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COMPLEX BEHAVIOR IN A DISCRETE COUPLED LOGISTIC MODEL FOR THE SYMBIOTIC INTERACTION OF TWO SPECIES

Ricardo López-Ruiz

Department of Computer Science and BIFI, Facultad de Ciencias-Edificio B
Universidad de Zaragoza - 50009 Zaragoza (Spain)

Danièle Fournier-Prunaret

Systèmes Dynamiques (SYD), L.E.S.I.A.
Institut National de Sciences Appliquées - 31077 Toulouse Cedex (France)

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Abstract. A symmetrical cubic discrete coupled logistic equation is proposed to model the symbiotic interaction of two isolated species. The coupling depends on the population size of both species and on a positive constant λ, called the *mutual benefit*. Different dynamical regimes are obtained when the mutual benefit is modified. For small λ, the species become extinct. For increasing λ, the system stabilizes in a synchronized state or oscillates in a two-periodic orbit. For the greatest permitted values of λ, the dynamics evolves into a quasiperiodic, into a chaotic scenario, or into extinction. The basins for these regimes are visualized as colored figures on the plane. These patterns suffer different changes as consequence of basins' bifurcations. The use of the critical curves allows us to determine the influence of the zones with different numbers of first-rank preimages in those bifurcation mechanisms.

1. Dynamics of isolated species: The logistic model.

Imagine an island with no contact with the exterior. Living species there cannot migrate in search of a new land with affordable resources. Thus, for instance, if initially the island has as inhabitants a couple of rabbits, they will reproduce exponentially. This expansion regime will colonize the whole island in a few generations. Hence, the island will become overpopulated. At that point a new dynamical regime will be present, with a natural population control mechanism because of the overcrowding.

If x_n represents the population after n generations, let us suppose this variable is bounded in the range $0 < x_n < 1$. The *activation or expanding phase* is controlled by the term μx_n proportional to the current population x_n and to the constant *growth rate* μ. Resource limitations bring the system to an *inhibition or contracting phase* directly related to overpopulation. The term can denote how far the system is from overcrowding. Therefore, if we take the product of both terms as the most simple approach to the population dynamics, the model

$$x_{n+1} = \mu\, x_n(1 - x_n) \tag{1}$$

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gives an account of its evolution. This is the so-called logistic map, where $0 < \mu < 4$ in order to assure $0 < x_n < 1$. This discrete equation has been a subject of study in the last century as a tool to be applied to the most diverse phenomenology [1] or as an object interesting to analyze by itself from a mathematical point of view [2, 3]. The continuous version of this model was originally introduced by Verhulst [4] in the nineteenth century as a counterpart to the Malthusian theories of human overpopulation.

When the growth rate is modified the dynamical behavior of the logistic equation is as follows:

(i) $\underline{0 < \mu < 1}$: The growth rate is not big enough to stabilize the population. It will drop and the species will become extinct.
(ii) $\underline{1 < \mu < 3}$: A drastic change is obtained when μ is greater than 1. A non-vanishing equilibrium between the two competing forces, reproduction on one hand and resource limitation on the other, is now possible. The population reaches, independent of its initial conditions, a fixed value that is maintained in time.
(iii) $\underline{3 < \mu < 3.57}$: A cascade of sudden changes causes the population to oscillate in cycles of period 2^n, where n increases from 1, when μ is close to 3, to infinity when μ is approaching the critical value 3.57. This is called *the period-doubling cascade*.
(iv) $\underline{3.57 < \mu < 3.82}$: When the parameter moves, the system alternates between periodical behaviors with high periods on parameter interval windows and *chaotic regimes* for parameter values not located in intervals. The population can be unpredictable although the system is deterministic. The chaotic regimes are observed for a given value of μ on sub-intervals of $[0, 1]$.
(v) $\underline{3.82 < \mu < 3.85}$: The orbit of period 3 appears for $\mu = 3.82$ after a regime where unpredictable bursts, called *intermittences*, have become rarer until they disappear in the three-periodic time signal. As the Sarkovskii theorem tell us, the existence of the period-3 orbit means, that all periods are possible for population dynamics, although, in this case, they are not observable due to their instability. What it is observed in this range is the period-doubling cascade $3 \cdot 2^n$.
(vi) $\underline{3.85 < \mu < 4}$: Chaotic behavior with periodic windows is observed in this interval.
(vii) $\underline{\mu = 4}$: The chaotic regime is obtained on the whole interval $[0, 1]$. This specific regime produces dynamics, that appears random. The dynamics has lost its determinism and the population evolves as a random number generator.

Therefore, there are essentially three remarkable dynamical behaviours in this system: the period-doubling route to chaos when μ is approximately 3.57 [5], the time signal complexification by intermittence when μ is approximately 3.82 [6], and the random-like dynamics when $\mu = 4$.

2. **Dynamics of two isolated species: A coupled logistic model.** Let us suppose now, under a similar scheme of expansion and contraction, that two symbiotic species (x_n, y_n) are living on the island. Each evolves following a logistic-type

dynamics,

$$x_{n+1} = \mu(y_n)\, x_n(1-x_n), \tag{2}$$
$$y_{n+1} = \mu(x_n)\, y_n(1-y_n). \tag{3}$$

In this model, the symbiotic interaction between species causes the growth rate $\mu(z)$ to vary with time. The interaction depends on the population size of the others and on a positive constant λ that we call the *mutual benefit*. As the equations show, we are thinking of a symmetrical interaction. Concretely, the particular dynamics of each species is a logistic map whose parameter μ_n is not fixed, $x_{n+1} = \mu_n\ x_n(1-x_n)$, but which itself is forced to remain in the interval $(1,4)$. The existence of a nontrivial fixed point at each step n ensures the nontrivial evolution of the system [7]. The simplest election for this growth rate is a linear function expanding at the interval $(1,4)$:

$$\mu(z) = \lambda\ (3z+1), \tag{4}$$

with the mutual benefit λ being a positive constant. The study has discovered to have sense in the range $0 < \lambda < 1.084$. Thus, the model obtained to mimic the dynamics of two isolated symbiotic species takes the form:

$$x_{n+1} = \lambda\ (3y_n+1)\, x_n(1-x_n), \tag{5}$$
$$y_{n+1} = \lambda\ (3x_n+1)\, y_n(1-y_n). \tag{6}$$

This application can be represented by $T_\lambda : [0,1]\times[0,1] \to [0,1]\times[0,1]$, $T_\lambda(x_n, y_n) = (x_{n+1}, y_{n+1})$, where λ is a real and adjustable parameter. In the following we shall write T instead of T_λ, as the dependence on the parameter λ is understood. Let us observe that when $y_n = 0$ or $x_n = 0$, the logistic dynamics for one isolated species is recovered. In this case the parameter λ takes the role of the parameter μ.

At this point we must comment that the different choices of μ_n give a wide variety of dynamical behaviours. For instance, the application of this idea produces the on-off intermittence phenomenon when is chosen random [8] or the adaptation to the edge of chaos when μ_n is a constant with a small time perturbation [9]. Other systems built under this mechanism are models (a), (b), and (c) presented in [7, 10]. Equations (5−6) correspond to model (a) of those works. Model (b) has been studied in detail in [11], and a similar investigation of model (c) is presented in [12].

In Sections 3 and 4, we study more accurately the model (5−6) from a dynamical point of view. To summarize, we explain first the dynamical behavior of the coupled logistic system (5−6). When λ is modified, this is as follows:

(i) <u>$0 < \lambda < 0.75$</u>: The mutual benefit is too small to allow a stable coexistence of both species and they will disappear.
(ii) <u>$0.75 < \lambda < 0.86$</u>: A sudden change is obtained when λ is greater than 0.75. Both populations are synchronized to a stable non-vanishing fixed quantity when the initial populations overcome certain critical values. If the initial species are under these limits both will become extinct.
(iii) <u>$0.86 < \lambda < 0.95$</u>: The system is now bi-stable. Each one of the species oscillates out-of-phase between the same two fixed values. This is a lag-synchronized state; that is, a stable two-period orbit. In this range, there is still the possibility of extinction when the initial populations are very small or close to the overcrowding.

(iv) <u>$0.95 < \lambda < 1.03$</u>: The system is no longer on a periodic orbit. It acquires a new frequency and the dynamics is now quasiperiodic. Both populations oscillate among infinitely many different states. Synchronization is lost. There are in this regime periodic windows where the system becomes lag-synchronized. Also, for initial populations nearly zero and for $\lambda < 1$, the species can not survive.

(v) <u>$1.03 < \lambda < 1.08$</u>: The system is now in a chaotic regime. It is characterized by a noisy-like small oscillation around a synchronized state with non-periodic unpredictable bursts. Periodic oscillations can be also obtained for some particular values of the mutual benefit. Some other initial conditions are not meaningful or interpretable in this scheme because the system is going outward from the square $[0,1] \times [0,1]$ and evolves toward infinity. The system "crashes." This sudden "damage" is interpreted as some kind of catastrophe, provoking the extinction of species.

Although the equations are formed by logistic-type components, the logistic effects have been lost and a completely new scenario emerges when they are coupled. In this case, the symbiotic interaction causes the species to reach different stable states. Depending on the mutual benefit, the system can reach extinction, a fixed synchronized state, a bi-stable lag-synchronized configuration, an oscillating dynamics among infinite possible states, or a chaotic regime. We must highlight in this model the phenomenon of synchronization in the periodic regime [13], the transition to chaos by the Ruelle-Takens route [14], and the bursting events around a noisy-like synchronized state in the chaotic regime [15]. All these behaviors are caused by the symbiotic coupling of the species and are not predictable from the properties of the individual logistic evolution of any of them. Moreover, this interaction implies a mutual profit for both species. In fact, when $\mu < 1$ one of the isolated species is extinct, but it can survive for $\lambda < 1$ if a small number of individuals of the other species is aggregated to the island. Hence, the symbiosis appears to be well held in this cubic model.

3. **Stable attractors: Symmetry and bifurcations.** First, for the sake of clarity, we summarize the dynamical behavior of model (1) when the mutual benefit λ is inside the interval $0 < \lambda < 1.0843$. The different parameter regions where the mapping T has stable attractors are given in the next table. The meanings of all

INTERVAL	NUMBER OF ATTRACTORS	ATTRACTORS
$0 < \lambda < 0.75$	1	p_0
$0.75 < \lambda < 0.866$	2	p_0, p_4
$0.866 < \lambda < 0.957$	2	$p_0, p_{5,6}$
$0.957 < \lambda < 1$	2	p_0, pair of invariant closed curves
$1 < \lambda < 1.03$	1	pair of invariant closed curves
$1.03 < \lambda < 1.032$	1	pair of weakly chaotic rings
$1.032 < \lambda < 1.0843$	1	symmetric chaotic atractor (or frequency lockings)

TABLE 1. Unfolding of T as function of λ.

these attractors are explained in the following subsections.

3.1. **Symmetry.** This model has reflection symmetry P through the diagonal $\Delta = \{(x,x), x \in \Re\}$. If $P(x,y) = (y,x)$ then T commutes with P:

$$T[P(x,y)] = P[T(x,y)]. \tag{7}$$

Note that the diagonal is T-invariant, $T(\Delta) = \Delta$. In general, if Ω is an invariant set of T, $T(\Omega) = \Omega$, so also is $P(\Omega)$, due to the commutation property: $T[P(\Omega)] = P[T(\Omega)] = P(\Omega)$. It means that if $\{p_i, i \in N\}$ is an orbit of T, so is $\{P(p_i), i \in N\}$. In fact, if some bifurcation happens in the half plane below the diagonal, it occurs in the above half plane, and vice versa. The dynamical properties of the two halves of phase space separated by the diagonal are interconnected by the symmetry. Also if the set Γ verifies $P(\Gamma) = \Gamma$, so is $T(\Gamma)$. Then the T-iteration of a reflection symmetrical set continues to keep the reflection symmetry through the diagonal. It is worth noting that the square $[0,1] \times [0,1]$ is invariant for $\mu < 1$, but not anymore for $\mu > 1$.

3.2. **Fixed points, two-cycles, and closed invariant curves.** We focus our attention on bifurcations playing an important role in the dynamics, those happening in the interval $0 < \lambda < 1.0843$. In this range, there exist stable attractors for each value of λ, and it will make sense to study their basins of attraction; that is, the initial populations leading to the each of the existing final asymptotic configurations.

The restriction of T to the diagonal is a one-dimensional cubic map, which is given by the equation $x_{n+1} = \lambda\,(3x_n + 1)x_n(1 - x_n)$. The restriction of the map T to the axes reduces to the logistic map $x_{n+1} = f(x_n)$ with $f(x) = \lambda\, x(1-x)$. Thus the solutions of $x_{n+1} = x_n$ are the fixed points p_0, p_3, p_4 on the diagonal and p_1, p_2 on the axes:

$$\begin{aligned}
p_0 &= (0,0),\\
p_1 &= \left(\frac{\lambda-1}{\lambda}, 0\right),\\
p_2 &= \left(0, \frac{\lambda-1}{\lambda}\right),\\
p_3 &= \frac{1}{3}\left\{1 - \left(4 - \frac{3}{\lambda}\right)^{\frac{1}{2}}, 1 - \left(4 - \frac{3}{\lambda}\right)^{\frac{1}{2}}\right\},\\
p_4 &= \frac{1}{3}\left\{1 + \left(4 - \frac{3}{\lambda}\right)^{\frac{1}{2}}, 1 + \left(4 - \frac{3}{\lambda}\right)^{\frac{1}{2}}\right\}.
\end{aligned}$$

For $0 < \lambda < 1$, p_0 is an attractive node. For all the rest of parameter values, p_0 is a repelling node. The points (p_1, p_2) exist for every parameter value, and they are unstable for every value of λ. For $0 < \lambda < 0.75$, $p_{3,4}$ are not possible solutions. When $\lambda = 0.75$, a saddle-node bifurcation on the diagonal generates $p_{3,4}$. For $0.75 < \lambda < 0.866$, p_3 is a saddle point and p_4 is an attractive node. In this parameter interval, the whole diagonal segment between p_3 and p_4 is a locus of points belonging to heteroclinic trajectories connecting the two fixed points. The point p_4 suffers a flip bifurcation when $\lambda = \sqrt{3}/2 \cong 0.866$. It generates a stable period-2 orbit $p_{5,6}$ outside the diagonal. These points are obtained by solving the quadratic equation $\lambda\,(4\lambda+3)x^2 - 4\lambda(\lambda+1)x + 1 + \lambda = 0$. The solutions are as follows:

$$p_5 = \left(\frac{2\lambda(\lambda+1) + \sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)}, \frac{2\lambda(\lambda+1) - \sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)} \right),$$

$$p_6 = \left(\frac{2\lambda(\lambda+1) - \sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)}, \frac{2\lambda(\lambda+1) + \sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)} \right).$$

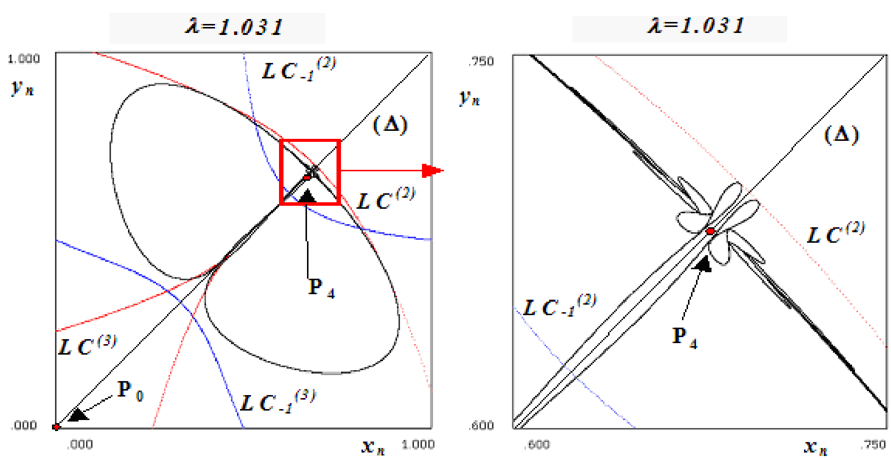
For $\lambda = 0.975$, these period-2 symmetric points lose stability through a Neimark-Hopf bifurcation. The set of points $p_{5,6}$ gives rise to a period 2 set of two stable closed invariant curves. These symmetric invariant curves grow in size when λ increases into the interval $0.957 < \lambda < 1$, and, for some values of λ, frequency locking windows are obtained.

The period-2 cycles on the axes appear by a period doubling bifurcation, and are found by solving the cubic equation: $\lambda^3x^3 - 2\lambda^3x^2 + (\lambda^3+\lambda^2)x + 1 - \lambda^2 = 0$. They have existence for $\lambda > 3$. The solutions are

$$p_7 = \left(\frac{(\lambda+1) - \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}, 0 \right) \quad \leftrightarrow \quad p_8 = \left(\frac{(\lambda+1) + \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}, 0 \right),$$

$$p_9 = \left(0, \frac{(\lambda+1) - \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda} \right) \quad \leftrightarrow \quad p_{10} = \left(0, \frac{(\lambda+1) + \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda} \right).$$

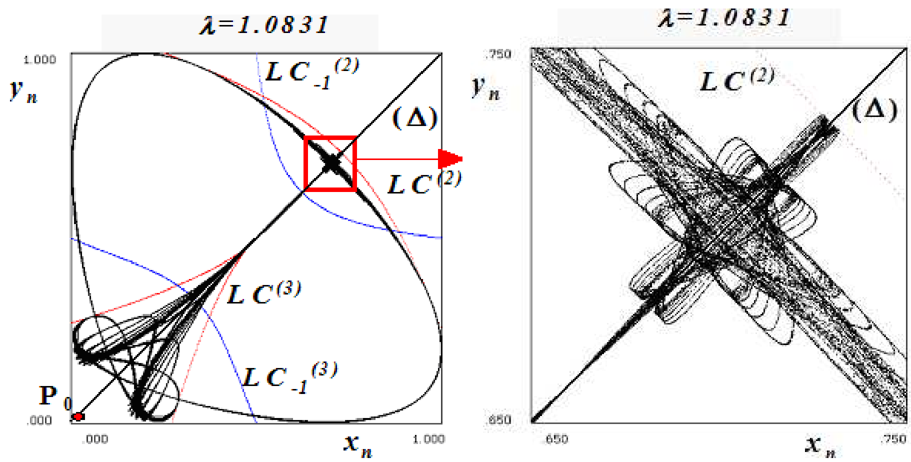
Observe that the restriction of the map T to the axes is the logistic map, so that its dynamics gives rise to the well known cyclic logistic behavior on the axes, as explained in section 1.



(a) (b)

FIGURE 1. (a) Attractive closed invariant curves for $\lambda = 1.031$. (b) Enlargement of (a), where weakly chaotic rings limited by segments of critical curves LC_n can be observed.

3.3. **Transition to chaos.** The two closed invariant curves approach the stable invariant set of the hyperbolic point p_4 on the diagonal when λ is slightly larger than 1 (fig. 1a). At first sight, for $\lambda \cong 1.029$, the system still seems quasi-periodic, but a finer analysis reveals the fingerprints of chaotic behavior. Effectively, a folding process takes place in the two invariant sets (cf. [16]), which gives rise to the phenomenon of weakly chaotic rings when the invariant set intersects itself (fig. 1b) (cf. [17] p.529). For $\lambda \cong 1.032$, the tangential contact of the two symmetric invariant sets with the stable set of the saddle p_4 on the diagonal leads to the disappearance of those two weakly chaotic rings. Just after the contact, infinitely many repulsive cycles appear due to the creation of homoclinic points and a single and symmetric chaotic attractor appears (fig. 2a). For $1.031 < \lambda < 1.0843$, this chaotic invariant set folds strongly around p_4, and the dynamics becomes very complex (fig. 2b). When the limit value $\lambda = 1.084322$ is reached, the chaotic area becomes tangent to its basin boundary, the mapping iterates can escape to infinite, and the attractor disappears by a contact bifurcation ([17], chap. 5). The time behavior of the system can be seen in figures 3a, 3b, and 3c.

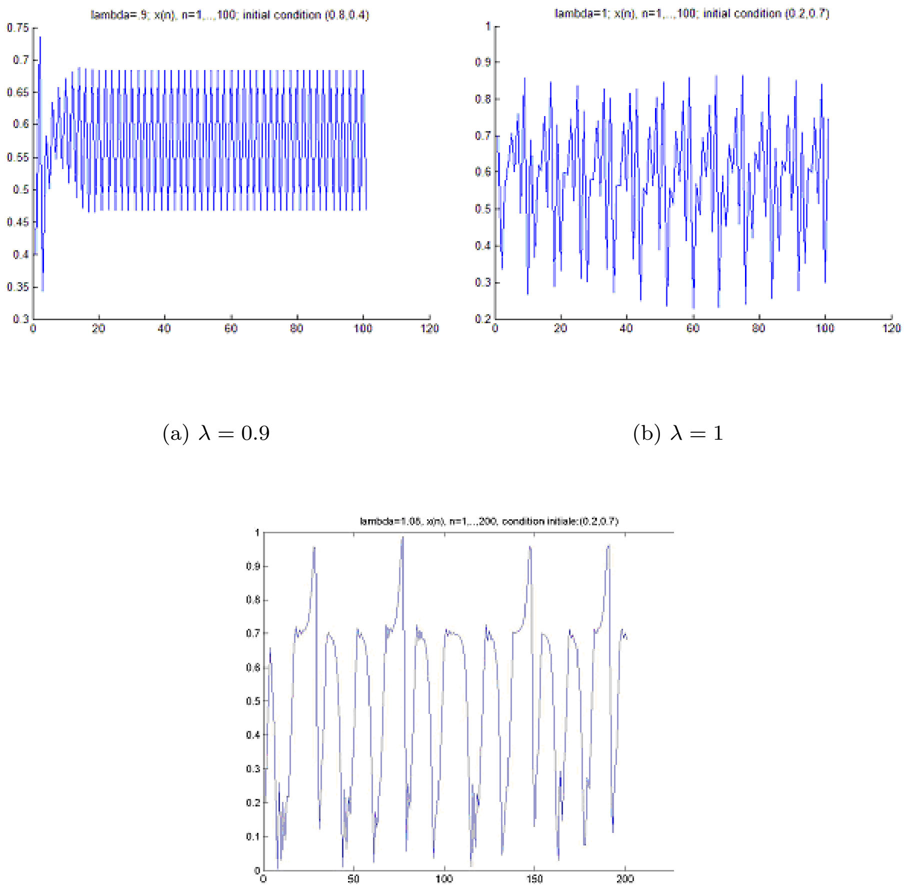


(a) (b)

Figure 2. (a) Symmetric chaotic attractor for $\lambda = 1.0831$. (b) Complex folding process around p_4 for same value of λ.

4. **Basin fractalization.** Let us now examine how the different initial populations evolve toward an asymptotic stable state. This is exactly the problem of considering the basins of the different attractors of model (5−6). For the sake of coherence, we consider the square $[0, 1] \times [0, 1]$ as the source of initial conditions making sense in our biological model; that is, in the map T. Basins constitute an interesting object of study themselves. If a color is given to the basin of each attractor, we obtain a colored figure, which is a phase-plane visual representation of the asymptotic behavior of the points of interest. The strong dependence on the parameters of this colored figure generates a rich variety of complex patterns on the plane and gives rise to different types of basin fractalization. See, for instance, the work done by

Gardini [18] and also by López-Ruiz and Fournier-Prunaret [11] in this direction. It is now our objective to analyze the parameter dependence of basin fractalization of model (5−6) by using the technique of critical curves.



(a) $\lambda = 0.9$

(b) $\lambda = 1$

(c) $\lambda = 1.08$

FIGURE 3. Asymptotic temporal behavior of the dynamics for different λ.

4.1. **Definitions and general properties of basins and critical curves.** The set D of initial conditions that converge towards an attractor at finite distance when the number of iterations of T tends toward infinity is the basin of the attracting set at finite distance. When only one attractor exists at finite distance, D is the basin of this attractor. When several attractors at finite distance exist, D is the union of the basins of each attractor. The set D is invariant under backward iteration T^{-1} but not necessarily invariant by T : $T^{-1}(D) = D$ and $T(D) \subseteq D$. A basin may be connected or non-connected. A connected basin may be simply connected or multiply connected, which means connected with holes. A non-connected basin consists of a finite or an infinite number of connected components, which may be simply or multiply connected. The closure of D includes also the points of the boundary ∂D, whose sequences of images are also bounded and lie on the boundary itself. If we consider the points at infinite distance as an attractor, its basin D_∞ is the complement of the closure of D. When D is multiply connected, D_∞ is non-connected, the holes (called lakes) of D being the non-connected parts (islands) of D_∞. Inversely, non-connected parts (islands) of D are holes of D_∞ [17].

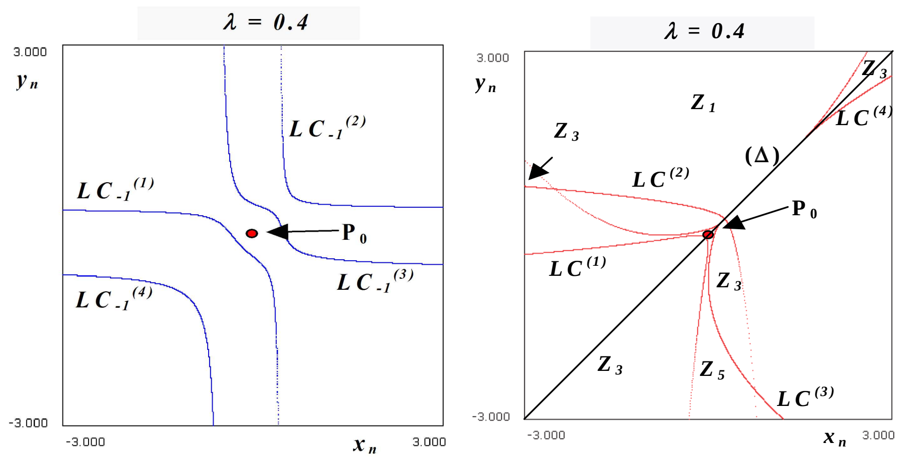
In section 3, we explained that the map (5−6) may possess one or two attractors at a finite distance. The points at infinity constitute the third attractor of T. Thus, if a different color for each different basin is given we obtain a colored pattern in the square $[0, 1] \times [0, 1]$ with a maximum of two colors. In the present case, the phenomena of finite basins' disappearance have their origin in the competition between the attractor at infinity (whose basin is D_∞) and the attractors at finite distance (whose basin is D). When a bifurcation of D takes place, some important changes appear in the colored figure representing the basins, and, although the dynamical causes cannot be clear, the colored pattern becomes an important visual tool to analyze those changes.

Critical curves are an important tool used to study basin bifurcations. They were introduced by Mira in 1964 (see [3] for further details). The map T is said to be noninvertible if points exist in state space that do not have a unique rank-one preimage under the map. Thus the state space is divided into regions, called Z_i, in which points have i rank-one preimages under T. These regions are separated by the so called critical curves LC, which are the images of the curves LC_{-1}. If the map T is continuous and differentiable, LC_{-1} is the locus of points where the determinant of the Jacobean matrix of T vanishes. When initial conditions are chosen to both sides of LC, the rank-one preimages appear or disappear in pairs. (See the glossary for technical terms used in this work.)

4.2. **Critical curves and Z_i - regions of T.** In our case, the map T defined in (5−6) is noninvertible. It has a non-unique inverse. As we know, $LC = T(LC_{-1})$. LC_{-1} is the curve verifying $\mid DT(x, y) \mid= 0$, where $DT(x, y)$ is the Jacobean matrix of T. It is formed by the points (x, y) that satisfy the equation

$$27x^2y^2 + 3x^2y + 3xy^2 - 6x^2 - 6y^2 - 8xy + x + y + 1 = 0. \tag{8}$$

Hence, LC_{-1} is independent of λ parameter and is quadratic in x and y. It can be seen that LC_{-1} is a curve of four branches, with two horizontal and two vertical asymptotes. The branches $LC_{-1}^{(1)}$ and $LC_{-1}^{(2)}$ have as horizontal asymptote the line $y = 0.419$ and the vertical asymptote in $x = 0.419$. The other two branches, $LC_{-1}^{(3)}$ and $LC_{-1}^{(4)}$, have the horizontal asymptote in $y = -0.530$ and the vertical one is the line $x = -0.530$. The values 0.419 and −0.530 are the roots of the polynomial factor, $27x^2+3x-6$, that multiplies the term y^2 in equation (8). It follows that the critical curve of rank-1, $LC^{(i)} = T(LC_{-1}^{(i)})$, $i = 1, 2, 3, 4$, consists of four branches. The shape of LC and LC_{-1} is shown in figures 4a-4b. LC depends on λ and separates the plane into three regions that are locus of points having 1, 3, or 5 distinct preimages of rank-1. They are named by Z_i, $i = 1, 3, 5$, respectively (figure 4b). Observe that the set of points with three preimages of rank-1, Z_3, is not connected and is formed by five disconnected zones in the plane. Let us note that has the reflection symmetry through the diagonal: $P(LC_{-1}) = LC_{-1}$. Then every



(a) (b)

FIGURE 4. (a) Critical curves $LC_{-1}^{(i)}$, $i = 1, 2, 3, 4$. (b) Critical curves $LC^{(i)}$, $i = 1, 2, 3, 4$, for $\lambda = 0.4$. Observe the different Z_j-zones, $j = 1, 3, 5$.

critical curve of rank-(k+1), $LC_k = T^{k+1}(LC_{-1})$, will conserve this symmetry: $P(LC_k) = LC_k$.

We see in figure 4b that the four-branched LC-curve divides the diagonal Δ in five intervals. If we know the number i of preimages of rank-1 of each segment on the diagonal, the number of preimages of rank-1 of each Z_i-zone of the plane is also determined. This calculation has been performed in [11]. The number of rank-1 preimages of a point (x', x') on the diagonal can be summarized in the following table: The coordinates of the points marking the frontier between the different Z_i-

INTERVAL	$x' < x'_{2d}$	$x'_{2d} < x' < x'_{2h}$	$x'_{2h} < x' < x'_{1d}$	$x'_{1d} < x' < x'_{1h}$	$x' > x_{1h}$
NUMBER OF PREIMAGES	3	5	3	1	3

TABLE 2. Number of T-preimages of a point (x', x') on the diagonal.

zones on the diagonal are $x'_{1d} \cong 0.65\lambda$, $x'_{2d} \cong -0.1\lambda$, $x'_{1h} \cong 4\lambda$, and $x'_{2h} \cong 0.44\lambda$. For example, the origin p_0 is always in the Z_5-zone. It is located into the interval limited by x'_{2d} and x'_{2h}. In fact, its preimages are $(1, 1)$, $(-1/3, -1/3)$ and p_0 itself on the diagonal, and $(1, 0)$ and $(0, 1)$ out of the diagonal. According to the nomenclature established in [17], the map (5–6) is of type $Z_3 - Z_5 \succ Z_3 - Z_1 \prec Z_3$.

4.3. **Types of basins in T.** Depending on λ, three different types of patterns are obtained in the square $[0, 1] \times [0, 1]$. We proceed to present them and to explain the role played by critical curves in the bifurcations giving rise to the third basin type.

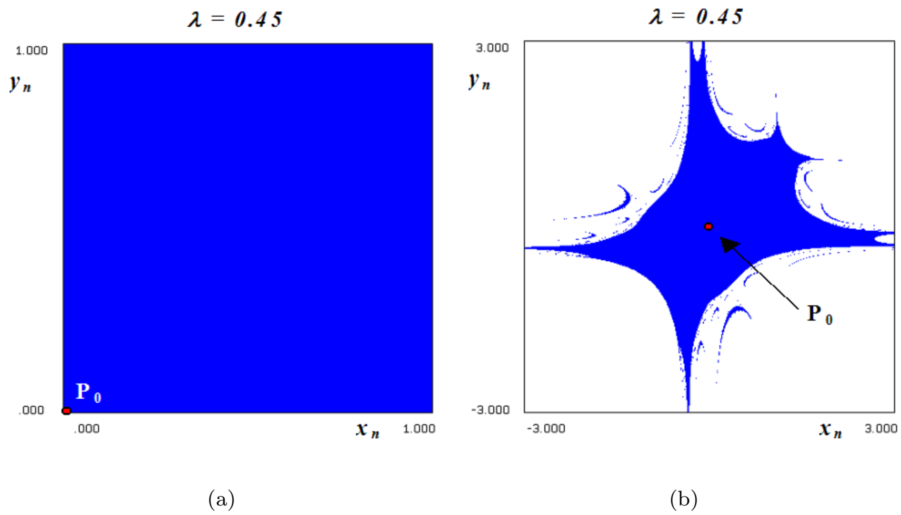


FIGURE 5. (a) One-colour basin for $\lambda = 0.45$. The only attractor is the origin. (b) Fractal pattern of islands when the whole plane is considered as a source of initial conditions.

4.3.1. *Extinction of Species,* $0 < \lambda < 0.75$. In this regime, any given initial population evolves toward the extinction. The mutual benefit is too small, then it is not possible the surviving of the species. Then, all initial conditions tend to zero under iteration of T. A pattern of only one color is obtained (fig. 5a).

If we regard the behavior of T in the whole plane $\Re^2$, D undergoes an interesting bifurcation consisting of the transition from a connected to a non-connected basin (fig. 5b). It takes place when λ increases from $\lambda \cong 0.39$ to $\lambda \cong 0.61$. When D becomes non-connected, it is made up of the immediate basin D_0 containing the single attractor p_0 and infinite small regions without connection (islands). This disaggregation is the result of infinitely many contact bifurcations, which are explained in [11]. Such phenomena can be also found in some quadratic $Z_0 - Z_2$ maps [19].

4.3.2. *Extinction or Non trivial Evolution of Species,* $0.75 < \lambda < 1$. A sudden change affects the basin for $\lambda = 0.75$. A second attractor p_4 appears and a ball of initial conditions is attracted by this synchronized state. When $0.75 < \lambda < 0.86$, the coexistence between both species can reach a non-null stable value in this regime. All the rest of initial conditions on the square $[0, 1] \times [0, 1]$ continue to shrink to the origin, then go extinct. The basin is a two-colour pattern (fig. 6a). When $0.86 < \lambda < 0.95$, p_4 bifurcates to a two-periodic orbit and the system becomes now a lag-synchronized oscillation. The colour corresponding to this last state has gained space on the zero state in the two-colour pattern (fig. 6b). When $0.95 < \lambda < 1$, synchronization is finally lost and the system becomes quasiperiodic. Only the corners of the square $[0, 1] \times [0, 1]$ lead to extinction in the two-color pattern (fig. 6c). If we regard the total basin in $\Re^2$, D seems to be formed by the square $D_0 \equiv [0, 1] \times [0, 1]$, which contains the attracting set at finite distance and four small like-triangled regions linked to the square by four narrow arms. These arms

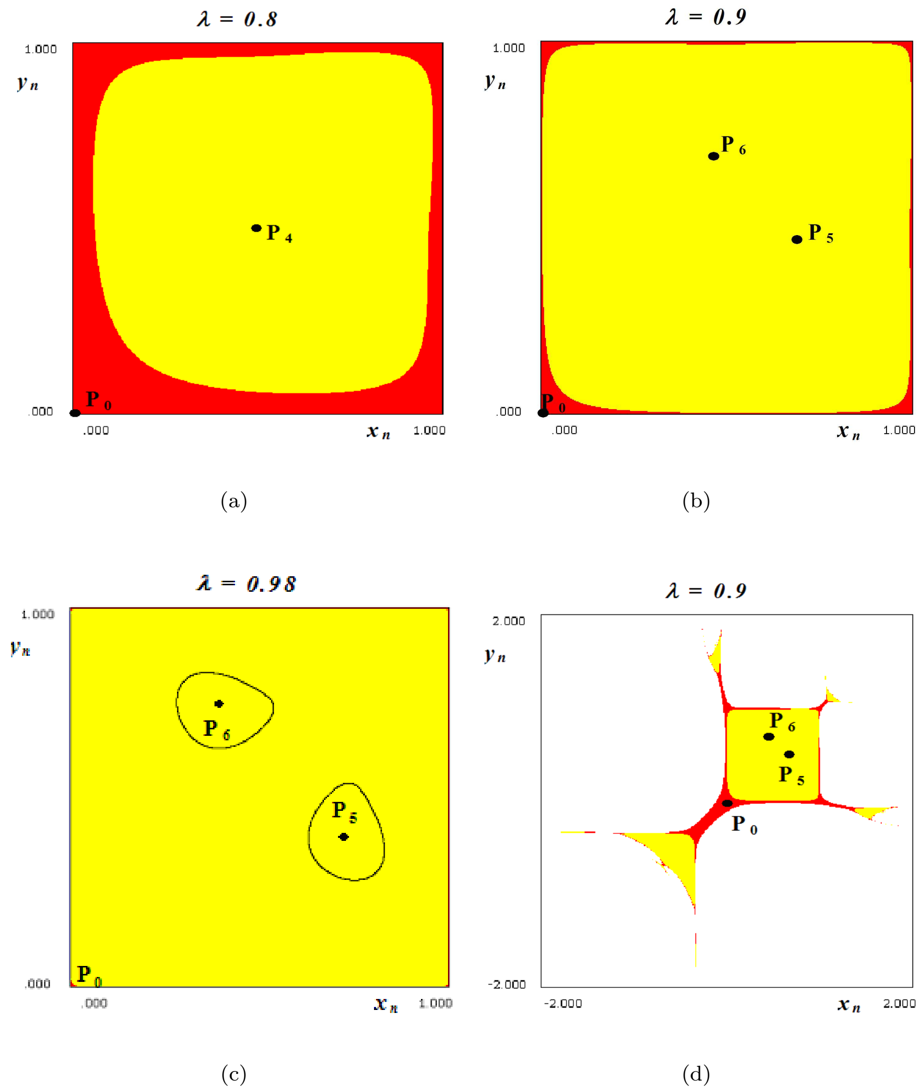


Figure 6. (a) Basin for $\lambda = 0.8$. The two colors correspond to the basins of the two existing attractors: the synchronized state on the diagonal and the origin. (b) For $\lambda = 0.9$, the central colored ball in the square is the basin of a 2-periodic orbit. (c) For $\lambda = 0.98$, the central colored area is the basin of two attractive closed invariant curves. (d) Pattern of the basin in the whole plane. It is formed by the square $(0,1) \times (0,1)$, which contains the attractors, and four small like-triangle regions linked to the square by four narrow arms for $\lambda = 0.9$.

shrink when λ approaches 1, and disappear for $\lambda = 1$ when the origin p_0 undergoes a transcritical bifurcation. The main part of D is then a disconnected pattern of five components: the square D_0, a triangle-shaped component located in a Z_3 neighborhood of the vertex point $(-1/3, -1/3)$ (preimage of rank-1 of the point p_0), and the three triangle-shaped regions that are preimages of rank-1 of the latter component (fig. 6d).

4.3.3. *Non-trivial Evolution or Catastrophe of Species,* $1 < \lambda < 1.0843$. A new phenomenon takes place in this range of the parameter. Some initial conditions can give rise to an evolution that surpass the boundaries of the square $[0,1] \times [0,1]$ and tends to infinity. We interpret this behavior as some kind of internal catastrophe (war, epidemics, etc.) leading to extinction. Although we are aware of its disconcerting meaning, this would imply that an internal catastrophe can follow in this model as a consequence of the population start from some particular initial conditions. All the rest of the initial conditions bring the system to a quasiperiodic state when $1 < \lambda < 1.03$ or to a chaotic dynamical regime when $1.03 < \lambda < 1.0843$ (fig. 7(a-d) and fig. 8(a-b)). Therefore, a two-color basin is also obtained in this range of λ parameter.

In a more detailed form, the basin bifurcations happen as follows. Points $(1,0)$ and $(0,1)$ cross through $LC^{(2)}$ when $\lambda = 1$. When $\lambda > 1$, it makes two regions appear, S_1^1 and S_2^1, inside $[0,1] \times [0,1]$, which are part of D_∞ and are located in a Z_3 zone (fig. 7a). The square is no longer invariant by T. The rank-one preimages of S_1^1 and S_2^1, -respectively S_1^{-1} and S_2^{-1}- are two new semicircular regions and intersect $LC_{-1}^{(2)}$. They are located in the vicinity of points $(1, 0.5)$ and $(0.5, 1)$, preimages of $(1,0)$ and $(0,1)$ (fig. 7a,7b). When λ increases, the two semicircular zones of D_∞, S_1^{-1} and S_2^{-1}, located in the immediate basin $D_0 \equiv [0,1] \times [0,1]$, in the neighborhood of points $(1, 0.5)$ and $(0.5, 1)$, grow in size. For $\lambda > 1.0801$, the basin undergoes a contact bifurcation. D_∞ crosses through $LC^{(2)}$ and two bays (headlands of D_∞), H_{01} and H_{02}, are created in a Z_3 area (fig. 7c). Their rank-1 preimages, $H_{01}^{(1)}$ and $H_{02}^{(1)}$, are holes (lakes) intersecting $LC_{-1}^{(2)}$ into the middle Z_3-region. Rank-1 preimages of the latter holes generate four new lakes in D_0, $H_{0i}^{(21)}$ and $H_{0i}^{(22)}$, $i = 1, 2$. Preimages with increasing rank give rise to an arborescent sequence of lakes. The accumulation points of this infinite sequence of holes are the two unstable foci $p_{5,6}$ and their rank-1 preimages inside the basin. When $\lambda \cong 1.0806$, $H_{01}^{(21)}$ and $H_{02}^{(21)}$ cross through $LC^{(2)}$ (fig. 7d). This new contact bifurcation is the germ of a new arborescent and spiraling sequence of lakes converging towards the same accumulation points. When λ increases values, new holes intersect $LC^{(2)}$ and give rise to new holes crossing through $LC_{-1}^{(2)}$ and new sequences of lakes converging towards the unstable foci $p_{5,6}$ and their preimages. Because the preimages have a finite number of accumulation points, the structure is not fractal. A similar phenomenon has been found and studied in $Z_0 - Z_2$ maps [20]. When λ increases ($\lambda \cong 1.0835$), the chaotic attractor, which is limited by arcs of LC_n curves, is destroyed by a contact bifurcation with its basin boundary (fig. 8a). A new dynamical state arises. The infinite number of unstable cycles and their rank-n images belonging to the existing chaotic area before the bifurcation define a strange repulsor that manifests itself by chaotic transients (fig. 8b). For $\lambda \cong 1.085$, the basin pattern disappears definitively.

5. **Conclusions.** One-dimensional and two-dimensional mappings are simple models that have been extensively studied as models of population dynamics [1, 21], as ingredients of other more complex systems [22, 23], or as independent objects of

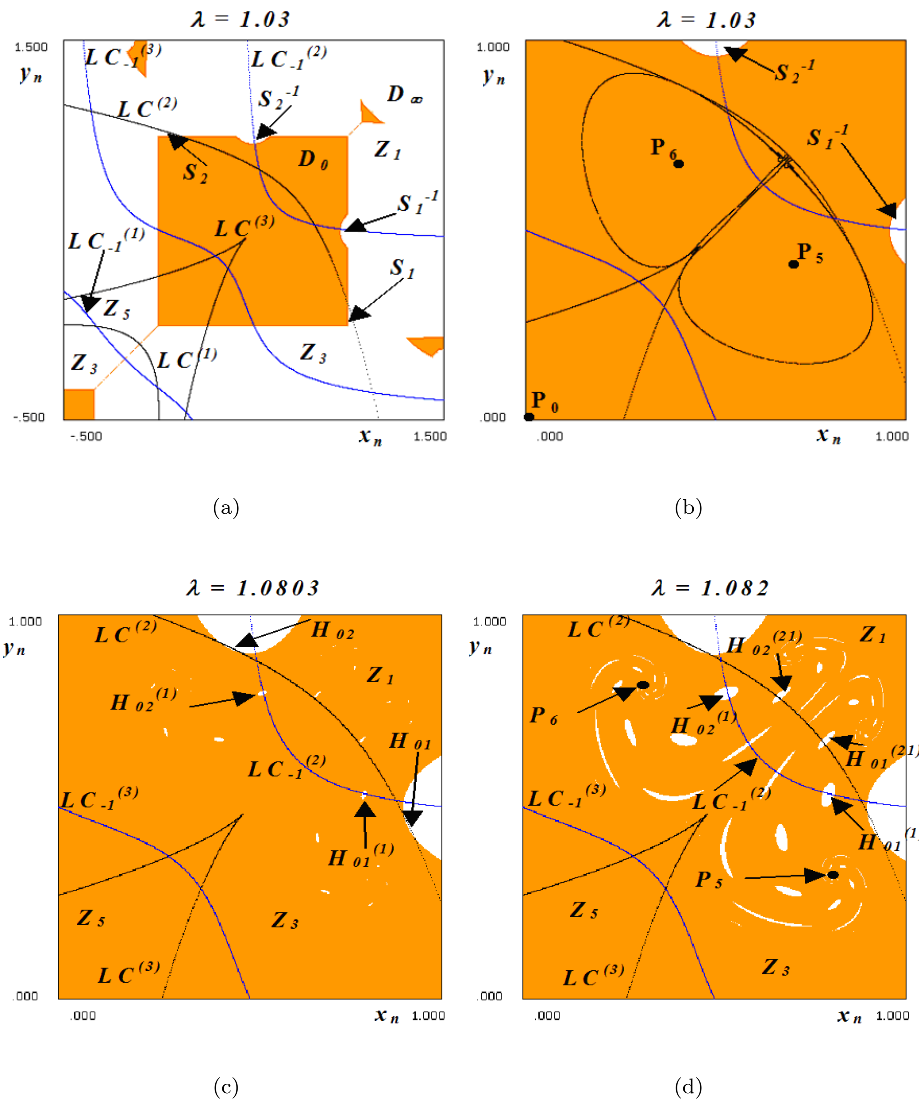
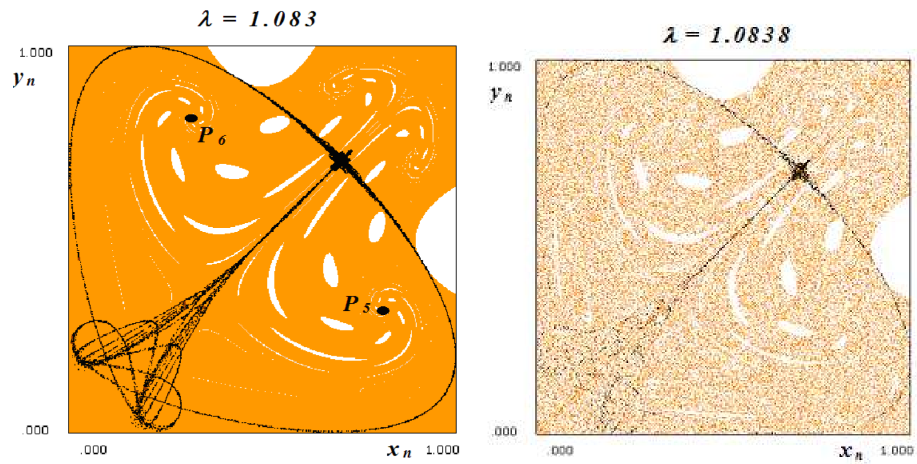


FIGURE 7. (a) Basin for $\lambda = 1.03$. One color corresponds to basin of the attractive invariant curves and the other one to basin of infinity. (b) Detail of basin and weakly chaotic rings for $\lambda = 1.03$. (c) For $\lambda = 1.0803$, first rank holes $H^{(1)}_{(01)}$ and $H^{(1)}_{(02)}$ (and higher rank preimages holes) of the bays $H_{(01)}$ and $H_{(02)}$, respectively. (d) New arborescent sequence of holes created from the crossing of $H^{(21)}_{(01)}$ and $H^{(21)}_{(02)}$ with $LC^{(2)}$ for $\lambda = 1.082$.

interest [17]. Specifically, different two-dimensional coupled logistic maps are found in the literature of several fields, such as physics, engineering, biology, ecology, and economics [24, 25, 26, 27, 28].



(a) (b)

FIGURE 8. (a) Chaotic attractor and its basin for $\lambda = 1.083$. (b) Chaotic transient for $\lambda = 1.0838$.

The models scattered in the literature on the three main types of population interaction, that is, predator-prey situation, competition, and mutualism among species, are usually stated as quadratic equations [29]. In this work, we have reinterpreted a cubic two-dimensional coupled logistic equation, which was proposed in reference [7], as a discrete model to explain the evolution of two symbiotically interacting species. The symbiotic interaction between both species is population-size dependent and is controlled by a positive constant λ that we call *the mutual benefit*. Depending on λ, the system can reach extinction due to the small mutual benefit or the lack of resources, it can stabilize in a synchronized state or oscillates in a 2-periodic orbit for intermediate λ or it can evolve in a quasiperiodic or chaotic regime for the greatest λ. In this last scenario, initial conditions also lead the system to extinction. This kind of extinction could be interpreted as an internal catastrophe caused, for instance, by political decisions or by a deficient health provision system in the case of human society, and not, in general, by the exhaustion of resources. Another remarkable property of the model is that when $\mu < 1$ one of the isolated species is extinct, but it can survive for $\lambda < 1$ when it interacts symbiotically with one of the other species. Then, symbiosis seems to be well held in this model.

Different complex color patterns on the plane have been obtained when the mutual benefit is modified. If $0 < \lambda < 0.75$, all the dynamics is attracted by the origin and a one-color pattern is found. When $0.75 < \lambda < 1$, the dynamics can settle down in two possible attractors and the basins are now characterized by two colors. Finally, if $1 < \lambda < 1.0843$, the two-color basins result from the two possible asymptotic states: a quasiperiodic or chaotic finite distance attractor and an additional one located at infinity.

Critical curves have been used to understand the basin bifurcations found in this system. Hence, common features with those present in the simplest and well-studied case of $Z_0 - Z_2$ maps are now more evident for this map of $Z_3 - Z_5 \succ Z_3 - Z_1 \prec Z_3$ type. A detailed study of the different fractalization mechanisms for the whole range of λ parameter for a similar coupled logistic equation was performed in [11]. The rich dynamics and the complex patterns produced on the plane in this model are controlled by a single parameter, in this case, the mutual benefit between the interacting species.

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GLOSSARY

INVARIANT: A subset of the plane is invariant under the iteration of a map if this subset is mapped exactly onto itself.

ATTRACTING: An invariant subset of the plane is attracting if it has a neighborhood every point of which tends asymptotically to that subset or arrives there in a finite number of iterations.

CHAOTIC AREA: An invariant subset that exhibits chaotic dynamics. A typical trajectory fills this area densely.

CHAOTIC ATTRACTOR: An attracting chaotic area.

BASIN: The basin of attraction of an attracting set is the set of all points that converge toward the attracting set.

IMMEDIATE BASIN: The largest connected part of a basin containing the attracting set.

ISLAND: Non-connected region of a basin, which does not contain the attracting set.

LAKE: Hole of a multiply connected basin. Such a hole can be an island of the basin of another attracting set.

HEADLAND: Connected component of a basin bounded by a segment of a critical curve and a segment of the immediate basin boundary of another attracting set, the preimages of which are islands.

BAY: Region bounded by a segment of a critical curve and a segment of the basin boundary, the successive images of which generate holes in this basin, which becomes multiply connected.

CONTACT BIFURCATION: Bifurcation involving the contact between the boundaries of different regions. For instance, the contact between the boundary of a chaotic attractor and the boundary of its basin of attraction or the contact between a basin boundary and a critical curve LC are examples of this kind of bifurcation.

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E-mail address: rilopez@unizar.es
E-mail address: Daniele.Fournier@insa-tlse.fr