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THE STABILITY OF AN SIR EPIDEMIC MODEL WITH TIME DELAYS

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Abstract. In this paper, an SIR epidemic model for the spread of an infectious disease transmitted by direct contact among humans and vectors (mosquitoes) which have an incubation time to become infectious is formulated. It is shown that a disease-free equilibrium point is globally stable if no endemic equilibrium point exists. Further, the endemic equilibrium point (if it exists) is globally stable with respect to a "weak delay". Some known results are generalized.

1. Introduction.

A model for the spread of an infectious disease transmitted by a vector (e.g, mosquitoes) was proposed by Cooke [1]. Extensions of his model, were considered by Busenberg and Cooke [2], Marcati and Pozio [3], Volz [4], and Beretta etc [5,7-9]. They considered only the spread of an infectious disease transmitted by a vector. In fact, the dynamics of some diseases (e.g, dengue fever, chagas disease, malaria) are influenced by many factors involving humans, the vector and the blood transfusion transmission, as well as the environment, which directly or indirectly affects these elements and the interrelations among them [10]. In this paper, we consider an SIR epidemic models for the spread of an infectious disease transmitted by direct contact among humans and vectors (mosquitoes) which have an incubation time to become infectious. The model still can be considered an extension of Cooke's model.

Let $S(t)$ be the number of members of a population susceptible to the disease, $I(t)$ be the number of infective members, and $R(t)$ be the number of members who have been removed from the possibility of infection through full immunity. Then the epidemic models for the spread of an infectious disease transmitted by vectors and humans can be described by the following system:

$$\begin{cases} S'(t) &= b-\beta_1 S(t)\int_0^h f(s)I(t-s)ds-\beta_2 S(t)I(t)-\mu S(t), \\ I'(t) &= \beta_1 S(t)\int_0^h f(s)I(t-s)ds+\beta_2 S(t)I(t)-(\mu+\lambda+c)I(t), \\ R'(t) &= \lambda I(t)-\mu R(t), \end{cases} \tag{1.1}$$

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where the constant coefficients are all positive. The f is usually nonnegative and continuous, and $\int_0^h f(\tau)d\tau = 1$.

As usual, the initial condition of (1.1) is given as

$$S(\theta) = \varphi_1(\theta), I(\theta) = \varphi_2(\theta), R(\theta) = \varphi_3(\theta), \ (-h \leq \theta \leq 0), \tag{1.2}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$ such that $\varphi_i(\theta) = \varphi_i(0) \geq 0$ $(-h \leq \theta \leq 0$ $i = 1, 3)$, $\varphi_2(\theta) \geq 0(-h \leq \theta \leq 0)$, and C denotes the Banach space $C = C([-h, 0], R^3)$ of continuous functions defined on $[-h, 0]$.

From [6], the solution (S(t), I(t), R(t)) of (1.1) with the initial condition (1.2) exists for all $t \geq 0$ and is unique. Furthermore, we can easily show that $S(t) > 0, I(t) > 0, R(t) > 0$.

2. **Preliminaries.** We consider stability properties of system (1.1) with biologically reasonable initial values ϕ belonging to the Banach space $C = C([-h, 0], R^3)$ of a continuum defined on $[-h, 0]$. Using the following result, we consider an autonomous system of delay differential equations

$$x'(t) = F(x_t) \tag{2.1}$$

Such that $F(0) = 0$ and $F : C([-h, 0], R^n) \to R^n$ is Lipschitzian. The following lemma is known.

Lemma 1 (see kuang [6]). Assume that $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are nonnegative continuous scalar functions: $R_{+0} \to R_{+0}$ such that $\omega_1(0) = \omega_2(0) = 0$, $lim_{r\to+\infty}\omega_1(r) = +\infty$ and that $V : C \to R$ is a continuous differentiable scalar functional such that for a special set S of solutions of (2.1) the following are satisfied:

$$V(\phi) \geq \omega_1(|\phi(0)|), \quad V'(\phi)|_{(2.1)} \leq -\omega_2(|\phi(0)|). \tag{2.2}$$

Then the solution $x = 0$ of (2.1) is uniformly stable and every solution is bounded. If in addition, $\omega_2(r) > 0$ for $r > 0$, then $x = 0$ is globally asymptotically stable.

3. **The stability analysis for the system (1.1).** Let us consider model (1.1). It is easy to show that $n = S + I + R$ is bounded. In fact,

$$n'(t) = S'(t) + I'(t) + R'(t) = b - \mu n(t) - cI(t) \leq b - \mu n(t), \tag{3.1}$$

and there exists $T = T(\epsilon, n(0)) > 0$ for any sufficiently small $\epsilon > 0$ such that

$$n(t) \leq \epsilon + \frac{b}{\mu} \qquad \text{for} \quad t > T. \tag{3.2}$$

Let

$$\Omega_\epsilon = \{(S, I, R) \in R^3_+ | n = S + I + R \leq \epsilon + \frac{b}{\mu}\}. \tag{3.3}$$

For any sufficiently small $\epsilon > 0$, the region Ω_ϵ is an attractive region of R^3_{+0} for any trajectory of (1.1) with $n(0) > \epsilon + \frac{b}{\mu}$. Accordingly, we will restrict the stability analysis of the equilibria of (1.1) with respect to the compact subset Ω_ε of R^3_{+0}.

By computation, it is easy to obtain the following two conclusions:

i. The endemic equilibrium is given by

$$E_+ = (S^*, I^*, R^*) = (\frac{\mu + \lambda + c}{\beta_1 + \beta_2}, \frac{(b - \mu S^*)}{(\beta_1 + \beta_2)S^*}, \frac{\lambda(b - \mu S^*)}{\mu(\beta_1 + \beta_2)S^*})$$

provided that

$$\frac{b}{\mu} > \frac{\mu + \lambda + c}{\beta_1 + \beta_2}. \tag{3.4}$$

ii. The disease-free equilibrium point is given by $E_0 = (S^*, I^*, R^*) = (\frac{b}{\mu}, 0, 0)$, which exists for all parameter values.

We can now prove the following theorem.

Theorem 3.1. (i) E_0 of (2.18) is globally asymptotically stable with respect to Ω_ε whenever

$$\frac{b}{\mu} < \frac{\mu + \lambda + c}{\beta_1 + \beta_2}. \tag{3.5}$$

(ii) When E_+ of (1.1) exists (that is, (3.4) is true), it is locally asymptotically stable.

Proof. (i) We consider the following Lyapunov functional:

$$V(x_t) = I(t) + w_1 R(t) + w_2 \int_0^h f(s) \int_{t-s}^t I(u) du ds + \frac{w_3}{2}(S(t) - S^*)^2,$$

where $w_i > 0$ $(i = 1, 2, 3)$ and $S^* = \frac{b}{\mu}$. Then $V(x_t) \geq \min\{1, w_1, \frac{w_3}{2}\}(I(t) + R(t) + (S(t) - S^*)^2)$. The time derivative of $V(x_t)$ along the solution of system (1.1) becomes

$$\begin{aligned}
\dot{V}(x_t)|_{(1.1)} =& \beta_1 S(t) \int_0^h f(s) I(t-s) ds + \beta_2 S(t) I(t) - (\mu + \lambda + c) I(t) \\
&+ w_1(\lambda I(t) - \mu R(t)) + w_2 I(t) - w_2 \int_0^h f(s) I(t-s) ds \\
&+ w_3(S - S^*)[-\beta_1 S(t) \int_0^h f(s) I(t-s) ds - \beta_2 S(t) I(t) \\
&- \mu(S(t) - S^*)] \\
=& - w_3 \mu (S - S^*)^2 + [w_1 \lambda + w_2 - (\mu + c + \lambda)] I(t) - w_1 \mu R(t) \\
&+ \beta_2 S(t) I(t)[1 - w_3(S - S^*)] \\
&+ [\beta_1 S - w_2 - w_3 \beta_1 S(S - S^*)] \int_0^h f(s) I(t-s) ds \\
=& - w_3 \mu (S - S^*)^2 + [w_1 \lambda + w_2 + \sigma - (\mu + c + \lambda)] I(t) - w_1 \mu R(t) \\
&+ I(t)[\beta_2 S(t)(1 - w_3(S - S^*)) - \sigma] \\
&+ [\beta_1 S - w_2 - w_3 \beta_1 S(S - S^*)] \int_0^h f(s) I(t-s) ds
\end{aligned} \tag{3.6}$$

where $\sigma > 0$ is some positive constant chosen later. Let us choose $\sigma > 0, w_i > 0$ $(i = 1, 2, 3)$ satisfying

$$w_1 \lambda + w_2 + \sigma < \mu + c + \lambda, \tag{3.7}$$
$$\beta_1 (1 + w_3 S^*)^2 < 4 w_2 w_3, \tag{3.8}$$
$$\beta_2 (1 + w_3 S^*)^2 < 4 w_3 \sigma. \tag{3.9}$$

The choice of (3.8) is possible if $w_2 > \beta_1 S^*$. In fact, (3.8) is equivalent to

$$\beta_1 {S^*}^2 w_3^2 + 2(\beta_1 S^* - 2w_2) w_3 + \beta_1 < 0,$$

which is true given that $w_3 = \frac{1}{S^*}$ if $\beta_1 S^* - 2w_2 < 0$ and $(\beta_1 S^* - 2w_2)^2 > (\beta_1 S^*)^2$. Similarly, the choice of (3.9) is possible if $\sigma > \beta_2 S^*$. Together $w_2 > \beta_1 S^*, \sigma > \beta_2 S^*$

with (3.7), we can choose $\sigma, w_i > 0$ $(i = 1, 2, 3)$, satisfying (3.7), (3.8) and (3.9) if

$$\mu + c + \lambda - w_1\lambda > w_2 + \sigma > (\beta_1 + \beta_2)S^* = \frac{b(\beta_1 + \beta_2)}{\mu},$$

which is possible because of assumption (3.5). Further, (3.8) and (3.9) ensure that the coefficients of the integral and $I(t)$ in (3.6) are negative definite for all $S > 0$. Hence from (3.6), $\dot{V}(x_t)|_{(1.1)}$ is negative definite and is equal to zero if and only if $(S, I, R) = E_0$. This completes the proof of (i).

For(ii),we change the variables to $u_1 = S - S^*, u_2 = I - I^*, u_3 = R - R^*$. From equation (1.1), we obtain

$$\begin{cases} u_1' = & -\beta_1(u_1 + S^*)\int_0^h f(s)u_2(t-s)ds - \beta_1(u_1 + S^*)I^* \\ & -\beta_2(u_1 + S^*)(u_2 + I^*) - \mu u_1 - \mu S^* + b, \\ u_2' = & \beta_1(u_1 + S^*)\int_0^h f(s)u_2(t-s)ds + \beta_1(u_1 + S^*)I^* \\ & +\beta_2(u_1 + S^*)(u_2 + I^*) - (\mu + \lambda + c)(u_2 + I^*), \\ u_3' = & \lambda(u_2 + I^*) - \mu(u_3 + R^*). \end{cases} \tag{3.10}$$

The corresponding linear part becomes

$$\begin{cases} u_1' = -[(\beta_1 + \beta_2)I^* + \mu]u_1 - \beta_1 S^* \int_0^h f(s)u_2(t-s)ds - \beta_2 S^* u_2, \\ u_2' = (\beta_1 + \beta_2)I^* u_1 + \beta_1 S^* \int_0^h f(s)u_2(t-s)ds + \beta_2 S^* u_2 - (\mu + \lambda + c)u_2, \\ u_3' = \lambda u_2 - \mu u_3. \end{cases} \tag{3.11}$$

Consider the Lyapunov functional

$$V(u_t) = \frac{1}{2}w_1(u_1 + u_2)^2 + \frac{1}{2}u_2^2 + \frac{1}{2}w_3 u_3^2 + \frac{1}{2}\beta_1 S^* \int_0^h f(s)\int_{t-s}^t u_2^2(v)dvds,$$

where $w_i > 0 (i = 1, 3)$. Note that

$$V(u_t) \geq \frac{1}{2}w_1(u_1 + u_2)^2 + \frac{1}{2}u_2^2 + \frac{1}{2}w_3 u_3^2.$$

The time derivative of $V(u_t)$ along the solution of (3.11) is

$$\begin{aligned} V'(u_t) = & w_1(u_1 + u_2)[-\mu u_1 - (\lambda + \mu + c)u_2] + (\beta_1 + \beta_2)I^* u_1 u_2 - \\ & (\lambda + \mu + c - \beta_2 S^*)u_2^2 \\ & + \beta_1 S^* u_2 \int_0^h f(s)u_2(t-s)ds + w_3 u_3 u_3' + \frac{1}{2}\beta_1 S^* u_2^2 \\ & - \frac{1}{2}\beta_1 S^* \int_0^h f(s)u_2^2(t-s)ds \\ = & -w_1\mu u_1^2 - [w_1(\lambda + \mu + c) + (\lambda + \mu + c - \beta_2 S^*)]u_2^2 \\ & + [-w_1\mu - w_1(\lambda + \mu + c) + (\beta_1 + \beta_2)I^*]u_1 u_2 + w_3\lambda u_2 u_3 - w_3\mu u_3^2 \\ & + \beta_1 S^* u_2 \int_0^h f(s)u_2(t-s)ds + \frac{1}{2}\beta_1 S^* u_2^2 - \frac{1}{2}\beta_1 S^* \int_0^h f(s)u_2^2(t-s)ds. \end{aligned} \tag{3.12}$$

Choose $w_1 > 0$ satisfying

$$w_1(\lambda + 2\mu + c) = (\beta_1 + \beta_2)I^*. \tag{3.13}$$

Note that $(\beta_1+\beta_2)S^*=\lambda+\mu+c$ and for any u_2,

$$\beta_1 S^* u_2\int_0^h f(s)u_2(t-s)ds\le \frac{1}{2}\beta_1 S^* u_2^2+\frac{1}{2}\beta_1 S^*\int_0^h f(s)u^2(t-s)ds.$$

Then (3.12) becomes

$$V'(u_t)\quad\le\quad -w_1\mu u_1^2-w_1(\lambda+\mu+c)u_2^2-w_3\mu u_3^2+w_3\lambda u_2u_3. \tag{3.14}$$

Further if we choose w_3 satisfying $(\lambda w_3)^2-4w_3w_1\mu(\lambda+\mu+c)<0$, then

$$0<w_3<\frac{4(\lambda+\mu+c)\mu w_1}{\lambda^2}=\frac{4(\lambda+\mu+c)\mu(\beta_1+\beta_2)I^*}{\lambda^2(\lambda+2\mu+c)}.$$

Form (3.14) we obtain that $V'(u_t)$ is negative definite for any u_2 and u_3. This completes the proof of (ii).

Theorem 3.2. The disease-free equilibrium E_0 of (1.1) is globally attractive with respect to Ω_ε whenever

$$\frac{b}{\mu}=\frac{\mu+\lambda+c}{\beta_1+\beta_2}.$$

Proof. for any solution $(S(t),I(t),R(t))$ of (1.1), let us first consider the case (a): $S(t)>S^*$ for all $t>t_0$. In this case, we see that for all $t\ge t_0$,

$$\begin{aligned}(S(t)-S^*)'+I'(t)+R'(t)\quad&=-\mu(S(t)-S^*)-(\mu+c)I(t)-\mu R(t)\\ &\le -\mu[(S(t)-S^*)+I(t)+R(t)].\end{aligned}$$

Thus,

$$\lim_{t\to+\infty}S(t)=S^*,\quad \lim_{t\to+\infty}I(t)=\lim_{t\to+\infty}R(t)=0.$$

Now let us consider the case (b): $\varphi_1<S^*$ and $S(t)<S^*$ for all $t\ge t_0$. Set

$$G=\{\varphi=(\varphi_1,\varphi_2,\varphi_3)\in C|0\le\varphi_1\le S^*,0\le\varphi_2,0\le\varphi_3\}.$$

We define

$$V(\varphi)=\varphi_2(0)+S^*\beta_1\int_0^h f(s)\int_{-s}^0\varphi_2(u)duds.$$

Then

$$\begin{aligned}V'(\varphi)|_{(1.1)}&=-(S^*-S(t))\beta_1\int_0^h f(s)\varphi_2(-\tau)d\tau+\beta_2S(t)I(t)+\beta_1S^*I(t)\\ &\quad-(\lambda+c+\mu)I(t)\\ &\le -(S^*-S(t))\beta_1\int_0^h f(s)\varphi_2(-\tau)d\tau\le 0.\end{aligned}\tag{3.15}$$

Since $S(t)\le n(t)\le\frac{b}{\mu}=S^*$, $V(\varphi)$ is a Liapunov function on the subset G in C. Let

$$Q=\{\varphi\in G|\dot V(\varphi)|_{(1.1)}=0\}$$

and M be the largest set in Q, which is invariant with respect to (1.1). Clearly, M is not empty, since $(S^*,0,0)\in M$. From (3.15) we see that that $V'(\varphi)|_{(1.1)}=0$ only if $S^*-\varphi_1(0)=0$ and $\varphi_2(0)=0$. Note that $S^*-\varphi_1(0)=S^*-S(t)=0$, and the equations of (1.1) imply that $\varphi_2=0$. Thus, we always have $\varphi_2=0$ if $V'(\varphi)|_{(1.1)}=0$. Observe that G is invariant with respect to (1.1) and that any solution of (1.1) is bounded by (3.1). Thus, it follows from the Liapunov-LaSalle invariance principle that $\lim_{t\to+\infty}I(t)=0$. Hence, $\lim_{t\to+\infty}R(t)=0$ by $\lim_{t\to+\infty}I(t)=0$ and the last equation of (1.1). Furthermore, note that boundedness of $S(t)$ and

$\int_0^h f(\tau)I(t-\tau)d\tau \to 0$ as $t \to +\infty$ by $\lim_{t\to+\infty} I(t) = 0$, we can also easily have that $\lim_{t\to+\infty} S(t) = S^*$ by the first equation of (1.1).

We now must consider case (c). There is $\hat{\tau}$ with $0 \le \hat{\tau} < h$ such that $\varphi_1(-\hat{\tau}) \ge S^*$ and $S(t) < S^*$ for all $t \ge t_0$, or there is some $\hat{t}_0 \ge t_0$ such that $S(\hat{t}_0) = S^*$.

If $S(t) < S^*$ for all $t \ge t_0$, observe that system (1.1) is autonomous and the solution of (1.1) with any initial function $\varphi \in C$ is unique, and by the same argument as that used in case (b) with $t_0 = t_0 + 2h$, we can show that $\lim_{t\to+\infty} S(t) = S^*$ and $\lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} R(t) = 0$.

If there is some $\hat{t}_0 \ge t_0$ such that $S(\hat{t}_0) = S^*$, by (1.1) we see that

$$\begin{aligned}(S-S^*)'(\hat{t}_0) &= -\beta_1 S(\hat{t}_0)\int_0^h f(\tau)I(\hat{t}_0-\tau)d\tau - \beta_2 S(\hat{t}_0)I(\hat{t}_0) - \mu S(\hat{t}_0) + b \\ &= -\beta_1 S(\hat{t}_0)\int_0^h f(\tau)I(\hat{t}_0-\tau)d\tau - \beta_2 S(\hat{t}_0)I(\hat{t}_0) < 0.\end{aligned}$$

Thus, for all $t > \hat{t}_0$, $S(t) - S^* < 0$, i.e. $S(t) < S^*$, Again by the same argument as used in case (b) with t_0 replaced by $\hat{t}_0 + 2h$, we can show that $\lim_{t\to+\infty} S(t) = S^*$ and $\lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} R(t) = 0$.

This completes the proof of theorem 3.2.

4. **The global asymptotic stability of the endemic equilibrium.**

In the following, we consider the global asymptotic stability of the endemic equilibrium E_+ by applying the same techniques as that used in [5]. Let us define

$$T \equiv \int_0^h \tau f(\tau)d\tau.$$

Theorem 4.1. If there is some $\tilde{S}$ satisfying $S^* < \tilde{S} < \frac{b}{\mu+c+\lambda} \stackrel{def}{=} \Delta$ such that the following conditions hold true:

$$\text{i. } h < \min\{(2\beta_1\tilde{S})^{-1}, \frac{\tilde{S}-S^*}{b-\mu S^*}\};$$

$$\text{ii. } b \le \tilde{S}[(\beta_1+\beta_2)(\Delta-\tilde{S})+\mu],$$

then the endemic equilibrium E_+ of system (1.1) is globally asymptotically stable.

Proof. For any positive constant $\tilde{S}$ satisfying $S^* < \tilde{S} < \Delta$, define

$$\Omega_{\varepsilon,\tilde{S}} \equiv \{(S,I,R) \in \Omega_\varepsilon | S < \tilde{S}\}.$$

We can show that the following two assertions are true.

Assertion A: For any positive constant $\tilde{S}$ satisfying $S^* < \tilde{S} < \Delta$, if $h < (2\beta\tilde{S})^{-1}$, then any solution of (1.1) will not ultimately stay in $\Omega_\varepsilon \backslash \Omega_{\varepsilon,\tilde{S}}$.

Assertion B: If conditions (i) and (ii) hold, then any solution of (1.1) will eventually stay in $\Omega_{\varepsilon,\tilde{S}}$.

The proof of assertion A is similar to that show in [5]. Thus, we show the proof of assertion B only.

In fact, if not, by assertion A, there is some solution $(S(t), I(t), R(t))$ of (1.1) such that, for any positive constant $\tilde{S}_1$ satisfying $S^* < \tilde{S}_1 < \tilde{S} < \Delta$, there are two time sequences $\{t_n\}$ and $\{t'_n\}$ with $t_n < t'_n < t_{n+1} < t'_{n+1}$, $t_n \to +\infty$ and $t'_n \to +\infty$, such that

$$S(t_n) = \tilde{S}_1,\ S(t'_n) = \tilde{S},\ \tilde{S}_1 \le S(t) \le \tilde{S} \text{ for } t_n \le t \le t'_n \tag{4.1}$$

and $\dot{S}(t_n') \geq 0$. From (1.1), we get

$$\begin{aligned} \tilde{S} - \tilde{S}_1 = \ & S(t_n') - S(t_n) \\ = \ & -\beta_1 \int_{t_n}^{t_n'} S(v) \int_0^h f(\tau) I(v-\tau) dv d\tau - \beta_2 \int_{t_n}^{t_n'} S(v) I(v) dv d\tau \\ & -\mu \int_{t_n}^{t_n'} S(v) dv + b(t_n' - t_n), \end{aligned} \tag{4.2}$$

which, together with (4.1), yields

$$\begin{aligned} b(t_n' - t_n) = & \tilde{S} - \tilde{S}_1 + \beta_1 \int_{t_n}^{t_n'} S(v) \int_0^h f(\tau) I(v-\tau) dv d\tau \\ & + \beta_2 \int_{t_n}^{t_n'} S(v) I(v) dv d\tau + \mu \int_{t_n}^{t_n'} S(v) dv \\ & \geq \tilde{S} - \tilde{S}_1 + \mu \tilde{S}_1 (t_n' - t_n). \end{aligned}$$

Thus,

$$t_n' - t_n \geq \frac{\tilde{S} - \tilde{S}_1}{b - \mu \tilde{S}_1} \tag{4.3}$$

and

$$\frac{\tilde{S} - \tilde{S}_1}{b - \mu \tilde{S}_1} \to \frac{\tilde{S} - \tilde{S}_1}{b - \mu \tilde{S}_1} > h \quad \text{as} \quad \tilde{S}_1 \to S^*. \tag{4.4}$$

By condition (i), from (1.1), we also have

$$\begin{aligned} (S(t) + I(t))' = & \ -\mu S(t) - (\mu + \lambda + c) I(t) + b \\ \geq & \ -(\mu + c + \lambda)(S(t) + I(t)) + b, \end{aligned} \tag{4.5}$$

which, together with $\tilde{S} < \Delta$, implies that, for any sufficiently small positive constant η, there is a large $T_1 > 0$ such that for $t \geq T_1$,

$$S(t) + I(t) \geq \Delta - \eta \stackrel{def}{=} H(\eta) > \tilde{S}. \tag{4.6}$$

Thus, for large t_n' and $\tilde{S}_1$ that is sufficiently close to S^*, from (4.6), we get

$$I(t_n' - \tau) \geq H(\eta) - S(t_n' - \tau). \tag{4.7}$$

It follows from (4.3) and (4.4) that we have

$$t_n \leq t_n' - \tau \leq t_n', \quad \text{for} \quad 0 \leq \tau \leq h. \tag{4.8}$$

Thus, it follows from (4.1) and (4.7) that

$$I(t_n' - \tau) \geq H(\eta) - \tilde{S} > 0, \quad 0 \leq \tau \leq h. \tag{4.9}$$

From (4.1) and (4.6), we also have that

$$I(t_n' - \tau) \geq H(\eta) - \tilde{S} > 0, \quad 0 \leq \tau \leq h. \tag{4.10}$$

Equations (4.9) and (4.10) and condition (ii) enable us to show that $\dot{S}(t_n') < 0$, which is a contradiction to $\dot{S}(t_n') \geq 0$.

In fact, from (1.1), (4.9) and (4.10) , we have that

$$\begin{aligned} S'(t_n') \ & = -\beta_1 S(t_n') \int_0^h f(\tau) I(t_n' - \tau) d\tau - \beta_2 S(t_n') I(t_n') - \mu S(t_n') + b \\ & = -\beta_1 \tilde{S} \int_0^h f(\tau) I(t_n' - \tau) d\tau - \beta_2 \tilde{S} I(t_n') - \mu \tilde{S} + b \\ & \leq -(\beta_1 + \beta_2) \tilde{S} [H(\eta) - \tilde{S}] - \mu \tilde{S} + b \\ & \equiv G(\tilde{S}, \eta). \end{aligned} \tag{4.11}$$

By condition (ii), we see that

$$G(\tilde{S}, 0) = -\tilde{S}[(\beta_1 + \beta_2)(\Delta - \tilde{S}) + \mu] + b < 0. \tag{4.12}$$

Thus, it follows from (4.11), (4.12) and the continuity of $G(\tilde{S},\eta)$ with respect to η that $S'(t'_n) \le G(\tilde{S},\eta) < 0$ for sufficiently small $\eta > 0$. This proves our second assertion.

Now, by assertions A and B, we can complete the proof of Theorem 4.1 by using the following Liapunov functional:

$$\begin{aligned} V(t,S,I_t,R) = \quad & S - S^* ln\tfrac{S}{S^*} + \tfrac{1}{2}w_1(S - S^* + I - I^*)^2 \\ & + \tfrac{1}{2}w_2 \int_0^h f(\tau) \int_{t-\tau}^t (I(u) - I^*)^2 du d\tau, \end{aligned}$$

where w_1 and w_2 are some positive constants chosen later. By assertion B, since $S^* < \tilde{S} < \Delta$, there is a sufficiently large time $T_2 > t_0$ such that for $t > T_2$,

$$S(t) \le \tilde{S}. \tag{4.13}$$

The derivative $\dot{V}(t,S,I_t,R)$ of $V(t,S,I_t,R)$ along the solution of (1.1) satisfies

$$\dot{V}(t,S,I_t,R) = -\delta[(S - S^*)^2 + (I - I^*)^2] - \frac{1}{2}\int_0^h f(\tau)[Q(t,\tau)B(t,S)Q^T(t,\tau)]d\tau \tag{4.14}$$

for all $t \ge T_2$, where δ is some positive constant chosen later:

$$B(t) = \begin{pmatrix} 2(w_1\mu - \delta + \frac{(\beta_1+\beta_2)I^*+\mu}{S}) & w_1[(\beta_1+\beta_2)S^* + \mu] + \beta_2 & \beta_1 \\ w_1[(\beta_1+\beta_2)S^* + \mu] + \beta_2 & 2[w_1(\beta_1+\beta_2)S^* - w_2 - \delta] & 0 \\ \beta_1 & 0 & 2w_2 \end{pmatrix},$$

$$Q(t,\tau) = (S(t) - S^*,\ I(t) - I^*,\ I(t-\tau)).$$

We can easily see that the symmetric matrix $B(S(t))$ is positive dominant diagonal for every $t \ge T_2$ if

$$\frac{2[(\beta_1+\beta_2)I^* + \mu]}{S} - 4\delta - (\beta_1+\beta_2) > w_1[(\beta_1+\beta_2)S^* - \mu] - 2\delta > 2w_2 > \beta_1 \tag{4.15}$$

Let us choose δ small enough such that

$$0 < \delta < \frac{(\beta_1+\beta_2)}{2\tilde{S}}(\Delta - \tilde{S}).$$

Then, for all $t \ge T_2$,

$$\frac{2[(\beta_1+\beta_2)I^* + \mu]}{S} - 4\delta - (\beta_1+\beta_2) > \beta_1 + \beta_2 > \beta_1.$$

Note that $(\beta_1+\beta_2)S^* - \mu = \lambda + c > 0$, thus, we can easily choose the positive constants w_1, w_2 and δ satisfying (4.15). Hence, it follows from (4.14) that for all $t > T_2$,

$$\dot{V}(t,S,I_t,R) \le -\delta[(S - S^*)^2 + (I - I^*)^2],$$

from which we have that for all $t > T_2$,

$$V(t,S,I_t,R) \le V(T_2, S(T_2), I_{T_2}, R(T_2)) - \delta\int_{T_2}^t [(S(u) - S^*)^2 + (I(u) - I^*)^2]du.$$

Thus,

$$\int_{t_0}^\infty (S(u) - S^*)^2 du < +\infty, \quad \int_{t_0}^\infty (I(u) - I^*)^2 du < +\infty.$$

By (1.1), we see that $d/dt(S(t) - S^*)^2$ and $d/dt(I(t) - I^*)^2$ are also uniformly bounded for $t \ge t_0$. Thus, the well-known Barbalat's lemma shows that

$$(S(t) - S^*)^2 + (I(t) - I^*)^2 \to 0 \quad \text{as } t \to +\infty. \tag{4.16}$$

That $R(t) \to R^*$ as $t \to +\infty$ is an immediate result of (4.16) and the third equation of (1.1).

The proof of Theorem 4.1 is completed.

Theorem 4.1 gives a sufficient condition for the endemic equilibrium of system (1.1) to be globally asymptotically stable and also needs that the system with sufficiently small delay h. Theorem 4.1 extends the results in [5].

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