Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 26260–26272.
DOI: 10.3934/math.20261053
Received: 10 June 2026
Revised: 15 August 2026
Accepted: 18 August 2026
Published: 24 August 2026

Research article

Evolution of the first eigenvalue of a class of geometric operators under the weighted Yamabe flow

Jingyu Xiao and Jiancheng Liu*

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: liujc@nwnu.edu.cn.

Abstract: This paper derived the evolution equation for the first eigenvalue of the operator $-\Delta_\phi+a(R^m_\phi)^\alpha$ under both the unnormalized and normalized weighted Yamabe flows. For constants $0 < \alpha < 1$, and $a>0$, we then establish monotone quantities associated with this eigenvalue under suitable assumptions on the weighted scalar curvature.

Keywords: weighted Laplacian operator; first eigenvalue; weighted Yamabe flow; monotonicity
Mathematics Subject Classification: 53C21, 53C25

1. Introduction

Eigenvalues of geometric operators play an important role in understanding the geometry and topology of Riemannian manifolds. In [1], Perelman introduced the F-entropy functional and proved that it is nondecreasing along the Ricci flow coupled to a backward heat-type equation, which implies the monotonicity of the first eigenvalue of the operator $-4\Delta + R$ along the Ricci flow. Motivated by this work, considerable effort has been devoted to studying the monotonicity for the first eigenvalue of geometric operators under various geometric flows, especially the Ricci flow and the Yamabe flow.

For example, in [2], Cao proved that all eigenvalues of $-\Delta + \frac{1}{2}R$ with a nonnegative curvature operator are nondecreasing along the Ricci flow in dimensions two and three. In particular, Li obtained in [3] that the first eigenvalue of $-\Delta + \frac{1}{2}R$ is nondecreasing without curvature assumptions under the Ricci flow. Furthermore, Cao showed in [4] that the first eigenvalue of $-\Delta + aR$ for a constant $a \geq \frac{1}{4}$ is nondecreasing under the Ricci flow. Similarly, Ho in [5] proved that the first eigenvalue of $-\Delta + aR$ is nondecreasing under the Yamabe flow if $0 \leq a < \frac{n-2}{4(n-1)}$ or $a \geq \frac{n-2}{4(n-1)}$ with the scalar curvature satisfying certain conditions.

Also, there are many results on the first eigenvalue of geometric operators involving the weighted Laplacian $\Delta_\phi = \Delta - \nabla\phi \cdot \nabla$. For instance, Fang et al. in [6] generalized Cao's results to eigenvalues of the geometric operator $-\Delta_\phi + aR$ with $a > \frac{1}{4}$. They showed that the eigenvalues are nondecreasing along the Ricci flow coupled to a heat equation on a Riemannian manifold under suitable Ricci curvature conditions. Afterward, this first-eigenvalue problem was extended to other geometric flows. For example, Zhang in [7] considered the evolution equations for the eigenvalues of the operator $-\Delta_\phi + aR$ and obtained some nondecreasing quantities along the Yamabe flow. More generally, Ho and Shin in [8] studied the evolution of the first eigenvalue of $-\Delta_\phi + aR^m_\phi$ and proved that its first eigenvalue is nondecreasing along the unnormalized weighted Yamabe flow, if

$$0 \leq a \leq \frac{m+n-2}{4(m+n-1)} \quad \text{or} \quad \frac{m+n-2}{4(m+n-1)} \leq a \leq \frac{m+n}{4(m+n-1)}$$

provided that the weighted scalar curvature R^m_ϕ satisfies certain restrictions.

Recently, Yang and Zhang in [9] derived evolution equations for the first eigenvalue of $-\Delta + aR^\alpha$ and proved that the first eigenvalue of $-\Delta + aR^\alpha$ for either $0 < \alpha < 1,\ a > 0$, or $\alpha < 0,\ a < 0$, is nondecreasing along the Ricci flow if

$$0 < R_{g(t_0)} < (4a\alpha)^{\frac{1}{1-\alpha}}.$$

In [10], Abolarinwa and Azami used the same technique to derive the evolution equations for the first eigenvalue of $-\Delta + aR^\alpha$ with $0 < \alpha < 1,\ a > 0$, and obtained some monotone quantities under the Yamabe flow.

Inspired by these recent results, we study in this paper the evolution equation for the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ with constants $0 < \alpha < 1,\ a > 0$ and construct a monotone quantity associated with the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ under the unnormalized (or normalized) weighted Yamabe flow. These results provide a framework for understanding spectral stability along the weighted Yamabe flow and may serve as a useful tool in studying convergence and rigidity phenomena of weighted geometric structures.

Before stating the main results, we need to introduce the geometric flow and some terminology. A *smooth metric measure space* is a quadruple $(M^n, g, e^{-\phi}dV_g, m)$ consisting of a Riemannian manifold (M^n, g), a smooth measure $e^{-\phi}dV_g$ determined by a function $\phi \in C^\infty(M)$ and the Riemannian volume element of g, and a dimensional parameter $m \in [0, \infty)$. In the case $m = 0$, we require $\phi = 0$. The weighted scalar curvature R^m_ϕ of a smooth metric measure space $(M^n, g, e^{-\phi}dV_g, m)$ is defined as

$$R^m_\phi := \begin{cases} R + 2\Delta\phi - \dfrac{m+1}{m}|\nabla\phi|^2, & m > 0, \\ R, & m = 0, \end{cases}$$

where R, Δ, and ∇ are the scalar curvature, the Laplacian, and the gradient with respect to g, respectively.

In [8], Ho and Shin first introduced the *unnormalized weighted Yamabe flow*, which is the evolution equation defined on $(M^n, g(t), e^{-\phi(t)}dV_{g(t)}, m)$ given by

$$\begin{cases} \frac{\partial g}{\partial t} = -R^m_\phi g, \\ \frac{\partial \phi}{\partial t} = \frac{m}{2} R^m_\phi, \end{cases} \tag{1.1}$$

where $R^m_\phi = R^m_{\phi(t)}$ is the weighted scalar curvature. For related recent work, see [11–13].

The main results of this paper are stated as follows. First, we obtain the evolution equation for the first eigenvalue of $-\Delta_\phi + a(R^m_\phi)^\alpha$ under the unnormalized weighted Yamabe flow (1.1).

Theorem 1.1. *Let (M^n, g) be a closed connected Riemannian manifold and let $(g(t), \phi(t))$, $t \in [0, T)$ be a solution of the unnormalized weighted Yamabe flow (1.1) on a smooth metric measure space $(M^n, g, e^{-\phi}dV_g, m)$ with $R^m_\phi > 0$. Suppose that $\lambda(t)$ is the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ for constants a and α, and that $f(t)$ is the corresponding eigenfunction. Then, under the flow (1.1), $\lambda(t)$ evolves by*

$$\begin{aligned}\frac{d}{dt}\lambda(t) = \lambda(t)\int_M \left(\frac{m+n}{2}R^m_\phi - 2a(m+n-1)(R^m_\phi)^\alpha\right) f^2 e^{-\phi}dV_g \\ + a\int_M \left(2a(m+n-1)(R^m_\phi)^\alpha - \frac{m+n-2\alpha}{2}R^m_\phi\right)(R^m_\phi)^\alpha f^2 e^{-\phi}dV_g \\ + a\alpha(1-\alpha)(m+n-1)\int_M (R^m_\phi)^{\alpha-2}|\nabla R^m_\phi|^2 f^2 e^{-\phi}dV_g \\ + \int_M \left(2a(m+n-1)(R^m_\phi)^{\alpha-1} - \frac{m+n-2}{2}\right) R^m_\phi |\nabla f|^2 e^{-\phi}dV_g.\end{aligned} \tag{1.2}$$

Remark 1.2. The linear weighted case $\alpha = 1$ was treated by Ho and Shin [8, Proposition 5.10] under the weighted Yamabe flow, while the nonlinear unweighted case $0 < \alpha < 1$ was studied by Abolarinwa and Azami [10, Theorem 1.1]. Our result connects these two settings: In the limit $\alpha \to 1$, our formula formally reduces to the linear weighted case studied in [8], and when $m = 0$ and $\phi = 0$, it recovers the corresponding unweighted setting in [10].

Using Theorem 1.1, we can establish the monotone quantity associated with the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ under the flow (1.1). Although the weighted Yamabe flow shares some similarities with the classical Yamabe flow, the combination of the weighted measure and a nonlinear power of the weighted scalar curvature introduces additional terms into the evolution equation, including both the gradient of the weighted scalar curvature and nonlinear curvature interactions. Therefore, new curvature control conditions are required to recover the monotonicity property. The pointwise curvature restrictions are chosen to control the signs of the integral terms involving $(R^m_\phi)^\alpha f^2$ and $R^m_\phi|\nabla f|^2$ in the evolution equation. In particular, the upper bound in (1.3) ensures the required coefficient inequalities, while the two-sided bound in (1.4) yields the monotonicity of $\lambda(t)$ itself.

Theorem 1.3. *Let (M^n, g) be a closed connected Riemannian manifold. Let $(g(t), \phi(t))$, $t \in [0, T)$, $T < \frac{1}{\max_M R^m_{\phi_0}}$ (where $\phi_0 = \phi(0)$) be a solution of the unnormalized weighted Yamabe flow (1.1) and $\lambda(t)$ be the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$. Suppose that for some constants $0 < \alpha < 1$, $a > 0$, the weighted scalar curvature R^m_ϕ satisfies*

$$0 < (R^m_\phi)^\alpha \le \left(\frac{4a(m+n-1)}{m+n-2\alpha}\right)^{\frac{\alpha}{1-\alpha}}. \tag{1.3}$$

Then, along the flow (1.1), the quantity

$$\lambda(t)\exp\left\{-\int_0^t \left(\frac{m+n}{2}y_1(s) - 2a(m+n-1)y_2^\alpha(s)\right)ds\right\}$$

is nondecreasing, where

$$y_1(s) = \frac{\min_M R^m_{\phi_0}}{1 - s\min_M R^m_{\phi_0}} \quad \textit{and} \quad y_2(s) = \frac{\max_M R^m_{\phi_0}}{1 - s\max_M R^m_{\phi_0}}.$$

In particular, if

$$\left(\frac{4a(m+n-1)}{m+n}\right)^{\frac{\alpha}{1-\alpha}} \leq (R^m_\phi)^\alpha \leq \left(\frac{4a(m+n-1)}{m+n-2\alpha}\right)^{\frac{\alpha}{1-\alpha}}, \tag{1.4}$$

then $\lambda(t)$ is nondecreasing under the flow (1.1).

Remark 1.4. (i) It is worth comparing our result for the case $m = 0$ and $\phi = 0$ with [10, Theorem 1.2]. In [10, Theorem 1.2], the monotonicity of the first eigenvalue requires either

$$a \geq \frac{n-2}{4(n-1)} \quad \text{and} \quad 0 \leq R_{g(t)} \leq 1,$$

or

$$0 \leq a \leq \frac{n-2}{4(n-1)} \quad \text{and} \quad 0 \leq R^\alpha_{g(t)} \leq \left(\frac{4a(n-1)}{n-2\alpha}\right)^{\frac{\alpha}{1-\alpha}}$$

on [0,T). In contrast, our result is obtained under the curvature condition (1.3), which directly couples the parameter a with the admissible range of the weighted scalar curvature. Therefore, the assumptions in the two results are not directly comparable. Our result provides an alternative curvature criterion for obtaining a monotone quantity associated with the first eigenvalue.

(ii) The case $\alpha = 1$ was studied by Ho and Shin [8], who showed that the first eigenvalue of the operator $-\Delta_\phi + aR^m_\phi$ is nondecreasing along the unnormalized weighted Yamabe flow (1.1) if

$$0 \leq a \leq \frac{m+n-2}{4(m+n-1)} \quad \text{and} \quad \min_M R^m_\phi \geq \frac{m+n-2}{m+n} \max_M R^m_\phi \geq 0,$$

or

$$\frac{m+n-2}{4(m+n-1)} \leq a \leq \frac{m+n}{4(m+n-1)} \quad \text{and} \quad \min_M R^m_\phi \geq 0.$$

Remark 1.5. We emphasize that the curvature conditions (1.3) and (1.4) are assumed to hold throughout the time interval under consideration. In the present paper, we do not establish a preservation result for these pointwise curvature conditions along the weighted Yamabe flow. Thus, these assumptions are understood as hypotheses imposed on the flow over the time interval considered in Theorem 1.3, rather than as conditions whose preservation from the initial data is claimed. Determining sufficient conditions on the initial data that ensure the preservation of these bounds along the flow, or constructing explicit non-stationary flows satisfying these conditions on a time interval, remains an interesting problem for future investigation.

2. Evolution equations under unnormalized weighted Yamabe flow

Let Δ_ϕ: $C^\infty(M) \to C^\infty(M)$ be the weighted Laplacian operator on M^n associated to $\phi \in C^\infty(M)$, i.e., for any $u \in C^\infty(M)$,

$$\Delta_\phi u = \Delta u - \langle \nabla\phi, \nabla u\rangle,$$

where ∇ is the gradient operator and Δ is the Laplace-Beltrami operator on M. When ϕ is a constant, Δ_ϕ is just the Laplace-Beltrami operator. It is formally self-adjoint with respect to the measure $e^{-\phi}dV_g$.

We assume that $(M, g(t))$ is a closed connected manifold. By eigenvalue perturbation theory [8, 14], we may assume that there exist C^1 functions $\lambda(t)$ and $f(t)$ such that $\lambda(t)$ is the first eigenvalue and $f(t)$ is a corresponding eigenfunction of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$, i.e.,

$$-\Delta_\phi f + a(R^m_\phi)^\alpha f = \lambda(t) f. \tag{2.1}$$

Throughout this paper, by rescaling, we may normalize $f(t)$ to satisfy

$$\int_M f^2 e^{-\phi} dV_g = 1. \tag{2.2}$$

Multiplying both sides of (2.1) by f, integrating on M, and then using (2.2), we obtain

$$\lambda(t) = \int_M (-\Delta_\phi f + a(R^m_\phi)^\alpha f) f e^{-\phi} dV_g. \tag{2.3}$$

In this section, we derive evolution equations for $\lambda(t)$ under the unnormalized weighted Yamabe flow (1.1). In fact, we can obtain the evolution equation of $\lambda(t)$ under the general geometric flow

$$\frac{\partial}{\partial t} g_{ij} = s_{ij},$$

where s_{ij} is a symmetric $(0, 2)$-tensor.

Lemma 2.1. *Let (M^n, g) be a closed connected Riemannian manifold. Let $\lambda(t)$ be the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ at time $t \in [0, T)$ and let $f(t)$ be the corresponding eigenfunction. Suppose that the metric $g(t)$ evolves by*

$$\frac{\partial}{\partial t} g_{ij} = s_{ij}, \tag{2.4}$$

where s_{ij} is a symmetric $(0, 2)$-tensor. Assume that $R^m_\phi > 0$ on $M \times [0, T)$. Then we have

$$\begin{aligned} \frac{d}{dt}\lambda(t) = &\int_M \left(s_{ij} f_{ij} - s_{ij}\phi_i f_j + a\alpha (R^m_\phi)^{\alpha-1} \frac{\partial R^m_\phi}{\partial t} f \right) f e^{-\phi} dV_g \\ &+ \int_M \left(s_{ij,i} - \frac{1}{2} S_j \right) f_j f e^{-\phi} dV_g + \int_M (\phi_t)_i f_i f e^{-\phi} dV_g, \end{aligned} \tag{2.5}$$

where

$$S = g^{ij} s_{ij}, \quad f_j = \nabla_j f, \quad f_{ij} = \nabla_i \nabla_j f, \quad s_{ij,i} = \nabla_i s_{ij}, \quad \phi_t = \frac{\partial}{\partial t}\phi.$$

Proof. It follows from [15, Proposition 3.1] that the weighted Laplacian evolves according to

$$\frac{\partial}{\partial t}\Delta_\phi f = -s_{ij} f_{ij} - (s_{ij,i} - \frac{1}{2} S_j) f_j + \Delta_\phi f_t + s_{ij}\phi_i f_j - (\phi_t)_i f_i.$$

Hence, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t) =&\frac{d}{dt}\int_M (-\Delta_\phi f + a(R^m_\phi)^\alpha f) f e^{-\phi} dV_g\\
=&\int_M \left(\frac{\partial}{\partial t}(-\Delta_\phi f) + a\left(\frac{\partial}{\partial t}(R^m_\phi)^\alpha\right) f + a(R^m_\phi)^\alpha f_t\right) f e^{-\phi} dV_g\\
&+\int_M (-\Delta_\phi f + a(R^m_\phi)^\alpha f)(f e^{-\phi} dV_g)_t\\
=&\int_M \left(s_{ij} f_{ij} - s_{ij}\phi_i f_j + a\alpha(R^m_\phi)^{\alpha-1}\frac{\partial R^m_\phi}{\partial t} f\right) f e^{-\phi} dV_g\\
&+\int_M \left(s_{ij,i} - \frac{1}{2}S_j\right) f_j f e^{-\phi} dV_g + \int_M (\phi_t)_i f_i f e^{-\phi} dV_g\\
&+\int_M (-\Delta_\phi f_t + a(R^m_\phi)^\alpha f_t) f e^{-\phi} dV_g\\
&+\int_M (-\Delta_\phi f + a(R^m_\phi)^\alpha f)(f e^{-\phi} dV_g)_t.\end{aligned}$$

Using integration by parts, we obtain that

$$\int_M (-\Delta_\phi f_t + a(R^m_\phi)^\alpha f_t) f e^{-\phi} dV_g = \int_M (-\Delta_\phi f + a(R^m_\phi)^\alpha f) f_t e^{-\phi} dV_g.$$

Then, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t) =&\int_M \left(s_{ij} f_{ij} - s_{ij}\phi_i f_j + a\alpha(R^m_\phi)^{\alpha-1}\frac{\partial R^m_\phi}{\partial t} f\right) f e^{-\phi} dV_g + \int_M \left(s_{ij,i} - \frac{1}{2}S_j\right) f_j f e^{-\phi} dV_g\\
&+\int_M (\phi_t)_i f_i f e^{-\phi} dV_g + \lambda(t)\int_M f(f_t e^{-\phi} dV_g + (f e^{-\phi} dV_g)_t).\end{aligned} \tag{2.6}$$

Meanwhile, by the normalization condition (2.2), we know

$$\frac{d}{dt}\int_M f^2 e^{-\phi} dV_g = \int_M f(f_t e^{-\phi} dV_g + (f e^{-\phi} dV_g)_t) = 0. \tag{2.7}$$

Combining (2.6) with (2.7), we complete the proof of Lemma 2.1. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall that under the unnormalized weighted Yamabe flow, the weighted scalar curvature evolves by ([8, Section 5.1])

$$\frac{\partial R^m_\phi}{\partial t} = (m+n-1)\Delta_\phi R^m_\phi + (R^m_\phi)^2. \tag{2.8}$$

Applying Lemma 2.1 with $s_{ij} = -R^m_\phi g_{ij}$, $\phi_t = \frac{m}{2}R^m_\phi$, and using (2.8), we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(t) =& -\int_M R^m_\phi(\Delta_\phi f) f e^{-\phi} dV_g + \frac{m+n-2}{2}\int_M \langle\nabla R^m_\phi, \nabla f\rangle f e^{-\phi} dV_g\\
&+ a\alpha\int_M (R^m_\phi)^{\alpha-1}((m+n-1)\Delta_\phi R^m_\phi + (R^m_\phi)^2) f^2 e^{-\phi} dV_g.\end{aligned} \tag{2.9}$$

Since M^n is closed, the boundary integral vanishes. Further, using integration by parts yields

$$\int_M \langle \nabla R_\phi^m, \nabla f\rangle f e^{-\phi} dV_g = -\int_M R_\phi^m (\Delta_\phi f) f e^{-\phi} dV_g - \int_M R_\phi^m |\nabla f|^2 e^{-\phi} dV_g \tag{2.10}$$

and

$$\begin{aligned}\int_M (R_\phi^m)^{\alpha-1}(\Delta_\phi R_\phi^m) f^2 e^{-\phi} dV_g =& -(\alpha-1)\int_M (R_\phi^m)^{\alpha-2}|\nabla R_\phi^m|^2 f^2 e^{-\phi} dV_g \\ & - 2\int_M (R_\phi^m)^{\alpha-1}\langle \nabla R_\phi^m, \nabla f\rangle f e^{-\phi} dV_g.\end{aligned} \tag{2.11}$$

By integration by parts and (2.10),

$$\int_M (R_\phi^m)^{\alpha-1}\langle \nabla R_\phi^m, \nabla f\rangle f e^{-\phi} dV_g = -\frac{1}{\alpha}\int_M (R_\phi^m)^{\alpha}((\Delta_\phi f) f + |\nabla f|^2) e^{-\phi} dV_g. \tag{2.12}$$

Combining (2.11) with (2.12), we get

$$\begin{aligned}\int_M (R_\phi^m)^{\alpha-1}(\Delta_\phi R_\phi^m) f^2 e^{-\phi} dV_g =& -(\alpha-1)\int_M (R_\phi^m)^{\alpha-2}|\nabla R_\phi^m|^2 f^2 e^{-\phi} dV_g \\ & + \frac{2}{\alpha}\int_M (R_\phi^m)^{\alpha}((\Delta_\phi f) f + |\nabla f|^2) e^{-\phi} dV_g.\end{aligned} \tag{2.13}$$

Substituting (2.10) and (2.13) into (2.9), we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(t) =& -\frac{m+n}{2}\int_M R_\phi^m (\Delta_\phi f) f e^{-\phi} dV_g + a\alpha\int_M (R_\phi^m)^{\alpha+1} f^2 e^{-\phi} dV_g \\ & - a\alpha(\alpha-1)(m+n-1)\int_M (R_\phi^m)^{\alpha-2}|\nabla R_\phi^m|^2 f^2 e^{-\phi} dV_g \\ & + 2a(m+n-1)\int_M (R_\phi^m)^{\alpha}(\Delta_\phi f) f e^{-\phi} dV_g \\ & + \int_M \left(2a(m+n-1)(R_\phi^m)^{\alpha-1} - \frac{m+n-2}{2}\right) R_\phi^m |\nabla f|^2 e^{-\phi} dV_g.\end{aligned} \tag{2.14}$$

According to (2.1),

$$-\Delta_\phi f = \lambda(t) f - a(R_\phi^m)^\alpha f. \tag{2.15}$$

Inserting (2.15) into (2.14) yields

$$\begin{aligned}\frac{d}{dt}\lambda(t) =& \frac{m+n}{2}\int_M R_\phi^m (\lambda(t) f - a(R_\phi^m)^\alpha f) f e^{-\phi} dV_g + a\alpha\int_M (R_\phi^m)^{\alpha+1} f^2 e^{-\phi} dV_g \\ & - a\alpha(\alpha-1)(m+n-1)\int_M (R_\phi^m)^{\alpha-2}|\nabla R_\phi^m|^2 f^2 e^{-\phi} dV_g \\ & - 2a(m+n-1)\int_M (R_\phi^m)^{\alpha}(\lambda(t) f - a(R_\phi^m)^\alpha f) f e^{-\phi} dV_g \\ & + \int_M \left(2a(m+n-1)(R_\phi^m)^{\alpha-1} - \frac{m+n-2}{2}\right) R_\phi^m |\nabla f|^2 e^{-\phi} dV_g.\end{aligned}$$

Simplifying the above equation completes the proof of Theorem 1.1. □

3. A monotone quantity along the unnormalized weighted Yamabe flow

In this section, using the evolution equation obtained in Theorem 1.1, we prove Theorem 1.3.

Proof of Theorem 1.3. Since $0 < \alpha < 1$, $a > 0$, raising both sides of (1.3) to the power $\frac{\alpha-1}{\alpha}$ yields

$$(R_\phi^m)^{\alpha-1} \geq \frac{m+n-2\alpha}{4a(m+n-1)}.$$

Furthermore, we have

$$2a(m+n-1)(R_\phi^m)^{\alpha-1} - \frac{m+n-2}{2} \geq 0, \tag{3.1}$$

$$2a(m+n-1)(R_\phi^m)^{\alpha} - \frac{m+n-2\alpha}{2} R_\phi^m \geq 0. \tag{3.2}$$

Inserting (3.1) and (3.2) into (1.2), and dropping the nonnegative terms, we get

$$\frac{d}{dt}\lambda(t) \geq \lambda(t) \int_M \Big(\frac{m+n}{2} R_\phi^m - 2a(m+n-1)(R_\phi^m)^\alpha\Big) f^2 e^{-\phi} dV_g. \tag{3.3}$$

Recall that the weighted scalar curvature R_ϕ^m satisfies the evolution Eq (2.8). By the maximum principle (see [8, 16]), R_ϕ^m can be compared with the solutions y_1 and y_2 of $y'(t) = y^2(t)$ with initial values

$$y_1(0) = \min_M R_{\phi_0}^m \quad \text{and} \quad y_2(0) = \max_M R_{\phi_0}^m,$$

respectively. Hence,

$$R_\phi^m \geq y_1(t) = \frac{\min\limits_M R_{\phi_0}^m}{1 - t \min\limits_M R_{\phi_0}^m}, \tag{3.4}$$

$$R_\phi^m \leq y_2(t) = \frac{\max\limits_M R_{\phi_0}^m}{1 - t \max\limits_M R_{\phi_0}^m}, \tag{3.5}$$

for $t \in [0, T)$, $T < \frac{1}{\max_M R_{\phi_0}^m}$. Substituting (3.4) and (3.5) into (3.3), we obtain

$$\frac{d}{dt}\lambda(t) \geq \lambda(t)\Big(\frac{m+n}{2} y_1(t) - 2a(m+n-1) y_2^\alpha(t)\Big).$$

Furthermore, we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\lambda(t) \exp\left\{-\int_0^t \Big(\frac{m+n}{2} y_1(s) - 2a(m+n-1) y_2^\alpha(s)\Big) ds\right\}\right\} \\
&= \exp\left\{-\int_0^t \Big(\frac{m+n}{2} y_1(s) - 2a(m+n-1) y_2^\alpha(s)\Big) ds\right\} \\
&\quad \times \left\{\frac{d}{dt}\lambda(t) - \lambda(t)\Big(\frac{m+n}{2} y_1(t) - 2a(m+n-1) y_2^\alpha(t)\Big)\right\} \\
&\geq 0.
\end{aligned} \tag{3.6}$$

Since $0 < \alpha < 1$, $a > 0$, condition (1.4) implies that

$$(R^m_\phi)^\alpha \geq \left(\frac{4a(m+n-1)}{m+n}\right)^{\frac{\alpha}{1-\alpha}}.$$

Raising both sides of the previous equation to the power $\frac{\alpha-1}{\alpha}$, after simplification, we obtain

$$\frac{m+n}{2}R^m_\phi - 2a(m+n-1)(R^m_\phi)^\alpha \geq 0.$$

By (2.3), we get $\lambda(t) \geq 0$. Hence, we have $\frac{d}{dt}\lambda(t) \geq 0$. This completes the proof of Theorem 1.3. □

4. Evolution equation under the normalized weighted Yamabe flow

In this section, we consider the evolution equation for the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ under the normalized weighted Yamabe flow.

The *normalized weighted Yamabe flow* was first introduced in [17] by Yan, which is the evolution equation defined on $(M^n, g, e^{-\phi}dV_g, m)$ given by

$$\begin{cases} \frac{\partial g}{\partial t} = -(R^m_\phi - r^m_\phi)g, \\ \frac{\partial \phi}{\partial t} = \frac{m}{2}(R^m_\phi - r^m_\phi), \end{cases} \tag{4.1}$$

where $r^m_\phi = r^m_\phi(t)$ is the average of the weighted scalar curvature R^m_ϕ, i.e.,

$$r^m_\phi = \frac{\int_M R^m_\phi e^{-\phi}dV_g}{\int_M e^{-\phi}dV_g}.$$

The normalized weighted Yamabe flow preserves the total weighted volume by incorporating r^m_ϕ as a damping term, whereas the unnormalized flow allows the volume to change. This distinction introduces additional r^m_ϕ-dependent terms in the eigenvalue evolution equation under the normalized flow, which require corresponding corrections in the associated monotone functionals.

In [17], Yan proved that the normalized weighted Yamabe flow exists for all time and converges to a metric with constant weighted scalar curvature. Ho et al. in [18] considered the convergence rate of the normalized weighted Yamabe flow and showed that if the flow converges to an integrable critical point, then the convergence is exponential; in general, the convergence cannot be worse than a polynomial rate.

Recall that under the normalized weighted Yamabe flow, the weighted scalar curvature R^m_ϕ evolves by the following equation (see [17]):

$$\frac{\partial R^m_\phi}{\partial t} = (m+n-1)\Delta_\phi R^m_\phi + R^m_\phi(R^m_\phi - r^m_\phi). \tag{4.2}$$

Using the same argument as in the previous section, we obtain the following evolution equation (Theorem 4.1) and monotonicity result (Theorem 4.3).

Theorem 4.1. *Let (M^n, g) be a closed connected Riemannian manifold, and let $(g(t), \phi(t))$, $t \in [0, T)$, be a solution of the normalized weighted Yamabe flow (4.1) on a smooth metric measure space*

$(M^n, g, e^{-\phi}dV_g, m)$ with $R^m_\phi > 0$. Suppose that $\lambda(t)$ is the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ for constants a and α, and $f(t)$ is the corresponding eigenfunction. Then, under the flow (4.1), $\lambda(t)$ satisfies the evolution equation

$$\begin{aligned}\frac{d}{dt}\lambda(t) =& \lambda(t)\int_M\left(\frac{m+n}{2}R^m_\phi - 2a(m+n-1)(R^m_\phi)^\alpha\right)f^2e^{-\phi}dV_g\\&+ a\int_M\left(2a(m+n-1)(R^m_\phi)^\alpha - \frac{m+n-2\alpha}{2}R^m_\phi\right)(R^m_\phi)^\alpha f^2e^{-\phi}dV_g\\&+ a\alpha(1-\alpha)(m+n-1)\int_M(R^m_\phi)^{\alpha-2}|\nabla R^m_\phi|^2f^2e^{-\phi}dV_g\\&+ \int_M\left(2a(m+n-1)(R^m_\phi)^{\alpha-1} - \frac{m+n-2}{2}\right)R^m_\phi|\nabla f|^2e^{-\phi}dV_g\\&+ a(1-\alpha)r^m_\phi\int_M(R^m_\phi)^\alpha f^2e^{-\phi}dV_g - r^m_\phi\lambda(t).\end{aligned}$$

Proof. Set $s_{ij} = -(R^m_\phi - r^m_\phi)g_{ij}$ and $\phi_t = \frac{m}{2}(R^m_\phi - r^m_\phi)$. Then, by Lemma 2.1, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t) =& -\int_M(R^m_\phi - r^m_\phi)(\Delta_\phi f)fe^{-\phi}dV_g + \frac{m+n-2}{2}\int_M\langle\nabla R^m_\phi, \nabla f\rangle fe^{-\phi}dV_g\\&+ a\alpha\int_M(R^m_\phi)^{\alpha-1}((m+n-1)\Delta_\phi R^m_\phi + R^m_\phi(R^m_\phi - r^m_\phi))f^2e^{-\phi}dV_g.\end{aligned}$$

Proceeding as in the derivation of (2.9), we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(t) =& -\frac{m+n}{2}\int_M(R^m_\phi - r^m_\phi)(\Delta_\phi f)fe^{-\phi}dV_g + a\alpha\int_M(R^m_\phi)^{\alpha+1}f^2e^{-\phi}dV_g\\&- a\alpha(\alpha-1)(m+n-1)\int_M(R^m_\phi)^{\alpha-2}|\nabla R^m_\phi|^2f^2e^{-\phi}dV_g\\&+ 2a(m+n-1)\int_M(R^m_\phi)^\alpha(\Delta_\phi f)fe^{-\phi}dV_g - a\alpha r^m_\phi\int_M(R^m_\phi)^\alpha f^2e^{-\phi}dV_g\\&+ \int_M\left(2a(m+n-1)(R^m_\phi)^{\alpha-1} - \frac{m+n-2}{2}\right)R^m_\phi|\nabla f|^2e^{-\phi}dV_g.\end{aligned}$$

Inserting (2.1) into the above equation and simplifying completes the proof of Theorem 4.1. □

Remark 4.2. When $m = 0$ and $\phi = 0$, Theorem 4.1 recovers [10, Theorem 4.1] as a special case.

Using Theorem 4.1, we can also establish the monotone quantity associated with the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ under the flow (4.1).

Theorem 4.3. *Let (M^n, g) be a closed connected Riemannian manifold. Let $(g(t), \phi(t))$, $t \in [0, T)$, $T < \frac{1}{\max_M R^m_{\phi_0}}$ (where $\phi_0 = \phi(0)$), be a solution of the normalized weighted Yamabe flow (4.1) and let $\lambda(t)$ be the first eigenvalue of the operator $-\Delta_\phi + a(R^m_\phi)^\alpha$ for constants $0 < \alpha < 1$ and $a > 0$. Suppose that the weighted scalar curvature R^m_ϕ satisfies*

$$0 < (R^m_\phi)^\alpha \le \left(\frac{4a(m+n-1)}{m+n-2\alpha}\right)^{\frac{\alpha}{1-\alpha}}.$$

Then, along the flow (4.1), the quantity

$$\lambda(t)\exp\left\{-\int_0^t\left(\frac{m+n}{2}z_2(s)-2a(m+n-1)z_1^{\alpha}(s)-r_{\phi}^m\right)ds\right\}$$

is nondecreasing, where $z_1(t)$ *and* $z_2(t)$ *are given by* (4.3) *and* (4.4).

Proof. We first establish upper and lower bounds of R_{ϕ}^m. Because M is closed, there exists a spatial maximum point x_1 of $R_{\phi}^m(\cdot,t)$ such that $\Delta R_{\phi}^m(x_1,t)\le 0$. Therefore, evaluating (4.2) at x_1 gives

$$\frac{d}{dt}R_{\phi}^m\le (R_{\phi}^m)^2-R_{\phi}^m r_{\phi}^m<(R_{\phi}^m)^2.$$

By the maximum principle, we may compare R_{ϕ}^m with the solution of an ordinary differential equation $z'(t)=z^2(t)$ for initial values

$$z_1(0)=\max_M R_{\phi_0}^m,$$

and have

$$R_{\phi}^m\le z_1(t)=\frac{\max\limits_M R_{\phi_0}^m}{1-t\max\limits_M R_{\phi_0}^m},\tag{4.3}$$

for $t\in[0,T)$, $T<\frac{1}{\max\limits_M R_{\phi_0}^m}$.

On the other hand, since r_{ϕ}^m is the average of R_{ϕ}^m, we get $r_{\phi}^m\le\max\limits_M R_{\phi}^m$. Combining this with the pointwise upper bound (4.3), we obtain $r_{\phi}^m\le z_1(t)$. Therefore, for the minimum point x_2, we have

$$\frac{d}{dt}R_{\phi}^m\ge (R_{\phi}^m)^2-R_{\phi}^m r_{\phi}^m\ge (R_{\phi}^m)^2-R_{\phi}^m z_1(t).$$

Similarly, we get

$$R_{\phi}^m\ge z_2(t)=\frac{(1-t\max\limits_M R_{\phi_0}^m)\min\limits_M R_{\phi_0}^m}{1-t\min\limits_M R_{\phi_0}^m+\frac{1}{2}t^2(\max\limits_M R_{\phi_0}^m)\min\limits_M R_{\phi_0}^m}.\tag{4.4}$$

Arguing as in the proof of Theorem 1.3 and applying Theorem 4.1, we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(t)\ge&\lambda(t)\int_M\left(\frac{m+n}{2}R_{\phi}^m-2a(m+n-1)(R_{\phi}^m)^{\alpha}\right)f^2e^{-\phi}dV_g-r_{\phi}^m\lambda(t)\\ \ge&\lambda(t)\left(\frac{m+n}{2}z_2(t)-2a(m+n-1)(z_1(t))^{\alpha}\right)-r_{\phi}^m\lambda(t).\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\lambda(t)\exp\left\{-\int_0^t\left(\frac{m+n}{2}z_2(s)-2a(m+n-1)z_1^{\alpha}(s)-r_{\phi}^m\right)ds\right\}\right\}\\ &=\exp\left\{-\int_0^t\left(\frac{m+n}{2}z_2(s)-2a(m+n-1)z_1^{\alpha}(s)-r_{\phi}^m\right)ds\right\}\\ &\quad\times\left\{\frac{d}{dt}\lambda(t)-\lambda(t)\left(\frac{m+n}{2}z_2(t)-2a(m+n-1)z_1^{\alpha}(t)-r_{\phi}^m\right)\right\}\\ &\ge 0.\end{aligned}$$

This completes the proof of Theorem 4.3. □

5. Conclusions and discussion

The results obtained in this paper provide a first step toward understanding the spectral behavior of the weighted Yamabe flow with nonlinear curvature terms. Several interesting questions remain open.

First, it would be interesting to investigate whether the curvature restrictions imposed in the monotonicity theorem can be weakened.

Second, the long-time behavior of the monotone eigenvalue functional deserves further study. In particular, one may ask whether the monotonicity result can be combined with convergence results for the weighted Yamabe flow to obtain spectral rigidity or a characterization of limiting weighted manifolds.

Finally, the techniques developed here may be applicable to other geometric flows involving weighted curvature quantities. Possible directions include weighted Ricci flows, weighted conformal flows, and other curvature-driven evolutions on smooth metric measure spaces.

Author contributions

J. Y. Xiao: writing-original draft; J. C. Liu: writing-review & editing. The authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere gratitude to the reviewers for their valuable comments and suggestions, and to Dr. Mengke Wu and Dr. Yanfang Ma for their generous assistance during the revision of this manuscript. This research was supported by the National Natural Science Foundation of China (No. 12161078) and the Funds for Innovative Fundamental Research Group Project of Gansu Province (No. 24JRRA778).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. G. Perelman, The entropy formula for the Ricci flow and its geometric applications, *ArXiv*, 2002. https://doi.org/10.48550/arXiv.math/0211159
2. X. Cao, Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator, *Math. Ann.*, **337** (2007), 435–441. https://doi.org/10.1007/s00208-006-0043-5

3. J. Li, Eigenvalues and energy functionals with monotonicity formulae under Ricci flow, *Math. Ann.*, **338** (2007), 927–946. https://doi.org/10.1007/s00208-007-0098-y
4. X. Cao, First eigenvalues of geometric operators under the Ricci flow, *Proc. Amer. Math. Soc.*, **136** (2008), 4075–4078. https://doi.org/10.1090/S0002-9939-08-09533-6
5. P. T. Ho, First eigenvalues of geometric operators under the Yamabe flow, *Ann. Global Anal. Geom.*, **54** (2018), 449–472. https://doi.org/10.1007/s10455-018-9608-2
6. S. Fang, H. Xu, P. Zhu, Evolution and monotonicity of eigenvalues under the Ricci flow, *Sci. China Math.*, **58** (2015), 1737–1744. https://doi.org/10.1007/s11425-014-4943-7
7. L. Zhang, Monotonicity of first eigenvalues along the Yamabe flow, *Czechoslovak Math. J.*, **71** (2021), 387–401. https://doi.org/10.21136/CMJ.2020.0392-19
8. P. T. Ho, J. Shin, On the problem of prescribing weighted scalar curvature and the weighted Yamabe flow, *Anal. Geom. Metric Spaces*, **11** (2023), 20220152. https://doi.org/10.1515/agms-2022-0152
9. F. Yang, L. Zhang, On the evolution and monotonicity of first eigenvalues under the Ricci flow, *Hokkaido Math. J.*, **51** (2022), 107–116. https://doi.org/10.14492/hokmj/2019-215
10. A. Abolarinwa, S. Azami, First eigenvalues evolution for some geometric operators along the Yamabe flow, *J. Geom.*, **115** (2024), 18. https://doi.org/10.1007/s00022-024-00717-6
11. S. Hamanaka, P. T. Ho, Convergence rate of the weighted conformal mean curvature flow, *Anal. Geom. Metric Spaces*, **13** (2025), 20250026. https://doi.org/10.1515/agms-2025-0026
12. P. T. Ho, J. Shin, Z. Yan, Some results on the weighted Yamabe problem with or without boundary, *Bull. Sci. Math.*, **203** (2025), 103658. https://doi.org/10.1016/j.bulsci.2025.103658
13. R. Mi, Classification of complete expanding gradient weighted Yamabe solitons, *Math. Phys. Anal. Geom.*, **28** (2025), 39. https://doi.org/10.1007/s11040-025-09536-w
14. M. C. Reed, B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, 1978.
15. S. Azami, A. Abolarinwa, Evolution of first eigenvalues of some geometric operators under the rescaled List's extended Ricci flow, *Rev. Unión Mat. Argentina*, **67** (2024), 529–543. https://doi.org/10.33044/revuma.3413
16. B. Chow, S. C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. L vey, et al., *The Ricci flow: techniques and applications. Part II: analtic aspects*, American Mathematical Society, 2008.
17. Z. Yan, Convergence of the weighted Yamabe flow, *Differ. Geom. Appl.*, **84** (2022), 101922. https://doi.org/10.1016/j.difgeo.2022.101922
18. P. T. Ho, J. Shin, Z. Yan, Convergence rate of the weighted Yamabe flow, *Differ. Geom. Appl.*, **93** (2024), 102119. https://doi.org/10.1016/j.difgeo.2024.102119

AIMS Press