AIMS Mathematics, 11(8): 26301–26307.
DOI: 10.3934/math.20261055
Received: 17 April 2026
Revised: 23 July 2026
Accepted: 05 August 2026
Published: 24 August 2026

https://www.aimspress.com/journal/Math

Research article

Remark on Itô and Tanaka formulas of multidimensional bifractional Brownian motion

Nenghui Kuang[1,*] **and Huantian Xie**[2]

[1] School of Mathematics and Statistics, Hunan University of Science and Technology, Xiangtan, Hunan 411201, China
[2] School of Mathematics and Statistics, Linyi University, Linyi, Shandong 276005, China

* **Correspondence:** Email: knh1552@163.com.

Abstract: In this paper, by using the idea of Proposition 3.3 in Chen et al. (2024), we correct the proof of Lemma 2 in Es-sebaiy and Tudor (2007) about multidimensional bifractional Brownian motion: Itô and Tanaka formulas.

Keywords: bifractional Brownian motion; Itô formula; Tanaka formula; Cauchy-Schwarz inequality; fractional Brownian motion
Mathematics Subject Classification: 60G12, 60G15, 60H05, 60H07

1. Introduction

Es-sebaiy and Tudor [1] derived Itô and Tanaka formulas for the d-dimensional bifractional Brownian motion using the Malliavin calculus with respect to Gaussian processes and the multiple stochastic integrals. But there exists a mistake in the proof of Lemma 2 in [1]. They said the function

$$g(x) = \left(1 + \frac{j^{2H}}{x^{2H}}\right)^{K-1} - \left(1 + \frac{(j-1)^{2H}}{x^{2H}}\right)^{K-1} \tag{1.1}$$

is decreasing on $(1, \infty)$ for $2HK = 1$. In fact, we can not obtain that $g(x)$ is decreasing on $(1, \infty)$. Because

$$g'(x) = \frac{2H(1-K)}{x^{2H+1}}\left[j^{2H}\left(1 + \frac{j^{2H}}{x^{2H}}\right)^{K-2} - (j-1)^{2H}\left(1 + \frac{(j-1)^{2H}}{x^{2H}}\right)^{K-2}\right]. \tag{1.2}$$

We can not get $g'(x) < 0$ on $(1, \infty)$. Moreover, if we take $H = \frac{5}{8}, K = \frac{4}{5}$ which satisfies $2HK = 1$, and $j = 4, x_1 = 2, x_2 = 3$, then we get by R software,

$$g(2) = \left(1 + \frac{4^{1.25}}{2^{1.25}}\right)^{-0.2} - \left(1 + \frac{3^{1.25}}{2^{1.25}}\right)^{-0.2} = -0.03839191, \tag{1.3}$$

and

$$g(3)=\left(1+\frac{4^{1.25}}{3^{1.25}}\right)^{-0.2}-\left(1+\frac{3^{1.25}}{3^{1.25}}\right)^{-0.2}=-0.0334415. \tag{1.4}$$

It is obvious that $g(2) < g(3)$, which shows that $g(x)$ is not decreasing on $(1,\infty)$. Thus it is wrong that Es-sebaiy and Tudor [1] deduced that the function $f_j(x)$ is increasing on $(1,\infty)$ by the decreasing property of $g(x)$ on $(1,\infty)$, where

$$f_j(x) := ((x-1)^{2H}+j^{2H})^K-((x-1)^{2H}+(j-1)^{2H})^K-(x^{2H}+j^{2H})^K+(x^{2H}+(j-1)^{2H})^K. \tag{1.5}$$

Similarly, if we take $H=\frac{5}{8}, K=\frac{4}{5}$ which satisfies $2HK=1$, and $j=4, x_1=2, x_2=3$, then we get by R software,

$$f_4(2)=(1+4^{1.25})^{0.8}-(1+3^{1.25})^{0.8}-(2^{1.25}+4^{1.25})^{0.8}+(2^{1.25}+3^{1.25})^{0.8}=0.04044688, \tag{1.6}$$

and

$$f_4(3)=(2^{1.25}+4^{1.25})^{0.8}-(2^{1.25}+3^{1.25})^{0.8}-(3^{1.25}+4^{1.25})^{0.8}+(3^{1.25}+3^{1.25})^{0.8}=0.03591274. \tag{1.7}$$

It is obvious that $f_4(2) > f_4(3)$, which shows that $f_j(x)$ is not increasing on $(1,\infty)$.

2. A correct proof of Lemma 2 in Es-sebaiy and Tudor [1]

In this section we correct the proof of Lemma 2 in Es-sebaiy and Tudor [1] by using the idea of Proposition 3.3 in Chen et al. [2].

Lemma 2 in Es-sebaiy and Tudor [1]. Suppose that $2HK=1$, then

$$V_t^n := \sum_{j=1}^n \left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)^2 \xrightarrow{n\to\infty} \frac{1}{2^{K-1}}t,$$

in $L^2(\Omega)$, where $\left\{t_j=\frac{jt}{n}; j=0,\cdots,n\right\}$ is a partition of $[0,t]$.

Proof. A straightforward calculation shows that,

$$\mathbf{E}(V_t^n)=\sum_{j=1}^n \mathbf{E}\left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)^2=\frac{t}{n}\sum_{j=1}^n h(j)+\frac{t}{2^{K-1}} \xrightarrow{n\to\infty} \frac{t}{2^{K-1}},$$

since $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n h(j)=0$ by (11) of Lemma 1 in [1], where

$$h(j):=j^{2HK}+(j-1)^{2HK}-\frac{2}{2^K}\left(j^{2H}+(j-1)^{2H}\right)^K.$$

To obtain the conclusion it suffices to show that

$$\lim_{n\to\infty}\mathbf{E}\left[(V_t^n)^2\right]=\left(\frac{t}{2^{K-1}}\right)^2. \tag{2.1}$$

In fact, we have

$$\begin{aligned}\mathbf{E}\left[(V_t^n)^2\right] =&\mathbf{E}\left[\left(\sum_{j=1}^{n}\left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)^2\right)^2\right]\\ =&\mathbf{E}\left\{\left[\sum_{i=1}^{n}\left(B^{H,K}(t_i)-B^{H,K}(t_{i-1})\right)^2\right]\cdot\left[\sum_{j=1}^{n}\left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)^2\right]\right\}\\ =&\sum_{i,j=1}^{n}\mathbf{E}\left[\left(B^{H,K}(t_i)-B^{H,K}(t_{i-1})\right)^2\left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)^2\right].\end{aligned} \tag{2.2}$$

Denote by

$$\mu_n(i,j) := \mathbf{E}\left[\left(B^{H,K}(t_i)-B^{H,K}(t_{i-1})\right)^2\left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)^2\right].$$

Using the fact

$$\mathbf{E}\left[X_1X_2X_3X_4\right] = \mathbf{E}\left[X_1X_2\right]\mathbf{E}\left[X_3X_4\right] + \mathbf{E}\left[X_1X_3\right]\mathbf{E}\left[X_2X_4\right] + \mathbf{E}\left[X_1X_4\right]\mathbf{E}\left[X_2X_3\right],$$

where $X_i (i = 1, 2, 3, 4)$ are normal random variables with zero mean, we get

$$\mu_n(i,j) = 2(\theta_n(i,j))^2 + \tau_n(i,j), \tag{2.3}$$

where

$$\begin{aligned}\theta_n(i,j) :=\ & \mathbf{E}\left[\left(B^{H,K}(t_i)-B^{H,K}(t_{i-1})\right)\left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)\right]\\ =\ & \frac{t}{2^K n}\left\{\left(i^{2H}+j^{2H}\right)^K - \left[i^{2H}+(j-1)^{2H}\right]^K - \left[(i-1)^{2H}+j^{2H}\right]^K\right.\\ & \left.+\left[(i-1)^{2H}+(j-1)^{2H}\right]^K - 2|i-j| + |i-j+1| + |i-j-1|\right\},\end{aligned} \tag{2.4}$$

and

$$\tau_n(i,j) := \mathbf{E}\left(B^{H,K}(t_i)-B^{H,K}(t_{i-1})\right)^2\mathbf{E}\left(B^{H,K}(t_j)-B^{H,K}(t_{j-1})\right)^2. \tag{2.5}$$

We will use the idea of the Proposition 3.3 in Chen et al. [2] to prove

$$\lim_{n\to\infty}\sum_{i,j=1}^{n}(\theta_n(i,j))^2 = 0. \tag{2.6}$$

Let

$$R^B(t,s) := \frac{1}{2^K}\left(t^{2HK} + s^{2HK} - |t-s|^{2HK}\right),$$

be the covariance function of the fractional Brownian motion with Hurst constant HK, and

$$R(t,s) := \mathbf{E}\left(B^{H,K}(t)B^{H,K}(s)\right) = \frac{1}{2^K}\left[\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right],$$

be the covariance function of the bifractional Brownian motion with indices $H \in (0,1)$ and $K \in (0,1]$.

Then we obtain

$$\Psi(t,s):=\frac{\partial^2 R(t,s)}{\partial t\partial s}-\frac{\partial^2 R^B(t,s)}{\partial t\partial s}=\frac{4H^2K(K-1)}{2^K}(t^{2H}+s^{2H})^{K-2}(ts)^{2H-1}.$$

When $2HK=1$, we have

$$|\Psi(t,s)|=\frac{2H(1-K)}{2^K}(t^{2H}+s^{2H})^{K-2}(ts)^{2H-1}\le\frac{H(1-K)}{2}(ts)^{-\frac{1}{2}}. \tag{2.7}$$

Similar to (27) in Chen et al. [2], we denote

$$\begin{aligned}\gamma_n(i,j):=&\frac{t}{2^Kn}\left\{\left(i^{2H}+j^{2H}\right)^K-\left[i^{2H}+(j-1)^{2H}\right]^K-\left[(i-1)^{2H}+j^{2H}\right]^K\right.\\&\left.+\left[(i-1)^{2H}+(j-1)^{2H}\right]^K\right\},\end{aligned} \tag{2.8}$$

and

$$\rho_n(i,j):=-\frac{t}{2^Kn}\left[2|i-j|-|i-j+1|-|i-j-1|\right]. \tag{2.9}$$

Hence, by (2.4), (2.8), (2.9) and (2.7), we have

$$\begin{aligned}|\gamma_n(i,j)|&=|\theta_n(i,j)-\rho_n(i,j)|\\&=\left|\int_{[0,t]^2}I_{(t_{i-1},t_i]}(r_1)I_{(t_{j-1},t_j]}(r_2)\Psi(r_1,r_2)dr_1dr_2\right|\\&\le\int_{[0,t]^2}I_{(t_{i-1},t_i]}(r_1)I_{(t_{j-1},t_j]}(r_2)\left|\Psi(r_1,r_2)\right|dr_1dr_2\\&\le\frac{H(1-K)}{2}\int_{[0,t]^2}I_{(t_{i-1},t_i]}(r_1)I_{(t_{j-1},t_j]}(r_2)(r_1r_2)^{-\frac{1}{2}}dr_1dr_2\\&=\frac{H(1-K)t}{n}\left[i^{1/2}-(i-1)^{1/2}\right]\left[j^{1/2}-(j-1)^{1/2}\right].\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i,j=1}^n(\gamma_n(i,j))^2\le&\frac{H^2(1-K)^2t^2}{n^2}\sum_{i,j=1}^n\left[i^{1/2}-(i-1)^{1/2}\right]^2\left[j^{1/2}-(j-1)^{1/2}\right]^2\\=&\frac{H^2(1-K)^2t^2}{n^2}\left\{\sum_{i=1}^n\left[i^{1/2}-(i-1)^{1/2}\right]^2\right\}^2.\end{aligned} \tag{2.10}$$

It is clear that the function $f(u)=(1+u)^{1/2}$ in $u\in[0,\infty)$ is concave, i.e., $f''(u)\le 0$, which implies that for any $u\ge 0, (1+u)^{1/2}\le 1+\frac{1}{2}u$. Hence, for any $i\ge 2$,

$$i^{1/2}-(i-1)^{1/2}=(i-1)^{1/2}\left[\left(1+\frac{1}{i-1}\right)^{1/2}-1\right]\le\frac{1}{2}(i-1)^{-1/2}. \tag{2.11}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{n}\left[i^{1/2}-(i-1)^{1/2}\right]^2 &= 1+\sum_{i=2}^{n}\left[i^{1/2}-(i-1)^{1/2}\right]^2\\ &\leq 1+\frac{1}{4}\sum_{i=2}^{n}(i-1)^{-1}\\ &= 1+\frac{1}{4}\sum_{i=1}^{n-1}\frac{1}{i}\\ &\sim \frac{1}{4}\log(n),\end{aligned} \tag{2.12}$$

since $\sum_{i=1}^{n-1}\frac{1}{i}\sim\log(n)$, where $a_n\sim b_n$ means $\lim_{n\to\infty}\frac{a_n}{b_n}=1$. By (2.10) and (2.12), we have, as $n\to\infty$,

$$\sum_{i,j=1}^{n}(\gamma_n(i,j))^2\leq\frac{H^2(1-K)^2t^2\left(\frac{1}{4}\log(n)\right)^2}{n^2}\to 0. \tag{2.13}$$

By (2.9), we obtain

$$\rho_n(i,j)=\begin{cases}\frac{t}{2^{K-1}n}, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases}$$

Hence, we have, as $n\to\infty$,

$$\sum_{i,j=1}^{n}(\rho_n(i,j))^2=\sum_{i=1}^{n}\frac{t^2}{2^{2K-2}n^2}=\frac{t^2}{2^{2K-2}n}\to 0. \tag{2.14}$$

The Cauchy-Schwarz inequality, (2.13) and (2.14) imply that, as $n\to\infty$,

$$\left|\sum_{i,j=1}^{n}\gamma_n(i,j)\rho_n(i,j)\right|\leq\left[\sum_{i,j=1}^{n}(\gamma_n(i,j))^2\sum_{i,j=1}^{n}(\rho_n(i,j))^2\right]^{1/2}\to 0. \tag{2.15}$$

So by (2.13)–(2.15), we get, as $n\to\infty$,

$$\begin{aligned}\sum_{i,j=1}^{n}(\theta_n(i,j))^2 &= \sum_{i,j=1}^{n}(\gamma_n(i,j)+\rho_n(i,j))^2\\ &= \sum_{i,j=1}^{n}(\gamma_n(i,j))^2+2\sum_{i,j=1}^{n}[\gamma_n(i,j)\rho_n(i,j)]+\sum_{i,j=1}^{n}(\rho_n(i,j))^2\\ &\to 0,\end{aligned}$$

thus (2.6) holds.

Next, we consider $\tau_n(i,j)$. When $2HK=1$, we have,

$$\begin{aligned}\mathbf{E}\left(B^{H,K}(t_i)-B^{H,K}(t_{i-1})\right)^2 &= \frac{t}{n}\left\{i^{2HK}+(i-1)^{2HK}-\frac{2}{2^K}\left[i^{2H}+(i-1)^{2H}\right]^K+\frac{1}{2^{K-1}}\right\}\\ &= \frac{t}{n}\left[h(i)+\frac{1}{2^{K-1}}\right],\end{aligned}$$

and

$$\mathbf{E}\left(B^{H,K}(t_j) - B^{H,K}(t_{j-1})\right)^2 = \frac{t}{n}\left[h(j) + \frac{1}{2^{K-1}}\right].$$

Hence, by (2.5),

$$\begin{aligned}
\sum_{i,j=1}^{n} \tau_n(i,j) = &\ \frac{t^2}{n^2}\sum_{i,j=1}^{n}\left[h(i) + \frac{1}{2^{K-1}}\right]\left[h(j) + \frac{1}{2^{K-1}}\right] \\
= &\ \frac{t^2}{n^2}\sum_{i=1}^{n}\left[h(i) + \frac{1}{2^{K-1}}\right]\sum_{j=1}^{n}\left[h(j) + \frac{1}{2^{K-1}}\right] \\
= &\ \frac{t^2}{n^2}\left(\sum_{i=1}^{n}\left[h(i) + \frac{1}{2^{K-1}}\right]\right)^2 \\
= &\ \frac{t^2}{n^2}\left(\sum_{i=1}^{n} h(i) + \frac{n}{2^{K-1}}\right)^2 \\
= &\ \left(\frac{t}{2^{K-1}}\right)^2 + \frac{t^2}{2^{K-2}n}\sum_{i=1}^{n} h(i) + \frac{t^2}{n^2}\left(\sum_{i=1}^{n} h(i)\right)^2.
\end{aligned}$$

By (11) of Lemma 1 in [1], we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} h(i) = 0.$$

Therefore, we have, as $n \to \infty$,

$$\sum_{i,j=1}^{n} \tau_n(i,j) \to \left(\frac{t}{2^{K-1}}\right)^2. \tag{2.16}$$

So (2.1) holds from (2.2), (2.3), (2.6) and (2.16). We finished the proof of Lemma 2 in [1]. □

3. Conclusions

By using the idea of Proposition 3.3 in [2], we maily correct the proof of Lemma 2 in [1] about multidimensional bifractional Brownian motion: Itô and Tanaka formulas. The idea of Proposition 3.3 in [2] is very important. It can be also applied to the calculation of the quadratic variation of other Gaussian processes such as the sub-fractional Brownian motion, the sub-bifractional Brownian motion and so on.

Author contributions

Huantian Xie has given some ideas about the correction of proof of Lemma 2 in Es-sebaiy and Tudor (2007) and has corrected some grammar mistakes in this paper. Nenghui Kuang has written this paper. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Nenghui Kuang was supported by the Natural Science Foundation of Hunan Province under Grant 2021JJ30233 and by the Scientific Research Project of the Education Department of Hunan Province under Grant 24A0345. Huantian Xie was partly supported by the NSF of China (No. 12271233), the Shandong SME Innovation Project (No. 2023TSGC0466), and the Horizontal Project of Linyi University (No. HX250341).

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. K. Es-sebaiy, C. A.Tudor, Multidimensional bifractional Brownian motion: Itô and Tanaka formulas, *Stoch. Dyn.*, **7** (2007), 365–388.
2. Y. Chen, Z. Ding, Y. Li, Berry-Esséen bounds and almost sure CLT for the quadratic variation of a class of Gaussian process, *Commun. Stat. Theory Methods*, **53** (2024), 3920–3939. https://doi.org/10.1080/03610926.2023.2167055

AIMS Press