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Research article

Optimal constants and equality conditions for a class of quadratic functional inequalities in real inner product spaces

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Abstract: In this paper, we systematically studied the optimal constants and equality conditions for a class of inequalities involving finitely many linear functionals on a real inner product space V. These inequalities arose naturally when combining several Cauchy-Schwarz inequalities, where the resulting optimal constants could not be obtained by simply adding or multiplying the individual bounds. By performing an orthogonal decomposition of V, we transformed the problem into the extremum of a quadratic functional, which was then solved completely using the Rayleigh quotient theory for real symmetric matrices. The main result, Theorem 3.1, provided a unified framework that reduced the determination of sharp constants to computing the extremal eigenvalues of a matrix associated with the Gram matrix. Complete characterizations of the equality cases were also provided. This framework was then applied to solve several illustrative problems, including Problem 12318 from the April 2022 issue of the American Mathematical Monthly. Furthermore, a brief exploration of a non-quadratic case was also included to demonstrate the broader applicability of the orthogonal decomposition approach. The method highlighted the role of the finite-dimensional reduction technique in infinite-dimensional settings and discussed potential extensions to complex inner product spaces, which may be of independent interest for further research.

Keywords: real inner product space; orthogonal decomposition; Gram matrix; Rayleigh quotient; optimal constants; equality conditions
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1. Introduction

It is well known that the Cauchy-Schwarz inequality is one of the most fundamental inequalities in analysis. Its discrete form states that for real sequences $\{a_n\}, \{b_n\} \in \ell^2$,

$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \leqslant \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2,$$

with equality if and only if $\{a_n\}$ and $\{b_n\}$ are linearly dependent. The analogous integral form holds in $L^2(\Omega)$ [1]. These inequalities can be regarded as special cases of the Cauchy-Schwarz inequality in general inner product spaces. Numerous refinements, generalizations, and reverses of this inequality have been established in various settings, including but not limited to normed spaces [2], inner product spaces [3], operator algebras [4,5], operator inequalities [6,7], as well as functional generalizations [8] and self-improvements [9] and references therein.

We now consider a problem related to the Cauchy-Schwarz inequality. We first introduce the following examples.

For any $f(x) \in L^2\,[0, \pi]$, the Cauchy-Schwarz inequality implies that

$$\left(\int_0^{\pi} f(x)\cos x\,dx\right)^2 \leq \int_0^{\pi} f^2(x)dx \int_0^{\pi} \cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi} f^2(x)dx, \tag{1.1}$$

$$\left(\int_0^{\pi} f(x)\sin x\,dx\right)^2 \leq \int_0^{\pi} f^2(x)dx \int_0^{\pi} \sin^2 x\,dx = \frac{\pi}{2}\int_0^{\pi} f^2(x)dx. \tag{1.2}$$

Combining inequalities (1.1) and (1.2), we have

$$\left(\int_0^{\pi} f(x)\cos x\,dx\right)^2 + \left(\int_0^{\pi} f(x)\sin x\,dx\right)^2 \leq \pi\int_0^{\pi} f^2(x)dx. \tag{1.3}$$

Equality holds in (1.3) if and only if equality holds in both (1.1) and (1.2), which would require f to be proportional to both $\cos x$ and $\sin x$ almost everywhere, impossible unless $f = 0$ almost everywhere. Hence, π cannot be asserted as the optimal constant in (1.3).

Similarly, it follows from the Cauchy-Schwarz inequality that

$$\left|\int_0^{1} f(x)dx\right| \leq \left(\int_0^{1} f^2(x)dx\right)^{\frac{1}{2}}, \tag{1.4}$$

$$\left|\int_0^{1} x^4 f(x)dx\right| \leq \left(\int_0^{1} x^8 dx\right)^{\frac{1}{2}}\left(\int_0^{1} f^2(x)dx\right)^{\frac{1}{2}} = \frac{1}{3}\left(\int_0^{1} f^2(x)dx\right)^{\frac{1}{2}}. \tag{1.5}$$

Combining inequalities (1.4) and (1.5), we have

$$\left|\int_0^{1} f(x)dx \int_0^{1} x^4 f(x)dx\right| \leq \frac{1}{3}\int_0^{1} f^2(x)dx. \tag{1.6}$$

Equality holds in (1.6) if and only if equality holds in both (1.4) and (1.5), which would require f to be proportional to both 1 and x^4 almost everywhere, impossible unless $f = 0$ almost everywhere. Hence, $\frac{1}{3}$ cannot be asserted as the optimal constant in (1.6).

Inequalities (1.3) and (1.6) are obtained by combining several inequalities, and their coefficients do not necessarily guarantee optimality. In the April 2022 issue of the American Mathematical Monthly, the following questions were posed [10]:

Let a be a positive real number, and let S_a be the set of functions $f : [-a, a] \to \mathbb{R}$ such that $\int_{-a}^{a} f^2(x)dx = 1$. Let $A(f) = \int_{-a}^{a} f(x)dx, B(f) = \int_{-a}^{a} xf(x)dx, C(f) = \int_{-a}^{a} x^2 f(x)dx.$

(a) What is $\sup_{f\in S_a} \left\{A^2(f) + B^2(f)\right\}$?

(b) What is $\sup_{f\in S_a} \left\{A^2(f) + B^2(f) + C^2(f)\right\}$?

There are many similar problems, and it is necessary to establish a unified rule for their solutions.

These examples point to a common structure: For a real inner product space V, a set of linearly independent vectors $\varepsilon_1, \ldots, \varepsilon_m \in V$, and a real symmetric matrix $A = (a_{ij})$, consider the quadratic combination

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(\alpha, \varepsilon_i)(\alpha, \varepsilon_j), \alpha \in V.$$

By the triangle inequality and the Cauchy-Schwarz inequality, we obtain the following simple inequality:

$$-\left[\sum_{i=1}^{m}\sum_{j=1}^{m} \left|a_{ij}\right| \sqrt{(\varepsilon_i, \varepsilon_i)(\varepsilon_j, \varepsilon_j)}\right](\alpha, \alpha) \leqslant \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(\alpha, \varepsilon_i)(\alpha, \varepsilon_j) \leqslant \left[\sum_{i=1}^{m}\sum_{j=1}^{m} \left|a_{ij}\right| \sqrt{(\varepsilon_i, \varepsilon_i)(\varepsilon_j, \varepsilon_j)}\right](\alpha, \alpha).$$

Thus, finding the optimal constants and equality conditions for the inequalities

$$C_1(\alpha, \alpha) \leq \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(\alpha, \varepsilon_i)(\alpha, \varepsilon_j) \leq C_2(\alpha, \alpha)$$

becomes a problem worth investigating.

This problem is related to several classical results. Bessel's inequality in inner product spaces states that for an orthonormal system $\{\varepsilon_i\}$, one has

$$\sum_{i} |(\alpha, \varepsilon_i)|^2 \leqslant (\alpha, \alpha).$$

In the finite-dimensional setting, this is a special case of the problem formulated above: The vectors $\varepsilon_1, \ldots, \varepsilon_m$ form an orthonormal system and the quadratic combination reduces to $\sum\limits_{i=1}^{m} (\alpha, \varepsilon_i)^2$. More generally, frame theory [11, 12] extends Bessel's inequality to systems of vectors that are not necessarily orthonormal, and the optimal frame bounds are precisely the smallest and largest eigenvalues of the associated Gram matrix. Massey and Ruiz studied the minimization of convex functionals over frame operators in finite-dimensional Hilbert spaces, where the associated quadratic forms also involve combinations of inner products [13]. Recent developments in this direction include two-sided inequalities for frames and fusion frames with new perspectives [14, 15].

However, none of the abovementioned works seems to provide a unified treatment for the optimal constants and equality conditions of the inequalities related to quadratic combinations of an arbitrary finite set of linearly independent vectors in a real inner product space. In this paper, by employing orthogonal decomposition, Gram matrices, and the Rayleigh quotient theory, we establish a general theorem that provides the optimal constants and equality conditions for such quadratic combination inequalities.

Although orthogonal decomposition, Gram matrices, and Rayleigh quotients are classical tools, the contribution of this paper is a systematic reduction of a broad class of sharp quadratic inequalities to a single eigenvalue problem. The framework yields both optimal bounds and complete equality conditions, which are determined by the extremal eigenvalues of the associated matrix and the orthogonal complement of the subspace spanned by the given vectors, a detail often overlooked in the standard Rayleigh quotient theory. The result applies to any real inner product space, bridging finite-dimensional linear algebra with infinite-dimensional analysis.

This paper is organized as follows. In Section 2, we introduce some necessary notations, definitions, and preliminary results. In Section 3, we establish and prove the main results of this article. In Section 4, we present some applications and illustrative examples, together with a preliminary discussion of the non-quadratic case.

2. Preliminaries

In what follows, we introduce some necessary notations, definitions, and preliminary results.

Throughout this paper, the prime $'$ denotes the transpose of a matrix or a vector. For vectors $\alpha_1, \alpha_2, \cdots, \alpha_m$ in a linear space V, the subspace spanned by these vectors is denoted by

$$\mathrm{span}\{\alpha_1, \alpha_2, \cdots, \alpha_m\}.$$

For a subspace W of a real inner product space V, the orthogonal complement of W is denoted by $W^{\perp}$. For a real symmetric matrix A, the largest eigenvalue of A is denoted by $\lambda_{\max}(A)$, and the smallest by $\lambda_{\min}(A)$.

Lemma 2.1. *[16] Let V be a real inner product space and W be a finite-dimensional subspace of V, then W has a unique orthogonal complement $W^{\perp}$, such that*

$$V = W \oplus W^{\perp}.$$

Definition 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $0 \neq x \in \mathbb{R}^n$. We call*

$$R(x) = \frac{x^T A x}{x^T x}$$

the Rayleigh quotient of A.

Lemma 2.2. *[16] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and $R(x) = \frac{x^T A x}{x^T x}$ is the Rayleigh quotient of A. Then, the following conclusions hold.*

(i) The maximum and minimum values of $R(x)$ are

$$\max_{0\neq x\in\mathbb{R}^n} R(x) = \lambda_{\max}(A),$$

$$\min_{0\neq x\in\mathbb{R}^n} R(x) = \lambda_{\min}(A).$$

(ii) $R(x)$ attains its maximum value if and only if x is an eigenvector of A corresponding to $\lambda_{\max}(A)$, and its minimum value if and only if x is an eigenvector of A corresponding to $\lambda_{\min}(A)$.

3. Main results

In this section, we assume that V is a real inner product space equipped with inner product $(\cdot,\cdot)$, $\varepsilon_1,\ldots,\varepsilon_m \in V$ are linearly independent, and $W = \text{span}\{\varepsilon_1,\ldots,\varepsilon_m\}$. Let $G = ((\varepsilon_i,\varepsilon_j))$ be the corresponding Gram matrix, and let $A = (a_{ij})$ be a real symmetric matrix. Choose any invertible matrix B such that $G = B'B$.

We seek the optimal constants C_1 and C_2 such that

$$C_1(\alpha,\alpha) \le \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(\alpha,\varepsilon_i)(\alpha,\varepsilon_j) \le C_2(\alpha,\alpha)$$

for all $\alpha \in V$. To this end, define the functional

$$J(\alpha) = \frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{m} a_{ij}(\alpha,\varepsilon_i)(\alpha,\varepsilon_j)}{(\alpha,\alpha)}, \alpha \neq 0.$$

It follows immediately that the optimal constants C_1 and C_2 are

$$C_1 = \inf_{\alpha\neq 0} J(\alpha) = \inf_{\|\alpha\|=1} J(\alpha), C_2 = \sup_{\alpha\neq 0} J(\alpha) = \sup_{\|\alpha\|=1} J(\alpha).$$

Thus, this is a functional extremum problem.

The matrix A may fall into one of the following categories: (1) $A = 0$; (2) A is positive definite; (3) A is positive semidefinite but not positive definite; (4) A is negative definite; (5) A is negative semidefinite but not negative definite; (6) A is indefinite.

Case (1) is trivial and will not be considered further. Cases (2) and (4) are equivalent up to a sign change, and so are cases (3) and (5). Thus, it suffices to study cases (2), (3), and (6), which share the common feature that A is not negative semidefinite.

The following theorem provides the solution to the above functional extremum problem.

Theorem 3.1. *With the notations introduced above, suppose additionally that A is not negative semidefinite. Then, the following conclusions hold.*

(i) The maximum value of $J(\alpha)$ is $\lambda_{\max}(BAB')$, with equality if and only if

$$\alpha = (\varepsilon_1,\ldots,\varepsilon_m)B^{-1}y,$$

where y is an eigenvector of BAB' corresponding to $\lambda_{\max}(BAB')$.

(ii) If $\lambda_{\min}(BAB') > 0$ and $W = V$, then the minimum value of $J(\alpha)$ is $\lambda_{\min}(BAB')$, with equality if and only if

$$\alpha = (\varepsilon_1,\ldots,\varepsilon_m)B^{-1}y,$$

where y is an eigenvector of BAB' corresponding to $\lambda_{\min}(BAB')$.

(iii) If $\lambda_{\min}(BAB') > 0$ and $W \neq V$, then the minimum value of $J(\alpha)$ is 0, with equality if and only if $\alpha \in W^{\perp}$.

(iv) If $\lambda_{\min}(BAB') = 0$, then the minimum value of $J(\alpha)$ is 0, with equality if and only if

$$\alpha = (\varepsilon_1, \ldots, \varepsilon_m)B^{-1}y + \gamma,$$

where $BAB'y = 0$, $\gamma \in W^{\perp}$, and y and γ are not both zero.

(v) If $\lambda_{\min}(BAB') < 0$, then the minimum value of $J(\alpha)$ is $\lambda_{\min}(BAB')$, with equality if and only if

$$\alpha = (\varepsilon_1, \ldots, \varepsilon_m)B^{-1}y,$$

where y is an eigenvector of BAB' corresponding to $\lambda_{\min}(BAB')$.

Proof. Observe that

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(\alpha, \varepsilon_i)(\alpha, \varepsilon_j) = K'AK,$$

where $K = ((\alpha, \varepsilon_1), (\alpha, \varepsilon_2), \cdots, (\alpha, \varepsilon_m))'$.

Consider the orthogonal decomposition of α:

$$\alpha = \sum_{j=1}^{m} c_j \varepsilon_j + \gamma,$$

with $c_1, \cdots, c_m \in \mathbb{R}$ and $\gamma \in W^{\perp}$. Let $c = (c_1, \cdots, c_m)'$, then for each i,

$$(\alpha, \varepsilon_i) = (\varepsilon_i, \sum_{j=1}^{m} c_j \varepsilon_j + \gamma) = \sum_{j=1}^{m} (\varepsilon_i, \varepsilon_j)c_j = (Gc)_i, i = 1, 2, \cdots, m.$$

Hence, $K = Gc$, and consequently

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(\alpha, \varepsilon_i)(\alpha, \varepsilon_j) = (Gc)'A(Gc) = c'(GAG)c.$$

On the other hand,

$$(\alpha, \alpha) = (\sum_{j=1}^{m} c_j \varepsilon_j + \gamma, \sum_{j=1}^{m} c_j \varepsilon_j + \gamma) = (\sum_{j=1}^{m} c_j \varepsilon_j, \sum_{j=1}^{m} c_j \varepsilon_j) + (\gamma, \gamma) = c'Gc + (\gamma, \gamma).$$

Therefore,

$$J(\alpha) = \frac{c'(GAG)c}{c'Gc + (\gamma, \gamma)}.$$

As B is a real invertible matrix with $G = B'B$, it follows that

$$J(\alpha) = \frac{c'(B'BAB'B)c}{c'B'Bc + (\gamma, \gamma)} = \frac{(Bc)'(BAB')(Bc)}{(Bc)'(Bc) + (\gamma, \gamma)} = \frac{y'(BAB')y}{y'y + (\gamma, \gamma)},$$

where $y = Bc$, and thus $c = B^{-1}y$. Since $\alpha \neq 0$ and $(\alpha, \alpha) = y'y + (\gamma, \gamma)$, it follows that y and γ are not both zero.

By Lemma 2.2, we have

$$\lambda_{\min}(BAB')y'y \leq y'(BAB')y \leq \lambda_{\max}(BAB')y'y.$$

Therefore,

$$J(\alpha) \leq \frac{\lambda_{\max}(BAB')y'y}{y'y + (\gamma, \gamma)}, \tag{3.1}$$

$$J(\alpha) \geq \frac{\lambda_{\min}(BAB')y'y}{y'y + (\gamma, \gamma)}. \tag{3.2}$$

Equality holds in (3.1) if and only if $BAB'y = \lambda_{\max}(BAB')y$. Equality holds in (3.2) if and only if $BAB'y = \lambda_{\min}(BAB')y$.

Since A is not negative semidefinite, BAB' is also not negative semidefinite, hence $\lambda_{\max}(BAB') > 0$. Moreover,

$$\frac{\lambda_{\max}(BAB')y'y}{y'y + (\gamma, \gamma)} \leq \lambda_{\max}(BAB'). \tag{3.3}$$

Equality holds in (3.3) if and only if $y \neq 0$ and $\gamma = 0$.

Combining (3.1) and (3.3) yields

$$J(\alpha) \leq \lambda_{\max}(BAB'). \tag{3.4}$$

Equality holds in (3.4) if and only if equality holds in both (3.1) and (3.3), i.e.,

$$\alpha = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)B^{-1}y,$$

where y is an eigenvector of BAB' corresponding to $\lambda_{\max}(BAB')$. This proves part (i).

If $\lambda_{\min}(BAB') > 0$, then BAB' is positive definite. Two cases are distinguished.

Case 1: $W = V$. Then, $\gamma = 0$ for every $\alpha \in V$, and hence

$$J(\alpha) = \frac{y'(BAB')y}{y'y} \geq \lambda_{\min}(BAB'). \tag{3.5}$$

Equality holds in (3.5) if and only if

$$\alpha = (\varepsilon_1, \ldots, \varepsilon_m)B^{-1}y,$$

where y is an eigenvector of BAB' corresponding to $\lambda_{\min}(BAB')$. This proves part (ii).

Case 2: $W \neq V$. Then, γ may be nonzero, and

$$J(\alpha) = \frac{y'(BAB')y}{y'y + (\gamma, \gamma)} \geq 0. \tag{3.6}$$

Equality holds in (3.6) if and only if $y = 0$, i.e., $\alpha = \gamma \in W^{\perp}$. This proves part (iii).

If $\lambda_{\min}(BAB') = 0$, then (3.2) gives

$$J(\alpha) \geq \frac{\lambda_{\min}(BAB')y'y}{y'y + (\gamma, \gamma)} = 0.$$

$J(\alpha) = 0$ if and only if $BAB'y = 0$. This proves part (iv).

If $\lambda_{\min}(BAB') < 0$, then

$$\frac{\lambda_{\min}(BAB')y^T y}{y^T y + (\gamma, \gamma)} \geqslant \lambda_{\min}(BAB'). \tag{3.7}$$

Equality holds in (3.7) if and only if $y \neq 0$ and $\gamma = 0$.

Combining (3.2) and (3.7) gives

$$J(\alpha) \geqslant \lambda_{\min}(BAB'). \tag{3.8}$$

Equality holds in (3.8) if and only if equality holds in both (3.2) and (3.7) , i.e.,

$$\alpha = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)B^{-1}y,$$

where y is an eigenvector of BAB' corresponding to $\lambda_{\min}(BAB')$. This proves part (v).

The proof of Theorem 3.1 is now complete. □

Remark 3.1. *It is important to clarify the novelty of Theorem 3.1 in relation to the classical Rayleigh quotient. The latter gives the extremal values of* $\frac{x'Mx}{x'x}$ *and the corresponding eigenvectors. Theorem 3.1 goes beyond this: The orthogonal decomposition* $\alpha = \sum_{j=1}^{m} c_j\varepsilon_j + \gamma$ *introduces an extra term* (γ, γ) *in the denominator, leading to refined lower-bound characterizations when* $W \neq V$*; it provides complete equality conditions when* $\lambda_{\min} = 0$*, including the nullspace of* BAB' *and the full orthogonal complement* $W^{\perp}$*; and it covers all cases of* A *in a unified manner. Thus, while the Rayleigh quotient provides the essential tool, the theorem offers a systematic framework for optimal constants and equality conditions of a broad class of quadratic functional inequalities.*

Remark 3.2. *When* $\varepsilon_1, \ldots, \varepsilon_m$ *form an orthonormal system and* $A = I$*, we have* $G = I$*. Taking* $B = I$*, then* $BAB' = I$*, and hence* $\lambda_{\max}(BAB') = 1$*. It follows from Theorem 3.1(i) that*

$$\sum_{i=1}^{m}(\alpha, \varepsilon_i)^2 \leq (\alpha, \alpha), \tag{3.9}$$

and equality holds in (3.9) if and only if $\alpha = (\varepsilon_1, \cdots, \varepsilon_m)y, \quad y \in \mathbb{R}^m$*, that is,* $\alpha \in W$*. The above inequality is precisely the finite-dimensional version of Bessel's inequality.*

Remark 3.3. *Let* $A = I$*. It follows from Theorem 3.1(i) that for any nonzero* $\alpha \in V$*,*

$$\sum_{i=1}^{m}(\alpha, \varepsilon_i)^2 \leqslant \lambda_{\max}(BB')(\alpha, \alpha) = \lambda_{\max}(G)(\alpha, \alpha), \tag{3.10}$$

and equality holds in (3.10) if and only if $\alpha = (\varepsilon_1, \cdots, \varepsilon_m)B^{-1}y$*, where* y *is an eigenvector of* BB' *corresponding to* $\lambda_{\max}(BB') = \lambda_{\max}(G)$*.*

If $W = V$*, it follows from Theorem 3.1(ii) that for any nonzero* $\alpha \in V$*,*

$$\sum_{i=1}^{m}(\alpha, \varepsilon_i)^2 \geq \lambda_{\min}(BB')(\alpha, \alpha) = \lambda_{\min}(G)(\alpha, \alpha), \tag{3.11}$$

and equality holds in (3.11) if and only if $\alpha = (\varepsilon_1, \cdots, \varepsilon_m)B^{-1}y$*, where* y *is an eigenvector of* BB' *corresponding to* $\lambda_{\min}(BB') = \lambda_{\min}(G)$*.*

These are precisely the classical frame bounds in frame theory.

Remark 3.4. *The main results of this paper are established in real inner product spaces. The framework can be naturally extended to complex inner product spaces, where $A = (a_{ij})$ is required to be Hermitian and the quadratic combination takes the form*

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}\overline{(\alpha,\varepsilon_i)}(\alpha,\varepsilon_j).$$

The results depend only on the fact that the number of vectors is finite. If infinitely many vectors are considered, the problem involves the spectral theory of operators and lies beyond the scope of this paper.

4. Applications and examples

We now solve the problems raised at the beginning of the paper.

Example 4.1. *The functions* $\cos x$ *and* $\sin x$ *are linearly independent in* $L^2[0,\pi]$. *Their Gram matrix is*

$$G = \begin{pmatrix} \int_0^{\pi} \cos^2 x\,dx & \int_0^{\pi} \cos x \sin x\,dx \\ \int_0^{\pi} \cos x \sin x\,dx & \int_0^{\pi} \sin^2 x\,dx \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} & 0 \\ 0 & \frac{\pi}{2} \end{pmatrix}.$$

Take $B = \begin{pmatrix} \sqrt{\frac{\pi}{2}} & 0 \\ 0 & \sqrt{\frac{\pi}{2}} \end{pmatrix}$, *then* $G = B'B$. *Let* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *then*

$$BAB' = G = \begin{pmatrix} \frac{\pi}{2} & 0 \\ 0 & \frac{\pi}{2} \end{pmatrix}.$$

The eigenvalues of BAB' *are both* $\frac{\pi}{2}$, *and every nonzero vector is an eigenvector. By Theorem 3.1,*

$$\left[\int_0^{\pi} \cos x\, f(x)dx\right]^2 + \left[\int_0^{\pi} \sin x\, f(x)dx\right]^2 \le \frac{\pi}{2}\int_0^{\pi} f^2(x)dx. \tag{4.1}$$

Equality holds in (4.1) if and only if

$$f(x) = (\cos x, \sin x)B^{-1}y = \sqrt{\frac{2}{\pi}}(\cos x, \sin x)y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2,$$

i.e., $f(x) \in \operatorname{span}\{\cos x, \sin x\}$. *On the other hand,*

$$\left[\int_0^{\pi} \cos x\, f(x)dx\right]^2 + \left[\int_0^{\pi} \sin x\, f(x)dx\right]^2 \ge 0. \tag{4.2}$$

Equality holds in (4.2) if and only if $f(x) \in (\operatorname{span}\{\cos x, \sin x\})^{\perp}$.

Example 4.2. *The functions* 1 *and* x^4 *are linearly independent in* $L^2[0,1]$. *Their Gram matrix is*

$$G = \begin{pmatrix} \int_0^1 1^2\,dx & \int_0^1 x^4\,dx \\ \int_0^1 x^4\,dx & \int_0^1 x^8\,dx \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{9} \end{pmatrix}.$$

Take $B = \begin{pmatrix} 1 & \frac{1}{5} \\ 0 & \frac{4}{15} \end{pmatrix}$, *then* $G = B'B$. *Let* $A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$, *then*

$$BAB' = \begin{pmatrix} \frac{1}{5} & \frac{2}{15} \\ \frac{2}{15} & 0 \end{pmatrix}.$$

The eigenvalues of BAB' *are* $\frac{4}{15}, -\frac{1}{15}$, *and the corresponding eigenvectors are* $y_1 = (2,1)'$ *and* $y_2 = (1,-2)'$, *respectively.*

$$B^{-1}y_1 = \begin{pmatrix} 1 & \frac{1}{5} \\ 0 & \frac{4}{15} \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

$$B^{-1}y_2 = \begin{pmatrix} 1 & \frac{1}{5} \\ 0 & \frac{4}{15} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

By Theorem 3.1,

$$\int_0^1 f(x)dx \int_0^1 x^4 f(x)dx \le \frac{4}{15}\int_0^1 f^2(x)dx, \tag{4.3}$$

$$\int_0^1 f(x)dx \int_0^1 x^4 f(x)dx \ge -\frac{1}{15}\int_0^1 f^2(x)dx. \tag{4.4}$$

Equality holds in (4.3) if and only if $f(x) = c(1+3x^4)$ *for some constant* c. *Equality holds in (4.4) if and only if* $f(x) = c(1-3x^4)$ *for some constant* c.

Example 4.3. *Recall the set* S_a *and the functionals* $A(f)$, $B(f)$, *and* $C(f)$ *defined in the introduction. For* $a > 0$, *let*

$$S_a = \left\{ f : [-a,a] \to \mathbb{R} \mid \int_{-a}^{a} f^2(x)\,dx = 1 \right\},$$

and define

$$A(f) = \int_{-a}^{a} f(x)\,dx, \quad B(f) = \int_{-a}^{a} x f(x)\,dx, \quad C(f) = \int_{-a}^{a} x^2 f(x)\,dx.$$

The functions 1 *and* x *are linearly independent in* $L^2[-a,a]$, *and their Gram matrix is*

$$G_1 = \begin{pmatrix} \int_{-a}^{a} 1^2\,dx & \int_{-a}^{a} x\,dx \\ \int_{-a}^{a} x\,dx & \int_{-a}^{a} x^2\,dx \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & \frac{2}{3}a^3 \end{pmatrix}.$$

Take $K = \begin{pmatrix} \sqrt{2a} & 0 \\ 0 & \sqrt{\frac{2}{3}a^3} \end{pmatrix}$, *then* $G_1 = K'K = KK'$. *The eigenvalues of matrix* G_1 *are* $2a$ *and* $\frac{2}{3}a^3$.

If $0 < a < \sqrt{3}$, *then* $\lambda_{\max}(G_1) = 2a$ *and a corresponding unit eigenvector of* $\lambda_{\max}(G_1)$ *is* $y = (1, 0)'$. *By Theorem 3.1, we have*

$$\sup\left\{A^2(f) + B^2(f) : f \in S_a\right\} = \lambda_{\max}(G_1) = 2a,$$

and when $f = (1, x)K^{-1}y = \frac{1}{\sqrt{2a}}$, *the sum* $A^2(f) + B^2(f)$ *attains its maximum.*

If $a = \sqrt{3}$, *then* $\lambda_{\max}(G_1) = 2\sqrt{3}$ *and every nonzero vector in* $\mathbb{R}^2$ *is an eigenvector corresponding to* $\lambda_{\max}(G_1)$. *By Theorem 3.1, we have*

$$\sup\left\{A^2(f) + B^2(f) : f \in S_a\right\} = \lambda_{\max}(G_1) = 2\sqrt{3},$$

and the supremum $2\sqrt{3}$ *is attained at every function of the form*

$$f(x) = (1, x)K^{-1}c = \frac{c_1 + c_2 x}{\sqrt{2\sqrt{3}}},$$

where $c = (c_1, c_2)'$ *is a unit vector in* $\mathbb{R}^2$.

If $a > \sqrt{3}$, *then* $\lambda_{\max}(G_1) = \frac{2}{3}a^3$ *and a corresponding unit eigenvector of* $\lambda_{\max}(G_1)$ *is* $y = (0, 1)'$. *By Theorem 3.1, we have*

$$\sup\left\{A^2(f) + B^2(f) : f \in S_a\right\} = \lambda_{\max}(G_1) = \frac{2}{3}a^3,$$

and when $f = (1, x)K^{-1}y = \frac{x}{\sqrt{\frac{2}{3}a^3}}$, *the sum* $A^2(f) + B^2(f)$ *attains its maximum.*

The functions $1, x, x^2$ *are linearly independent in* $L^2[-a, a]$, *and their Gram matrix is*

$$G_2 = \begin{pmatrix} \int_{-a}^{a} 1^2\,dx & \int_{-a}^{a} x\,dx & \int_{-a}^{a} x^2\,dx \\ \int_{-a}^{a} x\,dx & \int_{-a}^{a} x^2\,dx & \int_{-a}^{a} x^3\,dx \\ \int_{-a}^{a} x^2\,dx & \int_{-a}^{a} x^3\,dx & \int_{-a}^{a} x^4\,dx \end{pmatrix} = \begin{pmatrix} 2a & 0 & \frac{2}{3}a^3 \\ 0 & \frac{2}{3}a^3 & 0 \\ \frac{2}{3}a^3 & 0 & \frac{2}{5}a^5 \end{pmatrix}.$$

Take $L = \begin{pmatrix} \sqrt{2a} & 0 & \sqrt{\frac{2}{9}a^5} \\ 0 & \sqrt{\frac{2}{3}a^3} & 0 \\ 0 & 0 & \sqrt{\frac{8}{45}a^5} \end{pmatrix}$, *then* $G_2 = L'L$,

$$LL' = \begin{pmatrix} 2a + \frac{2}{9}a^5 & 0 & \frac{4}{9\sqrt{5}}a^5 \\ 0 & \frac{2}{3}a^3 & 0 \\ \frac{4}{9\sqrt{5}}a^5 & 0 & \frac{8}{45}a^5 \end{pmatrix}.$$

The eigenvalues of matrix G_2 *are*

$$\lambda_1 = \frac{2}{3}a^3, \lambda_2 = a\left(1 + \frac{1}{5}a^4 - \sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1}\right), \lambda_3 = a\left(1 + \frac{1}{5}a^4 + \sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1}\right).$$

Since $a > 0$, it is clear that $\lambda_3 > \lambda_2$. By the arithmetic-geometric mean inequality, we have

$$\lambda_3 = a(1 + \frac{1}{5}a^4 + \sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1}) > a(1 + \frac{1}{5}a^4) \geq \frac{2}{\sqrt{5}}a^3 > \frac{2}{3}a^3 = \lambda_1.$$

Therefore, $\lambda_{\max}(LL') = \lambda_{\max}(G_2) = \lambda_3$.

A direct computation shows that a unit eigenvector of LL' corresponding to λ_3 is:

$$y = \frac{\left(\frac{4}{9\sqrt{5}}a^4, 0, \sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1} - \frac{1}{45}a^4 - 1\right)'}{\sqrt{2(\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1) - 2(\frac{1}{45}a^4 + 1)\sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1}}}.$$

By Theorem 3.1, we have

$$\sup\left\{A^2(f) + B^2(f) + C^2(f) : f \in S_a\right\} = \lambda_{\max}(G_2) = a(1 + \frac{1}{5}a^4 + \sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1}),$$

and when

$$f(x) = (1, x, x^2)L^{-1}y = \frac{\frac{8}{9}a^6 + 5\left(\sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1} - \frac{1}{45}a^4 - 1\right)(3x^2 - a^2)}{a^2\sqrt{80a\left[(\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1) - (\frac{1}{45}a^4 + 1)\sqrt{\frac{1}{25}a^8 + \frac{2}{45}a^4 + 1}\right]}},$$

the sum $A^2(f) + B^2(f) + C^2(f)$ attains its maximum.

Remark 4.1. *The preceding results rely on the quadratic nature of the functional, which allows the use of orthogonal decomposition and Rayleigh quotient theory. When the functional is no longer a quadratic combination, the quadratic form approach is no longer applicable. Nevertheless, the orthogonal decomposition method remains valid.*

To illustrate this point, we consider a concrete cubic-type example. The argument proceeds as follows: We first use orthogonal decomposition to reduce the functional to an expression in two variables; we then transform the problem into the maximization of a single-variable function; finally, we obtain the optimal constant by elementary calculus.

The following cubic-type functional arises naturally when one attempts to estimate products of three linear functionals:

$$J(\alpha) = \frac{(\varepsilon_1, \alpha)^2(\varepsilon_2, \alpha)}{(\alpha, \alpha)^{\frac{3}{2}}}, \alpha \neq 0,$$

where V is a real inner product space and $\varepsilon_1, \varepsilon_2 \in V$ are two linearly independent unit vectors. Let $W = \mathrm{span}\{\varepsilon_1, \varepsilon_2\}$ and $\rho = (\varepsilon_1, \varepsilon_2)$. Then, $\rho \in (-1, 1)$.

Using the orthogonal decomposition $\alpha = c_1\varepsilon_1 + c_2\varepsilon_2 + \gamma$ with $\gamma \in W^\perp$, and setting

$$u = c_1 + \rho c_2, v = \sqrt{1 - \rho^2}\, c_2,$$

one obtains

$$J(\alpha) = \frac{u^2(\rho u + \sqrt{1 - \rho^2}\, v)}{[u^2 + v^2 + (\gamma, \gamma)]^{\frac{3}{2}}}.$$

When $u = 0$, $J(\alpha) = 0$. When $u \neq 0$,

$$\begin{aligned} J(\alpha) &= \frac{u^2(\rho u + \sqrt{1-\rho^2}\, v)}{[u^2 + v^2 + (\gamma,\gamma)]^{\frac{3}{2}}} \leq \frac{u^2(|\rho u| + \sqrt{1-\rho^2}\,|v|)}{[u^2 + v^2 + (\gamma,\gamma)]^{\frac{3}{2}}} \\ &\leq \frac{u^2(|\rho u| + \sqrt{1-\rho^2}\,|v|)}{(u^2+v^2)^{\frac{3}{2}}} = \sqrt{1-\rho^2}\,\frac{\frac{|\rho|}{\sqrt{1-\rho^2}} + \left|\frac{v}{u}\right|}{\left[1 + \left(\frac{v}{u}\right)^2\right]^{\frac{3}{2}}}, \end{aligned} \tag{4.5}$$

and equality holds in (4.5) if $\rho u \geq 0$, $v \geq 0$, and $\gamma = 0$. Therefore, the supremum of $J(\alpha)$ can be determined by reducing to the one-variable maximization of

$$f(x) = \frac{x + a}{(1 + x^2)^{\frac{3}{2}}}, x \geq 0,$$

where $a = \frac{|\rho|}{\sqrt{1-\rho^2}}$.

By elementary calculus, f attains its maximum at

$$x^* = -\frac{3}{4}a + \sqrt{\frac{9}{16}a^2 + \frac{1}{2}},$$

with maximum value

$$f_{\max} = \frac{\frac{1}{4}a + \sqrt{\frac{9}{16}a^2 + \frac{1}{2}}}{\left(\frac{9}{8}a^2 + \frac{3}{2} - \frac{3}{2}a\sqrt{\frac{9}{16}a^2 + \frac{1}{2}}\right)^{\frac{3}{2}}}.$$

For example, in $L^2[0,1]$, taking $\varepsilon_1 = 1$ and $\varepsilon_2 = \sqrt{3}x$, one obtains $\rho = \frac{\sqrt{3}}{2}$ and $a = \sqrt{3}$, which yields the optimal inequality

$$\left(\int_0^1 f(x)\,dx\right)^2 \int_0^1 xf(x)\,dx \leq \frac{2(\sqrt{70} + \sqrt{6})}{9(13 - \sqrt{105})^{\frac{3}{2}}} \left(\int_0^1 f^2(x)\,dx\right)^{\frac{3}{2}}. \tag{4.6}$$

Equality holds in (4.6) if $f(x) = c\left(4 + (\sqrt{105} - 9)(2x-1)\right)$ for some nonnegative constant c, and the constant on the righthand side is the best possible.

The above analysis suggests a natural generalization: When the combination of linear functionals is no longer quadratic, how should the optimal constants and equality conditions of the corresponding inequality be characterized? For higher-order cases, such as $\prod_{i=1}^{k}(\alpha, \varepsilon_i)$ with $k \geq 3$, the problem becomes much more involved. These extensions are left for future investigation.

5. Conclusions

In this paper, we have established a unified algebraic framework for determining the optimal constants and equality conditions of quadratic functional inequalities in real inner product spaces. The main result, Theorem 3.1, reduces the problem to a finite-dimensional eigenvalue problem associated with the Gram matrix of the given vectors. The framework covers all cases of the coefficient matrix and provides complete equality characterizations, including the role of the orthogonal complement when the given vectors do not span the whole space.

Several examples are provided, including the solution to Problem 12318 from the American Mathematical Monthly, to illustrate the practical utility of the framework. A preliminary extension to a non-quadratic cubic-type functional is also discussed, suggesting broader applicability of the orthogonal decomposition approach.

Future work may explore extensions to complex inner product spaces, infinite-dimensional settings, and higher-order combinations of linear functionals.

Use of Generative-AI tools declaration

The author declares that no generative AI tools were used in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

References

1. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, 2 Eds., Cambridge: Cambridge University Press, 1952. https://doi.org/10.1017/s0025557200027455
2. S. S. Dragomir, J. J. Koliha, The mapping $\delta_{x,y}$ in normed linear spaces and refinements of the Cauchy-Schwarz inequality, *Nonlinear Anal.-Theor.*, **41** (2000), 205–220. https://doi.org/10.1016/s0362-546x(98)00274-0
3. G. B. Wang, J. P. Ma, Some results on reverses of Cauchy-Schwarz inequality in inner product spaces, *Northeast. Math. J.*, **21** (2005), 207–211.
4. D. Ilisevic, S. Varosanec, On the Cauchy-Schwarz inequality and its reverse in semi-inner product C*-modules, *Banach J. Math. Anal.*, **1** (2007), 78–84. https://doi.org/10.15352/bjma/1240321557
5. S. S. Dragomir, Refinements of the Cauchy-Bunyakovsky-Schwarz inequality for functions of selfadjoint operators in Hilbert spaces, *Linear Multilinear A.*, **59** (2011), 711–717. https://doi.org/10.1080/03081081003743008
6. R. Sharma, R. Bhandari, M. Gupta, Inequalities related to the Cauchy-Schwarz inequality, *Sankhya A*, **74** (2012), 101–111. https://doi.org/10.1007/s13171-012-0013-9
7. T. Ando, Geometric mean and norm Schwarz inequality, *Ann. Funct. Anal.*, **7** (2016), 1–8. https://doi.org/10.1215/20088752-3158073

8. M. Masjed-Jamei, A functional generalization of the Cauchy-Schwarz inequality and some subclasses, *Appl. Math. Lett.*, **22** (2009), 1335–1339. https://doi.org/10.1016/j.aml.2009.03.001
9. S. G. Walker, A self-improvement to the Cauchy-Schwarz inequality, *Stat. Probabil. Lett.*, **122** (2017), 86–89. https://doi.org/10.1016/j.spl.2016.11.001
10. D. H. Ullman, D. J. Velleman, S. Wagon, D. B. West, P. Bracken, E. A. Brown, et al., Problems and solutions, *Am. Math. Mon.*, **129** (2022), 385–394. https://doi.org/10.1080/00029890.2022.2030153
11. R. J. Duffin, A. C. Schaeffer, A class of nonharmonic Fourier series, *T. Am. Math. Soc.*, **72** (1952), 341–366. https://doi.org/10.1090/s0002-9947-1952-0047179-6
12. O. Christensen, *An introduction to frames and Riesz bases*, 2 Eds., Cham: Birkhäuser, 2016. https://doi.org/10.1007/978-3-319-25613-9
13. P. Massey, M. Ruiz, Minimization of convex functionals over frame operators, *Adv. Comput. Math.*, **32** (2010), 131–153. https://doi.org/10.1007/s10444-008-9092-5
14. Z. Q. Xiang, C. X. Lin, Y. X. Chen, Z. S. Xu, A new view on some two-sided inequalities for frames in Hilbert spaces, *J. Math. Inequal.*, **20** (2026), 81–90. https://doi.org/10.7153/jmi-2026-20-06
15. Z. Q. Xiang, Y. X. Chen, C. X. Lin, New perspectives on bilateral inequalities for fusion frames in Hilbert spaces, *Math. Inequal. Appl.*, **28** (2025), 355–369. https://doi.org/10.7153/mia-2025-28-24
16. H. Dai, *Matrix theory*, Beijing: Science Press, 2001.