Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 26943–26985.
DOI: 10.3934/math.20261081
Received: 05 July 2026
Revised: 03 August 2026
Accepted: 14 August 2026
Published: 27 August 2026

Research article

Global existence and polynomial stability for a nonlinear clamped plate equation with logarithmic velocity damping

Iqra Kanwal[1,*], Jianghao Hao[1], Muhammad Fahim Aslam[2], Mohamed Balegh[3] and Zayd Hajjej[4]

[1] School of Mathematics and Statistics, Shanxi University, Taiyuan, China

[2] Department of Mathematics, University of Kotli Azad Jammu and Kashmir(UOKAJK), Kotli, Pakistan

[3] Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia

[4] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: ikanwal1703@gmail.com.

Abstract: This paper investigates a nonlinear clamped plate equation with logarithmic velocity damping and a polynomial source. The model combines the biharmonic operator, a nonstandard logarithmic damping mechanism and an energy-producing source term. Since the logarithmic damping is weak near the origin and the source may destabilize the system, the analysis requires a combination of monotonicity, potential well, and multiplier techniques. We first establish the local well-posedness of weak solutions by the Faedo–Galerkin method and compactness arguments. We then prove global existence for the initial data in the stable set. Finally, using logarithmic dissipation estimates and a Haraux-type integral inequality, we derive a polynomial decay estimate for the energy. To demonstrate the theoretical decay behavior and the impact of the damping and source factors on the plate's energy, numerical simulations are also provided.

Keywords: clamped plate equation; logarithmic velocity damping; polynomial source; well-posedness; global existence; potential well; polynomial stability; numerical simulation; energy decay
Mathematics Subject Classification: 74K20, 49K40, 93D20, 65M50

1. Introduction

Nonlinear partial differential equations provide a powerful mathematical framework for describing physical systems in which propagation, dissipation, elasticity and nonlinear effects interact. Among such models, hyperbolic-type evolution equations play a central role in wave propagation, structural vibration, elasticity theory, and continuum mechanics. A fundamental question in this area is to understand how the long-time behavior of solutions is influenced by the balance between dissipative mechanisms and energy-producing nonlinearities.

Plate equations constitute an important class of higher-order evolution problems arising in the mathematical theory of thin elastic structures [1,2]. In these models, the biharmonic operator describes bending effects and gives rise to a fourth-order spatial structure. This feature makes plate equations substantially different from classical second-order wave equations. In particular, for clamped plates, the natural energy space is based on $H_0^2(\Omega)$, and the control of nonlinear terms requires higher-order Sobolev embeddings. Consequently, the analysis of well-posedness, global existence, and stability in plate models is generally more delicate than in the standard wave equation setting.

The qualitative behavior of nonlinear evolution equations is strongly affected by the interaction between damping and the source terms. Damping mechanisms usually dissipate energy and support global existence or stabilization, whereas the source terms may increase the energy and, for large data, can lead to unstable dynamics. The competition between these two effects has been widely studied in nonlinear wave equations. The foundational works of Levine [3, 4] introduced the concavity method and established finite-time blow-up results for solutions with negative initial energy. Later, Georgiev and Todorova [5] investigated wave equations with nonlinear damping and source terms and obtained important criteria distinguishing stable and unstable regimes. Further developments based on potential well theory, Nehari-type functionals, and differential inequality methods can be found, for example, in [6–8].

For plate equations, the damping-source interaction is more involved because the elastic energy is governed by the $H_0^2(\Omega)$ structure. Fundamental contributions to the stabilization and control of plate models were given by Lagnese [9], Komornik [10], and Lasiecka and Triggiani [11]. These works developed important tools for energy decay, multiplier methods, and the stabilization of elastic systems. The long-time behavior of second-order evolution equations with nonlinear damping was systematically studied by Chueshov and Lasiecka [12]. Since then, several damping mechanisms have been considered in plate-type models, including frictional damping, structural damping, boundary feedback, viscoelastic damping, and memory-type dissipation. Nevertheless, the treatment of weak and nonstandard damping mechanisms in nonlinear fourth-order plate equations remains a challenging topic.

In recent years, logarithmic nonlinearities have attracted increasing attention because their growth is neither purely linear nor polynomial. In the present problem, the logarithmic velocity damping has a nonhomogeneous structure. Near the origin

$$s\ln(1+s^2) \sim s^3 \qquad \text{as } s \to 0,$$

so the damping is weak for small velocities. On the other hand, for large $|s|$, it behaves like

$$s\ln(1+s^2) \sim 2s\ln|s|,$$

and hence it has a logarithmic superlinear growth. Therefore, this damping is different from both standard linear damping and usual power-type damping and its treatment requires estimates adapted to the logarithmic dissipation. Li et al. [13] studied the energy decay of a wave equation with logarithmic feedback and obtained polynomial decay estimates. Li et al. [14] investigated blow-up and decay phenomena for wave equations with logarithmic damping. Messaoudi et al. [15] studied well-posedness, blow-up, and numerical behavior for a wave equation involving logarithmic damping and a polynomial source. The decay strategy used in the present paper is inspired by the logarithmic damping estimates developed for second-order wave equations. However, the present problem is substantially different from the source-free wave model and from wave equations with a polynomial source. Indeed, the biharmonic operator requires the clamped plate framework $H_0^2(\Omega)$, while the source term enters the energy with a negative sign. Therefore, the decay argument must be combined with a potential well mechanism in order to recover coercivity of the energy before applying the weighted multiplier method. This is the main point where the fourth-order plate structure differs from the corresponding wave equation setting.

Logarithmic damping has also appeared in Timoshenko-type systems and piezoelectric beam models; see [16, 17]. These works show that logarithmic feedback provides weak but effective dissipation, while also requiring refined estimates that differ from those used for classical polynomial damping. To the best of our knowledge, the combination of a logarithmic velocity damping mechanism and a polynomial source term has not been investigated for nonlinear clamped plate equations. While logarithmic damping has been studied extensively for wave, beam, and Timoshenko-type systems, the available techniques cannot be transferred directly to fourth-order plate equations. The presence of the biharmonic operator changes the natural energy space and requires higher-order Sobolev estimates, while the source term introduces an additional competition between energy production and dissipation. Consequently, the stabilization analysis must combine logarithmic dissipation estimates with a potential well framework adapted to the clamped plate setting.

Motivated by these developments, we study the following nonlinear clamped plate problem with logarithmic velocity damping and a polynomial source:

$$\begin{cases} \mathrm{u}_{tt} + \Delta^2 \mathrm{u} + \mathrm{u}_t \ln\left(1 + \mathrm{u}_t^2\right) = |\mathrm{u}|^{p-2}\mathrm{u}, & \text{in } \Omega \times (0,\infty), \\ \mathrm{u} = \dfrac{\partial \mathrm{u}}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,\infty), \\ \mathrm{u}(x,0) = \mathrm{u}_0(x), \qquad \mathrm{u}_t(x,0) = \mathrm{u}_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, ν denotes the outward unit normal vector to $\partial\Omega$, and $p > 2$ satisfies suitable restrictions depending on the dimension N. The boundary conditions in (1.1) correspond to the classical clamped plate case.

The polynomial source

$$|\mathrm{u}|^{p-2}\mathrm{u}$$

is an energy-producing nonlinearity whose admissible range is dictated by the Sobolev embedding properties of $H_0^2(\Omega)$. In contrast, the damping term

$$\mathrm{u}_t \ln(1 + \mathrm{u}_t^2)$$

has a nonstandard logarithmic structure. For small velocities, it behaves like a cubic damping term, since

$$u_t \ln(1 + u_t^2) \sim u_t^3 \quad \text{as } u_t \to 0.$$

For large velocities, it grows like $2u_t \ln |u_t|$, which is logarithmically stronger than a linear damping term but still is different from classical power-type feedbacks. Therefore, it is neither a standard linear damping nor a usual power-type damping. This nonhomogeneous growth prevents a direct application of classical decay methods and requires estimates adapted to the logarithmic structure.

The main mathematical difficulty in (1.1) comes from the simultaneous presence of the biharmonic operator, the weak logarithmic dissipation, and the polynomial source. The fourth-order operator requires the use of the clamped plate energy space $H_0^2(\Omega) \times L^2(\Omega)$. The source term must be controlled by higher-order Sobolev embeddings, while the logarithmic damping does not directly dominate the kinetic energy in the same way as linear or polynomial damping. These features require a careful combination of Galerkin approximations, compactness arguments, monotonicity methods, potential well techniques, weighted multiplier estimates, and logarithmic dissipation inequalities.

The purpose of this paper is to provide a rigorous qualitative analysis of Problem (1.1). First, we establish the local well-posedness of weak solutions by using the Faedo–Galerkin method together with compactness arguments and the monotonicity of the mapping $s \mapsto s\ln(1+s^2)$. Second, by introducing a potential well framework, we prove global existence for the initial data belonging to the stable set. This stable set condition ensures that the elastic energy dominates the polynomial source and that the total energy remains coercive. Third, we derive polynomial decay estimates for the global energy. The decay argument is based on a weighted multiplier identity, a decomposition of the domain according to the size of u_t, and an integral inequality of the Haraux type. Finally, numerical simulations are presented to illustrate the theoretical decay behavior and to demonstrate the influence of the logarithmic damping and the polynomial source on the evolution of the plate's energy.

The novelty of this work lies in extending the analysis of logarithmically damped hyperbolic equations to a nonlinear fourth-order clamped plate model with a polynomial source. The available results for second-order wave equations cannot be transferred directly to (1.1), since the biharmonic operator changes both the functional setting and the structure of the estimates. In particular, the decay analysis must combine the weak logarithmic damping mechanism with the potential well control of the source term in the $H_0^2(\Omega)$ framework. The results obtained here show that logarithmic damping is sufficient to stabilize the nonlinear clamped plate equation in the stable regime, although the corresponding decay rate is algebraic rather than exponential.

The paper is structured as follows. In Section 2, the functional setup is shown and the local well-posedness of Problem (1.1) is established. We present the energy functional and demonstrate its global existence for the initial data in the stable potential well in Section 3. The polynomial stability result is the focus of Section 4. We present numerical simulations in Section 5 to support the theoretical decay behavior. Section 6 concludes with some notes.

2. Well-posedness

In this section, we establish the local well-posedness of Problem (1.1). The proof is based on the Faedo–Galerkin method, compactness arguments, and the monotonicity of the logarithmic damping term. Throughout this section, $C > 0$ denotes a generic positive constant which may change from line to line.

For convenience, we also adopt the following notation throughout the paper. The constants C_ε denote positive constants depending only on the parameter $\varepsilon > 0$.

Let

$$\mathrm{H} = L^2(\Omega), \qquad \mathrm{V} = H_0^2(\Omega).$$

For the clamped plate problem, the natural norm on V is given by

$$\|\mathrm{u}\|_V = \|\Delta \mathrm{u}\|_2.$$

By the Poincaré inequality for clamped plates, this norm is equivalent to the usual $H^2(\Omega)$ norm on $H_0^2(\Omega)$. We denote the $L^2(\Omega)$ inner product by $(\cdot,\cdot)$ and the corresponding norm by $\|\cdot\|_2$.

Let A be the clamped biharmonic operator

$$A\mathrm{u} = \Delta^2 \mathrm{u},$$

with the domain

$$D(A) = \{\mathrm{u} \in H_0^2(\Omega) : \Delta^2 \mathrm{u} \in L^2(\Omega)\}.$$

If $\partial\Omega$ is sufficiently smooth, then

$$D(A) = H^4(\Omega) \cap H_0^2(\Omega).$$

The operator A is positive and self-adjoint and has compact inverse in $L^2(\Omega)$. Hence there is an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ of $L^2(\Omega)$ consisting of eigenfunctions of A, namely

$$\mathrm{A}\, w_j = \lambda_j w_j, \qquad 0 < \lambda_1 \le \lambda_2 \le \cdots, \qquad \lambda_j \to \infty.$$

Moreover, we have

$$(\Delta w_i,\ \Delta w_j) = \lambda_i \delta_{ij}.$$

We introduce the logarithmic damping function

$$g(s) = s\ln(1+s^2), \qquad s \in \mathbb{R}.$$

It is clear that $g(0) = 0$ and

$$g'(s) = \ln(1+s^2) + \frac{2s^2}{1+s^2} \ge 0, \qquad s \in \mathbb{R}.$$

Consequently, we have

$$(g(a) - g(b))(a-b) \ge 0, \qquad a, b \in \mathbb{R}. \tag{2.1}$$

In addition,

$$g(s)s = s^2\ln(1+s^2) \ge 0, \qquad s \in \mathbb{R}. \tag{2.2}$$

We also use the elementary estimate

$$|g(s)| \le |s|^2 + |s|^2\ln(1+s^2), \qquad s \in \mathbb{R}. \tag{2.3}$$

Indeed, if $|s| \le 1$, then

$$|s|\ln(1+s^2) \le |s|^3 \le |s|^2,$$

whereas, if $|s| > 1$, then

$$|s|\ln(1+s^2) \le |s|^2\ln(1+s^2).$$

The exponent p is chosen in accordance with the Sobolev embedding associated with $H_0^2(\Omega)$. Set

$$2_2^* = \begin{cases} \dfrac{2N}{N-4}, & N > 4, \\ +\infty, & 1 \le N \le 4. \end{cases}$$

The critical exponent 2_2^* follows from the Sobolev embedding theorem for $H_0^2(\Omega)$. Indeed, when $N > 4$, the embedding

$$H_0^2(\Omega) \hookrightarrow L^q(\Omega)$$

is characterized by

$$\frac{1}{q} = \frac{1}{2} - \frac{2}{N},$$

which yields

$$q = \frac{2N}{N-4} = 2_2^*.$$

When $1 \le N \le 4$, the embedding holds in every finite Lebesgue space; therefore, we write $2_2^* = +\infty$. Furthermore, to guarantee that the polynomial source

$$F(\mathrm{u}) = |\mathrm{u}|^{p-2}\mathrm{u}$$

belongs to $L^2(\Omega)$, it is sufficient to require

$$\mathrm{u} \in L^{2(p-1)}(\Omega).$$

Hence, for $N > 4$, the condition

$$2(p-1) \le 2_2^*$$

is equivalent to

$$p \le 1 + \frac{N}{N-4},$$

we assume that

$$2 < p < \infty, \quad 1 \le N \le 4, \qquad 2 < p \le 1 + \frac{N}{N-4}, \quad N > 4. \tag{2.4}$$

Then

$$2(p-1) \le 2_2^* \quad \text{if } N > 4,$$

and hence

$$H_0^2(\Omega) \hookrightarrow L^{2(p-1)}(\Omega).$$

This embedding ensures that the polynomial source

$$F(\mathrm{u}) = |\mathrm{u}|^{p-2}\mathrm{u}$$

is well defined as an $L^2(\Omega)$-valued mapping whenever $\mathrm{u} \in H_0^2(\Omega)$.

In what follows, we state and prove that the existence of a unique local solution can be established by adapting the arguments in [15].

We first study the following auxiliary clamped plate problem:

$$\begin{cases} \mathrm{u}_{tt} + \Delta^2 \mathrm{u} + g(\mathrm{u}_t) = f, & \text{in } \Omega \times (0, T), \\ \mathrm{u} = \dfrac{\partial \mathrm{u}}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, T), \\ \mathrm{u}(x, 0) = \mathrm{u}_0(x), \quad \mathrm{u}_t(x, 0) = \mathrm{u}_1(x), & \text{in } \Omega, \end{cases} \tag{2.5}$$

where f is a given function.

Theorem 2.1 (Auxiliary problem)**.** *Let*

$$(\mathrm{u}_0, \mathrm{u}_1) \in H_0^2(\Omega) \times L^2(\Omega), \qquad f \in L^2(0, T; L^2(\Omega)).$$

Then Problem (2.5) *admits a unique weak solution satisfying*

$$\mathrm{u} \in L^\infty(0, T; H_0^2(\Omega)), \qquad \mathrm{u}_t \in L^\infty(0, T; L^2(\Omega)),$$

and

$$\mathrm{u}_{tt} \in L^1(0, T; (H_0^2(\Omega) \cap L^\infty(\Omega))').$$

Moreover, we have

$$g(\mathrm{u}_t) \in L^1(\Omega \times (0, T)),$$

and, for every $\varphi \in H_0^2(\Omega) \cap L^\infty(\Omega)$,

$$\langle \mathrm{u}_{tt}(t), \varphi \rangle + \int_\Omega \Delta \mathrm{u}(t) \Delta \varphi \, dx + \int_\Omega g(\mathrm{u}_t(t)) \varphi \, dx = \int_\Omega f(t) \varphi \, dx \tag{2.6}$$

for a.e. $t \in (0, T)$.

Proof. We divide the proof into several steps.

Step 1: Galerkin approximation. For $m \in \mathbb{N}$, define

$$V_m = \operatorname{span}\{w_1, \ldots, w_m\}.$$

We look for an approximate solution of the form

$$\mathrm{u}_m(t) = \sum_{j=1}^m d_{jm}(t) w_j.$$

The coefficients d_{jm} are chosen such that, for every $\psi \in V_m$, we have

$$(\mathrm{u}_m''(t), \psi) + (\Delta \mathrm{u}_m(t), \Delta \psi) + (g(\mathrm{u}_m'(t)), \psi) = (f(t), \psi). \tag{2.7}$$

The initial conditions are

$$\mathrm{u}_m(0) = \mathrm{u}_{0m}, \qquad \mathrm{u}_m'(0) = \mathrm{u}_{1m},$$

where

$$\mathrm{u}_{0m} \to \mathrm{u}_0 \quad \text{in } H_0^2(\Omega), \qquad \mathrm{u}_{1m} \to \mathrm{u}_1 \quad \text{in } L^2(\Omega).$$

Since (2.7) is a finite-dimensional system of ordinary differential equations with continuous nonlinear terms, it admits a local solution on some interval $[0, T_m)$.

Step 2: First a priori estimate. Taking $\psi = \mathrm{u}_m'(t)$ in (2.7), we get

$$(\mathrm{u}_m'', \mathrm{u}_m') + (\Delta \mathrm{u}_m, \Delta \mathrm{u}_m') + (g(\mathrm{u}_m'), \mathrm{u}_m') = (f, \mathrm{u}_m').$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathrm{u}_m'(t)\|_2^2 + \|\Delta \mathrm{u}_m(t)\|_2^2\right) + \int_\Omega |\mathrm{u}_m'(t)|^2 \ln\left(1 + |\mathrm{u}_m'(t)|^2\right) dx = \int_\Omega f(t)\mathrm{u}_m'(t)\, dx. \tag{2.8}$$

By Young's inequality

$$\int_\Omega f(t)\mathrm{u}_m'(t)\, dx \le \frac{1}{2}\|f(t)\|_2^2 + \frac{1}{2}\|\mathrm{u}_m'(t)\|_2^2.$$

Thus

$$\frac{d}{dt}(\|\mathrm{u}_m'(t)\|_2^2 + \|\Delta \mathrm{u}_m(t)\|_2^2) \le \|f(t)\|_2^2 + \|\mathrm{u}_m'(t)\|_2^2.$$

By Gronwall's inequality and the convergence of the approximate initial data, there is a constant $C_T > 0$, independent of m, such that

$$\sup_{0 \le t \le T_m} (\|\mathrm{u}_m'(t)\|_2^2 + \|\Delta \mathrm{u}_m(t)\|_2^2) \le C_T. \tag{2.9}$$

Integrating (2.8) over $(0, t)$, using (2.9), we also obtain

$$\int_0^{T_m} \int_\Omega |\mathrm{u}_m'(s)|^2 \ln(1 + |\mathrm{u}_m'(s)|^2)\, dx\, ds \le C_T. \tag{2.10}$$

Consequently, the approximate solution can be extended to the whole interval $[0, T]$ and

$$\{\mathrm{u}_m\} \text{ is bounded in } L^\infty(0, T; H_0^2(\Omega)), \tag{2.11}$$

$$\{\mathrm{u}_m'\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{2.12}$$

Step 3: Bound for the logarithmic damping. Using (2.3), (2.10) and (2.12), we have

$$\begin{aligned}\int_0^T \int_\Omega |g(\mathrm{u}_m')|\, dx\, dt &\le \int_0^T \int_\Omega |\mathrm{u}_m'|^2\, dx\, dt + \int_0^T \int_\Omega |\mathrm{u}_m'|^2 \ln(1 + |\mathrm{u}_m'|^2)\, dx\, dt \\ &\le C_T.\end{aligned}$$

Hence

$$\{g(\mathrm{u}_m')\} \text{ is bounded in } L^1(\Omega \times (0, T)). \tag{2.13}$$

We now strengthen this bound to weak compactness in $L^1(\Omega \times (0, T))$. Since

$$g(s) = s \ln(1 + s^2),$$

one has

$$\frac{|g(s)|}{s^2\ln(1+s^2)} = \frac{1}{|s|} \longrightarrow 0 \qquad \text{as } |s| \to \infty.$$

Consequently, for every $\varepsilon > 0$, the constants $R_\varepsilon > 0$ and $C_\varepsilon > 0$ exist such that

$$|g(s)| \leq \varepsilon s^2 \ln(1+s^2) + C_\varepsilon, \qquad s \in \mathbb{R}.$$

Together with (2.10), this estimate implies, by the de la Vallée–Poussin criterion, that the sequence $\{g(\mathrm{u}_m')\}$ is uniformly integrable in $\Omega \times (0,T)$. Therefore, by the Dunford–Pettis theorem, $\{g(\mathrm{u}_m')\}$ is relatively weakly compact in $L^1(\Omega \times (0,T))$.

Step 4: Estimate for the second derivative. From (2.7), we have

$$\mathrm{u}_m'' = -A\mathrm{u}_m - g(\mathrm{u}_m') + P_m f$$

in V_m', where P_m denotes the L^2 projection onto V_m. Let $\varphi \in H_0^2(\Omega) \cap L^\infty(\Omega)$. Then

$$|\langle A\mathrm{u}_m, \varphi\rangle| = \left| \int_\Omega \Delta \mathrm{u}_m \Delta\varphi \, dx \right| \leq \|\Delta \mathrm{u}_m\|_2 \|\Delta\varphi\|_2.$$

Furthermore

$$\left| \int_\Omega g(\mathrm{u}_m')\, \varphi \, dx \right| \leq \|\varphi\|_\infty \int_\Omega |g(\mathrm{u}_m')| \, dx,$$

and

$$\left| \int_\Omega f \, \varphi \, dx \right| \leq \|f\|_2 \|\varphi\|_2.$$

Combining these estimates with (2.11) and (2.13), we obtain

$$\{\mathrm{u}_m''\} \text{ is bounded in } L^1(0,T; (H_0^2(\Omega) \cap L^\infty(\Omega))'). \tag{2.14}$$

Step 5: Compactness and identification of the logarithmic term. From (2.11) and (2.12), there is a subsequence, still denoted by $\{\mathrm{u}_m\}$ and a function u such that

$$\mathrm{u}_m \rightharpoonup^* \mathrm{u} \quad \text{in } L^\infty(0,T; H_0^2(\Omega)), \tag{2.15}$$

and

$$\mathrm{u}_m' \rightharpoonup^* \mathrm{u}_t \quad \text{in } L^\infty(0,T; L^2(\Omega)). \tag{2.16}$$

Moreover, by the Aubin–Lions–Simon compactness theorem [18, 19], together with (2.14), we may extract a subsequence such that

$$\mathrm{u}_m \to \mathrm{u} \quad \text{strongly in } C([0,T]; L^2(\Omega)). \tag{2.17}$$

By the uniform integrability established in Step 3 and the Dunford–Pettis theorem, there is a subsequence, still denoted by $\{\mathrm{u}_m\}$ and a function

$$\chi \in L^1(\Omega \times (0,T))$$

such that

$$g(\mathrm{u}_m') \rightharpoonup \chi \quad \text{weakly in } L^1(\Omega \times (0,T)). \tag{2.18}$$

We now identify the weak limit. Let $Q_T = \Omega \times (0,T)$. Since g is continuous and monotone, for every $\zeta \in L^\infty(Q_T)$, we have

$$\int_{Q_T} (g(\mathrm{u}_m') - g(\zeta))(\mathrm{u}_m' - \zeta)\, dx\, dt \ge 0.$$

Using

$$\mathrm{u}_m' \rightharpoonup \mathrm{u}_t \quad \text{weakly in } L^2(Q_T), \qquad g(\mathrm{u}_m') \rightharpoonup \chi \quad \text{weakly in } L^1(Q_T),$$

together with the usual limsup inequality obtained from the Galerkin energy identity, the standard Minty monotonicity argument yields

$$\chi = g(\mathrm{u}_t) \quad \text{a.e. in } Q_T.$$

Consequently,

$$g(\mathrm{u}_m') \rightharpoonup g(\mathrm{u}_t) \quad \text{weakly in } L^1(Q_T).$$

Step 6: Passage to the limit. Let w_j be one of the eigenfunctions introduced above. For every $m \ge j$, we may take $\psi = w_j$ in (2.7). Integrating over $(0,t)$, we obtain

$$\begin{aligned}(\mathrm{u}_m'(t), w_j) - (\mathrm{u}_{1m}, w_j) + \int_0^t \int_\Omega \Delta \mathrm{u}_m(s) \Delta w_j\, dx\, ds \\ + \int_0^t \int_\Omega g(\mathrm{u}_m'(s)) w_j\, dx\, ds \\ = \int_0^t \int_\Omega f(s) w_j\, dx\, ds.\end{aligned}$$

Using (2.15), (2.16), and the convergence of the initial data, we pass to the limit and get

$$\begin{aligned}(\mathrm{u}_t(t), w_j) - (\mathrm{u}_1, w_j) + \int_0^t \int_\Omega \Delta \mathrm{u}(s) \Delta w_j\, dx\, ds \\ + \int_0^t \int_\Omega g(\mathrm{u}_t(s)) w_j\, dx\, ds \\ = \int_0^t \int_\Omega f(s) w_j\, dx\, ds.\end{aligned}$$

By linearity, the same identity holds for every finite linear combination of the basis functions. Since such functions are dense in $H_0^2(\Omega)$ and since $g(\mathrm{u}_t) \in L^1(\Omega \times (0,T))$, the identity extends to every $\varphi \in H_0^2(\Omega) \cap L^\infty(\Omega)$ as follows:

$$\begin{aligned}(\mathrm{u}_t(t), \varphi) - (\mathrm{u}_1, \varphi) + \int_0^t \int_\Omega \Delta \mathrm{u}(s) \Delta \varphi\, dx\, ds \\ + \int_0^t \int_\Omega g(\mathrm{u}_t(s)) \varphi\, dx\, ds \\ = \int_0^t \int_\Omega f(s) \varphi\, dx\, ds.\end{aligned}$$

Differentiating this identity with respect to t, we obtain

$$\langle \mathrm{u}_{tt}(t), \varphi \rangle + \int_\Omega \Delta \mathrm{u}(t)\, \Delta\varphi\, dx + \int_\Omega g(\mathrm{u}_t(t))\, \varphi\, dx = \int_\Omega f(t)\, \varphi\, dx$$

for almost every $t \in (0, T)$. This is exactly (2.6). The initial condition $\mathrm{u}(0) = \mathrm{u}_0$ follows from (2.17), while $\mathrm{u}_t(0) = \mathrm{u}_1$ follows in the weak $L^2(\Omega)$ sense from (2.16) and the convergence of u_{1m}.

Step 7: Uniqueness. Let u and v be two weak solutions of (2.5) with the same initial data. Set

$$z = \mathrm{u} - v.$$

The following calculation is first justified by a standard time-regularization argument or, equivalently, at the Galerkin level and then passed to the limit. Then z satisfies

$$z_{tt} + \Delta^2 z + g(\mathrm{u}_t) - g(v_t) = 0$$

in the weak sense, with

$$z(0) = 0, \qquad z_t(0) = 0.$$

Testing this equation by z_t, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|z_t(t)\|_2^2 + \|\Delta z(t)\|_2^2) + \int_\Omega (g(\mathrm{u}_t) - g(v_t))(\mathrm{u}_t - v_t)\, dx = 0.$$

By the monotonicity property (2.1), we have

$$\int_\Omega (g(\mathrm{u}_t) - g(v_t))(\mathrm{u}_t - v_t)\, dx \geq 0.$$

Therefore

$$\frac{d}{dt}(\|z_t(t)\|_2^2 + \|\Delta z(t)\|_2^2) \leq 0.$$

Since the initial value of this quantity is zero, we conclude that

$$z_t(t) = 0, \qquad \Delta z(t) = 0, \qquad 0 \leq t \leq T.$$

Using the clamped boundary conditions, we obtain $z = 0$. Hence $\mathrm{u} = v$ and the solution of the auxiliary problem is unique. □

We now prove a regularity and local Lipschitz property for the polynomial source. This step is essential in order to apply the auxiliary result of Theorem 2.1 to the nonlinear Problem (1.1).

Lemma 2.2. *Assume that* (2.4) *holds. Then the mapping*

$$F(\mathrm{u}) = |\mathrm{u}|^{p-2}\mathrm{u}$$

is locally Lipschitz from $H_0^2(\Omega)$ into $L^2(\Omega)$. More precisely, for every $R > 0$, a constant $C_R > 0$ exists such that

$$\|F(\mathrm{u}) - F(v)\|_2 \leq C_R \|\Delta(\mathrm{u} - v)\|_2 \tag{2.19}$$

for all $u, v \in H_0^2(\Omega)$ *satisfying*

$$\|\Delta u\|_2 + \|\Delta v\|_2 \le R.$$

In particular, if

$$u \in L^\infty(0,T; H_0^2(\Omega)),$$

then

$$|u|^{p-2}u \in L^\infty(0,T; L^2(\Omega)).$$

Proof. For all $a, b \in \mathbb{R}$, the mean value theorem gives

$$\left||a|^{p-2}a - |b|^{p-2}b\right| \le C_p(|a|^{p-2} + |b|^{p-2})|a-b|.$$

Therefore, we have

$$\begin{aligned}\|F(u) - F(v)\|_2 &\le C_p \left\|(|u|^{p-2} + |v|^{p-2})|u-v|\right\|_2 \\ &\le C_p(\| \, |u|^{p-2}|u-v| \, \|_2 + \| \, |v|^{p-2}|u-v| \, \|_2).\end{aligned}$$

Using Hölder's inequality with the exponent $2(p-1)$, we obtain

$$\| \, |u|^{p-2}|u-v| \, \|_2 \le \|u\|_{2(p-1)}^{p-2}\|u-v\|_{2(p-1)}.$$

Similarly

$$\| \, |v|^{p-2}|u-v| \, \|_2 \le \|v\|_{2(p-1)}^{p-2}\|u-v\|_{2(p-1)}.$$

By the embedding

$$H_0^2(\Omega) \hookrightarrow L^{2(p-1)}(\Omega),$$

$C > 0$ exists such that

$$\|z\|_{2(p-1)} \le C\|\Delta z\|_2, \qquad z \in H_0^2(\Omega).$$

Hence

$$\|F(u) - F(v)\|_2 \le C\left(\|\Delta u\|_2^{p-2} + \|\Delta v\|_2^{p-2}\right)\|\Delta(u-v)\|_2.$$

If

$$\|\Delta u\|_2 + \|\Delta v\|_2 \le R,$$

then (2.19) follows.

Taking $v = 0$, we also get

$$\|F(u)\|_2 \le C\|\Delta u\|_2^{p-1}.$$

Therefore, if $u \in L^\infty(0,T; H_0^2(\Omega))$, then

$$\|F(u(t))\|_2 \le C\|\Delta u(t)\|_2^{p-1} \quad \text{for a.e. } t \in (0,T),$$

which proves

$$F(u) \in L^\infty(0,T; L^2(\Omega)).$$

The proof is complete. □

We now return to the original nonlinear clamped plate problem

$$u_{tt} + \Delta^2 u + u_t \ln(1 + u_t^2) = |u|^{p-2}u.$$

The local solution is constructed as a fixed point of the solution operator associated with the auxiliary problem.

Theorem 2.3 (Local existence)**.** *Assume that* (2.4) *holds and let*

$$(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega).$$

Then there exists $T > 0$ such that Problem (1.1) *has a weak solution satisfying*

$$u \in C([0, T]; H_0^2(\Omega)), \qquad u_t \in C([0, T]; L^2(\Omega)),$$

$$g(u_t) \in L^1(\Omega \times (0, T)).$$

and

$$u_{tt} \in L^1(0, T; (H_0^2(\Omega) \cap L^\infty(\Omega))').$$

Moreover, for every

$$\varphi \in H_0^2(\Omega) \cap L^\infty(\Omega),$$

the identity

$$\langle u_{tt}(t), \varphi\rangle + \int_\Omega \Delta u(t)\Delta\varphi\, dx + \int_\Omega u_t(t) \ln(1 + u_t^2(t))\varphi\, dx = \int_\Omega |u(t)|^{p-2}u(t)\varphi\, dx \tag{2.20}$$

holds for a.e. $t \in (0, T)$.

Proof. Let

$$Y_0 = \|u_1\|_2^2 + \|\Delta u_0\|_2^2.$$

Choose $R > 0$ such that

$$R > 4Y_0 + 1.$$

For $T > 0$, to be chosen later, define

$$\mathcal{X}_{T,R} = \left\{ w : \begin{array}{l} w \in C([0, T]; H_0^2(\Omega)), \quad w_t \in C([0, T]; L^2(\Omega)), \\ w(0) = u_0, \quad w_t(0) = u_1, \\ \sup\limits_{0\le t\le T} (\|w_t(t)\|_2^2 + \|\Delta w(t)\|_2^2) \le R \end{array} \right\}.$$

We endow $\mathcal{X}_{T,R}$ with the metric

$$d(w, z)^2 = \sup_{0\le t\le T} (\|w_t(t) - z_t(t)\|_2^2 + \|\Delta(w(t) - z(t))\|_2^2).$$

Since $C([0, T]; H_0^2(\Omega)) \times C([0, T]; L^2(\Omega))$ is complete and $\mathcal{X}_{T,R}$ is a closed subset with respect to the above metric, $(\mathcal{X}_{T,R}, d)$ is a complete metric space.

For each $w \in \mathcal{X}_{T,R}$, Lemma 2.2 implies that

$$|w|^{p-2}w \in L^\infty(0, T; L^2(\Omega)) \subset L^2(0, T; L^2(\Omega)).$$

Therefore, by Theorem 2.1, the auxiliary problem

$$\begin{cases} u_{tt} + \Delta^2 u + g(u_t) = |w|^{p-2}w, & \text{in } \Omega \times (0,T), \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2.21}$$

has a unique weak solution u. We define

$$\Phi(w) = u.$$

We first show that Φ maps $\mathcal{X}_{T,R}$ into itself for a $T > 0$ small enough. Taking u_t as a test function in (2.21), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) + \int_\Omega u_t^2(t)\ln\left(1 + u_t^2(t)\right)dx = \int_\Omega |w(t)|^{p-2}w(t)u_t(t)\,dx.$$

Since the damping contribution is non-negative, we have

$$\frac{1}{2}\frac{d}{dt}(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2) \le \||w(t)|^{p-2}w(t)\|_2\,\|u_t(t)\|_2.$$

Using Lemma 2.2 and the definition of $\mathcal{X}_{T,R}$, we get

$$\left\||w(t)|^{p-2}w(t)\right\|_2 \le C\|\Delta w(t)\|_2^{p-1} \le CR^{(p-1)/2}.$$

Therefore

$$\frac{d}{dt}(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2) \le CR^{p-1} + \|u_t(t)\|_2^2.$$

It follows that

$$\frac{d}{dt}(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2) \le CR^{p-1} + \|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2.$$

By Gronwall's inequality, we have

$$\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \le e^t\left[Y_0 + C\,R^{p-1}t\right]$$

for $0 \le t \le T$. Hence, choosing a $T > 0$ sufficiently small so that

$$e^T\left(Y_0 + CR^{p-1}T\right) \le R, \tag{2.22}$$

we obtain

$$\sup_{0\le t\le T}(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2) \le R.$$

Thus, we have

$$\Phi(\mathcal{X}_{T,R}) \subset \mathcal{X}_{T,R}.$$

We next prove that Φ is a contraction for a possibly smaller T. Let $w, z \in \mathcal{X}_{T,R}$ and set

$$u = \Phi(w), \qquad v = \Phi(z), \qquad y = u - v.$$

Then y satisfies

$$y_{tt} + \Delta^2 y + (g(\mathrm{u}_t) - g(v_t)) = |w|^{p-2}w - |z|^{p-2}z \tag{2.23}$$

with

$$y(0) = 0, \qquad y_t(0) = 0.$$

Testing (2.23) by y_t, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|y_t(t)\|_2^2 + \|\Delta y(t)\|_2^2\right) + \int_\Omega (g(\mathrm{u}_t) - g(v_t))(\mathrm{u}_t - v_t)\,dx = \int_\Omega \left(|w|^{p-2}w - |z|^{p-2}z\right) y_t\,dx.$$

By the monotonicity of g, the second term on the left-hand side is non-negative. Hence

$$\frac{1}{2}\frac{d}{dt}(\|y_t(t)\|_2^2 + \|\Delta y(t)\|_2^2) \leq \|F(w(t)) - F(z(t))\|_2\,\|y_t(t)\|_2.$$

Using Lemma 2.2, we have

$$\|F(w(t)) - F(z(t))\|_2 \leq C_R\|\Delta(w(t) - z(t))\|_2.$$

Thus

$$\frac{d}{dt}(\|y_t(t)\|_2^2 + \|\Delta y(t)\|_2^2) \leq C_R\|\Delta(w(t) - z(t))\|_2^2 + \|y_t(t)\|_2^2.$$

Since

$$\|y_t(t)\|_2^2 \leq \|y_t(t)\|_2^2 + \|\Delta y(t)\|_2^2,$$

Gronwall's inequality yields

$$\sup_{0\leq t\leq T}(\|y_t(t)\|_2^2 + \|\Delta y(t)\|_2^2) \leq C_R\,T\,e^T \sup_{0\leq t\leq T}\|\Delta(w(t) - z(t))\|_2^2.$$

Consequently, we have

$$d(\Phi(w), \Phi(z))^2 \leq C_R T e^T d(w,z)^2.$$

Choosing $T > 0$ to be smaller, if necessary, so that

$$C_R T e^T < 1, \tag{2.24}$$

we conclude that Φ is a contraction on $\mathcal{X}_{T,R}$.

By Banach's fixed-point theorem, there is a unique

$$\mathrm{u} \in \mathcal{X}_{T,R}$$

such that

$$\Phi(\mathrm{u}) = \mathrm{u}.$$

This fixed point is precisely a weak solution of Problem (1.1) on $[0, T]$. The proof of local existence is complete. □

The next theorem gives uniqueness and continuous dependence in the natural energy space.

Theorem 2.4 (Local well-posedness)**.** *Assume that* (2.4) *holds. For every*

$$(\mathrm{u}_0, \mathrm{u}_1) \in H_0^2(\Omega) \times L^2(\Omega),$$

$T > 0$ *exists such that Problem* (1.1) *admits a unique weak solution*

$$\mathrm{u} \in L^\infty(0, T; H_0^2(\Omega)), \qquad \mathrm{u}_t \in L^\infty(0, T; L^2(\Omega)).$$

Moreover

$$\mathrm{u} \in C([0, T]; H_0^2(\Omega)), \qquad \mathrm{u}_t \in C([0, T]; L^2(\Omega)),$$

and the solution depends continuously on the initial data. More precisely, if u *and* v *are two solutions corresponding to the initial data*

$$(\mathrm{u}_0, \mathrm{u}_1), \qquad (v_0, v_1),$$

and if both solutions remain bounded in $L^\infty(0, T; H_0^2(\Omega))$*, then*

$$\|\mathrm{u}_t(t) - v_t(t)\|_2^2 + \|\Delta(\mathrm{u}(t) - v(t))\|_2^2 \le C_T \left(\|\mathrm{u}_1 - v_1\|_2^2 + \|\Delta(\mathrm{u}_0 - v_0)\|_2^2\right) \tag{2.25}$$

for every $t \in [0, T]$.

Proof. Existence follows from Theorem 2.3. We prove the uniqueness and continuous dependence.

Let u and v be two weak solutions of (1.1). Set

$$z = \mathrm{u} - v.$$

Then z satisfies

$$z_{tt} + \Delta^2 z + \left(g(\mathrm{u}_t) - g(v_t)\right) = |\mathrm{u}|^{p-2}\mathrm{u} - |v|^{p-2}v.$$

Testing this equation with z_t, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|z_t(t)\|_2^2 + \|\Delta z(t)\|_2^2\right) + \int_\Omega (g(\mathrm{u}_t) - g(v_t))(\mathrm{u}_t - v_t)\, dx = \int_\Omega \left(|\mathrm{u}|^{p-2}\mathrm{u} - |v|^{p-2}v\right) z_t\, dx.$$

The logarithmic damping term is non-negative by (2.1). Suppose that

$$\|\Delta \mathrm{u}(t)\|_2 + \|\Delta v(t)\|_2 \le R, \qquad 0 \le t \le T.$$

Then Lemma 2.2 gives

$$\left\| |\mathrm{u}|^{p-2}\mathrm{u} - |v|^{p-2}v \right\|_2 \le C_R \|\Delta(\mathrm{u} - v)\|_2.$$

Therefore

$$\frac{d}{dt}(\|z_t(t)\|_2^2 + \|\Delta z(t)\|_2^2) \le C_R(\|\Delta z(t)\|_2^2 + \|z_t(t)\|_2^2).$$

By Gronwall's inequality, we have

$$\begin{aligned}\|z_t(t)\|_2^2 + \|\Delta z(t)\|_2^2 &\le e^{C_R t}(\|z_t(0)\|_2^2 + \|\Delta z(0)\|_2^2)\\ &= e^{C_R t}(\|\mathrm{u}_1 - v_1\|_2^2 + \|\Delta(\mathrm{u}_0 - v_0)\|_2^2).\end{aligned}$$

This proves (2.25). In particular, if the initial data coincide, then $z = 0$ and hence $\mathrm{u} = v$. The uniqueness follows.

It remains to justify the stated strong continuity in the energy space. From the uniform bounds and the equation, one first obtains

$$\mathrm{u} \in C_w([0,T]; H_0^2(\Omega)), \qquad \mathrm{u}_t \in C_w([0,T]; L^2(\Omega)).$$

Moreover, the energy identity obtained by the Galerkin approximation passes to the limit and gives the continuity of the energy functional. Since

$$\frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + \frac{1}{2}\|\Delta \mathrm{u}(t)\|_2^2 = E(t) + \frac{1}{p}\|\mathrm{u}(t)\|_p^p,$$

and since $\mathrm{u} \in C([0,T]; L^p(\Omega))$ follows from the compactness argument and the Sobolev embedding, the norm

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2$$

is continuous in t. Weak continuity together with continuity of the norm yields

$$\mathrm{u} \in C([0,T]; H_0^2(\Omega)), \qquad \mathrm{u}_t \in C([0,T]; L^2(\Omega)).$$

This completes the proof of local well-posedness. □

The local well-posedness result can be extended to a maximal interval of existence.

Theorem 2.5 (Continuation alternative)**.** *Assume that* (2.4) *holds and let*

$$(\mathrm{u}_0, \mathrm{u}_1) \in H_0^2(\Omega) \times L^2(\Omega).$$

Then there is a maximal time

$$T_{\max} \in (0, \infty]$$

such that Problem (1.1) *has a unique weak solution on every interval* $[0,T] \subset [0, T_{\max})$*. Moreover, if* $T_{\max} < \infty$*, then*

$$\limsup_{t \nearrow T_{\max}} \left(\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2 \right) = +\infty. \tag{2.26}$$

Proof. The local construction in Theorem 2.3 gives a solution on some interval $[0,T]$, where the length of T depends only on the size of the initial energy norm

$$\|\mathrm{u}_1\|_2^2 + \|\Delta \mathrm{u}_0\|_2^2.$$

Starting from the terminal data

$$\mathrm{u}(T), \qquad \mathrm{u}_t(T),$$

and repeating the same argument, the solution can be extended as long as

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2$$

remains finite. Hence, the union of all such intervals defines a maximal interval $[0, T_{\max})$.

Assume now that $T_{\max} < \infty$ and that (2.26) is false. Then $M > 0$ exists such that

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2 \le M, \qquad 0 \le t < T_{\max}.$$

Choose $t_0 < T_{\max}$. The local existence time obtained from Theorem 2.3, when applied at $t = t_0$, depends only on M, not on t_0. Therefore, for any t_0 sufficiently close to $T_{\max}$, the solution can be extended beyond $T_{\max}$. This contradicts the definition of $T_{\max}$. Hence, (2.26) must hold. □

3. Global existence

In this section, we prove that the local solution obtained in Theorem 2.4 extends globally in time for initial data lying in a suitable stable potential well. The proof relies on the energy identity, the Nehari functional, and the coercivity of the potential energy inside the stable set.

Throughout this section, we assume that the exponent p satisfies (2.4). In particular, the Sobolev embedding

$$H_0^2(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous. Hence a constant $B_p > 0$ exists such that

$$\|v\|_p \leq B_p \|\Delta v\|_2, \qquad \forall\, v \in H_0^2(\Omega). \tag{3.1}$$

We define the energy functional by

$$E(t) = \frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + \frac{1}{2}\|\Delta \mathrm{u}(t)\|_2^2 - \frac{1}{p}\|\mathrm{u}(t)\|_p^p. \tag{3.2}$$

We also introduce the potential functional

$$J(\mathrm{u}) = \frac{1}{2}\|\Delta \mathrm{u}\|_2^2 - \frac{1}{p}\|\mathrm{u}\|_p^p \tag{3.3}$$

and the Nehari functional

$$I(\mathrm{u}) = \|\Delta \mathrm{u}\|_2^2 - \|\mathrm{u}\|_p^p. \tag{3.4}$$

Thus

$$E(t) = \frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + J(\mathrm{u}(t)).$$

Lemma 3.1 (Energy identity)**.** *Let* u *be the local solution of Problem* (1.1) *on* $[0, T]$*. Then, for every* $0 \leq s \leq t \leq T$*, we have*

$$E(t) + \int_s^t \int_\Omega \mathrm{u}_t^2(\tau) \ln(1 + \mathrm{u}_t^2(\tau))\, dx\, d\tau = E(s). \tag{3.5}$$

In particular

$$E(t) \leq E(s) \leq E(0), \qquad 0 \leq s \leq t \leq T. \tag{3.6}$$

Equivalently, for a.e. $t \in (0, T)$*,*

$$E'(t) = -\int_\Omega \mathrm{u}_t^2(t) \ln(1 + \mathrm{u}_t^2(t))\, dx \leq 0. \tag{3.7}$$

Proof. Multiplying the equation in (1.1) by u_t and integrating over Ω, we obtain

$$\int_\Omega \mathrm{u}_{tt}\mathrm{u}_t\, dx + \int_\Omega \Delta^2 \mathrm{u}\, \mathrm{u}_t\, dx + \int_\Omega \mathrm{u}_t^2 \ln(1 + \mathrm{u}_t^2)\, dx = \int_\Omega |\mathrm{u}|^{p-2}\mathrm{u}\, \mathrm{u}_t\, dx.$$

The first term satisfies

$$\int_\Omega \mathrm{u}_{tt}\mathrm{u}_t\, dx = \frac{1}{2}\frac{d}{dt}\|\mathrm{u}_t(t)\|_2^2.$$

Using the clamped boundary conditions and Green's formula for the biharmonic operator, we have

$$\int_\Omega \Delta^2 \mathrm{u}\, \mathrm{u}_t\, dx = \int_\Omega \Delta \mathrm{u}\, \Delta \mathrm{u}_t\, dx = \frac{1}{2}\frac{d}{dt}\|\Delta \mathrm{u}(t)\|_2^2.$$

Moreover

$$\int_\Omega |\mathrm{u}|^{p-2}\mathrm{u}\, \mathrm{u}_t\, dx = \frac{1}{p}\frac{d}{dt}\, \|\mathrm{u}(t)\|_p^p\,.$$

Combining these identities gives

$$\frac{d}{dt}[\frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + \frac{1}{2}\|\Delta \mathrm{u}(t)\|_2^2 - \frac{1}{p}\|\mathrm{u}(t)\|_p^p] + \int_\Omega \mathrm{u}_t^2(t)\ln(1 + \mathrm{u}_t^2(t))\, dx = 0.$$

This proves (3.7). Integrating (3.7) over (s, t) yields (3.5). Since

$$\mathrm{u}_t^2 \ln(1 + \mathrm{u}_t^2) \geq 0,$$

the monotonicity estimate (3.6) follows.

For weak solutions, the identity is first derived at the Galerkin level. Passing to the limit is justified by the compactness of the approximate solutions, the identification of the weak limit of the logarithmic damping term and the convergence of the source term in the corresponding duality pairing. Hence, the limiting solution satisfies the energy identity (3.5). □

We now introduce the potential well depth. Define

$$d = \inf_{v \in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda \geq 0} J(\lambda v). \tag{3.8}$$

The Sobolev inequality (3.1) motivates the positive level

$$d_0 := \frac{p-2}{2p} B_p^{-\frac{2p}{p-2}} > 0. \tag{3.9}$$

The following lemma shows that d_0 is an admissible lower bound for the potential well depth.

Lemma 3.2 (Potential well estimate)**.** *Let $v \in H_0^2(\Omega) \setminus \{0\}$. If*

$$I(v) = 0,$$

then

$$J(v) \geq d_0. \tag{3.10}$$

Consequently

$$d \geq d_0.$$

Proof. Since $I(v) = 0$, we have

$$\|\Delta v\|_2^2 = \|v\|_p^p.$$

By (3.1)

$$\|v\|_p^p \leq B_p^p \|\Delta v\|_2^p.$$

Hence

$$\|\Delta v\|_2^2 \le B_p^p \|\Delta v\|_2^p.$$

Since $v \neq 0$, it follows that

$$\|\Delta v\|_2 \ge B_p^{-\frac{p}{p-2}}.$$

Therefore

$$\begin{aligned} J(v) &= \frac{1}{2}\|\Delta v\|_2^2 - \frac{1}{p}\|v\|_p^p = \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta v\|_2^2 = \frac{p-2}{2p}\|\Delta v\|_2^2 \\ &\ge \frac{p-2}{2p} B_p^{-\frac{2p}{p-2}} = d_0. \end{aligned}$$

This proves (3.10).

To prove $d \ge d_0$, let $v \in H_0^2(\Omega) \setminus \{0\}$ and consider

$$\Phi(\lambda) = J(\lambda v), \qquad \lambda \ge 0.$$

The maximum of Φ is reached at a positive value λ_v satisfying

$$I(\lambda_v v) = 0.$$

Applying the first part of the proof to $\lambda_v v$, we obtain

$$\sup_{\lambda \ge 0} J(\lambda v) = J(\lambda_v v) \ge d_0.$$

Taking the infimum over $v \neq 0$ gives $d \ge d_0$. □

Define the stable set by

$$\mathcal{W} = \left\{ v \in H_0^2(\Omega) : I(v) > 0,\ J(v) < d_0 \right\} \cup \{0\}. \tag{3.11}$$

Lemma 3.3 (Invariance of the stable set)**.** *Let* u *be the local solution of Problem* (1.1) *on* $[0, T]$*. Assume that*

$$E(0) < d_0, \qquad I(\mathrm{u}_0) > 0. \tag{3.12}$$

Then

$$\mathrm{u}(t) \in \mathcal{W}, \qquad 0 \le t \le T. \tag{3.13}$$

In particular, whenever $\mathrm{u}(t) \neq 0$*, one has*

$$I(\mathrm{u}(t)) > 0.$$

Proof. Since

$$E(0) = \frac{1}{2}\|\mathrm{u}_1\|_2^2 + J(\mathrm{u}_0) < d_0,$$

we have $J(\mathrm{u}_0) < d_0$. Together with $I(\mathrm{u}_0) > 0$, this gives $\mathrm{u}_0 \in \mathcal{W}$.

We show that the solution cannot leave $\mathcal{W}$. Since

$$\mathrm{u} \in C([0, T]; H_0^2(\Omega)),$$

the mappings

$$t \mapsto I(\mathrm{u}(t)), \qquad t \mapsto J(\mathrm{u}(t))$$

are continuous. Suppose, by contradiction, that $\mathrm{u}(t)$ leaves W and let $t_0 \in (0, T]$ be the first exit time. Thus

$$\mathrm{u}(t) \in \mathcal{W}, \qquad 0 \le t < t_0.$$

By the energy monotonicity (3.6), we have

$$E(t) \le E(0) < d_0, \qquad 0 \le t \le T.$$

Since

$$J(\mathrm{u}(t)) \le E(t),$$

we have

$$J(\mathrm{u}(t)) < d_0, \qquad 0 \le t \le T.$$

Therefore, the solution cannot leave $\mathcal{W}$ through the condition $J(\mathrm{u}(t)) < d_0$. Hence, the first exit must occur through the Nehari boundary, namely

$$I(\mathrm{u}(t_0)) = 0.$$

If $\mathrm{u}(t_0) = 0$, then $\mathrm{u}(t_0) \in \mathcal{W}$ by definition, which contradicts the choice of t_0. Thus $\mathrm{u}(t_0) \neq 0$. Applying Lemma 3.2, we obtain

$$J(\mathrm{u}(t_0)) \ge d_0.$$

Consequently

$$E(t_0) = \frac{1}{2}\|\mathrm{u}_t(t_0)\|_2^2 + J(\mathrm{u}(t_0)) \ge d_0.$$

This contradicts

$$E(t_0) \le E(0) < d_0.$$

Therefore, no such exit time exists and

$$\mathrm{u}(t) \in \mathcal{W}, \qquad 0 \le t \le T.$$

The final assertion follows immediately from the definition of $\mathcal{W}$. □

Lemma 3.4 (Coercivity in the stable set)**.** *Assume that* $\mathrm{u}(t) \in \mathcal{W}$ *on* $[0, T]$. *Then, for every* $t \in [0, T]$, *we have*

$$J(\mathrm{u}(t)) \ge \frac{p-2}{2p}\|\Delta \mathrm{u}(t)\|_2^2. \tag{3.14}$$

Consequently

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2 \le C_p E(0), \qquad 0 \le t \le T, \tag{3.15}$$

where

$$C_p = 2 + \frac{2p}{p-2}. \tag{3.16}$$

Proof. If $u(t) = 0$, then (3.14) is immediate. If $u(t) \neq 0$, then $u(t) \in \mathcal{W}$ implies

$$I(u(t)) = \|\Delta u(t)\|_2^2 - \|u(t)\|_p^p > 0.$$

Therefore

$$\|u(t)\|_p^p < \|\Delta u(t)\|_2^2.$$

Using the identity

$$J(u) = \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{1}{p}I(u),$$

we obtain (3.14).

Moreover

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)).$$

Thus

$$\frac{1}{2}\|u_t(t)\|_2^2 \leq E(t) \leq E(0),$$

and, by (3.14), we have

$$\frac{p-2}{2p}\|\Delta u(t)\|_2^2 \leq J(u(t)) \leq E(t) \leq E(0).$$

Consequently

$$\|u_t(t)\|_2^2 \leq 2E(0), \qquad \|\Delta u(t)\|_2^2 \leq \frac{2p}{p-2}E(0). \tag{3.17}$$

Combining these two estimates yields (3.15). □

We now state the global existence theorem.

Theorem 3.5 (Global existence)**.** *Assume that* (2.4) *holds. Let*

$$(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$$

and suppose that

$$E(0) < d_0, \qquad I(u_0) > 0. \tag{3.18}$$

Then the local solution of Problem (1.1) *exists globally in time. More precisely,* $T_{\max} = +\infty$ *and*

$$u \in L^\infty(0, \infty; H_0^2(\Omega)), \qquad u_t \in L^\infty(0, \infty; L^2(\Omega)).$$

Moreover

$$\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq C_p E(0), \qquad t \geq 0, \tag{3.19}$$

and

$$\int_0^\infty \int_\Omega u_t^2(s) \ln\left(1 + u_t^2(s)\right) dx\, ds \leq E(0). \tag{3.20}$$

Proof. By Theorem 2.4, Problem (1.1) admits a unique local solution on a maximal interval $[0, T_{\max})$. Let $T < T_{\max}$. Since (3.18) holds, Lemma 3.3 implies that

$$\mathrm{u}(t) \in \mathcal{W}, \qquad 0 \le t \le T.$$

Hence, by Lemma 3.4, we have

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2 \le C_p E(0), \qquad 0 \le t \le T.$$

Thus the solution exists globally and satisfies

$$\mathrm{u} \in L^\infty(0, \infty; H_0^2(\Omega)), \qquad \mathrm{u}_t \in L^\infty(0, \infty; L^2(\Omega)).$$

Finally, taking $s = 0$ in (3.5), we obtain

$$\int_0^t \int_\Omega \mathrm{u}_t^2(\tau) \ln\left(1 + \mathrm{u}_t^2(\tau)\right) dx\, d\tau = E(0) - E(t).$$

Since $\mathrm{u}(t) \in \mathcal{W}$, Lemma 3.4 yields $J(\mathrm{u}(t)) \ge 0$ and hence

$$E(t) = \frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + J(\mathrm{u}(t)) \ge 0.$$

Therefore

$$\int_0^t \int_\Omega \mathrm{u}_t^2(\tau) \ln\left(1 + \mathrm{u}_t^2(\tau)\right) dx\, d\tau \le E(0).$$

Letting $t \to \infty$, we obtain (3.20). The proof is complete. □

Remark 3.6. *The assumptions $E(0) < d_0$ and $I(\mathrm{u}_0) > 0$ mean that the initial data lie below the potential well level and on the stable side of the Nehari manifold. This invariant structure prevents the polynomial source from dominating the elastic energy. A simple sufficient condition for $I(\mathrm{u}_0) > 0$ is*

$$\|\Delta \mathrm{u}_0\|_2 < B_p^{-\frac{p}{p-2}}, \qquad \mathrm{u}_0 \neq 0.$$

Indeed, by (3.1)*, we have*

$$\|\mathrm{u}_0\|_p^p \le B_p^p \|\Delta \mathrm{u}_0\|_2^p < \|\Delta \mathrm{u}_0\|_2^2.$$

4. Polynomial stability

In this section, we establish the polynomial decay of the global solution obtained in Theorem 3.5. The proof is based on a weighted multiplier argument combined with suitable estimates for the logarithmic damping term. The presence of the polynomial source requires an additional use of the stable potential well structure, which ensures that the total energy remains positive and equivalent to the natural plate energy.

Throughout this section, we assume that the hypotheses of Theorem 3.5 are satisfied, namely

$$E(0) < d_0, \qquad I(\mathrm{u}_0) > 0. \tag{4.1}$$

Hence, the corresponding solution is global and remains in the stable set as follows:

$$I(\mathrm{u}(t)) > 0, \qquad t \geq 0. \tag{4.2}$$

We also assume

$$H_0^2(\Omega) \hookrightarrow L^3(\Omega). \tag{4.3}$$

This embedding is automatically valid in the usual low-dimensional plate cases, particularly for

$$1 \leq N \leq 4.$$

More generally, it is enough to assume that (4.3) holds.

Recall that the energy associated with Problem (1.1) is

$$E(t) = \frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + \frac{1}{2}\|\Delta \mathrm{u}(t)\|_2^2 - \frac{1}{p}\|\mathrm{u}(t)\|_p^p. \tag{4.4}$$

Moreover, by the energy identity, we have

$$E'(t) = -\int_\Omega \mathrm{u}_t^2(t)\ln\left(1 + \mathrm{u}_t^2(t)\right)\, dx \leq 0. \tag{4.5}$$

For simplicity, we define the dissipation functional

$$D(t) := \int_\Omega \mathrm{u}_t^2(t)\ln\left(1 + \mathrm{u}_t^2(t)\right)\, dx. \tag{4.6}$$

Then

$$E'(t) = -D(t). \tag{4.7}$$

In particular, $E(t)$ is non increasing on $[0, \infty)$. The first step is to show that inside the stable set, the energy is equivalent to the natural plate energy. This property is essential because the source term appears in the energy with a negative sign.

Set

$$\rho_* = B_p^{-\frac{p}{p-2}}, \qquad d_0 = \frac{p-2}{2p}\rho_*^2. \tag{4.8}$$

If $E(0) = 0$, then the coercivity of the energy in the stable set implies

$$u_t = 0, \qquad \Delta u = 0,$$

and hence the solution is trivial. Therefore, in the following, we assume $E(0) > 0$. Since $0 < E(0) < d_0$, $\rho_0 \in (0, \rho_*)$ exists such that

$$E(0) = \frac{p-2}{2p}\rho_0^2. \tag{4.9}$$

Lemma 4.1. *There is a constant* $\sigma_0 \in (0, 1)$ *such that*

$$I(\mathrm{u}(t)) \geq \sigma_0\|\Delta \mathrm{u}(t)\|_2^2, \qquad \forall\, t \geq 0. \tag{4.10}$$

Consequently, the positive constants $c_1, c_2 > 0$ *exist such that*

$$c_1(\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2) \leq E(t) \leq c_2(\|\mathrm{u}_t(t)\|_2^2 + I(\mathrm{u}(t))) \tag{4.11}$$

for all $t \geq 0$.

Proof. Since $I(\mathrm{u}(t)) > 0$ for all $t \geq 0$, we have

$$\|\mathrm{u}(t)\|_p^p < \|\Delta \mathrm{u}(t)\|_2^2. \tag{4.12}$$

From (3.17) and (4.9), it follows that

$$\|\Delta \mathrm{u}(t)\|_2^2 \leq \frac{2p}{p-2} E(0) = \rho_0^2, \qquad t \geq 0, \tag{4.13}$$

which implies, with (3.1), that

$$\|\mathrm{u}(t)\|_p^p \leq B_p^p \rho_0^{p-2} \|\Delta \mathrm{u}(t)\|_2^2.$$

Since

$$B_p^p \rho_*^{p-2} = 1,$$

we have

$$B_p^p \rho_0^{p-2} = \left(\frac{\rho_0}{\rho_*}\right)^{p-2} < 1.$$

Therefore, we have

$$\begin{aligned} I(\mathrm{u}(t)) &= \|\Delta \mathrm{u}(t)\|_2^2 - \|\mathrm{u}(t)\|_p^p \\ &\geq [1 - (\rho_0/\rho_*)^{p-2}] \|\Delta \mathrm{u}(t)\|_2^2. \end{aligned}$$

Thus (4.10) holds with

$$\sigma_0 = 1 - (\rho_0/\rho_*)^{p-2} \in (0, 1).$$

Next, using

$$E(t) = \frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + \frac{p-2}{2p}\|\Delta \mathrm{u}(t)\|_2^2 + \frac{1}{p} I(\mathrm{u}(t)),$$

we immediately obtain

$$E(t) \geq \frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + \frac{p-2}{2p}\|\Delta \mathrm{u}(t)\|_2^2.$$

This gives the lower bound in (4.11). For the upper bound, we write

$$E(t) \leq \frac{1}{2}\|\mathrm{u}_t(t)\|_2^2 + \frac{1}{2}\|\Delta \mathrm{u}(t)\|_2^2.$$

By (4.10), we get

$$\|\Delta \mathrm{u}(t)\|_2^2 \leq \frac{1}{\sigma_0} I(\mathrm{u}(t)).$$

Hence

$$E(t) \leq c_2 \left(\|\mathrm{u}_t(t)\|_2^2 + I(\mathrm{u}(t))\right),$$

for some $c_2 > 0$. The proof is complete. □

Remark 4.2. *Lemma 4.1 shows that the potential well condition prevents the polynomial source from destroying the positivity of the energy. Therefore, the decay of $E(t)$ is equivalent to the decay of the physical plate quantity*

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2.$$

The logarithmic damping is weak near the origin and, therefore, the dissipation $D(t)$ does not directly dominate $\|u_t(t)\|_2^2$. To handle this difficulty, we split the domain according to the size of the velocity as follows:

$$\Omega_-(t) = \{x \in \Omega : |u_t(x,t)| \le 1\}, \qquad \Omega_+(t) = \{x \in \Omega : |u_t(x,t)| > 1\}. \tag{4.14}$$

Lemma 4.3. *There is a constant $C > 0$ such that*

$$\|u_t(t)\|_2^2 \le C(D^{1/2}(t) + D(t)), \qquad \forall\, t \ge 0. \tag{4.15}$$

Consequently, for every $0 \le S < T < \infty$ and every $\varepsilon > 0$, $C_\varepsilon > 0$ exists such that

$$\int_S^T E(t)\|u_t(t)\|_2^2\, dt \le \varepsilon \int_S^T E^2(t)\, dt + C_\varepsilon E(S). \tag{4.16}$$

Proof. Let

$$r = u_t^2(t).$$

On $\Omega_-(t)$, we have $0 \le r \le 1$. Since

$$r\ln(1+r) \sim r^2 \quad \text{as } r \to 0,$$

$C > 0$ exists such that

$$r^2 \le Cr\ln(1+r), \qquad 0 \le r \le 1.$$

Therefore

$$\int_{\Omega_-(t)} u_t^4(t)\, dx \le C \int_{\Omega_-(t)} u_t^2(t)\ln\left(1 + u_t^2(t)\right) dx \le CD(t).$$

By Hölder's inequality, we have

$$\begin{aligned}\int_{\Omega_-(t)} u_t^2(t)\, dx &\le |\Omega|^{1/2}\Big(\int_{\Omega_-(t)} u_t^4(t)\, dx\Big)^{1/2}\\ &\le CD^{1/2}(t).\end{aligned}$$

On the other hand, on $\Omega_+(t)$, $r > 1$ and hence

$$r \le \frac{1}{\ln 2} r\ln(1+r).$$

Thus

$$\int_{\Omega_+(t)} u_t^2(t)\, dx \le C \int_{\Omega_+(t)} u_t^2(t)\ln\left(1 + u_t^2(t)\right) dx \le CD(t).$$

Adding the estimates over $\Omega_-(t)$ and $\Omega_+(t)$, we get

$$\|u_t(t)\|_2^2 \le C(D^{1/2}(t) + D(t)),$$

which proves (4.15).

Multiplying (4.15) by $E(t)$ and integrating over (S,T), we obtain

$$\int_S^T E(t)\|u_t(t)\|_2^2\,dt \le C\int_S^T E(t)D^{1/2}(t)\,dt + C\int_S^T E(t)D(t)\,dt.$$

By Young's inequality, we have

$$CE(t)D^{1/2}(t) \le \varepsilon E^2(t) + C_\varepsilon D(t).$$

Moreover, from $D(t) = -E'(t)$, we have

$$\int_S^T D(t)\,dt = E(S) - E(T) \le E(S),$$

and

$$\begin{aligned}\int_S^T E(t)D(t)\,dt &= -\int_S^T E(t)E'(t)\,dt\\ &= \frac{1}{2}(E^2(S) - E^2(T)) \le CE(S),\end{aligned}$$

because $E(t) \le E(0)$. Combining these estimates yields (4.16). $\square$

Lemma 4.4. *For every $\varepsilon > 0$, $C_\varepsilon > 0$ exists such that, for all $0 \le S < T < \infty$, we have*

$$\int_S^T E(t)\left|\int_\Omega u(t)u_t(t)\ln\left(1 + u_t^2(t)\right)dx\right|dt \le \varepsilon\int_S^T E^2(t)\,dt + C_\varepsilon E(S). \tag{4.17}$$

Proof. We estimate separately the integrals over $\Omega_-(t)$ and $\Omega_+(t)$.

First, on $\Omega_-(t)$, Young's inequality and the Poincaré inequality for clamped plates imply that

$$\left|\int_{\Omega_-(t)} u\,u_t\ln(1 + u_t^2)\,dx\right| \le \varepsilon\|\Delta u(t)\|_2^2 + C_\varepsilon\int_{\Omega_-(t)} u_t^2\ln^2(1 + u_t^2)\,dx.$$

Since $0 \le u_t^2 \le 1$ on $\Omega_-(t)$, we have

$$\ln^2(1 + u_t^2) \le C\ln(1 + u_t^2).$$

Hence

$$\left|\int_{\Omega_-(t)} u\,u_t\ln(1 + u_t^2)\,dx\right| \le \varepsilon\|\Delta u(t)\|_2^2 + C_\varepsilon D(t).$$

Using the energy equivalence in Lemma 4.1, we get

$$E(t)\left|\int_{\Omega_-(t)} u\,u_t\ln(1 + u_t^2)\,dx\right| \le C\varepsilon\,E^2(t) + C_\varepsilon\,E(t)D(t).$$

Integrating over (S,T) and using the fact that

$$\int_S^T E(t)D(t)\,dt \le CE(S),$$

we obtain

$$\int_S^T E(t)\left|\int_{\Omega_-(t)} \mathrm{u}\,\mathrm{u}_t \ln(1+\mathrm{u}_t^2)\,dx\right|dt \le C\varepsilon \int_S^T E^2(t)\,dt + C_\varepsilon E(S). \tag{4.18}$$

Next, we estimate the contribution over $\Omega_+(t)$. By Hölder's inequality, we have

$$\left|\int_{\Omega_+(t)} \mathrm{u}\,\mathrm{u}_t \ln(1+\mathrm{u}_t^2)\,dx\right| \le \|\mathrm{u}(t)\|_3 \Big(\int_{\Omega_+(t)} (|\mathrm{u}_t| \ln(1+\mathrm{u}_t^2))^{3/2} dx\Big)^{2/3}.$$

Let $r = \mathrm{u}_t^2$. On $\Omega_+(t)$, $r > 1$. Since

$$r^{3/4} \ln^{3/2}(1+r) \le C r \ln(1+r), \qquad r \ge 1,$$

we have

$$(|\mathrm{u}_t| \ln(1+\mathrm{u}_t^2))^{3/2} \le C\,\mathrm{u}_t^2 \ln(1+\mathrm{u}_t^2).$$

Therefore

$$\left|\int_{\Omega_+(t)} \mathrm{u}\,\mathrm{u}_t \ln(1+\mathrm{u}_t^2)\,dx\right| \le C\|\mathrm{u}(t)\|_3\, D^{2/3}(t).$$

By the embedding (4.3) and Lemma 4.1

$$\|\mathrm{u}(t)\|_3 \le C\|\Delta \mathrm{u}(t)\|_2 \le CE^{1/2}(t).$$

Consequently

$$E(t)\left|\int_{\Omega_+(t)} \mathrm{u}\,\mathrm{u}_t \ln(1+\mathrm{u}_t^2)\,dx\right| \le C\,E^{3/2}(t)\,D^{2/3}(t).$$

Applying Young's inequality with conjugate exponents 3 and $3/2$, we get

$$CE^{3/2}(t)D^{2/3}(t) \le \varepsilon E^{9/2}(t) + C_\varepsilon D(t).$$

Since $E(t) \le E(0)$, we have

$$E^{9/2}(t) = E^{5/2}(t)E^2(t) \le E^{5/2}(0)E^2(t).$$

Hence

$$CE^{3/2}(t)D^{2/3}(t) \le \varepsilon C(E(0))E^2(t) + C_\varepsilon D(t).$$

Integrating over (S,T) and using the fact that

$$\int_S^T D(t)\,dt \le E(S),$$

we infer that

$$\int_S^T E(t)\left|\int_{\Omega_+(t)} \mathrm{u}\,\mathrm{u}_t \ln(1+\mathrm{u}_t^2)\,dx\right|dt \le \varepsilon C(E(0)) \int_S^T E^2(t)\,dt + C_\varepsilon E(S). \tag{4.19}$$

Combining (4.18) and (4.19) and then renaming ε gives (4.17). □

Remark 4.5. *The preceding two lemmas are the core estimates associated with the logarithmic damping. The splitting into $\Omega_-(t)$ and $\Omega_+(t)$ is necessary because the damping behaves differently for small and large velocities. Near zero, the dissipation is comparable with u_t^4, while for a large velocity, it controls u_t^2 up to a logarithmic factor.*

We now derive the key integral estimate from which the polynomial decay follows. The main point is to estimate the quantity

$$\int_S^T E^2(t)\,dt$$

in terms of the energy at the initial time S. The source term is handled through the Nehari functional $I(\mathrm{u})$, while the logarithmic damping term is controlled by Lemmas 4.3 and 4.4.

Lemma 4.6. *There is a constant $C > 0$, independent of S and T, such that*

$$\int_S^T E^2(t)\,dt \le CE(S), \qquad 0 \le S < T < \infty. \tag{4.20}$$

Proof. The following computation is first performed for the Galerkin approximations, where all integrations by parts are justified. The resulting estimate is then passed to the weak limit by the compactness, lower semicontinuity, and monotonicity arguments used in the well-posedness section. By Lemma 4.1, the energy is equivalent to the positive plate quantity in the stable set. In particular, $C > 0$ exists such that

$$E(t) \le C(\|\mathrm{u}_t(t)\|_2^2 + I(\mathrm{u}(t))), \qquad t \ge 0. \tag{4.21}$$

Multiplying (4.21) by $E(t)$ and integrating over (S,T), we obtain

$$\int_S^T E^2(t)\,dt \le C\int_S^T E(t)\|\mathrm{u}_t(t)\|_2^2\,dt + C\int_S^T E(t)I(\mathrm{u}(t))\,dt. \tag{4.22}$$

The first integral on the right-hand side is estimated by Lemma 4.3: For every $\varepsilon > 0$, we have

$$\int_S^T E(t)\|\mathrm{u}_t(t)\|_2^2\,dt \le \varepsilon\int_S^T E^2(t)\,dt + C_\varepsilon E(S). \tag{4.23}$$

It remains to estimate the second integral in (4.22).

Multiplying the equation

$$\mathrm{u}_{tt} + \Delta^2\mathrm{u} + \mathrm{u}_t\ln(1+\mathrm{u}_t^2) = |\mathrm{u}|^{p-2}\mathrm{u}$$

by $E(t)\mathrm{u}$, integrating over $\Omega\times(S,T)$ and using the clamped boundary conditions, we get

$$\begin{aligned}\int_S^T E(t)I(\mathrm{u}(t))\,dt = &-\Big[E(t)\int_\Omega \mathrm{u}(t)\mathrm{u}_t(t)\,dx\Big]_S^T\\ &+\int_S^T E'(t)\int_\Omega \mathrm{u}(t)\mathrm{u}_t(t)\,dx\,dt\\ &+\int_S^T E(t)\|\mathrm{u}_t(t)\|_2^2\,dt\\ &-\int_S^T E(t)\int_\Omega \mathrm{u}(t)\mathrm{u}_t(t)\ln(1+\mathrm{u}_t^2(t))\,dx\,dt.\end{aligned} \tag{4.24}$$

Indeed, from the equation, we have

$$\int_\Omega \mathrm{u}\,\mathrm{u}_{tt}\,dx + \int_\Omega \mathrm{u}\,\Delta^2\mathrm{u}\,dx + \int_\Omega \mathrm{u}\,\mathrm{u}_t \ln(1+\mathrm{u}_t^2)\,dx - \int_\Omega |\mathrm{u}|^p\,dx = 0.$$

Since

$$\int_\Omega \mathrm{u}\,\Delta^2\mathrm{u}\,dx = \int_\Omega |\Delta\mathrm{u}|^2\,dx$$

under the clamped boundary conditions, it follows that

$$\int_\Omega \mathrm{u}\,\Delta^2\mathrm{u}\,dx - \int_\Omega |\mathrm{u}|^p\,dx = I(\mathrm{u}).$$

Furthermore, we have

$$\begin{aligned}\int_S^T E(t)\int_\Omega \mathrm{u}(t)\,\mathrm{u}_{tt}(t)\,dx\,dt = {} & \Big[E(t)\int_\Omega \mathrm{u}(t)\,\mathrm{u}_t(t)\,dx\Big]_S^T \\ & - \int_S^T E'(t)\int_\Omega \mathrm{u}(t)\,\mathrm{u}_t(t)\,dx\,dt \\ & - \int_S^T E(t)\|\mathrm{u}_t(t)\|_2^2\,dt.\end{aligned}$$

Substituting this identity into the previous relation gives (4.24).

We now estimate the four terms on the right-hand side of (4.24). By the Cauchy–Schwarz inequality and the Poincaré inequality for clamped plates, we have

$$\left|\int_\Omega \mathrm{u}(t)\,\mathrm{u}_t(t)\,dx\right| \le C\|\Delta\mathrm{u}(t)\|_2\,\|\mathrm{u}_t(t)\|_2.$$

Using Lemma 4.1, we obtain

$$\left|\int_\Omega \mathrm{u}(t)\,\mathrm{u}_t(t)\,dx\right| \le C\,E(t).$$

Hence

$$\begin{aligned}\left|\Big[E(t)\int_\Omega \mathrm{u}(t)\,\mathrm{u}_t(t)\,dx\Big]_S^T\right| &\le C(E^2(S)+E^2(T)) \\ &\le C\,E(S),\end{aligned} \tag{4.25}$$

because $E(T) \le E(S) \le E(0)$.

Since $E'(t) = -D(t) \le 0$, we also have

$$\begin{aligned}\left|\int_S^T E'(t)\int_\Omega \mathrm{u}(t)\,\mathrm{u}_t(t)\,dx\,dt\right| &\le C\int_S^T (-E'(t))E(t)\,dt \\ &= \frac{C}{2}(E^2(S)-E^2(T)) \\ &\le C\,E(S).\end{aligned} \tag{4.26}$$

Moreover, Lemma 4.3 gives

$$\int_S^T E(t)\,\|\mathrm{u}_t(t)\|_2^2\,dt \le \varepsilon\int_S^T E^2(t)\,dt + C_\varepsilon\,E(S). \tag{4.27}$$

Finally, by Lemma 4.4

$$\left|\int_S^T E(t)\int_\Omega \mathrm{u}(t)\,\mathrm{u}_t(t)\ln(1+\mathrm{u}_t^2(t))\,dx\,dt\right| \le \varepsilon\int_S^T E^2(t)\,dt + C_\varepsilon E(S). \tag{4.28}$$

Substituting (4.25)–(4.28) into (4.24), we obtain

$$\int_S^T E(t)\,I(\mathrm{u}(t))\,dt \le \varepsilon\int_S^T E^2(t)\,dt + C_\varepsilon E(S). \tag{4.29}$$

Combining (4.22), (4.23), and (4.29), we arrive at

$$\int_S^T E^2(t)\,dt \le C\varepsilon\int_S^T E^2(t)\,dt + C_\varepsilon E(S).$$

Choosing $\varepsilon > 0$ to be sufficiently small and absorbing the first term on the right-hand side into the left-hand side, we conclude that

$$\int_S^T E^2(t)\,dt \le C\,E(S), \qquad 0 \le S < T < \infty.$$

This proves (4.20). □

We shall use the following integral inequality ([10]).

Lemma 4.7. *Let $H : [0,\infty) \to [0,\infty)$ be a non increasing function. Suppose that $L > 0$ exists such that*

$$\int_S^\infty H^2(t)\,dt \le LH(S), \qquad \forall\, S \ge 0. \tag{4.30}$$

Then a constant $C > 0$ exists such that

$$H(t) \le \frac{C}{1+t}, \qquad \forall\, t \ge 0. \tag{4.31}$$

We are now ready to prove the main stability result.

Theorem 4.8 (Polynomial stability)**.** *Assume that the hypotheses of Theorem 3.5 hold. Assume, in addition, that*

$$H_0^2(\Omega) \hookrightarrow L^3(\Omega).$$

Then the global solution of Problem (1.1) *satisfies*

$$E(t) \le \frac{C}{1+t}, \qquad t \ge 0, \tag{4.32}$$

where $C > 0$ depends on $E(0)$, p, Ω, and the Sobolev constants but is independent of t. Moreover

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta\mathrm{u}(t)\|_2^2 \le \frac{C}{1+t}, \qquad \forall\, t \ge 0. \tag{4.33}$$

Proof. From Lemma 4.6, for every $0 \leq S < T < \infty$, we have

$$\int_S^T E^2(t)\,dt \leq CE(S).$$

Letting $T \to \infty$, we obtain

$$\int_S^\infty E^2(t)\,dt \leq CE(S), \qquad \forall\, S \geq 0. \tag{4.34}$$

Moreover, by the energy identity

$$E'(t) = -D(t) \leq 0,$$

and hence $E(t)$ is non increasing. Applying Lemma 4.7 with $H = E$, we obtain

$$E(t) \leq \frac{C}{1+t}, \qquad \forall\, t \geq 0.$$

This proves (4.32).

Finally, by the coercivity estimate in Lemma 4.1, $c_1 > 0$ exists such that

$$c_1\left(\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2\right) \leq E(t).$$

Therefore

$$\|\mathrm{u}_t(t)\|_2^2 + \|\Delta \mathrm{u}(t)\|_2^2 \leq \frac{1}{c_1}E(t) \leq \frac{C}{1+t}.$$

This proves (4.33). The proof is complete. □

Remark 4.9. *The polynomial rate in* (4.32) *is a consequence of the weak structure of the logarithmic damping. In contrast with linear damping, the term* $\mathrm{u}_t \ln(1+\mathrm{u}_t^2)$ *does not yield a direct uniform estimate of* $\|\mathrm{u}_t\|_2^2$ *by the dissipation* $D(t)$*. The decomposition into* $\Omega_-(t)$ *and* $\Omega_+(t)$ *compensates for this weakness.*

Remark 4.10. *The polynomial source is not treated as a small perturbation in the decay proof. It is controlled through the stable potential well condition, which gives* $I(\mathrm{u}(t)) > 0$ *and preserves the coercivity of the energy. This is why the weighted multiplier identity involves the Nehari functional* $I(\mathrm{u})$*, rather than only the bending energy* $\|\Delta \mathrm{u}\|_2^2$.

5. Numerical simulations

In this section, we present numerical experiments illustrating the theoretical results established in Theorem 4.8. In particular, we investigate the long-time behavior of solutions of the nonlinear clamped plate equation

$$\mathrm{u}_{tt} + \Delta^2 \mathrm{u} + \alpha\, \mathrm{u}_t \ln(1+\mathrm{u}_t^2) = |\mathrm{u}|^{p-2}\mathrm{u}, \qquad (x,y) \in \Omega,\ t > 0, \tag{5.1}$$

subject to the clamped boundary conditions

$$\mathrm{u} = 0, \qquad \frac{\partial \mathrm{u}}{\partial \nu} = 0, \qquad (x,y) \in \partial\Omega. \tag{5.2}$$

The objective of the simulations is to verify the global stability of solutions, the monotone decay of the total energy, and the polynomial decay rate predicted by Theorem 4.8.

5.1. Galerkin spatial approximation

The computational domain is chosen as

$$\Omega = (0, L_x) \times (0, L_y),$$

with

$$L_x = L_y = 1.$$

To approximate the solution, we use a tensor product Galerkin method based on admissible clamped beam eigenfunctions. Galerkin approximations constitute one of the standard tools for the numerical treatment of nonlinear evolution equations and have been successfully applied to plate-type problems; see Lions [18] and the recent work in [20].

The approximate solution is represented as

$$\mathrm{u}_M(x, y, t) = \sum_{m=1}^{M} q_m(t)\Phi_m(x, y), \tag{5.3}$$

where

$$M = N_x N_y$$

denotes the total number of modes and the coefficients $q_m(t)$ are unknown functions of time.

The basis functions are chosen so that the clamped boundary conditions are satisfied exactly. Consequently, no boundary approximation is required and the numerical solution remains within the admissible energy space throughout the computation.

5.2. Clamped beam basis functions

The one-dimensional basis functions are generated from the eigenfunctions of the clamped Euler–Bernoulli beam. The use of clamped beam eigenfunctions guarantees exact satisfaction of the boundary conditions and provides an efficient reduced-order approximation of the plate dynamics.

For each mode n, let β_n be the positive root of

$$\cos(\beta_n L)\cosh(\beta_n L) = 1. \tag{5.4}$$

The corresponding normalized beam function is

$$\phi_n(x) = \cosh(\beta_n x) - \cos(\beta_n x) - \sigma_n\left(\sinh(\beta_n x) - \sin(\beta_n x)\right), \tag{5.5}$$

where

$$\sigma_n = \frac{\cosh(\beta_n L) - \cos(\beta_n L)}{\sinh(\beta_n L) - \sin(\beta_n L)}. \tag{5.6}$$

The second derivative is given by

$$\phi_n''(x) = \beta_n^2\left[\cosh(\beta_n x) + \cos(\beta_n x) - \sigma_n\left(\sinh(\beta_n x) + \sin(\beta_n x)\right)\right]. \tag{5.7}$$

These functions satisfy

$$\phi_n(0) = \phi_n(L) = 0,$$

and

$$\phi_n'(0) = \phi_n'(L) = 0,$$

which correspond precisely to the clamped boundary conditions.

The two-dimensional basis is constructed through the tensor product

$$\Phi_{ij}(x, y) = \phi_i(x)\phi_j(y), \qquad i = 1, \ldots, N_x, \quad j = 1, \ldots, N_y. \tag{5.8}$$

The total number of basis functions is therefore

$$M = N_x N_y.$$

In all simulations, we choose

$$N_x = N_y = 5,$$

yielding

$$M = 25$$

Galerkin modes.

5.3. Semi discrete dynamic system

Substituting the approximation (5.3) into the governing equation and testing against each basis function Φ_n leads to the semi discrete system

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) + \mathbf{G}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{F}(\mathbf{q}), \tag{5.9}$$

where

$$\mathbf{q}(t) = (q_1(t), \ldots, q_M(t))^T.$$

The mass matrix is

$$M_{mn} = \int_\Omega \Phi_m \Phi_n \, dxdy, \tag{5.10}$$

while the biharmonic stiffness matrix is

$$K_{mn} = \int_\Omega (\Delta\Phi_m)(\Delta\Phi_n) \, dxdy. \tag{5.11}$$

The nonlinear source vector is

$$F_m(\mathbf{q}) = \int_\Omega |u_M|^{p-2} u_M \Phi_m \, dxdy, \tag{5.12}$$

and the logarithmic damping vector is

$$G_m(\mathbf{q}, \dot{\mathbf{q}}) = \alpha \int_\Omega u_{M,t} \ln\Big(1 + \mathrm{u}_{M,t}^2\Big) \Phi_m \, dxdy. \tag{5.13}$$

Introducing

$$\mathbf{r} = \dot{\mathbf{q}}.$$

System (5.9) can be rewritten in first-order form as

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{r}, \\ \mathbf{M}\,\dot{\mathbf{r}} = \mathbf{F}(\mathbf{q}) - \mathbf{K}\mathbf{q} - \mathbf{G}(\mathbf{q}, \mathbf{r}). \end{cases} \tag{5.14}$$

5.4. Numerical quadrature and time integration

All integrals are evaluated using tensor-product Gauss–Legendre quadrature. Such quadrature formulas provide highly accurate approximations for smooth integrands and are particularly effective for the evaluation of nonlinear terms arising in plate equations. Let

$$\{x_i,\ w_i\}_{i=1}^{N_q}$$

be the quadrature nodes and weights on $[0, L_x]$ and similarly

$$\left\{y_j,\ \omega_j\right\}_{j=1}^{N_q}$$

on $[0, L_y]$.

Then

$$\int_\Omega f(x,y)\,dx\,dy \approx \sum_{i=1}^{N_q}\sum_{j=1}^{N_q} w_i\omega_j\, f(x_i,y_j). \tag{5.15}$$

Throughout the computations, we use

$$N_q = 45$$

quadrature points in each spatial direction, which provides highly accurate evaluation of the nonlinear source and damping terms.

The resulting nonlinear system is stiff because of the fourth-order biharmonic operator and the logarithmic damping term. Therefore, temporal integration is performed using MATLAB's implicit solver ode15s.

The simulations are carried out on the interval

$$0 \le t \le 30,$$

with a relative tolerance of 10^{-7} and an absolute tolerance of 10^{-9}.

The values $p = 3$ and $\alpha = 1$ listed in Table 1 define the baseline simulation. Additional parameter sensitivity computations are presented below.

Table 1. Computational parameters used in the simulations.

Parameter	Value
Domain	$\Omega = (0,1)\times(0,1)$
Number of modes in x	$N_x = 5$
Number of modes in y	$N_y = 5$
Total Galerkin modes	$M = 25$
Quadrature points	$N_q = 45$
Source exponent	$p = 3$
Damping coefficient	$\alpha = 1$
Final time	$T = 30$
Relative tolerance	10^{-7}
Absolute tolerance	10^{-9}
Time integrator	MATLAB ode15s

5.5. Discrete energy and Nehari functional

To monitor the numerical solution, we compute the discrete counterpart of the total energy as follows:

$$E_M(t) = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\,\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T \mathbf{K}\,\mathbf{q} - \frac{1}{p}\int_\Omega |\mathrm{u}_M|^p \, dx\, dy. \tag{5.16}$$

The discrete Nehari functional is defined by

$$I_M(t) = \mathbf{q}^T\mathbf{K}\mathbf{q} - \int_\Omega |\mathrm{u}_M|^p \, dxdy. \tag{5.17}$$

The positivity of $I_M(t)$ indicates that the numerical trajectory remains inside the stable side of the Nehari manifold. This discrete quantity plays the same role as the continuous Nehari functional used in the potential well framework used in the proof of global existence and stability; see Komornik [10].

5.6. Convergence assessment

To verify the adequacy of the spatial discretization, we investigate the dependence of the numerical solution on the number of Galerkin modes. The final energy value at the terminal time $T = 30$ is computed for several choices of the truncation dimensions $N_x = N_y$, corresponding to $M = N_xN_y$ basis functions.

Table 2 reports the computed final energy together with the relative difference between two consecutive discretizations. The results show that the numerical approximation is essentially insensitive to further increases in the number of modes.

Table 2. Convergence assessment with respect to the number of Galerkin modes.

$N_x = N_y$	M	$E_M(30)$	Relative difference
4	16	$9.00209707 \times 10^{-3}$	—
5	25	$9.06191667 \times 10^{-3}$	6.6012×10^{-3}
6	36	$9.06294336 \times 10^{-3}$	1.1329×10^{-4}
7	49	$9.07659883 \times 10^{-3}$	1.5045×10^{-3}

In particular, the relative difference between the approximations with $M = 25$ and $M = 36$ is approximately 1.13×10^{-4}, while the corresponding final energy values differ only in the fourth decimal place. This indicates that the Galerkin approximation has essentially converged with respect to the spatial discretization. Consequently, the choice $N_x = N_y = 5$ ($M = 25$ modes) provides a reliable balance between computational efficiency and numerical accuracy, and is adopted throughout the subsequent simulations.

5.7. Numerical results

The numerical experiments are performed with the parameters summarized in Table 1. Specifically,

$$L_x = L_y = 1, \qquad N_x = N_y = 5, \qquad M = 25,$$

and

$$p = 3, \qquad \alpha = 1.$$

The initial data are chosen to be sufficiently small so that the corresponding solution remains inside the stable set associated with the Nehari manifold. Consequently, the assumptions of Theorem 4.8 are satisfied and global existence together with polynomial stability are expected.

To provide both qualitative and quantitative validation of the theoretical results, six complementary numerical experiments are performed. The corresponding figures illustrate the evolution of the displacement field, the decay of the total energy, the asymptotic stabilization rate, a comparison with the theoretical polynomial estimate, the sensitivity with respect to the damping coefficient, and the preservation of the stable side of the Nehari manifold.

In addition to the baseline computations, a quantitative parameter sensitivity study is carried out for several admissible values of the source exponent p. The corresponding results are summarized in an additional table, while all other numerical parameters and the initial spatial profile are kept fixed. This comparison is included to examine the robustness of the observed energy decay behavior with respect to variations in the polynomial source exponent.

Table 3 shows that $I_{M,p}(0) > 0$ for every tested value of p, so the corresponding initial states lie on the positive side of the discrete Nehari manifold. The normalized terminal energies and the fitted short-time decay slopes are very close for $p = 2.2$, $p = 3$, and $p = 5$. Thus, within the considered stable numerical regime, variations in the polynomial source exponent produce only minor quantitative changes, while the same qualitative energy decay behavior is preserved. This indicates that the observed stabilization mechanism is robust with respect to the admissible source exponents considered here.

Table 3. Sensitivity of the numerical solution to the source exponent p. The logarithmic damping coefficient is fixed at $\alpha = 1$, and the same initial spatial profile, normalized so that $\max_{(x,y)\in\Omega} |u_0(x, y)| = 2$, is used in all cases. The comparison is performed over $0 \le t \le T_p$, with $T_p = 5$.

p	$E_p(0)$	$I_{M,p}(0)$	$E_p(T_p)/E_p(0)$	Fitted slope
2.2	1.42962387×10^3	2.85920318×10^3	$7.66106708 \times 10^{-5}$	−2.8955
3.0	1.42964646×10^3	2.85909267×10^3	$7.66106689 \times 10^{-5}$	−2.8955
5.0	1.42957794×10^3	2.85834956×10^3	$7.66143862 \times 10^{-5}$	−2.8955

5.8. Discussion of the numerical results

The numerical simulations provide strong evidence supporting the theoretical results established in Section 4 and complement the analytical findings of [20]. In particular, the computations confirm the global stability of solutions and the polynomial decay of the associated energy.

Figure 1 illustrates the temporal evolution of the displacement field. The initial deformation gradually diminishes and the solution approaches the trivial equilibrium configuration. The reduction in the maximum displacement amplitude from 2.46×10^{-1} to 2.12×10^{-3}, reported in Table 4, demonstrates the effectiveness of the logarithmic damping term in dissipating mechanical energy from the system. No numerical instability or growth phenomenon is observed during the entire computation, which is consistent with the global existence result.

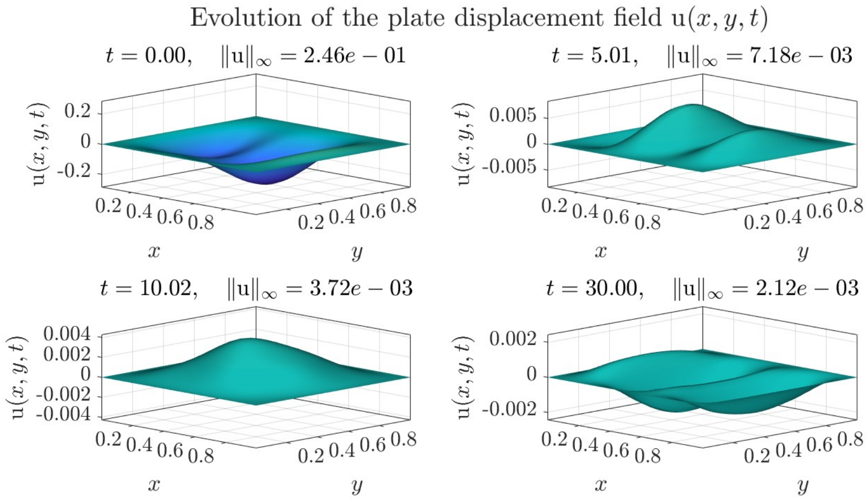

Figure 1. Snapshots of the plate displacement field $u_M(x, y, t)$ at selected times. The oscillation amplitude decreases continuously as time evolves, indicating convergence toward the equilibrium state under logarithmic velocity damping.

Table 4. Quantitative indicators of stabilization.

Indicator	Initial value	Final value	Reduction
$\|u_M\|_\infty$	2.46×10^{-1}	2.12×10^{-3}	99.14%
Observed log–log slope		-1.274	

The energy history displayed in Figure 2 shows a strictly decreasing behavior. This monotonicity agrees with the analytical energy identity derived in the proof of Theorem 4.8. Since

$$u_t \ln(1 + u_t^2)\, u_t = u_t^2 \ln(1 + u_t^2) \geq 0,$$

the damping mechanism continuously extracts energy from the system. Consequently, the numerical energy remains nonincreasing throughout the simulation.

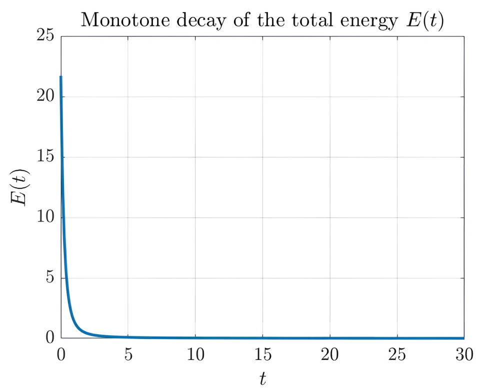

Figure 2. Evolution of the discrete energy $E_M(t)$. The energy decreases monotonically throughout the simulation, confirming the dissipative effect of the logarithmic damping mechanism.

To investigate the asymptotic decay rate, the energy is represented on a logarithmic scale in Figure 3. A least-squares fit yields an observed slope of approximately -1.274. Although numerical fitting alone cannot determine the exact decay exponent, this value is fully compatible with the theoretical estimate

$$E(t) \leq \frac{C}{1+t},$$

established in Theorem 4.8. The simulation therefore provides quantitative support for the predicted polynomial stabilization mechanism.

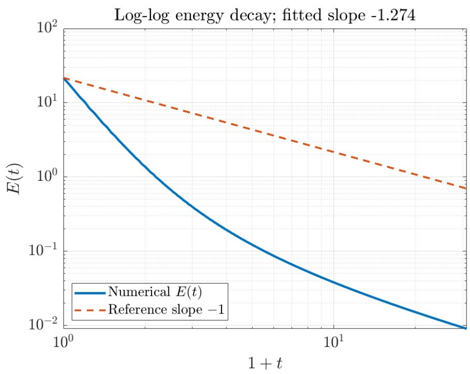

Figure 3. Log–log representation of the numerical energy. The observed slope is compatible with the polynomial decay predicted by Theorem 4.8.

A more direct validation of the analytical result is presented in Figure 4, where the numerical energy is compared with the envelope $C(1 + t)^{-1}$. The numerical curve remains below the theoretical bound throughout the computation, confirming that the estimate accurately captures the dominant long-time behavior of the solution.

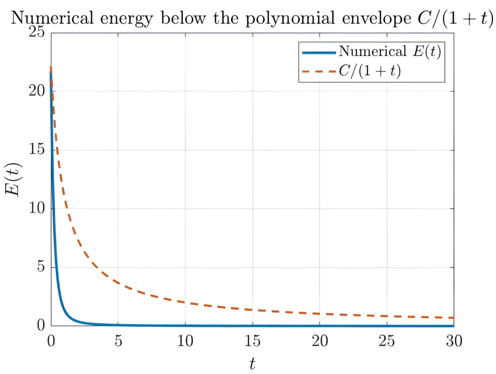

Figure 4. Comparison between the numerical energy and the theoretical upper bound $C(1 + t)^{-1}$. The numerical solution remains below the theoretical envelope throughout the simulation interval.

The influence of the damping coefficient is investigated in Figure 5 for $\alpha = 0.5$, $\alpha = 1$, and $\alpha = 2$, while $p = 3$ and all other numerical parameters are kept fixed. Larger values of α generate stronger dissipation and therefore faster stabilization, whereas smaller damping coefficients lead to slower energy decay. Nevertheless, all curves preserve the same qualitative stabilization behavior, demonstrating the robustness of the numerical results with respect to the damping intensity.

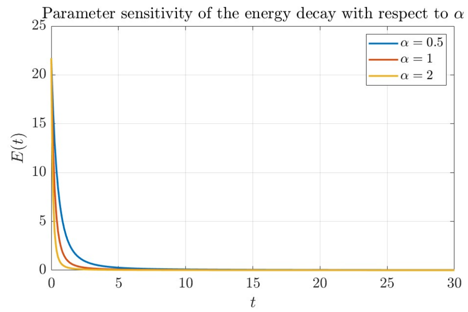

Figure 5. Influence of the damping coefficient α on the energy decay. Larger damping coefficients accelerate the stabilization process.

The effect of the source exponent is quantified in Table 3. For $p = 2.2$, $p = 3$, and $p = 5$, the normalized terminal energies and the fitted short-time decay slopes remain very close. Thus, changing the source exponent within the tested admissible range produces only minor quantitative variations in the stable numerical regime, while preserving the same qualitative energy decay behavior. In comparison, the damping coefficient has a more pronounced quantitative effect on the rate of energy reduction.

Figure 6 presents the evolution of the discrete Nehari functional. The quantity $I_M(t)$ remains strictly positive throughout the computation, indicating that the numerical trajectory never leaves the stable side of the Nehari manifold. This observation is fully consistent with the assumptions used in the proof of Theorem 4.8 and confirms preservation of the potential well structure at the discrete level.

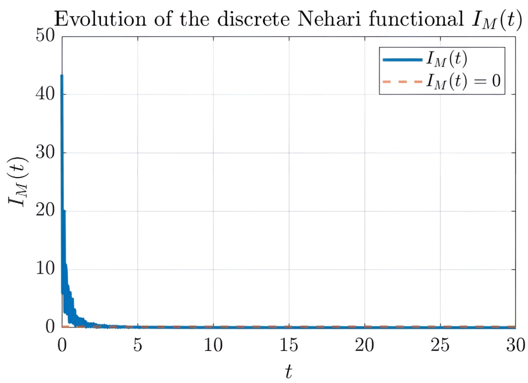

Figure 6. Evolution of the discrete Nehari functional $I_M(t)$. Positivity is preserved throughout the computation, indicating that the numerical solution remains inside the stable region.

Overall, the numerical experiments demonstrate excellent agreement with the analytical theory. The displacement converges toward equilibrium, the discrete energy decreases monotonically, the observed decay rate is compatible with the polynomial estimate established in Theorem 4.8, and the numerical trajectory remains confined to the stable side of the Nehari manifold. Together with the source exponent sensitivity results reported in Table 3 and the quantitative indicators reported in Table 4, these observations provide strong numerical evidence supporting the global existence and polynomial stability results proved in this work.

6. Conclusions

In this paper, we investigated a nonlinear clamped plate equation with logarithmic velocity damping and a polynomial source term. The main objective was to analyze the long-time behavior of the solutions and to establish rigorous stability properties for the associated dynamic system.

Using the Faedo–Galerkin approximation method, compactness arguments, and energy techniques, we first established the global existence of weak solutions under suitable assumptions on the initial data. The logarithmic damping mechanism was shown to provide sufficient dissipation to prevent instability and to guarantee the boundedness of solutions for all times.

A central contribution of this work is the derivation of a polynomial decay estimate for the total energy. More precisely, it was proved that the energy satisfies

$$E(t) \leq \frac{C}{1+t}, \qquad t \geq 0,$$

for a suitable positive constant C. This result demonstrates that the logarithmic damping term generates an effective stabilization mechanism, although it is weaker than the exponential decay typically associated with stronger damping laws. The analysis further highlights the role of the Nehari manifold and the potential well framework in characterizing the stable regime of the problem.

To complement the theoretical results, a comprehensive numerical study was performed using a tensor product Galerkin approximation based on clamped Euler–Bernoulli beam eigenfunctions. The numerical simulations confirmed the monotone dissipation of energy, the convergence of the displacement field toward equilibrium, the preservation of the stable side of the Nehari manifold, and the polynomial decay predicted by the theory. The observed numerical decay rate was found to be in good agreement with the analytical estimate, providing additional validation of the theoretical framework.

The results obtained in this work contribute to the growing theory of nonlinear plate equations with weak dissipation and nonstandard damping mechanisms. The combination of rigorous analytical results and the supporting numerical evidence demonstrates that logarithmic velocity damping is capable of ensuring the global existence and polynomial stabilization of the solutions. Future investigations may focus on optimal decay rates, fractional and viscoelastic extensions, coupled plate systems, and control-theoretic aspects of higher-order nonlinear evolution equations.

Author contributions

Iqra Kanwal, Jianghao Hao, Muhammad Fahim Aslam, Mohamed Balegh, and Zayd Hajjej: Conceptualization, methodology, investigation and writing original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article

Acknowledgments

Zayd Hajjej is supported by Ongoing Research Funding program, (ORF-2026-736), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no competing interests.

References

1. M. F. Aslam, J. Hao, S. Boulaaras, L. Bashir, Global existence and exponential stability of a fractionally damped plate equation with past history, *Electron. Res. Arch.*, **33** (2025), 4363–4381. https://doi.org/10.3934/era.2025199
2. I. Kanwal, J. Hao, L. Bashir, M. F. Aslam, A. Bchatnia, Exponential stability in a fractionally delayed and strongly damped logarithmic plate equation: theoretical and numerical analysis, *Arab. J. Math.*, **15** (2026), 1197–1214. https://doi.org/10.1007/s40065-026-00630-7
3. H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$, **51** (1973), 371–386. https://doi.org/10.1007/BF00263041
4. H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$, *Trans. Amer. Math. Soc.*, **192** (1974), 1–21. https://doi.org/10.2307/1996814
5. V. Georgiev, G. Todorova, Existence of solutions of the wave equation with nonlinear damping and source terms, *J. Differ. Equ.*, **109** (1994), 295–308. https://doi.org/10.1006/jdeq.1994.1051
6. M. Kafini, S. A. Messaoudi, On the decay and global nonexistence of solutions to a damped wave equation with variable-exponent nonlinearity and delay, *Ann. Polon. Math.*, **122** (2019), 49–70. https://doi.org/10.4064/ap180524-31-10
7. S. A. Messaoudi, Blow-up in a nonlinearly damped wave equation, *Math. Nachr.*, **231** (2001), 105–111.
8. E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, *Arch. Ration. Mech. Anal.*, **149** (1999), 155–182. https://doi.org/10.1007/s002050050171
9. J. E. Lagnese, *Boundary stabilization of thin plates*, Philadelphia: SIAM, 1989.
10. V. Komornik, *Exact controllability and stabilization: the multiplier method*, Paris: Masson-John Wiley, 1994.
11. I. Lasiecka, R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories*, Cambridge: Cambridge University Press, 2000.
12. I. Chueshov, I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, *Mem. Amer. Math. Soc.*, **195** (2008), 1–183. https://doi.org/10.1090/MEMO/0912
13. D. Li, C. Zhang, H. Zhang, Energy decay estimates for the wave equation with logarithmic feedback, *Math. Methods Appl. Sci.*, **48** (2025), 10110–10113. https://doi.org/10.1002/mma.10871

14. D. Li, G. Liu, H. Zhang, Decay estimates and blow-up of solutions to the wave equation with logarithmic damping, *Discrete Contin. Dyn. Syst. Ser. S*, **29** (2026), 418-435. https://doi.org/10.3934/dcdss.2026015

15. S. A. Messaoudi, M. Kafini, M. Zahri, On the wave equation with logarithmic damping: well-posedness, blow-up and numerical analysis, 2026, In press.

16. A. M. Al-Mahdi, M. M. Al-Gharabli, On the well-posedness and stability of piezoelectric beams with magnetic effects, logarithmic damping, and Fourier-type thermal conduction, *J. Therm. Stresses*, **49** (2026), 712–727. https://doi.org/10.1080/01495739.2025.2603426

17. S. A. Messaoudi, A. Al-Mahdi, Well-posedness and asymptotic stability of a one-dimensional Timoshenko system with a logarithmic damping, *Discrete Contin. Dyn. Syst. Ser. S*, **21** (2026), 1–16. https://doi.org/10.3934/dcdss.2025155

18. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Paris: Dunod, 1969.

19. J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65–96. https://doi.org/10.1007/BF01762360

20. I. Kanwal, J. Hao, M. F. Aslam, M. Sepúlveda-Cortés, Stability and finite-time blow-up for a fractionally damped nonlinear plate equation: numerical and analytical insights, *Commun. Nonlinear Sci. Numer. Simul.*, **162** (2026), 110269. https://doi.org/10.1016/j.cnsns.2026.110269