Mathematics

AIMS Mathematics, 11(8): 27231–27263.
DOI: 10.3934/math.20261090
Received: 19 June 2026
Revised: 11 August 2026
Accepted: 18 August 2026
Published: 28 August 2026

https://www.aimspress.com/journal/Math

Research article

Statistical inference for the Weibull distribution under improved adaptive Type-II progressive censoring with binomial removals

Hanefi Geze and İlhan Usta*

Department of Statistics, Faculty of Science, Eskişehir Technical University, Eskişehir, Türkiye

*** Correspondence:** Email: iusta@eskisehir.edu.tr.

Abstract: An improved adaptive type-II progressive censoring (IAT-II PC) scheme has been introduced to effectively control the duration of life-testing experiments within a prespecified limit. While this scheme has attracted considerable attention, statistical inference under random removals, in which the number of removals at each failure time is treated as a random variable, remains largely unexplored in the literature. In this paper, we focus on statistical inference for the two-parameter Weibull distribution based on the IAT-II PC scheme, where the number of removals at each failure time follows a binomial distribution. Point and interval estimation of the Weibull parameters was carried out using the maximum likelihood (ML), maximum product spacing (MPS), and Bayesian estimation methods. Asymptotic confidence intervals for the model parameters were constructed based on the asymptotic normality of the ML and MPS estimators. Since the posterior distributions are analytically intractable, Bayesian estimates and their corresponding credible intervals were obtained using the Markov Chain Monte Carlo method. The performance of the proposed estimators was evaluated through an extensive Monte Carlo simulation study. In addition, a real-world dataset was analyzed to demonstrate the practical applicability of the proposed inference methods. The results show that the Bayesian estimators yield more accurate point and interval estimates than their classical counterparts under the IAT-II PC scheme with binomial removals.

Keywords: improved adaptive Type-II progressive censoring; binomial random removals; Weibull distribution; maximum likelihood estimation; maximum product spacing estimation; Bayesian inference
Mathematics Subject Classification: 62N05, 62F15, 62F10

1. Introduction

In survival and reliability analysis, experiments are typically conducted over long durations, incur high costs, and are subject to various experimental constraints. As a result, full life testing is often impractical, and censoring is commonly used as an effective alternative. Among various censoring schemes, the type-II progressive censoring (T-II PC) scheme is widely used, as it allows the removal of surviving units at failure times according to a prespecified censoring plan [1]. Despite this flexibility, the T-II PC scheme may still result in long testing times and high experimental costs. To address this limitation, Ng et al. [2] introduced the adaptive type-II progressive censoring (AT-II PC) scheme. However, for high-reliability products, this scheme may not sufficiently reduce total testing time. To further reduce the total testing time, Yan et al. [3] proposed the IAT-II PC scheme. Specifically, in this scheme, two threshold time points are used to ensure that the experiment is completed within a specified time period, while at the same time, sufficient information is retained to enable statistical inference under censoring. Accordingly, the procedure of the IAT-II PC scheme can be described as follows. Suppose that n independent and identically distributed units, characterized by a probability density function $f(x)$ and a cumulative distribution function $F(x)$, are placed in a life-testing experiment at time zero. Before the experiment begins, the total number of failures, denoted by m $(m < n)$, is fixed in advance, and the censoring scheme $R = (R_1, R_2, \dots, R_m)$ with $R_i \geq 0$ is specified. The values of R_i may be predetermined or generated from a specified distribution. However, under the IAT-II PC scheme, some of these values may be changed during the experiment according to two prespecified time thresholds, T_1 and T_2, satisfying $0 < T_1 < T_2 < \infty$. The first threshold T_1 serves as a warning point, whereas the second threshold T_2 represents the maximum allowable testing time. During the experiment, when the first failure occurs at $X_{1:m:n}$, R_1 units are randomly selected from the $n-1$ surviving units and removed from the test. Upon the occurrence of the second failure at $X_{2:m:n}$, R_2 units are randomly selected from the remaining units and removed. This process continues iteratively for subsequent failures. Accordingly, an experiment conducted under the IAT-II PC scheme results in one of three possible cases, as summarized and illustrated below.

Case I: If the failure of the $m-th$ unit occurs before the first threshold T_1, then the experiment terminates at $X_{m:m:n}$. In this case, the IAT-II PC scheme coincides with the T-II PC scheme, and the corresponding sample consists of $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$. The experimental procedure for this case is illustrated in Figure 1.

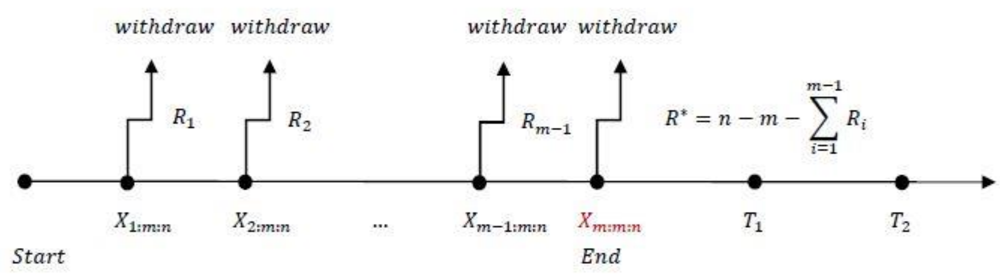

Figure 1. Case I of the IAT-II PC scheme.

Case II: If the failure of the $k_1 - th$ unit $(k_1 < m)$ occurs before the first threshold T_1, and the failure of the $(k_1+1) - th$ unit occurs after T_1, then the experiment is accelerated by retaining all surviving units. In this situation, $R_i = 0$ for $i = k_1 + 1, \dots, m - 1$, and the remaining removals are assigned to the final stage as $R_m = R^* = n - m - \sum_{i=1}^{k_1} R_i$. Furthermore, the experiment terminates at $X_{m:m:n}$. Consequently, the IAT-II PC scheme reduces to the AT-II PC scheme, and the resulting sample consists of $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$. The corresponding experimental procedure is illustrated in Figure 2.

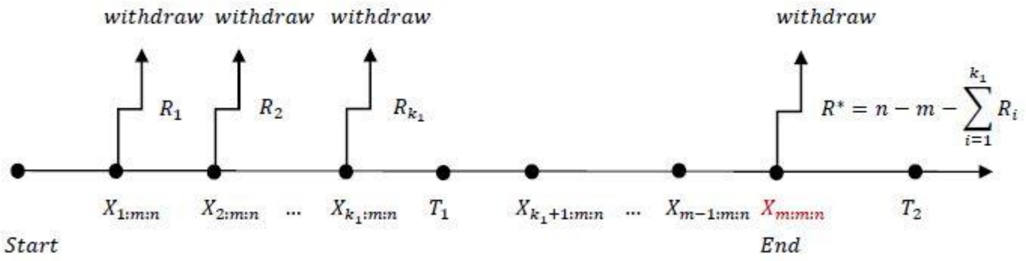

Figure 2. Case II of the IAT-II PC scheme.

Case III: If the failure of the $m-th$ unit occurs after the maximum allowable testing time, that is, $T_2 < X_{m:m:n}$, then the experiment is terminated at the second threshold T_2. In this case, let k_2 $(k_2 < m)$ denote the number of failures observed prior to T_2. The resulting IAT-II PC sample consists of $X_{1:m:n}, X_{2:m:n}, \dots, X_{k_2:m:n}$. The experimental procedure for this case is illustrated in Figure 3.

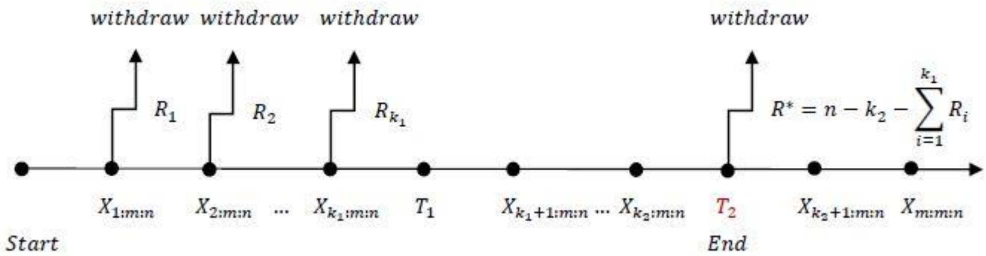

Figure 3. Case III of the IAT-II PC scheme.

In recent years, considerable attention has been devoted to the IAT-II PC scheme in the literature, with numerous studies investigating its use for modeling and inference for lifetime data. For instance, Dutta and Kayal [4] investigated a competing risks model with exponential lifetimes under the IAT-II PC scheme and derived both ML and Bayesian estimators for the model parameters. Zhang and Yan [5] focused on parameter estimation for the Chen distribution based on data obtained from the IAT-II PC scheme. Similarly, Nassar and Elshahhat [6] developed classical and Bayesian estimation procedures for the Weibull distribution, while Nassar et al. [7] extended the framework to Weibull constant-stress accelerated life tests and performed statistical inference for the associated model parameters. In addition, Dutta et al. [8] analyzed the logistic-exponential distribution under the IAT-II PC scheme and carried out both parameter and reliability estimation. Irfan et al. [9] addressed parametric inference for the Kumaraswamy-G family of distributions in the same setting. More recently, Alotaibi et al. [10] investigated parameter and reliability estimation for a competing risks model with Burr-X lifetimes under the IAT-II PC scheme using both ML and Bayesian approaches. Finally, Lone [11] considered estimating the parameters of the Burr III distribution based on an IAT-II PC sample.

A review of the literature indicates that most studies on the IAT-II PC scheme assume a predetermined censoring scheme, in which the number of units removed at each failure time is fixed in advance. However, in practical applications, such as certain clinical trials and reliability experiments, the number of units removed at each failure time cannot always be fixed in advance and may vary randomly over the course of the experiment. Therefore, there is a need for censoring schemes that allow random removals. In this context, Yuen and Tse [12], Tse et al. [13], and Singh et al. [14] considered censoring schemes in which the number of removed units follows discrete distributions such as the discrete uniform, binomial, or beta–binomial distributions. Subsequently, several studies have investigated statistical inference under censoring schemes with random removals in a variety of settings, including different lifetime distributions, competing risks models, and accelerated life tests; see, for example, Wu et al. [15], Usta and Gezer [16], Abushal and Abdel-Hamid [17], and the references therein.

Despite these studies, the IAT-II PC scheme with random removals has received limited attention. To the best of our knowledge, only one study has examined this setting: Gezer and Usta [18] investigated parameter estimation for the Rayleigh distribution under the IAT-II PC scheme with random removals, using both classical and Bayesian approaches. This limited attention is noteworthy given the importance of incorporating randomness into the censoring mechanism to better reflect real-world experimental conditions. Motivated by this gap, the present study has two main objectives, as follows:

- The first objective is to consider the IAT-II PC scheme in which the number of units removed at each failure time is assumed to follow a binomial distribution with parameter p. The binomial specification allows the random removal process to be modeled using a single-parameter distribution, which is computationally convenient. In contrast, alternative specifications such as the beta–binomial distribution introduce additional parameters, thereby increasing computational complexity. On the other hand, the discrete uniform distribution assumes equal probabilities for all possible numbers of removals, which may limit its ability to capture variability in the removal process.
- The second objective is to develop point and interval estimation procedures for the parameters of the underlying lifetime distribution based on samples obtained from the IAT-II PC scheme with binomial removals. To this end, the Weibull distribution is considered for its flexibility in modeling various types of failure-rate behavior. Owing to these properties, it is one of the most widely used lifetime distributions in survival and reliability analysis. Comprehensive treatments of the Weibull distribution can be found in Lawless [19] and Rinne [20], among others. In addition, statistical inference for the Weibull distribution under various censoring schemes has been extensively studied; see, for instance, Balakrishnan and Aggarwala [21], Hassan et al. [22], Nassar and Alotaibi [23], and the references therein.

In line with these objectives, this study focuses on statistical inference for the two-parameter Weibull distribution under the IAT-II PC scheme with binomial removals. To the best of our knowledge, statistical inference for the two-parameter Weibull distribution under the IAT-II PC scheme with binomial removals has not been investigated in the existing literature. Motivated by this research gap, the present study derives ML, MPS, and Bayesian estimators for the model parameters. Moreover, asymptotic confidence intervals (ACIs) based on the asymptotic normality of the ML and MPS estimators, as well as highest posterior density (HPD) credible intervals within the Bayesian framework, are also constructed. The performance of the proposed methods is evaluated through a

Monte Carlo simulation study, and a real-world data application is provided to illustrate their practical usefulness.

The remainder of this study is organized as follows: Section 2 introduces the Weibull distribution and the IAT-II PC scheme with binomial removals. Section 3 derives the ML, MPS, and Bayesian estimators of the model parameters, while Section 4 constructs asymptotic confidence intervals based on the asymptotic normality of the ML and MPS estimators, as well as HPD credible intervals within the Bayesian framework. Section 5 presents the simulation study and assesses the finite-sample performance of the estimators under various settings. Section 6 provides a real-world data analysis to illustrate the practical applicability of the proposed methods. Finally, Section 7 concludes the paper and summarizes the main findings.

2. Model description

Suppose that the lifetime of a particular unit, X, follows a Weibull distribution with the shape parameter α and the scale parameter β, denoted as $X \sim Weibull(\alpha,\beta)$. Then, the corresponding probability density function (pdf) and cumulative distribution function (cdf) are given as follows:

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1} exp(-\beta x^{\alpha}),\ x > 0, \alpha, \beta > 0 \tag{2.1}$$

and

$$F(x;\alpha,\beta) = 1 - exp(-\beta x^{\alpha}),\ x > 0, \alpha, \beta > 0. \tag{2.2}$$

Let $\boldsymbol{X} = \left(X_{1:m:n}, X_{2:m:n}, \ldots, X_{J_1:m:n}, \ldots, X_{J_2:m:n}\right)$ be an IAT-II PC sample from $Weibull(\alpha,\beta)$ with a predetermined number of removals, $\boldsymbol{R} = (R_1 = r_1,\ R_2 = r_2, \ldots, 0, \ldots 0, R^* = r^*)$. Then, the corresponding conditional likelihood function of the observed data can be written as follows:

$$L_1(\alpha,\beta;\boldsymbol{x}|\boldsymbol{R}=\boldsymbol{r}) = C\left[\prod_{i=1}^{J_2} f(x_{i:m:n})\right]\left\{\prod_{i=1}^{J_1}[1-F(x_{i:m:n})]^{r_i}\right\}[1-F(T^*)]^{r^*}, \tag{2.3}$$

where C is a constant. Substituting Eqs (2.1) and (2.2) into Eq (2.3) yields:

$$L_1(\alpha,\beta;\boldsymbol{x}|\boldsymbol{R}=\boldsymbol{r}) = C(\alpha\beta)^{J_2} exp\left[-\beta\left(\sum_{i=1}^{J_2} x_i^{\alpha} + \sum_{i=1}^{J_1} r_i x_i^{\alpha} + r^*(T^*)^{\alpha}\right)\right]\prod_{i=1}^{J_2} x_i^{\alpha-1}, \tag{2.4}$$

where x_i denotes $x_{i:m:n}$ for notational simplicity. Table 1 summarizes the values of C, J_1, J_2, r^*, and T^* corresponding to different cases of the IAT-II PC scheme.

In the present setup, suppose that each unit is independently removed from the IAT-II PC life test with the same probability p. Motivated by the binomial removal framework of Tse et al. [13], we incorporate this mechanism into the IAT-II PC scheme. Accordingly, the number of units removed at the i-th failure follows a binomial distribution with the following probability mass function. For $i = 1$,

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1} \tag{2.5}$$

and $i = 2, 3, \ldots, J_1$

$$P(R_i = r_i | R_{i-1} = r_{i-1}, \dots, R_1 = r_1) = \binom{n - m - \sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=1}^{i} r_j}, \tag{2.6}$$

where r_i are nonnegative integers satisfying $0 \le r_1 \le n - m$, and $0 \le r_i \le n - m - \sum_{j=1}^{i-1} r_j$ for $i = 2, 3, \dots, J_1$, with J_1 determined by the IAT-II PC scheme as given in Table 1.

$$L(\alpha, \beta, p; \boldsymbol{x}, \boldsymbol{r}) = L_1(\alpha, \beta; \boldsymbol{x} | \boldsymbol{R} = \boldsymbol{r}) P(p; \boldsymbol{R} = \boldsymbol{r}), \tag{2.7}$$

where

$$\begin{aligned} P(p; \boldsymbol{R} = \boldsymbol{r}) &= P\left(R_{J_1} = r_{J_1} | R_{J_1 - 1} = r_{J_1 - 1}, \dots, R_1 = r_1\right) \dots . P(R_2 = r_2 | R_1 = r_1) P(R_1 = r_1) \\ &= \frac{(n-m)!}{\left(n - m - \sum_{i=1}^{J_1} r_i\right)! \prod_{i=1}^{J_1} r_i!} p^{\sum_{i=1}^{J_1} r_i} (1-p)^{\sum_{i=1}^{J_1}\left(n-m-\sum_{j=1}^{i} r_j\right)}. \end{aligned} \tag{2.8}$$

Using Eqs (2.4), (2.7), and (2.8), the full likelihood function of the IAT-II PC scheme with binomial removals can be expressed as

$$L(\alpha, \beta, p; \boldsymbol{x}, \boldsymbol{r}) = K L_1(\alpha, \beta) L_2(p), \tag{2.9}$$

where

$$K = \frac{C(n-m)!}{\left(n - m - \sum_{j=1}^{J_1} r_j\right)! \prod_{j=1}^{J_1} r_j!}, \tag{2.10}$$

$$L_1(\alpha, \beta) = (\alpha\beta)^{J_2} exp\left[-\beta\left(\sum_{i=1}^{J_2} x_i^{\alpha} + \sum_{i=1}^{J_1} r_i x_i^{\alpha} + r^* (T^*)^{\alpha}\right)\right] \prod_{i=1}^{J_2} x_i^{\alpha - 1}, \tag{2.11}$$

$$L_2(p) = p^{\sum_{i=1}^{J_1} r_i} (1-p)^{\sum_{i=1}^{J_1}\left(n-m-\sum_{j=1}^{i} r_j\right)}. \tag{2.12}$$

Table 1. Values of C, J_1, J_2, r^*, and T^* in the likelihood function for cases of the IAT-II PC scheme.

Case	C	J_1	J_2	r^*	T^*
I	$\prod_{i=1}^{m}\left(n - i + 1 - \sum_{j=1}^{i-1} r_j\right)$	$m-1$	m	$n - m - \sum_{i=1}^{m-1} r_i$	X_m
II	$\prod_{i=1}^{m}\left(n - i + 1 - \sum_{j=1}^{k_1} r_j\right)$	k_1	m	$n - m - \sum_{i=1}^{k_1} r_i$	X_m
III	$\prod_{i=1}^{k_2}\left(n - i + 1 - \sum_{j=1}^{i-1} r_j\right)$	k_1	k_2	$n - k_2 - \sum_{i=1}^{k_1} r_i$	T_2

It is worth emphasizing once again that, in an experiment conducted under the IAT-II PC scheme with binomial removals, the researcher specifies the total number of test units n and the total number of failures m before the experiment, whereas the number of surviving units removed at each failure time cannot be predetermined. Instead, it is generated according to a binomial distribution with parameter p. Table 2 summarizes the sampling procedure that reflects this feature.

Table 2. Sampling procedure for a life test under the IAT-II PC scheme with binomial removals.

Case	Stage	Failure time	Number of removed units	Number of surviving units
I	1	X_1	$r_1 \sim Binom(n-m,p)$	$n-r_1-1$
	2	X_2	$r_2 \sim Binom(n-m-r_1,p)$	$n-r_1-r_2-2$
	⋮	⋮	⋮	⋮
	$m-1$	X_{m-1}	$r_{m-1} \sim Binom(n-m-\sum_{i=1}^{m-2} r_i\,,p)$	$n-m+1-\sum_{i=1}^{m-1} r_i$
	m	X_m	$r^* = n-m-\sum_{i=1}^{m-1} r_i$	0
II	1	X_1	$r_1 \sim Binom(n-m,p)$	$n-r_1-1$
	2	X_2	$r_2 \sim Binom(n-m-r_1,p)$	$n-r_1-r_2-2$
	⋮	⋮	⋮	⋮
	k_1	X_{k_1}	$r_{k_1} \sim Binom(n-m-\sum_{i=1}^{k_1-1} r_i\,,p)$	$n-k_1-\sum_{i=1}^{k_1} r_i$
	k_1+1	X_{k_1+1}	$r_{k_1+1}=0$	$n-k_1-1-\sum_{i=1}^{k_1} r_i$
	⋮	⋮	⋮	⋮
	$m-1$	X_{m-1}	$r_{m-1}=0$	$n-m+1-\sum_{i=1}^{k_1} r_i$
	m	X_m	$r^* = n-m-\sum_{i=1}^{k_1} r_i$	0
III	1	X_1	$r_1 \sim Binom(n-m,p)$	$n-r_1-1$
	2	X_2	$r_2 \sim Binom(n-m-r_1,p)$	$n-r_1-r_2-2$
	⋮	⋮	⋮	⋮
	k_1	X_{k_1}	$r_{k_1} \sim Binom(n-m-\sum_{i=1}^{k_1-1} r_i\,,p)$	$n-k_1-\sum_{i=1}^{k_1} r_i$
	k_1+1	X_{k_1+1}	$r_{k_1+1}=0$	$n-k_1-1-\sum_{i=1}^{k_1} r_i$
	⋮	⋮	⋮	⋮
	k_2-1	X_{k_2-1}	$r_{k_2-1}=0$	$n-k_2+1-\sum_{i=1}^{k_1} r_i$
	k_2	X_{k_2}	$r^* = n-k_2-\sum_{i=1}^{k_1} r_i$	0

3. Point estimation

In this section, we consider both classical and Bayesian approaches to estimate the shape parameter α and the scale parameter β of the Weibull distribution under the IAT-II PC scheme with binomial removals. Within the classical framework, we derive ML and MPS estimators. In addition, we obtain the ML estimator for the binomial removal parameter p.

3.1. Maximum likelihood estimation

In this subsection, the ML estimators (MLEs) for the parameters α, β, and p are derived based on data obtained under the IAT-II PC scheme with binomial removals. As seen from the full likelihood function in Eq (2.9), K is independent of all unknown parameters, $L_1(\alpha,\beta)$ depends only on α and β, and $L_2(p)$ involves only p. Consequently, the MLEs of α and β are directly obtained by maximizing $L_1(\alpha,\beta)$, whereas the MLE of p is directly derived by maximizing $L_2(p)$.

In this context, taking the natural logarithm of $L_1(\alpha,\beta)$ in Eq (2.11), we obtain

$$logL_1(\alpha,\beta) = J_2 log(\alpha\beta) - \beta\left(\sum_{i=1}^{J_2} x_i^{\alpha} + \sum_{i=1}^{J_1} r_i x_i^{\alpha} + r^*(T^*)^{\alpha}\right) + (\alpha-1)\sum_{i=1}^{J_2} log(x_i). \quad (3.1)$$

By setting the first-order partial derivatives of Eq (3.1) with respect to α and β to zero, the corresponding likelihood equations are obtained as follows:

$$\frac{\partial logL_1(\alpha,\beta)}{\partial\alpha} = \frac{J_2}{\alpha} - \beta\left(\sum_{i=1}^{J_2} \varphi_i + \sum_{i=1}^{J_1} r_i\varphi_i + r^*(T^*)^{\alpha} log(T^*)\right) + \sum_{i=1}^{J_2} log(x_i) = 0, \quad (3.2)$$

$$\frac{\partial logL_1(\alpha,\beta)}{\partial\beta} = \frac{J_2}{\beta} - \left(\sum_{i=1}^{J_2} x_i^{\alpha} + \sum_{i=1}^{J_1} r_i x_i^{\alpha} + r^*(T^*)^{\alpha}\right) = 0, \quad (3.3)$$

where $\varphi_i = {x_i}^{\alpha}\log(x_i)$.

If α is known, then the MLE of β, denoted by $\hat{\beta}_{MLE}$, can be easily obtained by solving Eq (3.3) as

$$\hat{\beta}_{MLE} = \frac{J_2}{\sum_{i=1}^{J_2} x_i^{\alpha} + \sum_{i=1}^{J_1} r_i x_i^{\alpha} + r^*(T^*)^{\alpha}}. \quad (3.4)$$

Conversely, when both α and β are unknown, the MLEs of α and β, denoted by $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$, respectively, cannot be obtained in closed form by solving Eqs (3.2) and (3.3) simultaneously. In such cases, numerical methods, such as the Newton–Raphson method, are required to compute $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$.

Similarly, the log-likelihood function of $L_2(p)$ in Eq (2.12) is given by

$$logL_2(p) = \left(\sum_{i=1}^{J_1} r_i\right) log(p) + \left(\sum_{i=1}^{J_1}\left(n - m - \sum_{j=1}^{i} r_j\right)\right) log(1-p). \quad (3.5)$$

Taking the first-order derivative of Eq (3.5) and setting it equal to zero yields the likelihood equation. Solving this equation gives the MLE of p, denoted by $\hat{p}_{\mathrm{MLE}}$, as follows:

$$\hat{p}_{MLE} = \frac{\sum_{i=1}^{J_1} r_i}{\sum_{i=1}^{J_1}\left(n - m - \sum_{j=1}^{i-1} r_j\right)}. \quad (3.6)$$

3.2. Maximum product spacing estimation

For continuous univariate distributions, the MPS method was introduced by Cheng and Amin [24] as an alternative to ML estimation. Around the same time, Ranneby [25] independently derived the MPS method by linking it to the Kullback–Leibler information. The MPS estimators have been shown to perform favorably compared with ML estimators in small samples, particularly for heavy-tailed or skewed distributions and in the presence of outliers. Moreover, the method possesses desirable properties such as consistency, asymptotic normality, and invariance; see Cheng and Iles [26] for further details**.** Owing to these properties, the MPS method has been widely used in reliability analysis and survival modeling, where it often performs competitively with ML estimation; see, for instance, Nassar and Elshahhat [6], Nassar and Alotaibi [23], and Anatolyev and Kosenok [27], among others. In this context, the following subsection presents the MPS estimation method for estimating the shape and scale parameters of the Weibull distribution under the IAT-II PC scheme.

Let $\boldsymbol{X} = \left(X_{1:m:n}, X_{2:m:n}, \dots, X_{J_1:m:n}, \dots, X_{J_2:m:n}\right)$ be an IAT-II PC sample from $Weibull(\alpha, \beta)$ with $\boldsymbol{R} = (R_1 = r_1, R_2 = r_2, \dots, 0, \dots 0, R^* = r^*)$. Following Nassar and Elshahhat [6], the product spacing (PS) function, without the normalizing constant, can be written as

$$PS(\alpha, \beta | \boldsymbol{x}, \boldsymbol{R} = \boldsymbol{r}) = \left\{\prod_{i=1}^{J_2^*}[F(x_i) - F(x_{i-1})]\right\}\left[\prod_{i=1}^{J_1}(1 - F(x_i))^{r_i}\right](1 - F(T^*))^{r^*}, \quad (3.7)$$

where $J_2^* = J_2 + 1$, $F(x_0) = 0$, and $F\left(x_{J_2^*}\right) = 1$. Substituting Eqs (2.1) and (2.2) into Eq (3.7), we obtain

$$PS(\alpha, \beta; \boldsymbol{x} | \boldsymbol{R} = \boldsymbol{r}) = \left\{\prod_{i=1}^{J_2^*}[\phi_{i-1} - \phi_i]\right\} exp\left[-\beta\left(\sum_{i=1}^{J_1} r_i x_i^{\alpha} + r^*(T^*)^{\alpha}\right)\right], \quad (3.8)$$

where $\phi_i = exp(-\beta x_i^{\alpha})$.

As in the ML framework, since the PS function does not involve the parameter p, the MPS estimators of α and β can be obtained by directly maximizing $PS(\alpha, \beta; \boldsymbol{x} | \boldsymbol{R} = \boldsymbol{r})$ with respect to α and β. For simplicity, we denote $PS(\alpha, \beta; \boldsymbol{x} \mid \boldsymbol{R} = \boldsymbol{r})$ by $PS(\alpha, \beta)$. For computational convenience, we consider the natural logarithm of $PS(\alpha, \beta)$, given by

$$logPS(\alpha, \beta) = \sum_{i=1}^{J_2^*} log(\phi_{i-1} - \phi_i) - \beta\left(\sum_{i=1}^{J_1} r_i x_i^{\alpha} + r^*(T^*)^{\alpha}\right). \quad (3.9)$$

Taking the first-order partial derivatives of Eq (3.9) with respect to α and β and setting them equal to zero yields the following nonlinear equations, respectively:

$$\frac{\partial logPS(\alpha, \beta)}{\partial \alpha} = \sum_{i=1}^{J_2^*} \frac{\omega_i - \omega_{i-1}}{\phi_{i-1} - \phi_i} - \beta\left(\sum_{i=1}^{J_1} r_i \varphi_i + r^*(T^*)^{\alpha} \, log(T^*)\right) = 0, \quad (3.10)$$

and

$$\frac{\partial logPS(\alpha,\beta)}{\partial\beta}=\sum_{i=1}^{J_2^*}\frac{\psi_i-\psi_{i-1}}{\phi_{i-1}-\phi_i}-\left(\sum_{i=1}^{J_1}r_ix_i^{\alpha}+r^*(T^*)^{\alpha}\right)=0, \tag{3.11}$$

where $\varphi_i = x_i{}^{\alpha}log(x_i)$, $\omega_i = \beta x_i^{\alpha}log(x_i)exp(-\beta x_i^{\alpha})$, and $\psi_i = x_i^{\alpha}exp(-\beta x_i^{\alpha})$.

The MPS estimators (MPSEs) of α and β, denoted by $\hat{\alpha}_{MPS}$ and $\hat{\beta}_{MPS}$, are obtained by solving Eqs (3.10) and (3.11) simultaneously. Because these equations do not admit closed-form solutions for α and β, numerical techniques, similar to those used for MLEs, are required to compute the MPSEs for the unknown parameters of the Weibull distribution under the IAT-II PC scheme.

3.3. Bayesian estimation

This subsection deals with a Bayesian framework for estimating the two unknown parameters of the Weibull distribution under the IAT-II PC scheme with binomial removals. An important aspect of Bayesian estimation is the specification of prior distributions that reflect available knowledge about the unknown parameters. In this context, the Weibull parameters are assumed to follow independent gamma distributions. This choice is motivated by several considerations. First, the support of the gamma distribution is consistent with the positive domains of the Weibull parameters. Second, it provides a flexible family that can accommodate a wide range of prior beliefs through its hyperparameters. Third, gamma priors are widely used in Bayesian inference for the Weibull distribution, which facilitates comparison with existing studies; see, for example, Lawless [19], Rinne [20], and Kundu [28]. Accordingly, we assume that the unknown parameters α and β are independent random variables, each following a gamma distribution, with $\alpha \sim Gamma(a_1, b_1)$ and $\beta \sim Gamma(a_2, b_2)$, respectively. The corresponding joint prior density is then written, ignoring the normalizing constant, as follows:

$$\pi(\alpha,\beta)\propto\alpha^{a_1-1}\beta^{a_2-1}e^{-b_1\alpha-b_2\beta},\qquad \alpha,\beta>0. \tag{3.12}$$

Here, the hyperparameters (a_i, b_i), $i = 1,2$, are assumed to be known and are selected to incorporate prior information. In particular, setting $a_i = b_i = 0$ leads to improper non-informative priors for α and β as the form $(\alpha\beta)^{-1}$.

Combining the likelihood function in Eq (2.11) with the joint prior in Eq (3.12), the joint posterior density of α and β is obtained, up to a normalizing constant, as

$$\begin{aligned}\pi(\alpha,\beta|\ \boldsymbol{x},\boldsymbol{r})\propto\ &\alpha^{a_1^*-1}\beta^{a_2^*-1}\prod_{i=1}^{J_2}x_i^{\alpha-1}\\&\times exp\left[-b_1\alpha-\beta\left(\sum_{i=1}^{J_2}x_i^{\alpha}+\sum_{i=1}^{J_1}r_ix_i^{\alpha}+r^*(T^*)^{\alpha}+b_2\right)\right],\end{aligned} \tag{3.13}$$

where $a_1^* = a_1 + J_2$ and $a_2^* = a_2 + J_2$.

In Bayesian estimation, the choice of loss function is another important aspect in deriving the optimal estimator of the parameter of interest from the posterior distribution. Among the available choices, the squared error loss function (SELF) is one of the most commonly employed in the literature, as it yields Bayesian estimators (BEs) in the form of posterior means; see, for instance, Gezer and Usta [18], Kundu [28], and Gelman et al. [30]. Accordingly, the BE of any function

$g(\alpha,\beta)$ under the SELF, denoted by $\hat{g}_{BE}(\alpha,\beta)$, is given by the posterior mean

$$\hat{g}_{BE}(\alpha,\beta) = E(g(\alpha,\beta)) = \int_0^\infty \int_0^\infty g(\alpha,\beta)\pi(\alpha,\beta|\ \boldsymbol{x},\boldsymbol{r})d\alpha d\beta\,. \tag{3.14}$$

From Eq (3.14), it follows that the $\hat{g}_{BE}(\alpha,\beta)$ is expressed as a ratio of two-dimensional integrals, for which no closed-form solution is available. Markov Chain Monte Carlo (MCMC) methods are widely used to compute Bayesian estimates of the distribution parameters and the corresponding HPD intervals. An alternative approach for obtaining Bayesian estimates is the Lindley approximation. However, this method is not suitable for constructing HPD intervals. Moreover, its implementation requires higher-order derivatives of the log-likelihood function, which increases computational complexity. For completeness, it should be noted that the Bayesian estimates of the parameters of the Weibull distribution based on the Lindley approximation were also obtained. However, due to space constraints and their inferior performance, these results are not reported in this study but can be provided upon request from the authors. Therefore, in this study, we use the MCMC approach, which allows us to compute Bayesian estimates for any function of the parameters and to construct HPD intervals. A detailed description of the MCMC procedure is provided below.

Samples of the unknown parameters α and β from the joint posterior distribution in Eq (3.13) are generated through the MCMC procedure. Before applying the MCMC methodology, it is necessary to examine whether the full conditional posterior distributions of the parameters α and β correspond to any well–known distributions. To this end, the full conditional posterior distributions of α and β are derived as follows:

$$\pi(\alpha|\ \beta,\boldsymbol{x},\boldsymbol{r}) \propto \alpha^{a_1^*-1}\prod_{i=1}^{J_2} {x_i}^{\alpha-1}\ \ exp\left[-b_1\alpha-\beta\left(\sum_{i=1}^{J_2} x_i^{\alpha}+\sum_{i=1}^{J_1} r_i x_i^{\alpha}+r^*(T^*)^{\alpha}\right)\right] \tag{3.15}$$

and

$$\pi(\beta|\ \alpha,\boldsymbol{x},\boldsymbol{r}) \propto \beta^{a_2^*-1}exp\left[-\beta\left(b_2+\sum_{i=1}^{J_2} x_i^{\alpha}+\sum_{i=1}^{J_1} r_i x_i^{\alpha}+r^*(T^*)^{\alpha}\right)\right]. \tag{3.16}$$

From Eq (3.16), it follows that the full conditional posterior distribution of β is a gamma distribution with the shape parameter a_2^*, and the scale parameter $b_2+\sum_{i=1}^{J_2} x_i^{\alpha}+\sum_{i=1}^{J_1} r_i x_i^{\alpha}+r^*(T^*)^{\alpha}$ is denoted as $\beta\sim Gamma\left(a_2^*,\ b_2+\sum_{i=1}^{J_2} x_i^{\alpha}+\sum_{i=1}^{J_1} r_i x_i^{\alpha}+r^*(T^*)^{\alpha}\right)$. Hence, samples of β can be generated directly from this distribution. In contrast, the full conditional posterior distribution of α in Eq (3.15) does not correspond to any well–known distribution. In such cases, the Metropolis–Hastings (M–H) algorithm provides an effective approach for generating samples of α. Therefore, to implement the MCMC procedure, we employ Gibbs sampling steps to update the unknown parameter β, in conjunction with M–H steps to update α. The MCMC procedure is implemented by updating β via Gibbs sampling, while α is updated using M–H steps. The steps for generating samples of α and β are summarized in Algorithm 1.

Algorithm 1:

Step 1. Start with the initial values of $\left(\alpha^{(0)},\beta^{(0)}\right)$ for (α,β).

Step 2. Set $j=1$.

Step 3. Generate $\beta^{(j)}$ from $Gamma\left(a_2^*,\ b_2+\sum_{i=1}^{J_2} x_i^{\alpha^{(j-1)}}+\sum_{i=1}^{J_1} r_i x_i^{\alpha^{(j-1)}}+r^*(T^*)^{\alpha^{(j-1)}}\right)$ for given m, n and the IAT-II PC sample with binomial removals.

Step 4. Generate $\alpha^{(j)}$ from $\pi\left(\alpha^{(j-1)}|\beta^{(j)},\boldsymbol{x},\boldsymbol{r}\right)$ using M–H steps with the proposal distribution $N\left(\alpha^{(j-1)},\sigma_\alpha^2\right)$:

(i) Generate α^* from $N\left(\alpha^{(j-1)},\sigma_\alpha^2\right)$.

(ii) If $\alpha^* < 0$, reject the proposal and retain the current state, i.e., set $\alpha^* = \alpha^{(j-1)}$.

(iii) Calculate $\delta\left(\alpha^*,\alpha^{(j-1)}\right) = min\left\{1,\frac{\pi\left(\alpha^*|\beta^{(j)},\boldsymbol{x},\boldsymbol{r}\right)}{\pi\left(\alpha^{(j-1)}|\beta^{(j)},\boldsymbol{x},\boldsymbol{r}\right)}\right\}$.

(iv) Generate u from the uniform distribution $U(0,1)$.

(v) Set $\alpha^{(j)} = \begin{cases} \alpha^*, & if\ \ u \le \delta\left(\alpha^*,\alpha^{(j-1)}\right) \\ \alpha^{(j-1)}, & otherwise \end{cases}$.

Step 5. Set $j = j + 1$.

Step 6. Repeat Steps 3–5 for M iterations to obtain $\left(\alpha^{(1)},\beta^{(1)}\right),\left(\alpha^{(2)},\beta^{(2)}\right),\dots,\left(\alpha^{(M)},\beta^{(M)}\right)$.

To reduce the effect of the initial values and ensure convergence of the Markov chain, the first B iterations are discarded as burn-in. The remaining samples, corresponding to iterations $j = B + 1, B + 2, \dots, M$, where M is sufficiently large, are used for statistical inference. Based on these post-burn-in samples, the Bayes estimates of the parameters are approximately calculated. Accordingly, the Bayes estimates of α and β under SELF are obtained as follows:

$$\hat{\alpha}_{BE} = \frac{\sum_{j=B+1}^{M}\alpha^{(j)}}{M - B} \tag{3.17}$$

and

$$\hat{\beta}_{BE} = \frac{\sum_{j=B+1}^{M}\beta^{(j)}}{M - B}, \tag{3.18}$$

where B is the burn-in period of the Markov chain.

4. Interval estimation

In this section, ACIs for the parameters of the Weibull distribution under the IAT-II PC scheme with binomial removals are constructed using the ML and MPS estimation methods. In addition, HPD credible intervals for the same parameters are obtained using the Bayesian method.

4.1. ACIs based on maximum likelihood estimation

As previously established, the MLEs of the parameters α and β of the Weibull distribution do not have closed-form expressions. This limitation complicates the derivation of their exact sampling distributions and the construction of exact confidence intervals. To overcome this issue, the ACIs for the parameters are constructed based on the Fisher information matrix and asymptotic theory. Furthermore, as seen from Eq (3.1), the binomial removal parameter p, which characterizes the random removal scheme, does not contain information about the Weibull parameters. Therefore, it is not considered in the construction of these confidence intervals. Accordingly, the Fisher information matrix for the parameters α and β of the Weibull distribution under the IAT-II PC sample with binomial removals is given by

$$I(\alpha,\beta) = E\begin{bmatrix} -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\alpha^2} & -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\beta\partial\alpha} & -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\beta^2} \end{bmatrix}, \tag{4.1}$$

where

$$\frac{\partial^2 \log L_1(\alpha,\beta)}{\partial\alpha^2} = -\frac{J_2}{\alpha^2} - \beta\left(\sum_{i=1}^{J_2}\varphi_i \log(x_i) + \sum_{i=1}^{J_1} r_i\varphi_i \log(x_i) + r^*(T^*)^\alpha(\log(T^*))^2\right), \tag{4.2}$$

$$\frac{\partial^2 \log L_1(\alpha,\beta)}{\partial\beta^2} = -\frac{J_2}{\beta^2}, \tag{4.3}$$

$$\frac{\partial^2 \log L_1(\alpha,\beta)}{\partial\alpha\partial\beta} = \frac{\partial^2 \log L_1(\alpha,\beta)}{\partial\beta\partial\alpha} = -\left(\sum_{i=1}^{J_2}\varphi_i + \sum_{i=1}^{J_1} r_i\varphi_i + r^*(T^*)^\alpha \log(T^*)\right), \tag{4.4}$$

where $\varphi_i = x_i^\alpha \log(x_i)$. However, the exact expression for the expectation in Eq (4.2) is not easily obtainable. As a practical alternative, the Fisher information matrix is approximated by the observed Fisher information matrix, given as

$$\hat{I}(\hat{\alpha}_{MLE},\hat{\beta}_{MLE}) = \left.\begin{bmatrix} -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\alpha^2} & -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\beta\partial\alpha} & -\dfrac{\partial^2 \log L_1(\alpha,\beta)}{\partial\beta^2} \end{bmatrix}\right|_{(\alpha,\beta)=(\hat{\alpha}_{MLE},\hat{\beta}_{MLE})}. \tag{4.5}$$

Furthermore, under certain regularity conditions, the joint distribution of the MLEs of α and β is asymptotically bivariate normal with mean (α,β) and variance-covariance matrix $\hat{I}^{-1}(\hat{\alpha}_{MLE},\hat{\beta}_{MLE})$; for further details, see Lawless [19] and Lehmann and Casella [31]. Accordingly, the inverse of the observed Fisher information matrix $\hat{I}^{-1}(\hat{\alpha}_{MLE},\hat{\beta}_{MLE})$ can be expressed as

$$\hat{I}^{-1}(\hat{\alpha}_{MLE},\hat{\beta}_{MLE}) = \begin{bmatrix} Var(\hat{\alpha}_{MLE}) & Cov(\hat{\alpha}_{MLE},\hat{\beta}_{MLE}) \\ Cov(\hat{\alpha}_{MLE},\hat{\beta}_{MLE}) & Var(\hat{\beta}_{MLE}) \end{bmatrix}. \tag{4.6}$$

Consequently, for any $0 < \tau < 1$, the $100(1-\tau)\%$ ACIs for the parameters α and β of the Weibull distribution are constructed, respectively, as

$$\left[\hat{\alpha}_{MLE} - z_{\tau/2}\sqrt{Var(\hat{\alpha}_{MLE})}, \hat{\alpha}_{MLE} + z_{\tau/2}\sqrt{Var(\hat{\alpha}_{MLE})}\right] \tag{4.7}$$

and

$$\left[\hat{\beta}_{MLE} - z_{\tau/2}\sqrt{Var(\hat{\beta}_{MLE})},\ \hat{\beta}_{MLE} + z_{\tau/2}\sqrt{Var(\hat{\beta}_{MLE})}\right], \tag{4.8}$$

where $z_{\tau/2}$ denotes the upper $\tau/2$ −th quantile of the standard normal distribution.

4.2. ACIs based on maximum product spacing estimation

In this subsection, the ACIs for the parameters α and β of the Weibull distribution are constructed based on the MPS estimation method under the IAT-II PC scheme with binomial removals. As noted by Cheng and Iles [26] and Anatolyev and Kosenok [27], under certain regularity conditions, the MPSEs possess the same asymptotic properties as the MLEs, including consistency and asymptotic normality. This equivalence provides a theoretical basis for using the asymptotic distribution of the MPSEs in constructing the ACIs. Accordingly, the ACIs for the Weibull parameters are constructed using the asymptotic normality of the MPSEs, with the variance–covariance matrix obtained from the Fisher information matrix derived from the log product spacing function. To this end, the second-order partial derivatives of the log product spacing function in Eq (3.9) with respect to α and β are derived as follows:

$$\begin{aligned}\frac{\partial^2 \log PS(\alpha,\beta)}{\partial\alpha^2} = {} & \sum_{i=1}^{J_2^*}\frac{\omega_i^* - \omega_{i-1}^*}{\phi_{i-1}-\phi_i} - \sum_{i=1}^{J_2^*}\left(\frac{\omega_i-\omega_{i-1}}{\phi_{i-1}-\phi_i}\right)^2 \\ & -\beta\left[\sum_{i=1}^{J_1} r_i\varphi_i \log(x_i) + r^*(T^*)^{\alpha}\log^2(T^*)\right],\end{aligned} \tag{4.9}$$

$$\frac{\partial^2 \log PS(\alpha,\beta)}{\partial\beta^2} = \sum_{i=1}^{J_2^*}\frac{\psi_i^* - \psi_{i-1}^*}{\phi_{i-1}-\phi_i} - \sum_{i=1}^{J_2^*}\left(\frac{\psi_i-\psi_{i-1}}{\phi_{i-1}-\phi_i}\right)^2, \tag{4.10}$$

$$\begin{aligned}\frac{\partial^2 \log PS(\alpha,\beta)}{\partial\beta\partial\alpha} = \frac{\partial^2 \log PS(\alpha,\beta)}{\partial\alpha\partial\beta} = {} & \sum_{i=1}^{J_2^*}\frac{\omega_i^{**} - \omega_{i-1}^{**}}{\phi_{i-1}-\phi_i} - \sum_{i=1}^{J_2^*}\frac{(\omega_i-\omega_{i-1})(\psi_i-\psi_{i-1})}{(\phi_{i-1}-\phi_i)^2} \\ & -\sum_{i=1}^{J_1} r_i\varphi_i + r^*(T^*)^{\alpha}\log(T^*),\end{aligned} \tag{4.11}$$

where $\phi_i = exp(-\beta x_i^{\alpha})$, $\varphi_i = x_i^{\alpha}log(x_i)$, $\omega_i = \beta x_i^{\alpha}log(x_i)exp(-\beta x_i^{\alpha})$, $\psi_i = x_i^{\alpha}exp(-\beta x_i^{\alpha})$, $\omega_i^* = \omega_i log(x_i)(1-\beta x_i^{\alpha})$, $\omega_i^{**} = x_i^{\alpha}log(x_i)exp(-\beta x_i^{\alpha})(1-\beta x_i^{\alpha})$, and $\psi_i^* = -x_i^{\alpha}\psi_i$.

As seen in Eqs (4.9)–(4.11), the resulting expressions are quite complex, which makes the evaluation of their expected values intractable. Therefore, as in the maximum likelihood approach, the observed information matrix derived from the log product spacing function is employed as an approximation to the Fisher information matrix. The observed Fisher information matrix is obtained by

$$\hat{I}(\hat{\alpha}_{MPSE},\hat{\beta}_{MPSE}) = \left.\begin{bmatrix} -\dfrac{\partial^2 \log PS(\alpha,\beta)}{\partial\alpha^2} & -\dfrac{\partial^2 \log PS(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\dfrac{\partial^2 \log PS(\alpha,\beta)}{\partial\beta\partial\alpha} & -\dfrac{\partial^2 \log PS(\alpha,\beta)}{\partial\beta^2}\end{bmatrix}\right|_{(\alpha,\beta)=(\hat{\alpha}_{MPSE},\hat{\beta}_{MPSE})}. \tag{4.12}$$

Following Cheng and Iles [26], under certain regularity conditions, the MPSEs of α and β are asymptotically distributed as a bivariate normal distribution with mean (α,β) and variance-covariance matrix given by the inverse of the observed Fisher information matrix, given by

$$\hat{I}^{-1}(\hat{\alpha}_{MPSE},\hat{\beta}_{MPSE}) = \begin{bmatrix} Var(\hat{\alpha}_{MPSE}) & Cov(\hat{\alpha}_{MPSE},\hat{\beta}_{MPSE}) \\ Cov(\hat{\alpha}_{MPSE},\hat{\beta}_{MPSE}) & Var(\hat{\beta}_{MPSE})\end{bmatrix}. \tag{4.13}$$

Hence, for any $0 < \tau < 1$, the $100(1-\tau)\%$ ACIs of the parameters α and β of the Weibull distribution are obtained, respectively, as

$$\left[\hat{\alpha}_{MPSE} - z_{\tau/2}\sqrt{Var(\hat{\alpha}_{MPSE})}, \hat{\alpha}_{MPSE} + z_{\tau/2}\sqrt{Var(\hat{\alpha}_{MPSE})}\right] \tag{4.14}$$

and

$$\left[\hat{\beta}_{MPSE} - z_{\tau/2}\sqrt{Var(\hat{\beta}_{MPSE})}, \hat{\beta}_{MPSE} + z_{\tau/2}\sqrt{Var(\hat{\beta}_{MPSE})}\right], \tag{4.15}$$

where $z_{\tau/2}$ denotes the upper $\tau/2$ −th quantile of the standard normal distribution.

4.3. HPD credible intervals based on Bayesian estimation

In Bayesian inference, an HPD interval is a credible interval defined as the region of highest posterior density containing the most probable values of a parameter. HPD intervals are typically shorter and more informative, particularly when the posterior distribution is skewed or multimodal; for more details, see Gelman et al. [30] and Chen and Shao [32]. For these reasons, HPD intervals are widely used in Bayesian inference for interval estimation of the parameters. However, HPD credible intervals generally cannot be derived in closed form and must therefore be computed numerically. In such cases, MCMC samples are used to estimate the HPD credible intervals of the parameters.

In this study, the MCMC samples of the Weibull parameters α and β generated via Algorithm 1 in Section 3.3 are used to compute numerical approximations of the HPD intervals. Based on these samples, the first B samples are discarded from the MCMC samples as burn-in, and the remaining post-burn-in samples $\alpha^{(j)}$ and $\beta^{(j)}$ for $j = B+1, B+2, \dots, M$ are retained. These samples are then sorted in ascending order and re-indexed with indices $i = 1,2,\dots, M-B$, as follows: $\alpha^{[1]} \le \alpha^{[2]} \le \cdots \le \alpha^{[i]} \le \cdots \le \alpha^{[M-B]}$ and $\beta^{[1]} \le \beta^{[2]} \le \cdots \le \beta^{[i]} \le \cdots \le \beta^{[M-B]}$. The HPD credible intervals for the parameters α and β are subsequently computed following the approach suggested by Chen and Shao [32], as described below.

Let δ denote any parameter of interest. Then, for any $0 < \tau < 1$, the $100(1-\tau)\%$ HPD credible interval for δ is given by

$$\left[\delta^{[i^*]}, \delta^{[i^*+k]}\right], \tag{4.16}$$

where $k = \lfloor (1-\tau)(M-B) \rfloor$ and $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. Here, the index i^* is determined such that

$$\delta^{[i^*+k]} - \delta^{[i^*]} = \min_{1 \le i \le (M-B)-k} \left(\delta^{[i+k]} - \delta^{[i]}\right). \tag{4.17}$$

5. Simulation study

In this section, a comprehensive Monte Carlo simulation study is conducted to evaluate the performance of the point and interval estimators derived in this study for the two-parameter Weibull distribution under the proposed IAT-II PC scheme with binomial removals. To this end, the IAT-II PC samples are generated by considering various combinations of n, m, T_1, T_2, and p, where the true parameter values of the Weibull distribution are set to be either $\alpha = 1.5$, $\beta = 1$ or $\alpha = 2.5$, $\beta = 1$. For the sake of brevity and due to space constraints, simulation results are presented only for $\alpha = 1.5$, $\beta = 1$; however, results for $\alpha = 2.5$, $\beta = 1$ are available upon request from the authors.

In the simulation, for each pair of time thresholds $(T_1, T_2) = (0.4, 0.7)$ and $(0.7, 1)$, different values of n and m are considered, where n assumes the values 40, 60, and 80, and m is set according to the ratios $m/n = 40\%$, 60%, and 80% for each n. Furthermore, for each combination of n and m, the number of units removed at the i-th failure time, R_i, is generated from a binomial distribution with parameter $p = 0.3, 0.5,$ and 0.7. To generate the samples for the Monte Carlo simulation study, we extend the IAT-II PC sample generation procedure proposed by Yan et al. [3] by incorporating the binomial removal mechanism described in Section 2. The procedure is presented in the following algorithm for generating IAT-II PC samples with binomial random removals from the Weibull distribution.

Algorithm 2:

Step 1. Set the values of $n, m, T_1, T_2, \alpha, \beta,$ and p.

Step 2. Generate r_1 from $R_1 \sim Binom(n - m, p)$.

Step 3. Generate r_i from

$$R_i \sim Binom\left(n - m - \sum_{j=1}^{i-1} r_j, p\right), i = 2,3, \dots, m-1.$$

Step 4. Compute r^* as

$$r^* = \begin{cases} n - m - \sum_{i=1}^{m-1} r_i\,, & \text{if} \quad n - m - \sum_{i=1}^{m-1} r_i > 0, \\ 0, & \text{for otherwise.} \end{cases}$$

Step 5. Generate a T-II PC sample using the censoring scheme $\boldsymbol{R} = (r_1, \dots, r_m)$, obtained in Steps 2–4, according to the procedure of Balakrishnan and Sandhu [33].

Step 6. Determine the value of k_1 that occurs before the first-time threshold T_1. Then:

(i) If $X_m < T_1$, the experiment is terminated at X_m. Set $X_1, X_2, \dots, X_m$ as the final IAT-II PC sample, and $(R_1 = r_1, \dots, R_m = r^*)$ as the final removal scheme. This yields Case I. Otherwise, continue with Step 6(ii).

(ii) If $X_{k_1} < T_1 < X_{k_1+1}$, then discard the sample $X_{k_1+2}, \dots, X_m$, and continue to the next step.

Step 7. Generate the first $m - k_1 - 1$ order statistics from a truncated distribution $f(x)/[1 - F(x_{k_1+1})]$ with sample size $n - k_1 - 1 - \sum_{i=1}^{k_1} r_i$. Denote the newly generated sample as $X_{k_1+2}, X_{k_1+3}, \dots, X_m$, which differ from those generated under the T-II PC scheme in Step 5.

Step 8. Determine the value of k_2 that occurs before the second time threshold T_2. Then:

(i) If $X_m < T_2$, the experiment is terminated at X_m. Set $X_1, X_2, \dots, X_{k_1}, \dots, X_m$ as the final IAT-II PC sample and $(R_1 = r_1, \dots, R_{k_1} = r_{k_1}, R_{k_1+1} = 0, \dots, R_{m-1} = 0, R_m = r^*)$ as the final removal scheme, where $r^* = n - m - \sum_{i=1}^{k_1} r_i$. This yields Case II.

(ii) If $T_2 < X_m$, the experiment is terminated at T_2. Set $X_1, \dots, X_{k_1}, \dots, X_{k_2}$ as the final IAT-II PC sample and $\left(R_1 = r_1, \dots, R_{k_1} = r_{k_1}, R_{k_1+1} = 0, \dots, R_{k_2-1} = 0, R_{k_2} = r^*\right)$ as the final removal scheme, where $r^* = n - k_2 - \sum_{i=1}^{k_1} r_i$. This yields Case III.

In the Bayesian estimation, 12,000 MCMC samples are generated using Algorithm 1 in Section 3.3. The first 2000 iterations are discarded as burn-in to ensure convergence, and the remaining 10,000 post-burn-in samples are used to compute the Bayesian point estimates of the parameters α and β, as well as to construct their corresponding HPD credible intervals. To assess the convergence of the MCMC algorithm, trace plots, autocorrelation plots, effective sample sizes, and acceptance rates are examined for all simulation settings. These diagnostics consistently indicate satisfactory mixing and stationarity of the Markov chains. Furthermore, the proposal standard deviation is carefully tuned under each prior specification to achieve appropriate acceptance rates for the M–H algorithm, as recommended by Robert and Casella [29]. The corresponding acceptance rates are monitored throughout the simulation study and are observed to fall remain an optimal range (e.g., between 0.25 and 0.45), ensuring the efficiency of the sampling process.

Bayesian inference for α and β is conducted under non-informative and informative prior specifications. As in Kundu [28], the hyperparameters of informative prior distributions are selected such that the corresponding prior means match the true parameter values. This specification is suitable for evaluating the estimators under correctly specified prior information. However, it should be noted that it naturally favors Bayesian estimation over classical methods. Specifically, for the first informative prior (Prior 1), the hyperparameters are set as $(a_1, b_1) = (3,2)$ for $\alpha = 1.5$ and $(a_2, b_2) = (2,2)$ for $\beta = 1$. For the second informative prior (Prior 2), the hyperparameters are specified as $(a_1, b_1) = (6,4)$ for $\alpha = 1.5$ and $(a_2, b_2) = (4,4)$ for $\beta = 1$. For the non-informative prior (Prior 0), all hyperparameters (a_1, b_1, a_2, b_2) are set to zero, yielding an improper prior distribution that is commonly used in Bayesian inference. As the estimates obtained under Prior 0 are nearly identical to the corresponding MLEs, only the results corresponding to Prior 1 and Prior 2 are presented here for brevity. The complete results for Prior 0 can be provided upon request.

The performance of the point estimators derived in this study is evaluated numerically in terms of their average estimates (AEs), mean squared errors (MSEs), and deficiencies (DEFs). The deficiency, denoted by $\mathrm{DEF}(\hat{\theta}_1, \hat{\theta}_2)$, provides a measure of the joint efficiency for a pair of estimators, such as $(\hat{\theta}_1, \hat{\theta}_2)$, and is defined as the sum of their MSEs, namely, $\mathrm{MSE}(\hat{\theta}_1) + \mathrm{MSE}(\hat{\theta}_2)$; for further details, see Usta [34] and Kantar and Şenoğlu [35]. In addition, the performance of the interval estimators is assessed based on their average lengths (ALs) and coverage probabilities (CPs).

For each case in the simulation study, the AEs, MSEs, DEFs, ALs, and CPs are computed over 1000 replications based on the point and interval estimates of all parameters obtained using the estimators derived in Sections 3 and 4. The computational implementation is carried out in the R statistical computing environment (version 4.2.2). Specifically, the *maxLik* package by Henningsen and Toomet [36] was used to obtain the ML and MPS estimates using the *L-BFGS* optimization algorithm implemented in the *maxLik* package. The *coda* package by Plummer et al. [37] is employed to compute the Bayesian estimates and the corresponding HPD credible intervals. All optimization procedures converged successfully, and no convergence problems were encountered during either simulation study.

The point estimation results for the parameters α and β are presented in Tables 3 and 4, while the interval estimation results are reported in Tables 5 and 6. In Tables 3 and 4, BE_{Prior_1} and BE_{Prior_2} denote the Bayesian estimates obtained under the corresponding informative prior distributions, respectively. In Tables 5 and 6, HPD_{Prior_1} and HPD_{Prior_2} represent the corresponding HPD credible intervals, whereas ACI_{ML} and ACI_{MPS} denote the asymptotic confidence intervals based on the ML and MPS estimation methods, respectively. In all tables, the estimators with the smallest MSE and DEF values and the intervals with the smallest AL are highlighted in bold.

Table 3. AEs, MSEs, and DEFs for the estimates of α and β at $(T_1, T_2) = (0.4, 0.7)$.

p	n, m		MLE		MPS		BE_{Prior_1}		BE_{Prior_2}	
			α	β	α	β	α	β	α	β
0.3	40,16	**AE**	1.6089	1.1566	1.7350	1.1610	1.5545	1.0668	1.5389	1.0370
		MSE	0.2006	0.2870	0.2862	0.3445	0.0993	0.1025	**0.0669**	**0.0587**
		DEF	0.4875		0.6306		0.2018		**0.1256**	
	40,24	**AE**	1.5991	1.1104	1.7150	1.1137	1.5563	1.0580	1.5407	1.0351
		MSE	0.1773	0.1762	0.2528	0.2011	0.0963	0.0846	**0.0659**	**0.0532**
		DEF	0.3535		0.4540		0.1808		**0.1191**	
	40,32	**AE**	1.5852	1.0829	1.6890	1.0853	1.5506	1.0493	1.5367	1.0315
		MSE	0.1586	0.1293	0.2157	0.1442	0.0923	0.0731	**0.0648**	**0.0495**
		DEF	0.2879		0.3599		0.1654		**0.1143**	
	60,24	**AE**	1.5620	1.0913	1.6635	1.1007	1.5355	1.0498	1.5256	1.0304
		MSE	0.1243	0.1478	0.1719	0.1746	0.0780	0.0759	**0.0572**	**0.0486**
		DEF	0.2721		0.3464		0.1540		**0.1058**	
	60,36	**AE**	1.5712	1.0849	1.6599	1.0922	1.5474	1.0570	1.5371	1.0407
		MSE	0.1126	0.1086	0.1510	0.1209	0.0745	0.0660	**0.0559**	**0.0456**
		DEF	0.2212		0.2719		0.1405		**0.1014**	
	60,48	**AE**	1.5485	1.0470	1.6289	1.0535	1.5332	1.0341	1.5256	1.0240
		MSE	0.0875	0.0715	0.1141	0.0777	0.0624	0.0497	**0.0484**	**0.0372**
		DEF	0.1590		0.1918		0.1121		**0.0856**	
	80,32	**AE**	1.5427	1.0825	1.6222	1.0894	1.5260	1.0530	1.5187	1.0366
		MSE	0.0967	0.1263	0.1215	0.1399	0.0663	0.0748	**0.0510**	**0.0472**
		DEF	0.2230		0.2613		0.1411		**0.0982**	
	80,48	**AE**	1.5604	1.0610	1.6326	1.0675	1.5460	1.0449	1.5386	1.0335
		MSE	0.0851	0.0809	0.1072	0.0866	0.0602	0.0558	**0.0464**	**0.0416**
		DEF	0.1660		0.1939		0.1160		**0.0879**	
	80,64	**AE**	1.5424	1.0452	1.6077	1.0530	1.5321	1.0367	1.5262	1.0287
		MSE	0.0712	0.0596	0.0885	0.0651	0.0553	0.0458	**0.0446**	**0.0363**
		DEF	0.1308		0.1536		0.1010		**0.0809**	
0.5	40,16	**AE**	1.6043	1.1620	1.7513	1.1684	1.5518	1.0634	1.5370	1.0337
		MSE	0.2111	0.3200	0.3237	0.3943	0.1006	0.1074	**0.0678**	**0.0598**
		DEF	0.5311		0.7180		0.2081		**0.1276**	
	40,24	**AE**	1.5928	1.1100	1.7185	1.1136	1.5518	1.0554	1.5367	1.0328
		MSE	0.1876	0.1924	0.2713	0.2228	0.0993	0.0890	**0.0669**	**0.0552**
		DEF	0.3800		0.4940		0.1883		**0.1221**	
	40,32	AE	1.5928	1.0821	1.7021	1.0853	1.5566	1.0471	1.5417	1.0292
		MSE	0.1705	0.1349	0.2383	0.1520	0.0973	0.0744	**0.0657**	**0.0499**
		DEF	0.3054		0.3903		0.1717		**0.1155**	
	60,24	**AE**	1.5760	1.1075	1.6949	1.1193	1.5484	1.0585	1.5371	1.0366
		MSE	0.1334	0.1783	0.2021	0.2124	0.0815	0.0891	**0.0592**	**0.0564**
		DEF	0.3118		0.4145		0.1705		**0.1156**	

Continued on next page

p	n,m		MLE		MPS		BE_{Prior_1}		BE_{Prior_2}	
			α	β	α	β	α	β	α	β
	60,36	**AE**	1.5671	1.0741	1.6586	1.0800	1.5469	1.0479	1.5365	1.0323
		MSE	0.1160	0.1205	0.1552	0.1364	0.0765	0.0737	**0.0570**	**0.0512**
		DEF	0.2365		0.2916		0.1502		**0.1082**	
	60,48	**AE**	1.5545	1.0421	1.6357	1.0475	1.5387	1.0281	1.5298	1.0174
		MSE	0.1088	0.0863	0.1396	0.0952	0.0747	0.0578	**0.0568**	**0.0428**
		DEF	0.1951		0.2348		0.1325		**0.0996**	
	80,32	**AE**	1.5572	1.0828	1.6546	1.0943	1.5418	1.0531	1.5329	1.0358
		MSE	0.0972	0.1325	0.1334	0.1504	0.0661	0.0779	**0.0500**	**0.0525**
		DEF	0.2297		0.2838		0.1440		**0.1025**	
	80,48	**AE**	1.5363	1.0388	1.6144	1.0461	1.5272	1.0275	1.5212	1.0184
		MSE	0.0828	0.0744	0.1053	0.0814	0.0615	0.0523	**0.0486**	**0.0391**
		DEF	0.1572		0.1867		0.1138		**0.0877**	
	80,64	**AE**	1.5525	1.0423	1.6194	1.0493	1.5421	1.0340	1.5359	1.0262
		MSE	0.0677	0.0597	0.0865	0.0645	0.0523	0.0461	**0.0426**	**0.0369**
		DEF	0.1274		0.1510		0.0985		**0.0795**	
0.7	40,16	**AE**	1.6029	1.1548	1.7656	1.1610	1.5535	1.0571	1.5392	1.0293
		MSE	0.2237	0.3317	0.3533	0.4208	0.1034	0.1093	**0.0678**	**0.0602**
		DEF	0.5554		0.7741		0.2126		**0.1281**	
	40,24	**AE**	1.5791	1.0969	1.7151	1.1031	1.5432	1.0476	1.5297	1.0272
		MSE	0.1851	0.1802	0.2815	0.2225	0.0969	0.0851	**0.0663**	**0.0533**
		DEF	0.3652		0.5040		0.1821		**0.1196**	
	40,32	**AE**	1.5765	1.0812	1.6973	1.083	1.5548	1.0475	1.5401	1.0296
		MSE	0.1656	0.1374	0.2222	0.154	0.0951	0.0761	**0.065**	**0.0514**
		DEF	0.3030		0.3763		0.1712		**0.1164**	
	60,24	**AE**	1.5552	1.0891	1.6786	1.0964	1.5351	1.0450	1.5259	1.0259
		MSE	0.1404	0.1734	0.2074	0.2040	0.0833	0.0856	**0.0593**	**0.0536**
		DEF	0.3138		0.4115		0.1689		**0.1129**	
	60,36	**AE**	1.5523	1.0571	1.6507	1.0621	1.5367	1.0354	1.5276	1.0220
		MSE	0.1129	0.1143	0.1486	0.1268	0.0761	0.0699	**0.0569**	**0.0491**
		DEF	0.2272		0.2754		0.1460		**0.1060**	
	60,48	**AE**	1.5584	1.0539	1.6391	1.0588	1.5415	1.0390	1.5321	1.0275
		MSE	0.1055	0.0809	0.1386	0.0882	0.0730	0.0552	**0.0557**	**0.0409**
		DEF	0.1864		0.2269		0.1282		**0.0966**	
	80,32	**AE**	1.5407	1.0493	1.6473	1.0589	1.5325	1.0295	1.5250	1.0169
		MSE	0.0954	0.1024	0.1354	0.1168	0.0657	0.0631	**0.0499**	**0.0436**
		DEF	0.1978		0.2522		0.1288		**0.0935**	
	80,48	**AE**	1.5393	1.0449	1.6220	1.0530	1.5308	1.0333	1.5234	1.0232
		MSE	0.0850	0.0768	0.1099	0.0841	0.0631	0.0542	**0.0496**	**0.0407**
		DEF	0.1617		0.1940		0.1173		**0.0903**	
	80,64	**AE**	1.5283	1.0319	1.5937	1.0374	1.5209	1.0260	1.5162	1.0195
		MSE	0.0671	0.0557	0.0821	0.0591	0.0522	0.0428	**0.0424**	**0.0342**
		DEF	0.1228		0.1411		0.0950		**0.0766**	

Table 4. AEs, MSEs, and DEFs for the estimates of α and β at $(T_1, T_2) = (0.7, 1)$.

p	n, m		MLE		$MPSE$		BE_{Prior_1}		BE_{Prior_2}	
			α	β	α	β	α	β	α	β
0.3	40,16	**AE**	1.5880	1.1076	1.6973	1.1133	1.5558	1.0555	1.5429	1.0374
		MSE	0.1483	0.1722	0.2076	0.2171	0.0961	0.0889	**0.0664**	**0.0583**
		DEF	0.3204		0.4248		0.1850		**0.1247**	
	40,24	**AE**	1.5782	1.0494	1.6696	1.0424	1.5583	1.0277	1.5478	1.0199
		MSE	0.1265	0.0899	0.1671	0.0991	0.0914	0.0623	**0.0645**	**0.0469**
		DEF	0.2164		0.2662		0.1537		**0.1114**	
	40,32	**AE**	1.5590	1.0379	1.6368	1.0297	1.5448	1.0254	1.5368	1.0204
		MSE	0.0969	0.0638	0.1221	0.0684	0.0742	0.0491	**0.0598**	**0.0398**
		DEF	0.1607		0.1904		0.1234		**0.0996**	
	60,24	**AE**	1.5523	1.0598	1.6354	1.0603	1.5387	1.0354	1.5321	1.0267
		MSE	0.0880	0.0904	0.1169	0.1007	0.0671	0.0623	**0.0539**	**0.0470**
		DEF	0.1784		0.2177		0.1294		**0.1008**	
	60,36	**AE**	1.5417	1.0424	1.6073	1.0381	1.5335	1.0300	1.5285	1.0253
		MSE	0.0708	0.0516	0.0892	0.0552	0.0577	0.0407	**0.0485**	**0.0334**
		DEF	0.1223		0.1444		0.0984		**0.0819**	
	60,48	**AE**	1.5469	1.0166	1.6076	1.0121	1.5393	1.0103	1.5356	1.0084
		MSE	0.0607	0.0384	0.0774	0.0394	0.0509	0.0327	**0.044**	**0.0285**
		DEF	0.0991		0.1168		0.0836		**0.0725**	
	80,32	**AE**	1.5265	1.0291	1.5967	1.0299	1.5211	1.0159	1.5180	1.0115
		MSE	0.0630	0.0611	0.0777	0.0668	0.0517	0.0466	**0.0438**	**0.0373**
		DEF	0.1241		0.1444		0.0984		**0.0811**	
	80,48	**AE**	1.5345	1.0229	1.5900	1.0203	1.5302	1.0156	1.5267	1.0131
		MSE	0.0551	0.0385	0.0658	0.0405	0.0472	0.0324	**0.0410**	**0.0279**
		DEF	0.0936		0.1063		0.0796		**0.0690**	
	80,64	**AE**	1.5342	1.0267	1.5816	1.0242	1.5298	1.0219	1.5271	1.0199
		MSE	0.0486	0.0283	0.0570	0.0295	0.0427	0.0249	**0.0380**	**0.0223**
		DEF	0.0768		0.0865		0.0676		**0.0603**	
0.5	40,16	**AE**	1.5723	1.0834	1.6952	1.0817	1.5490	1.0379	1.5380	1.0235
		MSE	0.1423	0.1672	0.2065	0.2030	0.0936	0.0908	**0.0686**	**0.0604**
		DEF	0.3095		0.4095		0.1844		**0.1290**	
	40,24	**AE**	1.5634	1.0458	1.6616	1.0371	1.5477	1.0257	1.5383	1.0186
		MSE	0.1125	0.0866	0.1539	0.0959	0.0818	0.0605	**0.0630**	**0.0459**
		DEF	0.1992		0.2497		0.1423		**0.1090**	
	40,32	AE	1.5597	1.0357	1.6446	1.0270	1.5460	1.0237	1.5374	1.0191
		MSE	0.0956	0.0587	0.1210	0.0629	0.0737	0.0455	**0.0593**	**0.0369**
		DEF	0.1543		0.1839		0.1192		**0.0962**	
	60,24	**AE**	1.5435	1.0752	1.6341	1.0735	1.5339	1.0495	1.5264	1.0394
		MSE	0.0874	0.0989	0.1158	0.1115	0.0674	0.0659	**0.0542**	**0.0487**
		DEF	0.1864		0.2273		0.1334		**0.1029**	

Continued on next page

p	n,m		MLE		$MPSE$		BE_{Prior_1}		BE_{Prior_2}	
			α	β	α	β	α	β	α	β
	60,36	**AE**	1.5481	1.0355	1.6215	1.0304	1.5408	1.0242	1.5352	1.0200
		MSE	0.0757	0.0524	0.0976	0.0560	0.0613	0.0414	**0.0510**	**0.0342**
		DEF	0.1281		0.1536		0.1027		**0.0851**	
	60,48	**AE**	1.5481	1.0159	1.6115	1.0113	1.5414	1.0099	1.5365	1.0078
		MSE	0.0656	0.0395	0.0814	0.0417	0.0549	0.0335	**0.0468**	**0.0291**
		DEF	0.1051		0.1231		0.0885		**0.0759**	
	80,32	**AE**	1.5155	1.0185	1.5923	1.0161	1.5139	1.0072	1.5114	1.0046
		MSE	0.0594	0.0562	0.0779	0.0616	0.0491	0.0439	**0.0413**	**0.0356**
		DEF	0.1156		0.1395		0.0930		**0.0770**	
	80,48	**AE**	1.5324	1.0189	1.5920	1.0154	1.5291	1.0125	1.5255	1.0103
		MSE	0.0547	0.0361	0.0663	0.0379	0.0471	0.0306	**0.0409**	**0.0265**
		DEF	0.0909		0.1043		0.0778		**0.0675**	
	80,64	**AE**	1.5222	1.0168	1.5708	1.0136	1.5189	1.0127	1.5168	1.0113
		MSE	0.0432	0.0285	0.0505	0.0296	0.0380	0.0252	**0.0339**	**0.0226**
		DEF	0.0717		0.0801		0.0632		**0.0565**	
0.7	40,16	**AE**	1.5698	1.0614	1.7017	1.0517	1.5520	1.0239	1.5405	1.0130
		MSE	0.1386	0.1438	0.2130	0.1689	0.0914	0.0821	**0.0662**	**0.0557**
		DEF	0.2824		0.3819		0.1735		**0.1219**	
	40,24	**AE**	1.5597	1.0382	1.6626	1.0279	1.5460	1.0197	1.5363	1.0135
		MSE	0.1137	0.0876	0.1557	0.0974	0.0818	0.0621	**0.0636**	**0.0474**
		DEF	0.2013		0.2531		0.1439		**0.1110**	
	40,32	**AE**	1.5617	1.0385	1.6416	1.0289	1.5482	1.0261	1.5397	1.0210
		MSE	0.0987	0.0654	0.1273	0.0702	0.0751	0.0504	**0.0598**	**0.0407**
		DEF	0.1641		0.1975		0.1255		**0.1005**	
	60,24	**AE**	1.5509	1.0460	1.6588	1.0414	1.5440	1.0253	1.5358	1.0181
		MSE	0.0963	0.0908	0.1422	0.1009	0.0734	0.0632	**0.0579**	**0.0477**
		DEF	0.1871		0.2431		0.1367		**0.1056**	
	60,36	**AE**	1.5533	1.0351	1.6329	1.0301	1.5469	1.0244	1.5402	1.0201
		MSE	0.0748	0.0533	0.1003	0.0573	0.0606	0.0425	**0.0501**	**0.0351**
		DEF	0.1281		0.1576		0.1032		**0.0852**	
	60,48	**AE**	1.5509	1.0306	1.6114	1.0254	1.5438	1.0236	1.5385	1.0206
		MSE	0.0643	0.0425	0.0792	0.0445	0.0537	0.0359	**0.0458**	**0.0312**
		DEF	0.1067		0.1236		0.0896		**0.0770**	
	80,32	**AE**	1.5353	1.0378	1.6171	1.0340	1.5334	1.0244	1.5282	1.0195
		MSE	0.0712	0.0643	0.0921	0.0695	0.0583	0.0496	**0.0486**	**0.0400**
		DEF	0.1355		0.1616		0.1079		**0.0886**	
	80,48	**AE**	1.5318	1.0370	1.5947	1.0339	1.5285	1.0296	1.5245	1.0265
		MSE	0.0556	0.0370	0.0705	0.0392	0.0476	0.0312	**0.0412**	**0.0270**
		DEF	0.0927		0.1097		0.0788		**0.0681**	
	80,64	**AE**	1.5164	1.0194	1.5664	1.0164	1.5136	1.0155	1.5116	1.0140
		MSE	0.0413	0.0269	0.0490	0.0281	0.0365	0.0239	**0.0327**	**0.0215**
		DEF	0.0682		0.0771		0.0604		**0.0542**	

Table 5. ALs and CPs for the estimates of α and β at $(T_1, T_2) = (0.4, 0.7)$.

p	n, m		ACI_{ML}		ACI_{MPS}		HPD_{Prior_1}		HPD_{Prior_2}	
			α	β	α	β	α	β	α	β
0.3	40,16	**AL**	1.6334	1.8710	1.8838	2.0195	1.3614	1.3772	**1.2286**	**1.1945**
		CP	0.9525	0.9426	0.9694	0.9353	0.9727	0.9726	0.9813	0.9843
	40,24	**AL**	1.5388	1.5571	1.7442	1.6614	1.3117	1.2390	**1.1940**	**1.0990**
		CP	0.9570	0.9485	0.9696	0.9443	0.9703	0.9709	0.9810	0.9801
	40,32	**AL**	1.4547	1.3470	1.6224	1.4237	1.2617	1.1241	**1.1577**	**1.0146**
		CP	0.9533	0.9479	0.9678	0.9437	0.9680	0.9662	0.9775	0.9771
	60,24	**AL**	1.3374	1.4961	1.4931	1.5911	1.1718	1.2170	**1.0805**	**1.0863**
		CP	0.9570	0.9560	0.9590	0.9520	0.9690	0.9790	0.9750	0.9910
	60,36	**AL**	1.2508	1.2495	1.3718	1.3129	1.1175	1.0720	**1.0400**	**0.9771**
		CP	0.9425	0.9525	0.9590	0.9495	0.9570	0.9665	0.9660	0.9755
	60,48	**AL**	1.1664	1.0623	1.2637	1.109	1.0579	0.9456	**0.995**	**0.8768**
		CP	0.9590	0.9610	0.9650	0.9630	0.9720	0.9730	0.9780	0.9780
	80,32	**AL**	1.1706	1.3094	1.2789	1.3741	1.0553	1.1143	**0.9851**	**1.0113**
		CP	0.9620	0.9590	0.9660	0.9560	0.9730	0.9760	0.9790	0.9830
	80,48	**AL**	1.0872	1.0724	1.1702	1.1152	0.9962	0.9558	**0.9401**	**0.8870**
		CP	0.9430	0.9520	0.9560	0.9510	0.9540	0.9620	0.9640	0.9660
	80,64	**AL**	1.0074	0.9176	1.0730	0.9508	0.9337	0.8416	**0.8880**	**0.7925**
		CP	0.9540	0.9440	0.9530	0.9440	0.9530	0.9630	0.9660	0.9660
0.5	40,16	**AL**	1.6947	1.9884	2.0150	2.1714	1.3993	1.4293	**1.2562**	**1.2319**
		CP	0.9535	0.9402	0.9713	0.9304	0.9741	0.9695	0.9843	0.9824
	40,24	**AL**	1.5664	1.5916	1.8011	1.7067	1.3295	1.2572	**1.2084**	**1.1129**
		CP	0.9517	0.9463	0.9671	0.9413	0.9704	0.9707	0.9788	0.9802
	40,32	**AL**	1.4774	1.3595	1.6576	1.4419	1.2768	1.1299	**1.1706**	**1.0186**
		CP	0.9519	0.9488	0.9657	0.9463	0.9667	0.9710	0.9782	0.9796
	60,24	**AL**	1.3969	1.5670	1.5945	1.6812	1.2164	1.2580	**1.1150**	**1.1171**
		CP	0.9480	0.9460	0.9660	0.9400	0.9660	0.9680	0.9780	0.9820
	60,36	**AL**	1.2743	1.2582	1.4056	1.3229	1.1365	1.0780	**1.0563**	**0.9827**
		CP	0.9410	0.9420	0.9600	0.9410	0.9590	0.9630	0.9740	0.9770
	60,48	**AL**	1.1870	1.0648	1.2893	1.1117	1.0750	0.9467	**1.0069**	**0.8774**
		CP	0.9440	0.9360	0.9600	0.9420	0.9580	0.9560	0.9640	0.9710
	80,32	**AL**	1.2115	1.3332	1.3480	1.4099	1.0878	1.1306	**1.0138**	**1.0249**
		CP	0.9540	0.9450	0.9650	0.9430	0.9670	0.9630	0.9750	0.9720
	80,48	**AL**	1.0932	1.0591	1.1870	1.1051	1.0019	0.9486	**0.9448**	**0.8814**
		CP	0.9620	0.9510	0.9630	0.9520	0.9700	0.9610	0.9720	0.9690
	80,64	**AL**	1.0262	0.9173	1.0955	0.9506	0.9491	0.8412	**0.9027**	**0.7922**
		CP	0.9590	0.9420	0.9660	0.9450	0.9700	0.9560	0.9790	0.9610

Continued on next page

p	n, m		ACI_{ML}		ACI_{MPS}		HPD_{Prior_1}		HPD_{Prior_2}	
			α	β	α	β	α	β	α	β
0.7	40,16	**AL**	1.7376	2.0127	2.1189	2.2128	1.4277	1.4408	**1.2786**	**1.2412**
		CP	0.9580	0.9400	0.9770	0.9350	0.9750	0.9740	0.9880	0.9870
	40,24	**AL**	1.5773	1.5791	1.8393	1.7078	1.3397	1.2531	**1.2163**	**1.1116**
		CP	0.9530	0.9430	0.9660	0.9370	0.9650	0.9670	0.9780	0.9860
	40,32	**AL**	1.4825	1.3566	1.6671	1.4375	1.2813	1.1299	**1.1736**	**1.0193**
		CP	0.9630	0.9450	0.9780	0.9390	0.9780	0.9630	0.9850	0.9690
	60,24	**AL**	1.4085	1.5531	1.6303	1.6657	1.2292	1.2510	**1.1269**	**1.1125**
		CP	0.9400	0.9500	0.9650	0.9490	0.9600	0.9690	0.9700	0.9800
	60,36	**AL**	1.2941	1.2637	1.4423	1.3347	1.1509	1.0816	**1.0692**	**0.9850**
		CP	0.9550	0.9465	0.9630	0.9465	0.9660	0.9665	0.9740	0.9765
	60,48	**AL**	1.1898	1.0734	1.2923	1.1205	1.0760	0.9542	**1.0084**	**0.8842**
		CP	0.9570	0.9550	0.9660	0.9550	0.9690	0.9690	0.9770	0.9740
	80,32	**AL**	1.2351	1.2964	1.3953	1.3746	1.1097	1.1116	**1.0328**	**1.0107**
		CP	0.9580	0.9480	0.9720	0.9460	0.9740	0.9630	0.9820	0.9710
	80,48	**AL**	1.1115	1.0564	1.2128	1.1043	1.0168	0.9467	**0.9579**	**0.8795**
		CP	0.9460	0.9520	0.9640	0.9490	0.9600	0.9660	0.9650	0.9700
	80,64	**AL**	1.0159	0.9050	1.0849	0.9364	0.9404	0.8323	**0.8964**	**0.7858**
		CP	0.9580	0.9510	0.9690	0.9500	0.9690	0.9600	0.9760	0.9630

Table 6. ALs and CPs for the estimates of α and β at $(T_1, T_2) = (0.7, 1)$.

p	n, m		ACI_{ML}		ACI_{MPS}		HPD_{Prior_1}		HPD_{Prior_2}	
			α	β	α	β	α	β	α	β
0.3	40,16	**AL**	1.3595	1.3967	1.5189	1.4724	1.2197	1.1604	**1.1321**	**1.0545**
		CP	0.9380	0.9360	0.9440	0.9290	0.9570	0.9520	0.9700	0.9620
	40,24	**AL**	1.2518	1.0847	1.3721	1.1137	1.1466	0.9691	**1.0768**	**0.9067**
		CP	0.9430	0.9470	0.9480	0.9360	0.9550	0.9580	0.9640	0.9630
	40,32	**AL**	1.1576	0.9283	1.2501	0.9448	1.0723	0.8547	**1.0166**	**0.8126**
		CP	0.9470	0.9350	0.9570	0.9320	0.9560	0.9400	0.9630	0.9500
	60,24	**AL**	1.1085	1.0881	1.2056	1.1224	1.0287	0.9701	**0.9761**	**0.9069**
		CP	0.9440	0.9350	0.9520	0.9280	0.9530	0.9510	0.9610	0.9610
	60,36	**AL**	1.0133	0.8695	1.0833	0.8850	0.9530	0.8065	**0.9124**	**0.7709**
		CP	0.9590	0.9440	0.9610	0.9370	0.9610	0.9530	0.9640	0.9560
	60,48	**AL**	0.9452	0.7423	1.0006	0.7517	0.8944	0.7026	**0.8624**	**0.6796**
		CP	0.9420	0.9470	0.9500	0.9430	0.9460	0.9490	0.9620	0.9520
	80,32	**AL**	0.9640	0.9362	1.0346	0.9592	0.9099	0.8593	**0.8733**	**0.8150**
		CP	0.9460	0.9480	0.9580	0.9500	0.9590	0.9580	0.9610	0.9620
	80,48	**AL**	0.8716	0.7529	0.9222	0.7641	0.8304	0.7116	**0.8032**	**0.6869**
		CP	0.9490	0.9520	0.9520	0.9480	0.9520	0.9580	0.9630	0.9600
	80,64	**AL**	0.8094	0.6492	0.8457	0.6558	0.7754	0.6223	**0.7541**	**0.6054**
		CP	0.9390	0.9520	0.9450	0.9480	0.9400	0.9530	0.9480	0.9550

Continued on next page

p	n,m		ACI_{ML}		ACI_{MPS}		HPD_{Prior_1}		HPD_{Prior_2}	
			α	β	α	β	α	β	α	β
0.5	40,16	**AL**	1.3884	1.3862	1.5840	1.4522	1.2463	1.1618	**1.1557**	**1.0571**
		CP	0.9438	0.9311	0.9579	0.9212	0.9620	0.9456	0.9725	0.9603
	40,24	**AL**	1.2592	1.0856	1.3931	1.1129	1.1520	0.9717	**1.0826**	**0.9100**
		CP	0.9509	0.9443	0.9631	0.9397	0.9605	0.9532	0.9704	0.9649
	40,32	**AL**	1.1681	0.9263	1.2663	0.9422	1.0802	0.8537	**1.0236**	**0.8117**
		CP	0.9469	0.9525	0.9556	0.9460	0.9558	0.9594	0.9670	0.9660
	60,24	**AL**	1.1267	1.1220	1.2404	1.1561	1.0454	0.9988	**0.9902**	**0.9313**
		CP	0.9410	0.9500	0.9520	0.9420	0.9490	0.9600	0.9620	0.9650
	60,36	**AL**	1.0265	0.8772	1.1045	0.8924	0.9649	0.8139	**0.9230**	**0.7777**
		CP	0.9520	0.9530	0.9480	0.9470	0.9540	0.9600	0.9640	0.9640
	60,48	**AL**	0.9540	0.7431	1.0127	0.7525	0.9018	0.7034	**0.8690**	**0.6801**
		CP	0.9510	0.9360	0.9560	0.9270	0.9540	0.9410	0.9680	0.9510
	80,32	**AL**	0.9840	0.9243	1.0672	0.9444	0.9284	0.8513	**0.8906**	**0.8099**
		CP	0.9550	0.9340	0.9630	0.9270	0.9570	0.9380	0.9640	0.9490
	80,48	**AL**	0.8922	0.7481	0.9464	0.7580	0.8482	0.7079	**0.8202**	**0.6837**
		CP	0.9485	0.9575	0.9600	0.9550	0.9585	0.9640	0.9650	0.9680
	80,64	**AL**	0.8133	0.6439	0.8514	0.6502	0.7782	0.6175	**0.7572**	**0.6014**
		CP	0.9530	0.9490	0.9610	0.9430	0.9590	0.9530	0.9620	0.9580
0.7	40,16	**AL**	1.4074	1.3809	1.6323	1.4378	1.2628	1.1647	**1.1693**	**1.0613**
		CP	0.9390	0.9470	0.9555	0.9430	0.9585	0.9605	0.9685	0.9735
	40,24	**AL**	1.2830	1.0858	1.4279	1.1112	1.1715	0.9738	**1.0978**	**0.9120**
		CP	0.9550	0.9400	0.9650	0.9390	0.9630	0.9560	0.9740	0.9660
	40,32	**AL**	1.1727	0.9281	1.2710	0.9434	1.0837	0.8552	**1.0274**	**0.8129**
		CP	0.9500	0.9500	0.9550	0.9450	0.9580	0.9540	0.9670	0.9610
	60,24	**AL**	1.1678	1.0891	1.3068	1.1177	1.0808	0.9748	**1.0204**	**0.9125**
		CP	0.9570	0.9520	0.9650	0.9510	0.9620	0.9570	0.9680	0.9690
	60,36	**AL**	1.0454	0.8737	1.1313	0.8890	0.9809	0.8123	**0.9372**	**0.7762**
		CP	0.9550	0.9450	0.9660	0.9370	0.9660	0.9540	0.9710	0.9610
	60,48	**AL**	0.9560	0.7517	1.0131	0.7608	0.9039	0.7113	**0.8707**	**0.6865**
		CP	0.9530	0.9350	0.9600	0.9320	0.9600	0.9420	0.9640	0.9520
	80,32	**AL**	1.0146	0.9351	1.1083	0.9544	0.9540	0.8606	**0.9133**	**0.8171**
		CP	0.9560	0.9380	0.9610	0.9240	0.9560	0.9480	0.9730	0.9550
	80,48	**AL**	0.8983	0.7572	0.9558	0.7676	0.8524	0.7162	**0.8246**	**0.6914**
		CP	0.9520	0.9640	0.9560	0.9600	0.9570	0.9660	0.9640	0.9700
	80,64	**AL**	0.8145	0.6446	0.8536	0.6510	0.7790	0.6183	**0.7582**	**0.6021**
		CP	0.9540	0.9530	0.9660	0.9470	0.9550	0.9570	0.9650	0.9650

The results summarized in Tables 3–6 lead to the following conclusions:

- Based on the point estimation results, the DEF and MSE values of all estimators exhibit a decreasing trend as m/n increases for fixed values of n, p, T_1, and T_2. Furthermore, for fixed m/n, p, T_1, and T_2, a similar pattern is observed as n increases. Conversely, as T_1 and T_2 increase, the DEF and MSE values of the estimators for the parameters α and β decrease.

- As prior information is taken into account, the Bayesian estimators demonstrate superior performance compared to the MLEs and MPSEs in terms of both DEF and MSE across all cases. In addition, the MLEs outperform the MPSEs with respect to MSE and DEF. Among the Bayesian estimators, $BE_{Prior\ 2}$ exhibits the best performance, as it achieves the lowest DEF and MSE values in all examined cases. This finding is consistent with expectations, as although Prior 1 and Prior 2 have the same prior means, Prior 2 contains more prior information due to its smaller prior variance. It should be noted that the informative priors considered in the simulation study were correctly specified, with prior means equal to the true parameter values. Therefore, the superior performance of the Bayesian estimators should be interpreted within the context of correctly specified prior information.
- In terms of AE, all estimators tend to slightly overestimate the true parameter values in all cases. However, as n or m/n increases, their estimates converge toward the true parameter values.
- Regarding interval estimation, as n/m increases, the AL values decrease for all considered interval estimators, while the CP values remain close to the nominal confidence level when n, p, T_1, and T_2 are fixed. Similar trends are also observed when other quantities are held constant and either n or T increases. Moreover, the HPD credible intervals for α and β outperform the $ACIs_{ML}$ and $ACIs_{MPS}$ in terms of AL. On the other hand, the HPD_{Prior_2} provides the best interval performance by achieving the smallest AIL values across all cases.
- When evaluating the point estimation results with respect to the parameter p under fixed settings, it is observed that for $(T_1, T_2) = (0.4, 0.7)$, when $n = 40$ and $n = 60$, the parameters α and β are estimated with the lowest MSE and DEF values at $p = 0.3$. In contrast, when $n = 80$, the most efficient estimates are predominantly obtained at $p = 0.7$. Futhermore, for $(T_1, T_2) = (0.7, 1)$, when $n = 40$ and $n = 80$, the estimates of α and β generally perform better at $p = 0.5$, although $p = 0.7$ also yields satisfactory results in some cases. For $n = 60$, the best estimates are again observed at $p = 0.3$.
- Considering the interval estimation results in terms of p, the interval estimates for α perform best at $p = 0.3$, with the smallest AL values across all cases. For β, the shortest interval estimates are obtained at $p = 0.3$ when $n = 40$ and $n = 60$, whereas for $n = 80$, the best results are achieved at $p = 0.7$ for $(T_1, T_2) = (0.4, 0.7)$, and at $p = 0.5$ for $(T_1, T_2) = (0.7, 1)$.
- In summary, when data are collected under the IAT-II PC scheme with binomial random removals, it is recommended to employ the Bayesian estimation approach implemented via the M-H algorithm including Gibbs steps for both point and interval estimation of the Weibull parameters. It should also be noted that the estimation performance is influenced by both the time thresholds (T_1, T_2) and the parameter p, and that more efficient results are generally obtained for moderate values of p when the length of threshold points is relatively large.

6. Real-life data analysis

To demonstrate the implementation of the proposed IAT-II PC scheme with binomial removals and to illustrate the applicability of the proposed inference procedures, a complete real dataset is used to generate IAT-II PC samples with binomial removals, which are then analyzed in this section. The complete dataset consists of breaking stress measurements (in GPa) of carbon fibers of a certain type used in fibrous composite materials, as reported in Table 7. This dataset was originally introduced by Padgett and Spurrier [38] and was later reanalyzed by Okasha et al. [39].

Table 7. Breaking stresses (in GPa) of carbon fibers used in fibrous composite materials.

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.98, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.7, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

Before conducting the inference under the proposed censoring scheme, the suitability of the Weibull distribution for the complete dataset is assessed by comparing it with several commonly used lifetime distributions, namely the gamma, lognormal, and exponential distributions. Table 8 presents the goodness-of-fit results, including the log-likelihood, AIC, BIC, and the Kolmogorov–Smirnov (K–S), Anderson–Darling (A–D), and Cramér–von Mises (CvM) goodness-of-fit statistics with their corresponding p-values. The goodness-of-fit tests are used to assess the adequacy of each candidate distribution, whereas the log-likelihood, AIC, and BIC values facilitate comparison among the distributions that adequately fit the data.

Table 8. Goodness-of-fit comparison of candidate lifetime distributions for the complete carbon fiber breaking stress dataset.

Distribution	**Log-likelihood**	**AIC**	**BIC**	**K–S statistic (p-value)**	**A–D statistic (p-value)**	**CvM statistic (p-value)**
Weibull	-141.5435	287.0869	292.2973	0.0605 (0.8569)	0.4178 (0.8306)	0.0633 (0.7945)
Gamma	-143.2520	290.5041	295.7144	0.0934 (0.3469)	0.7608 (0.5098)	0.1506 (0.3883)
Lognormal	-148.4382	300.8764	306.0867	0.1177 (0.1251)	1.4663 (0.1847)	0.2755 (0.1587)
Exponential	-196.3747	394.7493	399.3545	0.3206 (0.0001)	17.2974 (0.0001)	3.4328 (0.0001)

As shown in Table 8, the Weibull, gamma, and lognormal distributions adequately fit the data, as all three distributions yielded nonsignificant K–S, A–D, and CvM test results ($p > 0.05$). In contrast, the exponential distribution was rejected by all goodness-of-fit tests ($p < 0.001$), which suggests that it does not adequately fit the observed data. Among the distributions that satisfied the goodness-of-fit criteria, the Weibull distribution achieved the highest log-likelihood value (−141.5435) and the lowest AIC (287.0869) and BIC (292.2973) values, which indicates that it provides the best overall fit among the candidate distributions. Therefore, the Weibull distribution is selected as the underlying lifetime model for the subsequent analyses under the IIAT-II PC scheme with binomial removals.

Following the selection of the Weibull distribution, the ML estimation is performed using the complete dataset. The resulting parameter estimates provide reference values for subsequent comparisons with the IAT-II PC scheme with binomial removals. The ML estimates of the Weibull shape and scale parameters are obtained as $\hat{\alpha} = 2.7925$ (0.2141) and $\hat{\beta} = 0.0490$ (0.0139), respectively, where the values in parentheses denote the corresponding standard errors. These estimates are reported for comparison with the results obtained under the IAT-II PC scheme with binomial removals.

Based on the complete dataset in Table 7, three IAT-II PC samples with binomial random removals are generated for $m = 30$ under different combinations of the binomial removal probability p and the time thresholds (T_1, T_2). The resulting IAT-II PC samples, together with their corresponding censoring schemes, are presented in Table 9. For notational convenience, the censoring scheme $R = (25,12,0,0,0,1,1)$ is denoted by $R = (25,12,0^*3,1^*2)$.

Point estimates (Est.) of the unknown parameters α and β of the Weibull distribution, together with their corresponding standard errors (St. Er.), are obtained using ML, MPS, and Bayesian estimation methods based on the IAT-II PC samples with binomial removals presented in Table 9. In addition, the 95% ACIs and HPD credible intervals are constructed for both parameters, and the associated interval lengths (Len.) are computed. Since prior information on the Weibull distribution parameters α and β is unavailable for the carbon fiber breaking stress dataset, non-informative priors (Prior 0) are employed for the parameters. Bayesian point estimates and HPD credible intervals are obtained from the generated MCMC samples using Algorithm 1, with a total of $M = 80{,}000$ iterations, where the first $B = 10{,}000$ iterations are discarded as burn-in. The point and interval estimation results are summarized in Tables 10 and 11, respectively.

Table 9. Generated IAT-II PC samples with binomial removals from the carbon fiber breaking stress dataset.

p	Scheme	IAT-II PC sample	$T_1(d_1)$	$T_2(d_2)$	R*	T*
0.3	(38,16,10,2*3,0*24)	0.39, 0.85, 1.08, 1.36, 1.41, 1.61, 1.61, 1.69, 1.73, 1.84	1 (2)	2 (10)	52	1.84
	(14,12,8*2,11,2,6, 2*2,3,1*2,0*18)	0.81, 1.08, 1.18, 1.22, 1.41, 1.47, 1.57, 1.59, 1.61, 1.61, 2.17, 2.17, 2.35, 2.41, 2.55, 2.59, 2.77, 2.81, 2.83, 2.87	2 (10)	3 (20)	12	2.87
0.5	(30,17,9,2,5,3, 2,1,0,1,0*20)	0.39, 1.17, 1.18, 1.36, 1.47, 1.57, 1.57, 1.59, 1.69, 1.84, 1.89, 2	1 (1)	2 (12)	58	2
	(41,14,7,3,4,0*2,1,0*22)	0.39, 1.41, 1.69, 1.73, 1.92, 2, 2.03, 2.17, 2.35, 2.38, 2.48, 2.5, 2.53, 2.55, 2.56, 2.76, 2.88, 2.93, 2.97	2 (6)	3 (19)	12	3
0.7	(44,20,4,1*2,0*25)	0.39, 1.12, 1.36, 1.47, 1.57, 1.57, 1.59, 1.61, 1.69, 1.69, 1.84	1 (1)	2 (11)	45	1.84
	(55,12,0,2,1,0*25)	1.18, 1.36, 1.47, 1.61, 1.69, 1.84, 1.87, 2, 2.17, 2.41, 2.43, 2.48, 2.56, 2.59, 2.73, 2.93, 2.97, 2.98	2 (8)	3 (18)	12	3

Table 10. Point estimates of the Weibull parameters based on the IIAT-II PC samples with binomial removals generated from the carbon fiber breaking stress data.

p	T_1, T_2	Par.	MLE Est.	St. Er.	$MPSE$ Est.	St. Er.	BE_{Prior_0} Est.	St. Er.
0.3	1, 2	α	2.4361	0.68249	2.6076	0.83556	2.4529	0.00268
		β	0.0315	0.01690	0.0229	0.01480	0.0335	0.00007
	2, 3	α	3.3680	0.51810	3.4212	0.56984	3.3648	0.00201
		β	0.0218	0.01132	0.0184	0.01056	0.0239	0.00005
0.5	1, 2	α	2.9023	0.78752	3.4342	1.07553	2.9307	0.00307
		β	0.0250	0.01504	0.0148	0.01178	0.0277	0.00006
	2, 3	α	4.0160	0.71184	4.6019	0.90579	4.0229	0.00274
		β	0.0104	0.00763	0.0057	0.00538	0.0127	0.00004
0.7	1, 2	α	2.8483	0.77678	3.1607	1.06607	2.8914	0.00304
		β	0.0303	0.01795	0.0185	0.01469	0.0329	0.00007
	2, 3	α	4.2371	0.68295	4.2825	0.72521	4.2605	0.00268
		β	0.0084	0.00594	0.0078	0.00588	0.0099	0.00003

Table 11. Interval estimates of the Weibull parameters based on the IIAT-II PC samples with binomial removals generated from the carbon fiber breaking stress data.

p	T_1, T_2	Par.	ACI_{ML} Low.	Upp.	Len.	ACI_{MPS} Low.	Upp.	Len.	HPD_{Prior_0} Low.	Upp.	Len.
0.3	1, 2	α	1.0984	3.7737	2.6753	0.9700	4.2453	3.2753	1.1962	3.8265	2.6302
		β	0.0000	0.0646	0.0646	0.0000	0.0520	0.0520	0.0056	0.0676	0.0620
	2, 3	α	2.3526	4.3835	2.0309	2.3043	4.5381	2.2338	2.3673	4.3966	2.0293
		β	0.0000	0.0440	0.0440	0.0000	0.0391	0.0391	0.0045	0.0476	0.0431
0.5	1, 2	α	1.3588	4.4458	3.0870	1.3262	5.5422	4.2160	1.5106	4.5402	3.0296
		β	0.0000	0.0545	0.0545	0.0000	0.0379	0.0379	0.0031	0.0586	0.0555
	2, 3	α	2.6204	5.4107	2.7903	2.8265	6.3772	3.5507	2.6891	5.4323	2.7432
		β	0.0000	0.0254	0.0254	0.0000	0.0162	0.0162	0.0006	0.0307	0.0301
0.7	1, 2	α	1.3258	4.3707	3.0449	1.0713	5.2502	4.1789	1.4501	4.4348	2.9848
		β	0.0000	0.0654	0.0654	0.0000	0.0473	0.0473	0.0040	0.0687	0.0647
	2, 3	α	2.8989	5.5760	2.6771	2.8612	5.7039	2.8428	2.9491	5.6117	2.6626
		β	0.0000	0.0200	0.0200	0.0000	0.0193	0.0193	0.0006	0.0234	0.0229

The results presented in Tables 10 and 11 indicate that the Bayesian estimates consistently exhibit smaller standard errors than those obtained using the ML and MPS methods. In addition, the HPD credible intervals are comparable to the corresponding asymptotic confidence intervals. Overall, these findings demonstrate the superiority of the Bayesian approach in terms of point estimation, while its interval estimation performance remains comparable to that of the ML and MPS methods under the proposed IIAT-II PC scheme with binomial removals.

To assess the convergence of the MCMC chains, Figure 4 presents the posterior density and trace plots of α and β based on the IAT-II PC samples reported in Table 8. In each trace plot, the solid and dashed horizontal lines represent the posterior mean and the corresponding 95% HPD credible limits, respectively. The density plots further indicate that the posterior samples of α are approximately symmetric, whereas those of β are positively skewed. As a result, these plots show that the MCMC procedure converges for both parameters.

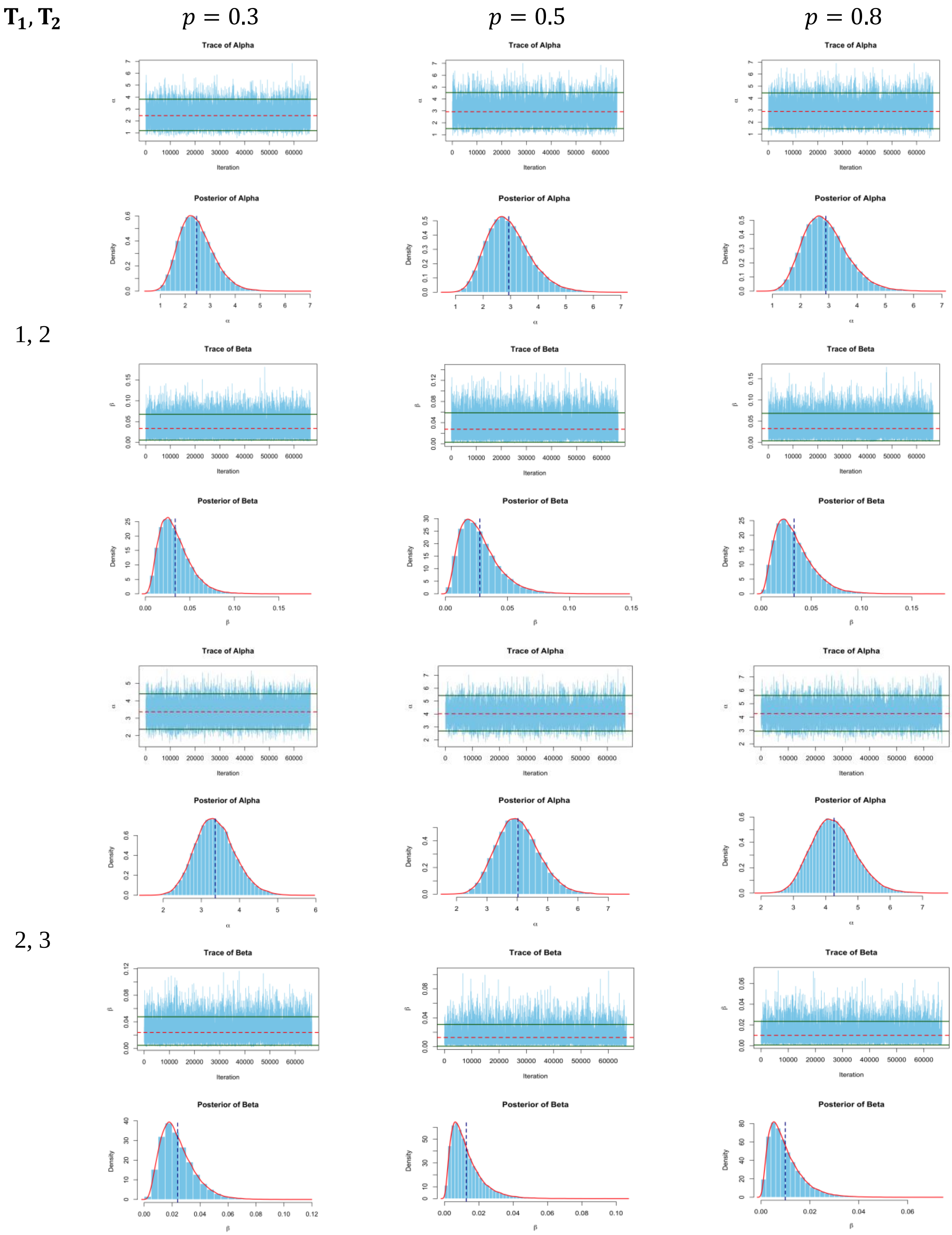

Figure 4. MCMC trace and posterior density plots of α and β based on the IIAT-II PC samples with binomial removals generated from the carbon fiber breaking stress data.

7. Conclusions

This study presents a comprehensive inferential framework for estimating the parameters of the Weibull distribution based on data obtained from an improved adaptive type-II progressive censoring scheme with binomial random removals. Maximum likelihood and maximum product spacing methods are employed for classical inference, whereas Bayesian inference is carried out using Markov Chain Monte Carlo techniques. Point estimates are obtained using all three methods, while interval estimation is conducted through asymptotic confidence intervals for the maximum likelihood and maximum product spacing estimation methods and highest posterior density credible intervals for the Bayesian method. The proposed point and interval estimation procedures are evaluated through Monte Carlo simulations, and their practical applicability is demonstrated using a real carbon fiber breaking stress dataset.

The simulation results demonstrate that, as the sample size and the proportion of observed failures increase, the estimation accuracy improves in terms of mean squared error, deficiency, and average interval length. Furthermore, under the correctly specified informative prior distributions considered in this study, the Bayesian approach generally outperforms the maximum likelihood and maximum product spacing methods in both point and interval estimation. Among the Bayesian estimators, those obtained under more informative priors exhibit the best overall performance. It should be noted that the informative priors used in the simulation study were specified such that their prior means coincided with the true parameter values. Therefore, the superior performance of the Bayesian estimators should be interpreted within the context of correctly specified prior information. The real-data analysis further demonstrates the practical applicability of the proposed inferential procedures. In particular, the Bayesian approach yields smaller standard errors than the ML and MPS methods for point estimation, while the corresponding highest posterior density credible intervals provide interval estimation performance comparable to that of the asymptotic confidence intervals.

Overall, the proposed inferential procedures provide effective tools for analyzing Weibull-distributed lifetime data obtained under the improved adaptive type-II progressive censoring scheme with binomial removals. The findings from both the simulation study and the real-data application demonstrate the effectiveness of the proposed inferential framework and support the use of the Bayesian estimation method for practical statistical inference under the considered censoring scheme in reliability and survival analysis.

Possible extensions of the present study include applying the proposed inferential procedures to other lifetime distributions under the improved adaptive type-II progressive censoring scheme with binomial random removals and considering alternative distributions for the random removal variables.

Author contributions

Hanefi Gezer: Conceptualization, software, validation, formal analysis, writing–original draft preparation and visualization; İlhan Usta: Conceptualization, methodology, validation, writing–review and editing, supervision, project administration, and funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this paper.

Acknowledgments

The authors are grateful to the Editors and the anonymous reviewers for their valuable comments and suggestions, which have significantly improved the manuscript. This study was supported by the Eskişehir Technical University Scientific Research Projects Commission (grant no: 25DRP042) and was based on the first author's doctoral dissertation.

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. C. Cohen, Progressively censored samples in life testing, *Technometrics*, **5** (1963), 327–339. https://doi.org/10.1080/00401706.1963.10490102
2. H. K. T. Ng, D. Kundu, P. S. Chan, Statistical analysis of exponential lifetimes under an adaptive Type-II progressive censoring scheme, *Nav. Res. Log*., **56** (2009), 687–698. https://doi.org/10.1002/nav.20371
3. W. Yan, P. Li, Y. Yu, Statistical inference for the reliability of Burr-XII distribution under improved adaptive Type-II progressive censoring, *Appl. Math. Model*., **95** (2021), 38–52. https://doi.org/10.1016/j.apm.2021.01.050
4. S. Dutta, S. Kayal, Inference of a competing risks model with partially observed failure causes under improved adaptive Type-II progressive censoring, *Proc. I. Mech. Eng. Part*, **237** (2023), 765–780. https://doi.org/10.1177/1748006x221104555
5. L. Zhang, R. Yan, Parameter estimation of Chen distribution under improved adaptive Type-II progressive censoring, *J. Stat. Comput. Sim*., **94** (2024), 2830–2861. https://doi.org/10.1080/00949655.2024.2358828
6. M. Nassar, A. Elshahhat, Estimation procedures and optimal censoring schemes for an improved adaptive progressively Type-II censored Weibull distribution, *J. Appl. Stat*., **51** (2024), 1664–1688. https://doi.org/10.1080/02664763.2023.2230536
7. M. Nassar, R. Alotaibi, A. Elshahhat, Reliability analysis at usual operating settings for Weibull constant-stress model with improved adaptive Type-II progressively censored samples, *AIMS Math*., **9** (2024), 16931–16965. https://doi.org/10.3934/math.2024823
8. S. Dutta, H. N. Alqifari, A. Almohaimeed, Bayesian and non-Bayesian inference for logistic-exponential distribution using improved adaptive Type-II progressively censored data, *PLoS ONE*, **19** (2024), e0298638. https://doi.org/10.1371/journal.pone.0298638
9. M. Irfan, S. Dutta, A. K. Sharma, Statistical inference and optimal plans for improved adaptive Type-II progressive censored data following Kumaraswamy-G family of distributions, *Phys. Scripta*, **100** (2025), 025213. https://doi.org/10.1088/1402-4896/ada216
10. R. Alotaibi, M. Nassar, A. Elshahhat, Reliability analysis of independent Burr-X competing risks model based on improved adaptive progressively Type-II censored samples with applications, *AIMS Math*., **10** (2025), 15302–15332. https://doi.org/10.3934/math.2025686
11. S. A. Lone, On estimation of Burr Type III model using improved adaptive progressive censoring with application to engineering data, *Qual. Reliab. Eng. Int.*, **42** (2026), 1–10. https://doi.org/10.1002/qre.70072

12. H. K. Yuen, S. K. Tse, Parameter estimation for Weibull distributed lifetimes under progressive censoring with random removals, *J. Stat. Comput. Sim.*, **55** (1996), 55–71. https://doi.org/10.1080/00949659608811749
13. S. K. Tse, C. Yang, H. Yuen, Statistical analysis of Weibull distributed lifetime data under Type-II progressive censoring with binomial removals, *J. Appl. Stat.*, **27** (2000), 1033–1043. https://doi.org/10.1080/02664760050173355
14. S. K. Singh, U. Singh, V. K. Sharma, Expected total test time and Bayesian estimation for generalized Lindley distribution under progressively Type-II censored sample where removals follow beta-binomial removals, *Appl. Math. Comput.*, **222** (2013), 402–419. https://doi.org/10.1016/j.amc.2013.07.058
15. S. J. Wu, Y. J. Chen, C. T. Chang, Statistical inference based on progressively censored samples with random removals from the Burr Type XII distribution, *J. Stat. Comput. Sim.*, **77** (2007), 19–27. https://doi.org/10.1080/10629360600569204
16. I. Usta, H. Gezer, Parameter estimation in Weibull distribution on progressively Type-II censored samples with beta-binomial removals, *Economy Business*, **10** (2016), 505–515.
17. T. A. Abushal, A. H. Abdel-Hamid, Inference on a new distribution under progressive-stress accelerated life tests and progressive Type-II censoring based on a series-parallel system, *AIMS Math.*, **7** (2021), 425–454. https://doi.org/10.3934/math.2022028
18. H. Gezer, I. Usta, Parameter and reliability estimation for the Rayleigh distribution under improved adaptive Type-II progressive censoring scheme with binomial removals, *Cumhuriyet Sci. J.*, **47** (2026), 182–193. https://doi.org/10.17776/csj.1728252
19. J. F. Lawless, *Statistical models and methods for lifetime data*, 2 Eds., New York: John Wiley & Sons, 2003. https://doi.org/10.1002/9781118033005
20. H. Rinne, *The Weibull distribution*: *A Handbook*, Boca Raton: Chapman & Hall/CRC, 2008. https://doi.org/10.1201/9781420087444
21. N. Balakrishnan, R. Aggarwala, *Progressive censoring: Theory, methods, and applications*, Boston: Birkhäuser, 2000. https://doi.org/10.1007/978-1-4612-1334-5
22. A. S. Hassan, N. Alsadat, O. S. Balogun, B. A. Helmy, Bayesian and non-Bayesian estimation of some entropy measures for a Weibull distribution, *AIMS Math.*, **9** (2024), 32646–32673. https://doi.org/10.3934/math.20241563
23. M. Nassar, R. Alotaibi, A new block censoring scheme: comparative assessment of likelihood and spacings methods for Weibull distribution with an application to cancer data, *AIMS Math.*, **11** (2026), 6834–6865. https://doi.org/10.3934/math.2026282
24. R. C. H. Cheng, N. A. K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, *J. Roy. Stat. Soc. B*, **45** (1983), 394–403. https://doi.org/10.1111/j.2517-6161.1983.tb01268.x
25. B. Ranneby, The maximum spacing method: An estimation method related to the maximum likelihood method, *Scand. J. Stat.*, **11** (1984), 93–112.
26. R. C. H. Cheng, T. C. Iles, Corrected maximum likelihood in non-regular problems, *J. R. Stat. Soc. Ser. B*, **49** (1987), 95–101. https://doi.org/10.1111/j.2517-6161.1987.tb01430.x
27. S. Anatolyev, G. Kosenok, An alternative to maximum likelihood based on spacings, *Economet. Theor.*, **21** (2005), 472–476. https://doi.org/10.1017/s0266466605050255
28. D. Kundu, Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring, *Technometrics*, **50** (2008), 144–154. https://doi.org/10.1198/004017008000000217

29. C. P. Robert, G. Casella, *Monte Carlo statistical methods*, 2 Eds., New York: Springer, 2004. https://doi.org/10.1007/978-1-4757-4145-2
30. A. Gelman, J. B. Carlin, H. S. Stern, D. B. Dunson, A. Vehtari, D. B. Rubin, *Bayesian data analysis*, 3 Eds., Boca Raton, FL: Chapman & Hall/CRC, 2013. https://doi.org/10.1201/b16018
31. E. L. Lehmann, G. Casella, *Theory of point estimation*, 2 Eds., New York: Springer, 1998. https://doi.org/10.1007/b98854
32. M. H. Chen, Q. M. Shao, Monte Carlo estimation of Bayesian credible and HPD intervals, *J. Comput. Graph. Stat.*, **8** (1999), 69–92. https://doi.org/10.1080/10618600.1999.10474802
33. N. Balakrishnan, R. Sandhu, A simple simulation algorithm for generating progressively Type-II censored samples, *Am. Stat.*, **49** (1995), 229–230. https://doi.org/10.1080/00031305.1995.10476150
34. I. Usta, Different estimation methods for the parameters of the extended Burr XII distribution, *J. Appl. Stat.*, **40** (2013), 397–414. https://doi.org/10.1080/02664763.2012.743974
35. Y. M. Kantar, B. Şenoğlu, A comparative study for the location and scale parameters of the Weibull distribution with a given shape parameter, *Comput. Geosci.*, **34** (2008), 1900–1909. https://doi.org/10.1016/j.cageo.2008.04.004
36. A. Henningsen, O. Toomet, MaxLik: A package for maximum likelihood estimation in R, *Comput. Stat.*, **26** (2011), 443–458. https://doi.org/10.1007/s00180-010-0217-1
37. M. Plummer, N. Best, K. Cowles, K. Vines, CODA: Convergence diagnosis and output analysis for MCMC, *R News*, **6** (2006), 7–11.
38. W. J. Padgett, J. D. Spurrier, Shewhart-type charts for percentiles of strength distributions, *J. Qual. Technol.*, **22** (1990), 283–288. https://doi.org/10.1080/00224065.1990.11979260
39. H. M. Okasha, H. S. Mohammed, Y. Lio, E-Bayesian estimation of reliability characteristics of a Weibull distribution with applications, *Mathematics*, **9** (2021), 1261. https://doi.org/10.3390/math9111261