\ \Delta_2>0$ (see Fig. 1 (1-6)). There are a homoclinic orbit and two heteroclinic orbits of (2.1) to the equilibrium point $O(\phi_1,0)$ with the level curve $y^2=\frac{D_0}{\Omega}\left(\frac{(-A_0)}{D_0}-\frac{\alpha\rho_2p(-A_0)}{D_0}\phi-\frac{C_0}{D_0}\phi^2-\phi^3\right)=\frac{2\alpha\rho_1}{3\beta(p+3)(-\rho_2)}(\phi_1-$

$\phi)^2(\phi-\phi_m)$, where $\phi_m = -\frac{1}{2}(\frac{C_0}{D_0}+\phi_1-\sqrt{\frac{4\alpha\rho_2 kA_0}{D_0}+\frac{C_0^2}{D_0^2}-\frac{2C_0}{D_0}\phi_1-3\phi_1^3}) < 0$ and $\phi_1$ is given by (2.3). Thus, we have a smooth solitary wave solution of equation (1.1) of valley form

$$u(x,t)=\phi(x-ct)=\phi_m+(\phi_1-\phi_m)tanh^2(\omega(x-ct)), \tag{4.6}$$

where $\omega=\sqrt{\frac{\alpha\rho_1(\phi_1-\phi_m)}{-6\beta\rho_2(p+3)}}$. On the other hand, for $\frac{1}{\omega}tanh^{-1}\sqrt{\frac{\phi_s-\phi_m}{\phi_1-\phi_m}} < |x-ct| < \infty$,

$$u(x,t)=\phi(x-ct)=-[\phi_m+(\phi_1-\phi_m)tanh^2(\omega(x-ct))] \tag{4.7}$$

describes the solitary cusp wave solution of peak type of (1.1), which corresponding to two heteroclinic orbits and a segment of $\phi=\phi_s$. 4.3.2. Suppose that $\rho_1>0,\ c>1,\ \phi_2<0<\phi_1<\phi_s,\ A_0=0,\ \Delta_2>0$,, that is, $h_0=h_s=0$ (see Fig. 1 (1-11)). There are a homoclinic orbit and two heteroclinic orbits of (2.1) to the equilibrium point $O(0,0)$ with the level curve $y^2=\frac{2\alpha\rho_1}{3\beta\rho_2(p+3)}\phi^2(\phi-\phi_m)$, where $\phi_m=\frac{3(2\rho_1-(p+3)\rho_2)}{2\alpha\rho_1\rho_2(p+2)}<0$. Thus, we obtain a smooth solitary wave solution of valley form

$$u(x-ct)=\phi(x-ct)=\phi_m sech^2(\omega_1(x-ct)), \tag{4.8}$$

where $\omega_1=\sqrt{\frac{\alpha\rho_1\phi_m}{6\beta\rho_2(p+3)}}$. On the other hand, when $sech^{-1}\left(\sqrt{\frac{\phi_s}{\phi_m}}\right)/\omega_1 < |x-ct| < \infty$,

$$u(x-ct)=\phi(x-ct)=-\phi_m sech^2(\omega_1(x-ct)), \tag{4.9}$$

describes the solitary cusp wave solution of peak type of (1.1), which corresponding to two heteroclinic orbits and a segment of $\phi=\phi_s$. 4.4. **Periodic cusp wave**. 4.4.1. Suppose that $\rho_1>0$. We see 8 cases in Figure 1 (1-1), (1-2), (1-4), (1-5), (1-7), (1-10), (1-12), (1-13). A branch of the level curve $H(\phi,y)=0$ is an arch of heteroclinic orbit of (2.1), which defines a periodic cusp wave solution of peak type of (1.1). In fact, in these cases we have $y^2=-\frac{D_0}{\Omega}\left(\frac{A_0}{D_0}-\frac{\alpha\rho_2 pA_0}{D_0}\phi+\frac{C_0}{D_0}\phi^2+\phi^3\right)=\frac{2\alpha\rho_1}{3\beta(p+3)(-\rho_2)}(\phi-\phi_m)[(\phi-b_1)^2+a_1^2]$, where $(\phi_M,0)$ is the intersection point the arch and $\phi-$axis. Thus, we obtain the parametric representation of these wave solutions as follows:

$$u(x,t)=\phi(x-ct)=\frac{(A+\phi_m)-(A-\phi_m)cn(\Omega_6(x-ct),k_0)}{1+cn(\Omega_6(x-ct),k_0)}, \tag{4.10}$$

where $0\le|x-ct|<\frac{1}{\Omega_6}cn^{-1}(\frac{A+\phi_m-\phi_s}{A+\phi_m+\phi_s}),\ A^2=(b_1-\phi_m)^2+a_1^2,\ k_0^2=\frac{A+b_1-\phi_m}{2A},\ \Omega_6=\sqrt{\frac{2\rho_1 A}{3\beta(p+3)(-\rho_2)}}$. 4.4.2. Suppose that $\rho_1<0$. We see 3 cases in Figure 3 (3-3), (3-10), (3-16). A branch of the level curve $H(\phi,y)=0$ is an arch of heteroclinic orbit of (2.1), which defines a periodic cusp wave solution of the valley type of equation (1.1). In fact, in these cases we have $y^2=\frac{D_0}{\Omega}\left(-\frac{A_0}{D_0}+\frac{\alpha\rho_2 kA_0}{D_0}\phi-\frac{C_0}{D_0}\phi^2-\phi^3\right)=-\frac{2\alpha\rho_1}{3\beta(k+3)\rho_2}(\phi_M-\phi)[(\phi-b_2)^2+a_2^2]$, where $(\phi_M,0)$ is the intersection point of the arch and $\phi-$axis. Thus, we obtain the parametric representation of these waves as follows:

$$u(x,t)=\phi(x-ct)=\frac{(A_1+\phi_M)cn(\Omega_7(x-ct),\tilde{k_0})-(A_1-\phi_M)}{1+cn(\Omega_7(x-ct),\tilde{k_0})}, \tag{4.11}$$

where $0\le|x-ct|<\frac{1}{\Omega_7}cn^{-1}(\frac{A_1-\phi_M+\phi_s}{A1+\phi_M-\phi_s}),\ A_1^2=(b_2-\phi_M)^2+a_2^2,\ \tilde{k_0}^2=\frac{A_1-b_2+\phi_M}{2\phi_M}$, $\Omega_7=\sqrt{\frac{2\rho_1 A_1}{3\beta(p+3)\rho_2}}$. 4.4.3. Suppose that $\rho_1=0$. We see two cases in Figure 2 (2-1) and (2-7). A branch of the level curve $H(\phi,y)=0$ is an arch of heteroclinic orbit of

(2.1), which defines a periodic cusp wave solution of peak type of equation (1.1). In fact, in these cases we have $y^2 = -\frac{C_0}{\Omega}\left(\frac{A_0}{C_0} + \frac{\alpha(-\rho_2)pA_0}{C_0}\phi + \phi^2\right) = \frac{1}{\beta(p+2)(-\rho_2)}(\phi - \phi_M)(\phi - \phi_m)$. Thus, we have the following parametric representations of two periodic cusp wave solutions of equation (1.1):

$$u(x,t) = \phi(x-ct) = \frac{1}{2}[(\phi_M + \phi_m) + (\phi_M - \phi_m)\cos\omega_2(x-ct)] \tag{4.12}$$

where $\omega_2 = \sqrt{\frac{1}{\beta(p+2)(-\rho_2)}}$, $\phi_m = \frac{\alpha\rho_2 pA_0 + \sqrt{\alpha^2\rho_2^2p^2A_0^2 - 4A_0C_0}}{2C_0}$, $\phi_M = \frac{\alpha\rho_2 pA_0}{2C_0} - \frac{\sqrt{\alpha^2\rho_2^2p^2A_0^2 - 4A_0C_0}}{2C_0}$.

5. **Coexistence of a solitary cusp wave of the peak type and periodic cusp waves of the valley type.** 5.1. Suppose that $\rho_1 < 0$, $0 < c < 1$, $\phi_1 < 0 < \phi_s < \phi_2$, $h_1 < 0$, that is, $A_0 - \alpha\rho_2 kA_0\phi_1 + C_0\phi_1^2 + D_0\phi_1^3 = 0$, $\Delta_2 > 0$ (see Fig. 3 (3-11). In this case, $H(\phi, y) = 0$ becomes $(1 + \alpha\rho_2\phi)^p = 0$ or $y^2 = \frac{D_0}{\Omega}\left(\frac{(-A_0)}{D_0} - \frac{\alpha\rho_2 p(-A_0)}{D_0}\phi - \frac{C_0}{D_0}\phi^2 - \phi^3\right) = \frac{2\alpha(-\rho_1)}{3\beta(p+3)(-\rho_2)}(\phi_M - \phi)(\phi - \phi_1)^2$. Hence, we have the following parametric representations of a solitary cusp wave solution for $\frac{1}{\omega_3}tanh^{-1}\sqrt{\frac{\phi_M - \phi_s}{\phi_M - \phi_1}} < |x - ct| < \infty$ and a periodic cusp wave solution for $0 \le |x - ct| < \frac{1}{\omega_3}tanh^{-1}\sqrt{\frac{\phi_M - \phi_s}{\phi_M - \phi_1}}$ of (1.1):

$$u(x,t) = \phi(x-ct) = \phi_M - (\phi_M - \phi_1)tanh^2(\omega_3(x-ct)), \tag{4.13}$$

where $\omega_3 = \sqrt{\frac{\alpha(-\rho_1)(\phi_M - \phi_1)}{6\beta(p+3)(-\rho_2)}}$ and $(\phi_M, 0)$ is the intersection point of the arch and $\phi$-axis, $\phi_1$ is defined by (2.3).

5.2. Suppose that $\rho_1 < 0$, $1 < c < 1 - \frac{3}{16\rho_1}$, $0 < \phi_1 < \phi_s < \phi_2$, $A_0 = 0$, $\Delta_2 > 0$ (see Fig. 3 (3-17). In this case, $H(\phi, y) = 0$ becomes $(1 + \alpha\rho_2\phi)^p = 0$ or $y^2 = \frac{D_0}{\Omega}\phi^2(\phi_M - \phi) = \frac{2\alpha(-\rho_1)}{3\beta(p+3)(-\rho_2)}\phi^2(\phi_M - \phi)$, where $\phi_M = -\frac{C_0}{D_0}$. Thus, we have the following parametric representations of a solitary cusp wave solution for $\frac{1}{\omega_4}sech^{-1}\sqrt{\frac{\phi_s}{\phi_M}} < |x - ct| < \infty$ and a periodic cusp wave solution for $0 \le |x - ct| < \frac{1}{\omega_4}sech^{-1}\sqrt{\frac{\phi_s}{\phi_M}}$ of equation (1.1):

$$u(x,t) = \phi(x-ct) = \phi_M sech^2(\omega_4(x-ct)) \tag{4.14}$$

where $\omega_4 = \sqrt{\frac{\alpha\rho_1 C_0}{6\beta(p+3)(-\rho_2)D_0}}$.

6. **Coexistence of periodic cusp waves of the peak type and the valley type.** 6.1. Suppose that $\rho_1 < 0$. We see two kinds of cases in Figure 3 (3-12) and (3-18). A branch of the level curve $H(\phi, y) = 0$ are two arches of heteroclinic orbits of (2.1), which defines a periodic cusp wave solutions of peak and valley type of equation (1.1). In fact, in these cases we have $y^2 = -\frac{D_0}{\Omega}\left(\frac{A_0}{D_0} - \frac{\alpha\rho_2 pA_0}{D_0}\phi + \frac{C_0}{D_0}\phi^2 + \phi^3\right) = \frac{2\alpha(-\rho_1)}{3\beta(p+3)(-\rho_2)}(\phi_M - \phi)(\phi_s - \phi)(\phi - \phi_m)$. and $y^2 = \frac{D_0}{\Omega}\left(\frac{(-A_0)}{D_0} - \frac{\alpha\rho_2 p(-A_0)}{D_0}\phi - \frac{C_0}{D_0}\phi^2 - \phi^3\right) = \frac{2\alpha(-\rho_1)}{3\beta(p+3)(-\rho_2)}(\phi_M - \phi)(\phi - \phi_s)(\phi - \phi_m)$. Corresponding to these two cases, respectively, we have the following parametric representations of a periodic cusp wave solution of the peak type for $0 \le |x - ct| < \frac{K(k_0)}{\Omega_8}$ :

$$u(x,t) = \phi(x-ct) = \phi_m + (\phi_s - \phi_m)sn^2(\Omega_8(x-ct), k_0) \tag{4.15}$$

and a periodic cusp wave solution of valley type for $0 \le |x-ct| < \frac{K(k_{10})}{\Omega_8}$ :

$$u(x,t) = \phi(x-ct) = \phi_M - (\phi_M - \phi_s)sn^2(\Omega_8(x-ct), k_{10}), \tag{4.16}$$

where $\Omega_8 = \sqrt{\frac{\alpha(-\rho_1)(\phi_M-\phi_m)}{6\beta(p+3)(-\rho_2)}}$, $k_0 = \sqrt{\frac{\phi_s-\phi_m}{\phi_M-\phi_m}}$, $k_{10} = \sqrt{\frac{\phi_M-\phi_s}{\phi_M-\phi_m}}$ and $(\phi_M, 0)$ and $(\phi_m, 0)$ are two intersection points of two arch and $\phi$-axis; $K(k)$ is the first type of complete elliptic integral.

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