Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 26927–26942.
DOI: 10.3934/math.20261080
Received: 24 May 2026
Revised: 19 August 2026
Accepted: 24 August 2026
Published: 26 August 2026

Research article

Maximal Rosenthal transfer principles and complete moment criteria for weighted triangular arrays of measurable operators

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Abstract: This paper studied complete moment convergence and polynomially weighted summability criteria for scalar weighted triangular arrays of measurable operators in tracial von Neumann algebras. The main result was a maximal Rosenthal transfer theorem for weighted triangular arrays whose rows admitted martingale filtrations. Starting from a uniform q-moment estimate for each terminal row sum, it controlled all weighted partial sums in $L_q(M;\ell_\infty)$ and, for $q > 2$, in $L_q(M;\ell_\infty^c)$. Under the associated scalar summability condition, it yielded row-uniform bilateral almost uniform convergence and, for $q > 2$, row-uniform almost uniform convergence. As an application of the general Rosenthal-type criterion, we derived a high-moment Marcinkiewicz–Zygmund normalization for freely independent rows and martingale-difference rows satisfying a conditional variance domination. The terminal estimates were verified directly from the inequalities of Junge and Xu. The Rosenthal summability criterion was then developed as a technical tool: Under explicit summability conditions on scalar weights and normalizing constants, together with uniform second and qth moments such as those supplied by stochastic domination, it yielded complete p-moment convergence for all $0 < p < q$. We also clarified the equivalent positive-part formulation of complete moment convergence and included a sharpness benchmark for freely independent semicircular rows, showing that the scalar variance summability condition was necessary and sufficient at the q-moment summability level. The resulting weighted triangular-array criterion complemented sharp Baum–Katz theorems for successively independent or adapted sequences. Finally, we formulated a tracial C*-probability version through the Gelfand–Naimark–Segal (GNS) representation and the generated von Neumann algebra.

Keywords: complete moment convergence; maximal partial sums; measurable operators; noncommutative Rosenthal inequality; triangular array; vector-valued noncommutative L_p space; almost uniform convergence
Mathematics Subject Classification: 46L53, 60F15, 60G42, 46L51

1. Introduction

Complete convergence and complete moment convergence are classical topics in probability limit theory. Since the seminal works of Hsu and Robbins [6], Erdös [4], and Baum and Katz [1], many extensions have been obtained for independent random variables, triangular arrays, weighted sums, martingales, and Banach-space-valued random variables; for rowwise independent triangular arrays we mention the strong laws of Hu, Móricz, and Taylor [7]. In particular, weighted complete moment convergence has become an effective tool for obtaining refined forms of the strong law of large numbers. Representative results based on generalized Rosenthal inequalities include Wang, Hu, and Volodin [19], Yi, Chen, and Sung [20], and Zheng, Wang, and Wang [22].

In recent years, limit theorems in noncommutative probability have attracted increasing attention. In a tracial von Neumann algebra (M, τ), measurable operators in the sense of Nelson [14] play the role of noncommutative random variables, and almost uniform convergence as well as bilateral almost uniform convergence serve as natural analogues of almost sure convergence [18]. Strong laws for weighted sums of measurable operators have been investigated by Huan and Quang [8], while convergence rates and array laws were studied by Choi and Ji [3] and by Son, Giap, and Quang [16]. Noncommutative Baum–Katz theorems have been studied by Stoica [17] and, more recently, by Ji and Quang [9]. The noncommutative Burkholder–Rosenthal theory [10, 13] provides useful moment tools for sums of noncommutative random variables.

Recent work of Cao, Jiao, Luo, and Zhou [2] establishes quantitative laws of large numbers in noncommutative probability, including a Baum–Katz theorem for successively independent variables, and extends the method to noncommutative martingales via asymmetric maximal inequalities. Ji and Quang [9] obtain a Baum–Katz theorem for adapted sequences. The present paper focuses on the complementary problem of weighted triangular arrays: Once a terminal Rosenthal estimate is available, we seek a practical transfer principle whose scalar hypotheses can be checked directly.

The terminal criterion controls only the last sum in each row. For martingale rows, however, every earlier partial sum is a conditional expectation of the terminal sum. Combining this observation with the strong and asymmetric noncommutative Doob inequalities transfers the terminal estimate to all partial sums in bilateral and column form. The admissibility estimate is then verified from specific theorems of Junge and Xu for freely independent rows and martingale-difference rows. In this way, one scalar series controls terminal moments, complete maximal moments, and row-uniform b.a.u./a.u. convergence.

The scope is precise. The polynomially weighted result is formulated as a sufficient complete-convergence criterion. The normalization $n^{1/s}$ below follows from the imposed moment $q > 2s/(2 - s)$ and is therefore a high-moment Rosenthal corollary. The semicircular calculation identifies an exact necessity-and-sufficiency condition at the level of q-moment summability.

Table 1 places this contribution among selected complete-moment, noncommutative Baum–Katz, and quantitative law-of-large-numbers results.

Table 1. Selected related results and the scope of the present contribution.

Reference	Setting	Main feature	Relation to the present contribution
Wang–Hu–Volodin [19]; Yi–Chen–Sung [20]; Zheng–Wang–Wang [22]	classical weighted sums	complete moments and generalized Rosenthal bounds	a noncommutative weighted-array and maximal formulation
Choi–Ji [3]; Son–Giap–Quang [16]; Huan–Quang [8]	measurable operators	rates, array laws, and weighted strong laws	a unified admissible-row transfer criterion
Cao–Jiao–Luo–Zhou [2]	successive independence and martingales	sharp quantitative laws of large numbers	weights, arrays, free models, and maximal complete moments
Ji–Quang [9]	adapted sequences	Baum–Katz theorem	a sufficient weighted triangular-array criterion
Present paper	weighted noncommutative triangular arrays	terminal and maximal Rosenthal transfer	free/martingale verification and row-uniform b.a.u./a.u. conclusions

The main contributions are as follows:

(i) We formulate a q-Rosenthal-admissibility hypothesis for triangular arrays of centered measurable operators and prove a complete p-moment convergence criterion for all $0 < p < q$, including its equivalent positive-part formulation.

(ii) We prove a maximal transfer theorem that controls all partial sums in $L_q(M; \ell_\infty)$ and, when $q > 2$, in $L_q(M; \ell_\infty^c)$, yielding complete maximal moments and convergence uniformly over each row.

(iii) We derive a polynomially weighted complete-convergence consequence by adding a factor n^α to the same summability condition.

(iv) We provide concrete calculations showing how the abstract conditions reduce to elementary restrictions on weights and normalizing constants.

(v) We verify the admissibility hypothesis directly from Theorem 2.1(i) of Junge and Xu [13] for freely independent rows and from Theorem 5.1 of Junge and Xu [10] for martingale-difference rows with dominated conditional variances.

(vi) As a concrete application we derive a high-moment Marcinkiewicz–Zygmund normalization, while explicitly distinguishing it from the sharp theorem under the minimal sth moment, and we give a semicircular q-moment benchmark.

(vii) We give a tracial C*-probability version through the Gelfand–Naimark–Segal (GNS) representation, emphasizing that the relevant spectral projections generally live in the generated von Neumann algebra rather than in the original C*-algebra.

The novelty of the paper lies in the triangular-array formulation, the explicit scalar-weight summability criterion, and the maximal transfer from terminal Rosenthal estimates, together with verification in concrete operator models. The resulting framework is a practical, transferable sufficient criterion under checkable high-moment hypotheses and provides simultaneous maximal control for weighted noncommutative arrays.

The author's related paper with Hou and Zhu [21] concerns classical extended negatively dependent random variables under slowly varying weights and proves equivalence or necessity statements using that dependence structure. Here the variables are measurable operators, the relevant maximal events are projection-valued, and the model classes are free or martingale rows. The hypotheses, conclusions, and proofs are therefore distinct.

The paper is organized as follows. Section 2 collects notation and preliminary facts, including maximal distributions and the positive-part equivalence. Section 3 states and proves the terminal and maximal transfer principles. Section 4 verifies their hypotheses in the free and martingale models. Section 5 presents applications, the semicircular benchmark, the GNS observation, and the limitations of the method.

2. Preliminaries

Let (M, τ) be a von Neumann algebra equipped with a faithful normal tracial state τ, so that $\tau(\mathbf{1}) = 1$. Denote by $S(M, \tau)$ the $*$-algebra of τ-measurable operators affiliated with M in the sense of Nelson [14]. For $0 < p < \infty$, the noncommutative L_p-space is

$$L_p(M, \tau) = \{x \in S(M, \tau) : \tau(|x|^p) < \infty\};$$

we refer to [5, 14, 15] for the construction and basic properties of these spaces and of the generalized singular numbers.

For $x \in S(M, \tau)$, define its tail distribution function by

$$\lambda_t(x) = \tau(\mathbf{1}_{(t,\infty)}(|x|)), \qquad t > 0,$$

where $\mathbf{1}_{(t,\infty)}(|x|)$ denotes the spectral projection of $|x|$ corresponding to the interval (t, ∞).

Lemma 2.1 (Layer-cake formula)**.** *Let* $0 < p < \infty$ *and* $x \in L_p(M, \tau)$*. Then,*

$$\tau(|x|^p) = p \int_0^\infty t^{p-1} \lambda_t(x)\, dt.$$

Moreover, for every $\varepsilon > 0$*,*

$$\tau\left(|x|^p \mathbf{1}_{(\varepsilon,\infty)}(|x|)\right) = \varepsilon^p \lambda_\varepsilon(x) + p \int_\varepsilon^\infty t^{p-1} \lambda_t(x)\, dt.$$

Proof. Let E denote the spectral measure of $|x|$ and $\mu = \tau \circ E$ the resulting Borel measure on $[0, \infty)$, so that $\lambda_t(x) = \mu((t, \infty))$ and $\tau(|x|^p) = \int_{[0,\infty)} s^p\, d\mu(s)$. By Tonelli's theorem,

$$p \int_0^\infty t^{p-1} \mu((t, \infty))\, dt = \int_{[0,\infty)} \Big(p \int_0^s t^{p-1}\, dt\Big) d\mu(s) = \int_{[0,\infty)} s^p\, d\mu(s) = \tau(|x|^p),$$

which is the first identity. For the second, the same computation restricted to (ε, ∞) gives

$$p\int_{\varepsilon}^{\infty} t^{p-1}\mu((t,\infty))\,dt = \int_{(\varepsilon,\infty)} (s^p - \varepsilon^p)\,d\mu(s) = \tau(|x|^p \mathbf{1}_{(\varepsilon,\infty)}(|x|)) - \varepsilon^p \lambda_\varepsilon(x). \qquad \square$$

Definition 2.2 (Complete p-moment convergence)**.** *A sequence $\{Z_n\} \subset S(M,\tau)$ is said to converge completely in p-moment to* 0 *if, for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \tau\left(|Z_n|^p \mathbf{1}_{(\varepsilon,\infty)}(|Z_n|)\right) < \infty.$$

The preceding truncation agrees, at the level of convergence for every threshold, with the positive-part convention often used in the classical literature.

Lemma 2.3. *Let $x \in S(M,\tau)$, $p > 0$, and $\varepsilon > 0$. Then,*

$$\tau((|x| - \varepsilon)_+^p) \leq \tau(|x|^p \mathbf{1}_{(\varepsilon,\infty)}(|x|))$$

and

$$\tau(|x|^p \mathbf{1}_{(2\varepsilon,\infty)}(|x|)) \leq 2^p \tau((|x| - \varepsilon)_+^p).$$

Consequently, for a sequence (Z_n), complete p-moment convergence in Definition 2.2 is equivalent to $\sum_n \tau((|Z_n| - \varepsilon)_+^p) < \infty$ for every $\varepsilon > 0$.

Proof. For $t \geq 0$ one has $(t-\varepsilon)_+^p \leq t^p \mathbf{1}_{(\varepsilon,\infty)}(t)$. If $t > 2\varepsilon$, then $t \leq 2(t-\varepsilon)$, whence $t^p \mathbf{1}_{(2\varepsilon,\infty)}(t) \leq 2^p (t-\varepsilon)_+^p$. Functional calculus and the trace give both inequalities. Applying the second one with threshold $\varepsilon/2$ proves the converse implication. $\square$

Definition 2.4 (Complete convergence in tracial measure)**.** *A sequence $\{Z_n\} \subset S(M,\tau)$ is said to converge completely in tracial measure to* 0 *if, for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \lambda_\varepsilon(Z_n) < \infty.$$

Definition 2.5 (Almost uniform and bilateral almost uniform convergence)**.** *A sequence $\{Z_n\} \subset S(M,\tau)$ converges to* 0 almost uniformly *(*a.u.*) if for every $\delta > 0$ there exists a projection $e \in M$ with $\tau(\mathbf{1} - e) < \delta$ such that $\|Z_n e\| \to 0$. It converges to* 0 bilaterally almost uniformly *(*b.a.u.*) if for every $\delta > 0$ there exists a projection $e \in M$ with $\tau(\mathbf{1} - e) < \delta$ such that $\|eZ_n e\| \to 0$. Since $\|eZ_n e\| \leq \|Z_n e\|$,* a.u. *convergence implies* b.a.u. *convergence.*

Lemma 2.6 (Noncommutative Borel–Cantelli lemma)**.** *Let $\{Z_n\} \subset S(M,\tau)$ and assume that for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \lambda_\varepsilon(Z_n) < \infty.$$

Then, $Z_n \to 0$ almost uniformly and, in particular, bilaterally almost uniformly.

Proof. Fix $\delta > 0$. For each integer $j \geq 1$ the series $\sum_n \lambda_{1/j}(Z_n)$ converges, so we may choose N_j with

$$\sum_{n \geq N_j} \lambda_{1/j}(Z_n) < \delta\, 2^{-j}.$$

Put $p_j = \bigvee_{n \geq N_j} \mathbf{1}_{(1/j,\infty)}(|Z_n|)$, a projection in M, and set $e = \mathbf{1} - \bigvee_{j \geq 1} p_j \in M$. Using subadditivity of τ on suprema of projections,

$$\tau(\mathbf{1} - e) = \tau\Big(\bigvee_{j \geq 1} p_j\Big) \leq \sum_{j \geq 1} \tau(p_j) \leq \sum_{j \geq 1} \sum_{n \geq N_j} \lambda_{1/j}(Z_n) < \delta \sum_{j \geq 1} 2^{-j} = \delta.$$

If $n \geq N_j$, set $p_{n,j} = \mathbf{1}_{[0,1/j]}(|Z_n|)$. Then, $e \leq p_{n,j}$ and hence $e = p_{n,j}e$. The spectral calculus therefore gives

$$\big\| |Z_n| e \big\|_\infty = \big\| |Z_n| p_{n,j} e \big\|_\infty \leq \big\| |Z_n| p_{n,j} \big\|_\infty \leq \frac{1}{j}.$$

Consequently, by the polar decomposition $Z_n = u_n |Z_n|$,

$$\|Z_n e\|_\infty = \big\| u_n |Z_n| e \big\|_\infty \leq \big\| |Z_n| e \big\|_\infty \leq \frac{1}{j}.$$

Thus, $\|Z_n e\|_\infty \to 0$, i.e., $Z_n \to 0$ almost uniformly. The bilateral assertion follows from Definition 2.5. This construction is standard in the theory of almost uniform convergence of measurable operators; see [8, 18]. □

Definition 2.7 (Stochastic domination)**.** *A family $\{X_{n,k} : 1 \leq k \leq k_n,\ n \geq 1\}$ of τ-measurable operators is said to be stochastically dominated by $Y \in S(M, \tau)$ if there exists a constant $C_Y > 0$ such that*

$$\lambda_t(X_{n,k}) \leq C_Y \lambda_t(Y), \qquad t > 0,\ n \geq 1,\ 1 \leq k \leq k_n.$$

Lemma 2.8 (Uniform moment control)**.** *Assume that $\{X_{n,k}\}$ is stochastically dominated by some $Y \in L_r(M, \tau)$, where $r > 0$. Then, for every $0 < s \leq r$,*

$$\tau(|X_{n,k}|^s) \leq C_Y \tau(|Y|^s), \qquad \textit{and in particular} \qquad \sup_{n,k} \tau(|X_{n,k}|^s) < \infty.$$

Proof. Since $\tau(\mathbf{1}) = 1$, the inclusion $L_r(M, \tau) \subset L_s(M, \tau)$ holds for every $0 < s \leq r$, so $\tau(|Y|^s) < \infty$. For $0 < s \leq r$, Lemma 2.1 gives

$$\tau(|X_{n,k}|^s) = s \int_0^\infty t^{s-1} \lambda_t(X_{n,k})\, dt \leq C_Y\, s \int_0^\infty t^{s-1} \lambda_t(Y)\, dt = C_Y \tau(|Y|^s). \qquad \square$$

Definition 2.9 (q-Rosenthal-admissible triangular array)**.** *Let $q \geq 2$. A rowwise centered triangular array*

$$\{X_{n,k} : 1 \leq k \leq k_n,\ n \geq 1\} \subset L_q(M, \tau)$$

is called q-Rosenthal-admissible if there exists a constant $R_q > 0$, independent of n and of the scalar coefficients, such that for every $n \geq 1$ and every *family of scalars $\{a_{n,k}\}_{k=1}^{k_n} \subset \mathbb{C}$,*

$$\tau\left(\left| \sum_{k=1}^{k_n} a_{n,k} X_{n,k} \right|^q \right) \leq R_q \left[\sum_{k=1}^{k_n} |a_{n,k}|^q \tau(|X_{n,k}|^q) + \left(\sum_{k=1}^{k_n} |a_{n,k}|^2 \tau(|X_{n,k}|^2) \right)^{q/2} \right].$$

Here, rowwise centered means $\tau(X_{n,k}) = 0$ for all admissible pairs (n, k).

Remark 2.10. *Admissibility is a named structural class, not an independence notion and not a theorem by itself. Section 4 proves that two concrete classes belong to it. The constant R_q must be independent of the row length and of every scalar coefficient family.*

For the maximal result let $\mathcal{P}(M)$ denote the projection lattice. For a finite family $x = (x_j)_{j=1}^m \subset S(M, \tau)$, define

$$\Lambda_t^{\mathrm{b}}(x) := \inf_{e\in\mathcal{P}(M)} \left\{ \tau(\mathbf{1} - e) : \max_j \|ex_je\|_\infty \le t \right\}, \tag{2.1}$$

$$\Lambda_t^{\mathrm{c}}(x) := \inf_{e\in\mathcal{P}(M)} \left\{ \tau(\mathbf{1} - e) : \max_j \|x_je\|_\infty \le t \right\}. \tag{2.2}$$

For $p > 0$, $\varepsilon > 0$, and $\sharp \in \{\mathrm{b}, \mathrm{c}\}$, put

$$\mathfrak{C}_{p,\varepsilon}^\sharp(x) = \varepsilon^p \Lambda_\varepsilon^\sharp(x) + p \int_\varepsilon^\infty t^{p-1} \Lambda_t^\sharp(x)\, dt. \tag{2.3}$$

In the commutative case this is exactly the truncated p-moment of $\max_j |x_j|$.

We use the vector-valued spaces of noncommutative Doob theory [11, 12]. For finite $x = (x_j)$,

$$\|x\|_{L_q(M;\ell_\infty)} = \inf_{x_j = aw_jb} \|a\|_{2q} \big(\max_j \|w_j\|_\infty \big) \|b\|_{2q},$$
$$\|x\|_{L_q(M;\ell_\infty^{\mathrm{c}})} = \inf_{x_j = w_jb} \big(\max_j \|w_j\|_\infty \big) \|b\|_q.$$

Lemma 2.11. *Let $q \ge 1$. If $x \in L_q(M; \ell_\infty)$, then*

$$\Lambda_t^{\mathrm{b}}(x) \le 2t^{-q} \|x\|_{L_q(M;\ell_\infty)}^q,$$

and, for $0 < p < q$,

$$\mathfrak{C}_{p,\varepsilon}^{\mathrm{b}}(x) \le \frac{2q}{q-p} \varepsilon^{p-q} \|x\|_{L_q(M;\ell_\infty)}^q.$$

If $x \in L_q(M; \ell_\infty^{\mathrm{c}})$, the corresponding bounds are

$$\Lambda_t^{\mathrm{c}}(x) \le t^{-q} \|x\|_{L_q(M;\ell_\infty^{\mathrm{c}})}^q, \qquad \mathfrak{C}_{p,\varepsilon}^{\mathrm{c}}(x) \le \frac{q}{q-p} \varepsilon^{p-q} \|x\|_{L_q(M;\ell_\infty^{\mathrm{c}})}^q.$$

Proof. For $\eta > 0$, choose a factorization $x_j = aw_jb$ such that $\max_j \|w_j\|_\infty \le 1$ and

$$\|a\|_{2q} \|b\|_{2q} \le \|x\|_{L_q(M;\ell_\infty)} + \eta.$$

Rescaling a and b by reciprocal positive constants, we may also arrange $\|a\|_{2q} = \|b\|_{2q}$. With $e_a = \mathbf{1}_{[0,\sqrt{t}]}(|a^*|)$, $e_b = \mathbf{1}_{[0,\sqrt{t}]}(|b|)$, and $e = e_a \wedge e_b$, one has $\max_j \|ex_je\|_\infty \le t$ and

$$\tau(\mathbf{1} - e) \le t^{-q} (\|a\|_{2q}^{2q} + \|b\|_{2q}^{2q}) = 2t^{-q} (\|a\|_{2q} \|b\|_{2q})^q.$$

Letting $\eta \downarrow 0$ gives the bilateral tail estimate. For the column case, choose $x_j = w_jb$ with $\max_j \|w_j\|_\infty \le 1$ and $\|b\|_q \le \|x\|_{L_q(M;\ell_\infty^{\mathrm{c}})} + \eta$, and use $e = \mathbf{1}_{[0,t]}(|b|)$. Then, $\tau(\mathbf{1} - e) \le t^{-q} \|b\|_q^q$; letting $\eta \downarrow 0$ proves the column tail estimate. Integrating either tail bound in (2.3) gives the displayed constants. □

3. Terminal and maximal transfer principles

Throughout this section, (M,τ) is a finite tracial von Neumann algebra. For weights $c_{n,k} \in \mathbb{C}$ and normalizations $b_n > 0$, write

$$S_{n,j} = \sum_{k=1}^{j} c_{n,k} X_{n,k}, \qquad S_n = S_{n,k_n},$$

and

$$A_n = \frac{\sum_{k=1}^{k_n} |c_{n,k}|^q}{b_n^q} + \left(\frac{\sum_{k=1}^{k_n} |c_{n,k}|^2}{b_n^2} \right)^{q/2}. \tag{3.1}$$

Proposition 3.1. *Let $q \geq 2$ and let $(X_{n,k})$ be a q-Rosenthal-admissible triangular array. Assume*

$$U_2 := \sup_{n,k} \tau(|X_{n,k}|^2) < \infty, \qquad U_q := \sup_{n,k} \tau(|X_{n,k}|^q) < \infty.$$

If (ω_n) is nonnegative and $\sum_n \omega_n A_n < \infty$, then

$$\sum_{n=1}^{\infty} \omega_n \tau \left(\left| \frac{S_n}{b_n} \right|^q \right) < \infty. \tag{3.2}$$

Consequently, for every $0 < p < q$ and $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \omega_n \tau \left(\left| \frac{S_n}{b_n} \right|^p \mathbf{1}_{(\varepsilon,\infty)} \left(\left| \frac{S_n}{b_n} \right| \right) \right) < \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} \omega_n \lambda_\varepsilon \left(\frac{S_n}{b_n} \right) < \infty. \tag{3.4}$$

For $\omega_n = 1$, $S_n/b_n \to 0$ almost uniformly. The same conclusions hold under stochastic domination by $Y \in L_q(M,\tau)$.

Proof. Admissibility and the uniform moments give, with $K_q = R_q \max\{U_q, U_2^{q/2}\}$,

$$\tau \left(\left| \frac{S_n}{b_n} \right|^q \right) \leq K_q A_n.$$

This proves (3.2). Functional calculus applied to $t^p \mathbf{1}_{(\varepsilon,\infty)}(t) \leq \varepsilon^{p-q} t^q$ and $\mathbf{1}_{(\varepsilon,\infty)}(t) \leq \varepsilon^{-q} t^q$ gives (3.3) and (3.4). When $\omega_n = 1$, Lemma 2.6 yields almost uniform convergence. Stochastic domination supplies $U_2, U_q < \infty$ by Lemma 2.8. □

The terminal criterion above sees only S_n. For a martingale row, the identity $S_{n,j} = \mathbb{E}_{n,j}(S_n)$ makes the whole row accessible to Doob factorization. The next theorem records the resulting maximal transfer.

Theorem 3.2 (Maximal Rosenthal transfer)**.** *Let $q \geq 2$ and let $(X_{n,k})$ satisfy the assumptions of Proposition 3.1. Suppose in addition that, for each n, there is a filtration*

$$M_{n,0} \subset M_{n,1} \subset \cdots \subset M_{n,k_n} \subset M$$

for which $(X_{n,k})_{k=1}^{k_n}$ is a martingale-difference row. Let (ω_n) be nonnegative and let A_n be given by (3.1). *Let $D_q = c(q, 2q, 2q)$ be the factorization constant in [11, Corollary 4.6] for the exponent triple $(q, 2q, 2q)$. When $q > 2$, let $D_q^{\mathrm{c}} = c(q, \infty, q)$ be its asymmetric constant for (q, ∞, q). These constants depend only on q, and the following assertions hold.*

(i)

$$\sum_n \omega_n \left\| \left(\frac{S_{n,j}}{b_n} \right)_{j=1}^{k_n} \right\|_{L_q(M;\ell_\infty)}^q \leq D_q^q K_q \sum_n \omega_n A_n, \tag{3.5}$$

where $K_q = R_q \max\{U_q, U_2^{q/2}\}$. Hence, for $0 < p < q$,

$$\sum_n \omega_n \mathfrak{C}_{p,\varepsilon}^{\mathrm{b}} \left(\left(\frac{S_{n,j}}{b_n} \right)_{j=1}^{k_n} \right) \leq \frac{2qD_q^q K_q}{q-p} \varepsilon^{p-q} \sum_n \omega_n A_n. \tag{3.6}$$

(ii) If $q > 2$, then

$$\sum_n \omega_n \left\| \left(\frac{S_{n,j}}{b_n} \right)_{j=1}^{k_n} \right\|_{L_q(M;\ell_\infty^{\mathrm{c}})}^q \leq (D_q^{\mathrm{c}})^q K_q \sum_n \omega_n A_n, \tag{3.7}$$

and

$$\sum_n \omega_n \mathfrak{C}_{p,\varepsilon}^{\mathrm{c}} \left(\left(\frac{S_{n,j}}{b_n} \right)_{j=1}^{k_n} \right) \leq \frac{q(D_q^{\mathrm{c}})^q K_q}{q-p} \varepsilon^{p-q} \sum_n \omega_n A_n. \tag{3.8}$$

(iii) If $\sum_n A_n < \infty$, the normalized partial sums converge to zero bilaterally almost uniformly, uniformly over each row: for every $\delta > 0$ there is $e \in \mathcal{P}(M)$ with $\tau(\mathbf{1} - e) < \delta$ such that

$$\max_{1 \leq j \leq k_n} \left\| e \frac{S_{n,j}}{b_n} e \right\|_\infty \longrightarrow 0. \tag{3.9}$$

If $q > 2$, e can be chosen so that

$$\max_{1 \leq j \leq k_n} \left\| \frac{S_{n,j}}{b_n} e \right\|_\infty \longrightarrow 0. \tag{3.10}$$

Stochastic domination by an element of L_q may again replace the two uniform moment assumptions.

Proof. Fix n. Since the coefficients are scalar, $(S_{n,j})_{j=1}^{k_n}$ is an L_q-martingale and $S_{n,j} = \mathbb{E}_{n,j}(S_n)$. Corollary 4.6 of Junge [11], applied with exponent triples $(q, 2q, 2q)$ and, when $q > 2$, (q, ∞, q), respectively, gives exactly the bilateral factorization and its asymmetric column endpoint:

$$\|(S_{n,j})_j\|_{L_q(M;\ell_\infty)} \leq D_q \|S_n\|_q, \tag{3.11}$$

$$\|(S_{n,j})_j\|_{L_q(M;\ell_\infty^{\mathrm{c}})} \leq D_q^{\mathrm{c}} \|S_n\|_q \quad (q > 2). \tag{3.12}$$

The terminal estimate in the proof of Proposition 3.1 is

$$\|S_n/b_n\|_q^q \leq K_q A_n.$$

Combining it with (3.11), multiplying by ω_n, and summing proves (3.5); Lemma 2.11 gives (3.6). Equations (3.12) and Lemma 2.11 give the column assertions.

For (iii), apply (3.5) with $\omega_n = 1$. Together with Lemma 2.11, this implies, for every $m \geq 1$,

$$\sum_n \Lambda^{\mathrm{b}}_{1/m}\left((S_{n,j}/b_n)_{j=1}^{k_n}\right) < \infty.$$

Given $\delta > 0$, choose the integers N_m recursively so that they are increasing and

$$\sum_{n \geq N_m} \Lambda^{\mathrm{b}}_{1/m}\left((S_{n,j}/b_n)_{j=1}^{k_n}\right) < \delta 2^{-m-2}.$$

For every $n \geq N_m$, the defining infimum provides a projection $e_{n,m}$ such that

$$\tau(\mathbf{1} - e_{n,m}) \leq \Lambda^{\mathrm{b}}_{1/m}\left((S_{n,j}/b_n)_{j=1}^{k_n}\right) + \delta 2^{-m-n-2},$$
$$\max_{1 \leq j \leq k_n} \|e_{n,m} S_{n,j} e_{n,m}/b_n\|_\infty \leq 1/m.$$

Set $e = \bigwedge_{m \geq 1} \bigwedge_{n \geq N_m} e_{n,m}$. Normality and trace subadditivity give

$$\tau(\mathbf{1} - e) \leq \sum_{m \geq 1} \sum_{n \geq N_m} \tau(\mathbf{1} - e_{n,m}) < \sum_{m \geq 1} \delta 2^{-m-2} + \sum_{m \geq 1} \sum_{n \geq N_m} \delta 2^{-m-n-2} < \delta.$$

Since $e \leq e_{n,m}$, compression preserves the last norm bound. Thus, for each m, every $n \geq N_m$ satisfies $\max_j \|e S_{n,j} e/b_n\|_\infty \leq 1/m$, which proves (3.9). Replacing Λ^{b} by Λ^{c} in the same construction and using right compressions gives (3.10). □

Corollary 3.3 (Polynomially weighted maximal criterion)**.** *Under the assumptions of Theorem 3.2, let $\alpha \geq 0$. If*

$$\sum_{n=1}^{\infty} n^\alpha A_n < \infty,$$

then (3.5)–(3.8) *hold with $\omega_n = n^\alpha$. In particular,*

$$\sum_n n^\alpha \Lambda^{\mathrm{b}}_\varepsilon\left((S_{n,j}/b_n)_{j=1}^{k_n}\right) < \infty,$$

and, if $q > 2$, the same assertion holds with Λ^{c}.

4. Verification in two operator models

We now derive admissibility rather than leaving it as an additional model assumption.

Theorem 4.1 (Freely independent rows)**.** *Let $q \geq 2$. If every row $(X_{n,k})_{k=1}^{k_n}$ consists of centered elements of $L_q(M, \tau)$ that are freely independent over $\mathbb{C}$, then the array is q-Rosenthal-admissible. Moreover, each row is a martingale-difference row for its natural filtration, so Theorem 3.2 applies whenever its uniform moment and scalar summability hypotheses hold.*

Proof. By [13, Example 1.5], freeness over $\mathbb{C}$ is independence in the sense used in that paper. More precisely, the freely independent row is independent over $\mathbb{C}\mathbf{1}$, its conditional expectation is $\mathbb{E}(x) =$

$\tau(x)\mathbf{1}$, and centering gives $\mathbb{E}(d_k) = 0$ for $d_k = a_k X_{n,k}$. Thus, all hypotheses of [13, Theorem 2.1(i)] are satisfied. If $c > 0$ is the absolute constant in that theorem, then

$$\left\|\sum_k d_k\right\|_q \leq \frac{q}{c} \max\left\{ \left(\sum_k \|d_k\|_q^q\right)^{1/q}, \left\|\left(\sum_k \mathbb{E}|d_k|^2\right)^{1/2}\right\|_q, \right.$$
$$\left. \left\|\left(\sum_k \mathbb{E}|d_k^*|^2\right)^{1/2}\right\|_q \right\}.$$

The three terms have precisely the form required here. The diagonal term is

$$\left(\sum_k \|d_k\|_q^q\right)^{1/q} = \left(\sum_k |a_k|^q \tau(|X_{n,k}|^q)\right)^{1/q},$$

while

$$\mathbb{E}|d_k|^2 = |a_k|^2 \tau(X_{n,k}^* X_{n,k})\mathbf{1}, \qquad \mathbb{E}|d_k^*|^2 = |a_k|^2 \tau(X_{n,k} X_{n,k}^*)\mathbf{1}.$$

Traciality makes the two scalar traces equal, so both square-function norms are exactly $(\sum_k |a_k|^2 \tau(|X_{n,k}|^2))^{1/2}$. Raising the inequality to the qth power and bounding the maximum by the sum yields Definition 2.9 with $R_q = (q/c)^q$, uniformly in n and (a_k).

For $M_{n,j} = W^*(X_{n,1}, \ldots, X_{n,j})$, freeness and centering give $\mathbb{E}_{n,j-1}(X_{n,j}) = \tau(X_{n,j})\mathbf{1} = 0$. Thus, the row is a martingale-difference row. □

Theorem 4.2 (Martingale rows)**.** *Let $q \geq 2$ and let $(X_{n,k})$ be self-adjoint martingale-difference rows. Assume that a constant $\kappa \geq 1$, independent of n, k, satisfies*

$$\mathbb{E}_{n,k-1}(X_{n,k}^2) \leq \kappa \tau(X_{n,k}^2)\mathbf{1}. \tag{4.1}$$

Then, the array is q-Rosenthal-admissible with $R_q = \eta_q^q \max\{1, \kappa^{q/2}\}$, where η_q is the constant in [10, Theorem 5.1]. Hence, the terminal and maximal transfer principles apply under their remaining hypotheses.

Proof. For $d_k = a_k X_{n,k}$, Theorem 5.1 of [10] states

$$\left\|\sum_k d_k\right\|_q \leq \eta_q \max\left\{ \left(\sum_k \|d_k\|_q^q\right)^{1/q}, \left\|\left(\sum_k \mathbb{E}_{n,k-1}|d_k|^2\right)^{1/2}\right\|_q, \right.$$
$$\left. \left\|\left(\sum_k \mathbb{E}_{n,k-1}|d_k^*|^2\right)^{1/2}\right\|_q \right\}.$$

Self-adjointness and scalar coefficients make the last two terms equal. By (4.1), their qth powers are at most

$$\kappa^{q/2} \left(\sum_k |a_k|^2 \tau(|X_{n,k}|^2)\right)^{q/2}.$$

The first term to the qth power is $\sum_k |a_k|^q \tau(|X_{n,k}|^q)$. Raising the displayed Burkholder–Rosenthal inequality to the qth power proves the claimed admissibility constant. □

Definition 4.3 (Successive independence)**.** *For each fixed n, put* $M_{n,0} = \mathbb{C}\mathbf{1}$ *and*

$$M_{n,k} = W^*(X_{n,1}, \ldots, X_{n,k}), \qquad 1 \le k \le k_n,$$

where $W^*(X_{n,k})$ *denotes the von Neumann algebra generated by the spectral projections of* $X_{n,k}$*. The nth row is* successively independent *with respect to* τ *if* $W^*(X_{n,k})$ *is independent of* $M_{n,k-1}$ *for every k, in the sense that*

$$\tau(ab) = \tau(a)\tau(b), \qquad a \in M_{n,k-1}, \quad b \in W^*(X_{n,k}).$$

Here is the connection with the hypothesis of Theorem 4.2. Let $\mathbb{E}_{n,k-1} : M \to M_{n,k-1}$ be the trace-preserving conditional expectation. For bounded $b \in W^*(X_{n,k})$ and $a \in M_{n,k-1}$, successive independence gives

$$\tau(a\mathbb{E}_{n,k-1}(b)) = \tau(ab) = \tau(a)\tau(b) = \tau(a\tau(b)\mathbf{1}).$$

The uniqueness of the trace-preserving conditional expectation implies $\mathbb{E}_{n,k-1}(b) = \tau(b)\mathbf{1}$. If $X_{n,k} \in L_2(M, \tau)$ is self-adjoint, apply this identity to the bounded spectral truncations $b_r = X_{n,k}^2\mathbf{1}_{[0,r]}(X_{n,k}^2)$ and then use the L_1-continuity of $\mathbb{E}_{n,k-1}$ to obtain

$$\mathbb{E}_{n,k-1}(X_{n,k}^2) = \tau(X_{n,k}^2)\mathbf{1}.$$

The same truncation argument in L_1 shows that a centered successively independent variable is a martingale difference relative to its natural filtration.

Remark 4.4. *Condition* (4.1) *is restrictive: It asks the conditional variance to be dominated by the scalar variance. The preceding calculation shows that it holds with* $\kappa = 1$ *for centered successively independent variables. More generally, it holds whenever the conditional variance is essentially deterministic with a row-uniform scalar bound.*

5. Consequences, benchmarks, and scope

The first calculation shows how the abstract scalar series is read. If $k_n = n$, $c_{n,k} = 1$, and $b_n = n^a$, then

$$A_n = n^{1-aq} + n^{q/2-aq}.$$

Thus, $\sum_n A_n < \infty$ when

$$a > \max\left\{\frac{2}{q}, \frac{1}{2} + \frac{1}{q}\right\}. \tag{5.1}$$

For $q > 2$, the second bound is decisive. With a polynomial factor n^α, the sufficient condition is

$$a > \max\left\{\frac{\alpha+2}{q}, \frac{1}{2} + \frac{\alpha+1}{q}\right\}.$$

Corollary 5.1 (High-moment $n^{1/s}$ normalization)**.** *Let* $1 \le s < 2$ *and* $q > 2s/(2-s)$*. Let* $(X_k) \subset L_q(M, \tau)$ *be centered and either freely independent, or a self-adjoint martingale-difference sequence, with respect to a filtration* (M_k)*, satisfying*

$$\mathbb{E}_{k-1}(X_k^2) \le \kappa\tau(X_k^2)\mathbf{1}$$

for a constant κ independent of k. Assume $\sup_k \tau(|X_k|^2) < \infty$ *and* $\sup_k \tau(|X_k|^q) < \infty$. *Put* $S_j = \sum_{k=1}^{j} X_k$. *Then, for* $0 < p < q$ *and* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathfrak{C}^{\mathrm{b}}_{p,\varepsilon}\left((S_j/n^{1/s})_{j=1}^{n}\right) < \infty,$$

and

$$\max_{1\le j\le n} \|eS_j n^{-1/s}e\|_\infty \longrightarrow 0$$

outside a projection of arbitrarily small trace. In fact, because the hypothesis forces $q > 2$, the column complete-moment estimate and the almost-uniform version with $\max_{j\le n} \|S_j n^{-1/s}e\|_\infty \to 0$ *also hold. If*

$$0 \le \alpha < \left(\frac{1}{s} - \frac{1}{2}\right)q - 1,$$

all maximal distribution and complete-moment series remain finite after multiplication by n^α.

Proof. Use the first n variables as row n and take $c_{n,k} = 1$ and $b_n = n^{1/s}$. The free or martingale theorem in Section 4 gives admissibility and the required filtration. Since $2 - s > 0$, a direct calculation gives

$$q > \frac{2s}{2-s} \iff q(2-s) > 2s \iff 2q > s(q+2) \iff \frac{1}{s} > \frac{1}{2} + \frac{1}{q}.$$

Thus, (5.1) holds. The polynomial range follows by inserting $a = 1/s$ into the preceding calculation; for $q > 2$, the square-function exponent is the restrictive one. □

This is a high-moment Rosenthal–Doob corollary: The hypothesis deliberately places q strictly above s. Sharp results under the minimal sth moment require different rate arguments.

Proposition 5.2 (Semicircular q-moment benchmark)**.** *Let each row $(s_{n,k})$ consist of freely independent standard semicircular variables and let $c_{n,k} \in \mathbb{R}$. Put $T_n = \sum_k c_{n,k}s_{n,k}$ and $A_n^{(2)} = \sum_k c_{n,k}^2$. Then, for every $q > 0$,*

$$\tau\left(\left|\frac{T_n}{b_n}\right|^q\right) = m_q\left(\frac{A_n^{(2)}}{b_n^2}\right)^{q/2},$$

where m_q is the qth absolute moment of a standard semicircular variable. Consequently,

$$\sum_n \tau(|T_n/b_n|^q) < \infty \iff \sum_n (A_n^{(2)}/b_n^2)^{q/2} < \infty.$$

For $q \ge 2$, the latter also dominates the diagonal scalar term. This is a necessity-and-sufficiency statement only for terminal q-moment summability; no converse for truncated p-moments or maximal convergence is asserted.

Proof. The free sum T_n is semicircular with variance $A_n^{(2)}$. Scaling the semicircular law gives the identity and, hence, the equivalence. Finally, $\sum_k |c_{n,k}|^q \le (\sum_k c_{n,k}^2)^{q/2}$ for $q \ge 2$. □

Corollary 5.3 (GNS formulation)**.** *Let (A, ϕ) be a unital C^*-algebra with a faithful tracial state and let $N = \pi_\phi(A)''$ carry the faithful normal trace $\widetilde{\phi}$ extending ϕ. If the represented array satisfies either transfer theorem in $(N, \widetilde{\phi})$, then its conclusions hold in N.*

The content of Corollary 5.3 is the location of the result: Spectral cut projections and the a.u./b.a.u. projections belong to $N = \pi_\phi(A)''$ and need not belong to $\pi_\phi(A)$. It is therefore a von Neumann algebra formulation attached to a tracial C*-probability space, rather than an intrinsic C*-algebraic spectral theorem.

Scalar weights are a key structural input to the present argument: They preserve the martingale-difference structure and allow the terminal estimate to pass directly to partial sums. Noncentral operator weights would require genuinely new row/column multiplier tools, while a nontracial extension would require Haagerup L_p spaces and modular methods. The scalar series in (3.1) is therefore used as a transferable sufficient criterion. Sharp Baum–Katz results for successively independent variables and adapted sequences remain those of [2, 9] under their respective hypotheses.

6. Conclusions

This paper established a maximal Rosenthal transfer principle for scalar weighted triangular arrays of measurable operators. It showed that a uniform terminal q-moment estimate, together with an explicit scalar summability condition, controls all weighted partial sums and yields complete moment convergence as well as row-uniform b.a.u./a.u. convergence. The required terminal estimates were verified for freely independent rows and for martingale-difference rows under conditional variance domination, and the resulting criteria were applied to high-moment Marcinkiewicz–Zygmund normalizations and a semicircular sharpness benchmark.

The results provide checkable sufficient criteria rather than optimal rate theorems. Scalar weights and traciality are essential to the present transfer argument. Extending the method to noncentral operator weights or to nontracial von Neumann algebras would require new multiplier estimates or Haagerup L_p techniques and remains a direction for future work.

Use of Generative-AI tools declaration

OpenAI Codex was used to assist with language editing and manuscript organization. The author reviewed and takes full responsibility for the final content.

Acknowledgments

This work was supported by the General Program of Basic Science (Natural Science) Research of Jiangsu Higher Education Institutions (No. 25KJD110002).

Conflict of interest

The author declares that there is no conflict of interest.

References

1. L. E. Baum, M. Katz, Convergence rates in the law of large numbers, *T. Am. Math. Soc.*, **120** (1965), 108–123. http://dx.doi.org/10.1090/S0002-9947-1965-0198524-1

2. J. Cao, Y. Jiao, S. Luo, D. Zhou, Quantitative laws of large numbers in noncommutative probability, *Can. J. Math.*, 2025. http://dx.doi.org/10.4153/S0008414X25101946
3. B. J. Choi, U. C. Ji, Convergence rates for weighted sums in noncommutative probability space, *J. Math. Anal. Appl.*, **409** (2014), 963–972. http://dx.doi.org/10.1016/j.jmaa.2013.07.074
4. P. Erdös, On a theorem of Hsu and Robbins, *Ann. Math. Stat.*, **20** (1949), 286–291. http://dx.doi.org/10.1214/aoms/1177730037
5. T. Fack, H. Kosaki, Generalized s-numbers of τ-measurable operators, *Pac. J. Math.*, **123** (1986), 269–300. http://dx.doi.org/10.2140/pjm.1986.123.269
6. P. L. Hsu, H. Robbins, Complete convergence and the law of large numbers, *Proc. Natl. Acad. Sci. USA*, **33** (1947), 25–31. http://dx.doi.org/10.1073/pnas.33.2.25
7. T. C. Hu, F. Móricz, R. L. Taylor, Strong laws of large numbers for arrays of rowwise independent random variables, *Acta Math. Hungar.*, **54** (1989), no. 1–2, 153–162. http://dx.doi.org/10.1007/BF01950716
8. N. V. Huan, N. V. Quang, Some strong limit theorems for weighted sums of measurable operators, *Infin. Dimens. Anal. Qu.*, **24** (2021), 2150022. http://dx.doi.org/10.1142/S0219025721500223
9. U. C. Ji, N. V. Quang, Baum–Katz theorem for adapted sequences in noncommutative probability, *Bull. Korean Math. Soc.*, **63** (2026), 567–584. http://dx.doi.org/10.4134/BKMS.b250095
10. M. Junge, Q. Xu, Noncommutative Burkholder/Rosenthal inequalities, *Ann. Probab.*, **31** (2003), 948–995. http://dx.doi.org/10.1214/aop/1048516542
11. M. Junge, Doob's inequality for non-commutative martingales, *J. Reine Angew. Math.*, **549** (2002), 149–190. http://dx.doi.org/10.1515/crll.2002.061
12. Y. Jiao, F. Sukochev, D. Zhou, Maximal inequalities of noncommutative martingale transforms, *Canad. J. Math.*, **73** (2021), 221–248. http://dx.doi.org/10.4153/S0008414X19000580
13. M. Junge, Q. Xu, Noncommutative Burkholder/Rosenthal inequalities. II. Applications, *Israel J. Math.*, **167** (2008), 227–282. http://dx.doi.org/10.1007/s11856-008-1048-4
14. E. Nelson, Notes on non-commutative integration, *J. Funct. Anal.*, **15** (1974), 103–116. http://dx.doi.org/10.1016/0022-1236(74)90014-7
15. G. Pisier, Q. Xu, *Non-commutative L^p-spaces*, In: Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1459–1517. http://dx.doi.org/10.1016/S1874-5849(03)80041-4
16. D. T. Son, D. X. Giap, N. V. Quang, Chung-type strong laws and almost complete convergence for arrays of measurable operators, *J. Theor. Probab.*, **35** (2022), 1391–1411. http://dx.doi.org/10.1007/s10959-021-01108-2
17. G. Stoica, Noncommutative Baum–Katz theorems, *Stat. Probabil. Lett.*, **79** (2009), 320–323. http://dx.doi.org/10.1016/j.spl.2008.08.013
18. F. J. Yeadon, Ergodic theorems for semifinite von Neumann algebras. I, *J. London Math. Soc.*, **16** (1977), 326–332. http://dx.doi.org/10.1112/jlms/s2-16.2.326
19. X. Wang, S. Hu, A. Volodin, General results of complete convergence and complete moment convergence for weighted sums of some class of random variables, *Commun. Stat.-Theor. M.*, **45** (2016), 4494–4508. http://dx.doi.org/10.1080/03610926.2014.921308

20. Y. Yi, P. Chen, S. H. Sung, Strong laws for weighted sums of random variables satisfying generalized Rosenthal type inequalities, *J. Inequal. Appl.*, 2020, 43. http://dx.doi.org/10.1186/s13660-020-02311-1
21. S. Zhang, S. Hou, Y. Zhu, A complete moment convergence theorem for extended negatively dependent random variables under slowly varying weights, *Mathematics*, **14** (2026), 2092. http://dx.doi.org/10.3390/math14122092
22. X. Zheng, C. Wang, X. Wang, Complete convergence and complete moment convergence for weighted sums of random variables satisfying generalized Rosenthal type inequalities and an application, *Stochastics*, **96** (2024), 1475–1506. http://dx.doi.org/10.1080/17442508.2023.2298861