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Research article

Singularities of concircular surfaces in Euclidean 3-space

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Abstract: In this paper, we study concircular surfaces, defined as surfaces whose unit normal vector field $\mathcal{N}$ and a given concircular vector field $\mathcal{Y}$ satisfy $\langle \mathcal{N}, \mathcal{Y} \rangle$ = constant. Applying singularity theory, we provide a classification of singularities on such surfaces, including cuspidal edges and swallowtails. We find that the order of contact between the curve $(\alpha, \mathcal{N})$ associated with the concircular surface and the affine tangent bundle of the unit sphere is closely related to these singularities. Furthermore, from the perspective of spherical Legendrian duality, we establish a direct relationship between the rulings and the normal vector fields on the surface. Finally, several examples are presented to illustrate the main conclusions.

Keywords: concircular surfaces; singularities; contact; spherical Legendrian duality
Mathematics Subject Classification: 57R45, 58K05

1. Introduction

The notion of a concircular vector field on a Riemannian manifold was first presented by A. Fialkow in [1]. Concircular vector fields arise naturally in the study of concircular mappings, which are conformal mappings that preserve geodesic circles [2]. Apart from playing significant roles in projective and conformal transformations, these fields are also applied in general relativity. For instance, it has been shown that the trajectories of timelike concircular fields in the de Sitter model determine the worldlines of receding or colliding galaxies, in accordance with the Weyl hypothesis [3]. Based on concircular vector fields, the authors in [4, 5] defined concircular submanifolds in Euclidean space, including concircular helices, concircular surfaces and concircular hypersurfaces. Generalized helices, slant helices, and rectifying curves are special cases of concircular helices. Analogously, classical surfaces such as the cylindrical surface, helix surface, and conical surface arise as particular examples of concircular surfaces. These special curves can be described as geodesics on their corresponding surfaces: the generalized helix lies on the cylindrical surface, the slant helix on the helix surface, and the rectifying curve on the conical surface. Motivated by this correspondence, the authors of [4] generalized the above relationships. They provided a complete characterization that concircular surfaces in Euclidean 3-space $\mathbb{R}^3$ form a special class of ruled surfaces. Moreover, they proved that concircular helices are geodesics of concircular surfaces.

In recent decades, singularity theory has become an effective approach for studying the geometric properties of submanifolds embedded in various ambient spaces [6–8]. In [9], P. Hartman and L. Nirenberg proved that the cylindrical surface is the only developable surface without singularities. In [10], Wang et al. provided a classification of singularities on helical surfaces. All of the ruled surfaces mentioned above can be classified as special cases of concircular surfaces; in other words, concircular surfaces are their generalizations. To the best of the author's knowledge, singularities on concircular surfaces have not yet been studied. Hence, this paper will classify the singularities on these surfaces. The geometric meaning of singularities on concircular surfaces is characterized by the order of contact between the curve $(\boldsymbol{\alpha}, \mathcal{N})$ (where $\boldsymbol{\alpha}$ is the generating curve and $\mathcal{N}$ is the unit normal field) and the affine tangent bundle of the unit sphere. Finally, based on spherical Legendrian duality, we demonstrate that the unit normal vector field of a concircular surface and its generating curves are dual to each other under this correspondence.

This paper is structured as follows: In Section 2, we introduce the foundational background knowledge required for concircular surfaces. In Section 3, we define the support function of a curve on a totally umbilical surface and provide a classification of singularities on concircular surfaces using the theory of unfolding. In Section 4, the geometric meaning of these singularities is described. In Section 5, we reveal the spherical Legendrian duality relation on concircular surfaces. In Section 6, we conclude with some examples that illustrate our main results.

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2. Preliminaries

On a Riemannian manifold M, a vector field $\mathcal{Y} \in \mathfrak{X}(M)$ with Levi-Civita connection ∇ is defined as concircular if $\nabla\mathcal{Y} = \mu I$, where $\mu \in C^\infty(M)$ is a differentiable function named the concircular factor [1, 11]. Let $\mathrm{Con}(M)$ denote the set of concircular vector fields on M. A submanifold $M \subset \mathbb{R}^n$ with a chosen concircular vector field $\mathcal{Y} \in \mathrm{Con}(\mathbb{R}^n)$ is called a concircular submanifold (with axis $\mathcal{Y}$) under the following condition: the function $\langle \mathcal{N}, \mathcal{Y} \rangle$ is constant along M, where $\mathcal{N}$ is any unit normal vector field in the first normal space of M (see [4]). As shown in [11], a concircular vector field $\mathcal{Y} \in \mathrm{Con}(\mathbb{R}^n)$ can be defined as $\mathcal{Y}(p) = \mu p + \boldsymbol{v}$, where $\mu \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^n$. Let $\mathcal{N}$ be the unit normal vector field of a surface M and let $\mathcal{Y} \in \mathrm{Con}(\mathbb{R}^n)$. Then, M is called a concircular surface (with axis $\mathcal{Y}$) if $\langle \mathcal{N}, \mathcal{Y} \rangle = \text{const.}$ along M. Planes and spheres are trivial examples of concircular surfaces, while all other surfaces are nontrivial concircular surfaces. The well-known cylindrical, conical, and helix surfaces are examples of nontrivial concircular surfaces.

Example 2.1. *Let $F\colon I \times \mathbb{R} \to \mathbb{R}^3$ be a cylindrical surface parametrized by $F(s, u) = \boldsymbol{\alpha}(s) + u\boldsymbol{\delta}$, where $\boldsymbol{\alpha}$ is a plane curve with Frenet frame $\{\boldsymbol{T}_\alpha, \boldsymbol{N}_\alpha\}$ and $\boldsymbol{\delta}$ is the unit constant vector perpendicular to the plane of $\boldsymbol{\alpha}$. The unit normal vector field of the surface F is given by $\mathcal{N} = \boldsymbol{T}_\alpha \times \boldsymbol{\delta}$. Since this unit normal vector field is collinear with $\boldsymbol{N}_\alpha$, we may simply set $\mathcal{N}(s, u) = \boldsymbol{N}_\alpha(s)$. If we take $\mathcal{Y} = \boldsymbol{\delta}$ as the concircular vector field, then we have $\langle \mathcal{N}, \mathcal{Y} \rangle = 0$ at every point of F.*

Example 2.2. *Let $F\colon I \times \mathbb{R} \to \mathbb{R}^3$ be a conical surface parametrized by $F(s, u) = u\boldsymbol{\alpha}(s)$, where $\boldsymbol{\alpha}$ is a curve on the unit sphere $S^2(1)$ and its Darboux frame is $\{\boldsymbol{T}_\alpha, \boldsymbol{N}_\alpha\}$. In this case, the unit normal vector* *field $\mathcal{N} = \boldsymbol{T}_\alpha \times \boldsymbol{\alpha}$ of the surface F is also collinear with $\boldsymbol{N}_\alpha$; hence we set $\mathcal{N}(s,u) = \boldsymbol{N}_\alpha(s)$, while the position vector field itself can be taken as the concircular vector field $\mathcal{Y}$. The conical surface is easily inferred to be a concircular surface.*

Example 2.3. *Let $F\colon I \times \mathbb{R} \to \mathbb{R}^3$ be a helix surface parametrized by*

$$F(s,u) = \boldsymbol{\alpha}(s) + u(\cos\phi\, \boldsymbol{N}_\alpha(s) + \sin\phi\, \boldsymbol{\delta}),$$

where $\phi \in (0, \pi/2]$, and $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ are the same as those in Example 2.1. Here, the unit normal vector field is represented as $\mathcal{N}(s,u) = -\sin\phi \boldsymbol{N}_\alpha(s) + \cos\phi\boldsymbol{\delta}$. By setting $\mathcal{Y} = \boldsymbol{\delta}$ again, we obtain $\langle \mathcal{N}, \mathcal{Y}\rangle = \cos\phi$. Surfaces of this type are also known as constant angle surfaces [12].

A nontrivial concircular surface $F \subset \mathbb{R}^3$ that is not a cylindrical, conical, or helix surface is termed a proper concircular surface. In [4], the authors provided a construction of nontrivial concircular surfaces from curves on a totally umbilical surface. Let $S \subset \mathbb{R}^3$ denote a totally umbilical surface with constant curvature c (for example, a plane or a sphere centered at p with radius r). Consider a curve $\boldsymbol{\alpha}\colon I \subset \mathbb{R} \to S \subset \mathbb{R}^3$ parametrized by arc length s, whose curvature is $\kappa_\alpha(s)$ and which has a Darboux frame $\{\boldsymbol{T}_\alpha(s), \boldsymbol{N}_\alpha(s), \boldsymbol{v}(s) = \boldsymbol{T}_\alpha(s) \times \boldsymbol{N}_\alpha(s)\}$. Then, we have the following Frenet-Darboux equations of $\boldsymbol{\alpha}$:

$$\begin{cases} \boldsymbol{T}'_\alpha(s) = -\sqrt{c}\,\boldsymbol{v}(s) + \kappa_\alpha(s)\boldsymbol{N}_\alpha(s), \\ \boldsymbol{N}'_\alpha(s) = -\kappa_\alpha(s)\boldsymbol{T}_\alpha(s), \\ \ \boldsymbol{v}'(s) = \sqrt{c}\,\boldsymbol{T}_\alpha(s). \end{cases}$$

Let F be a ruled surface generated by the curve $\boldsymbol{\alpha}$, which can be expressed as

$$F(s,u) = \boldsymbol{\alpha}(s) + u(\cos\phi\, \boldsymbol{N}_\alpha(s) + \sin\phi\, \boldsymbol{v}(s)), \tag{2.1}$$

where $\phi \in (0, \pi/2]$. By a direct calculation, we obtain

$$\begin{aligned} F_s(s,u) &= [1 + u(\sin\phi\,\sqrt{c} - \cos\phi\,\kappa_\alpha(s))]\boldsymbol{T}_\alpha(s), \\ F_u(s,u) &= \cos\phi\,\boldsymbol{N}_\alpha(s) + \sin\phi\,\boldsymbol{v}(s). \end{aligned}$$

Since

$$F_s(s,u) \times F_u(s,u) = [1 + u(\sin\phi\,\sqrt{c} - \cos\phi\kappa_\alpha(s))](\cos\phi\boldsymbol{v}(s) - \sin\phi\boldsymbol{N}_\alpha(s)), \tag{2.2}$$

the unit normal vector field of the surface $F(s,u)$ can be expressed as

$$\mathcal{N}(s) = \cos\phi\boldsymbol{v}(s) - \sin\phi\boldsymbol{N}_\alpha(s).$$

Theorem 2.4 ([4]). *A nontrivial concircular surface F associated with a concircular vector field $\mathcal{Y}$ can be described by the local parameterization given by (2.1), where $\boldsymbol{\alpha}$ is a curve on the totally umbilical surface $S \subset \mathbb{R}^3$, and its normal vector field $\boldsymbol{v}$ along S is parallel to $\mathcal{Y}$.*

Set $\mathcal{R}_\phi(s) = \cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s)$. According to $\det(\boldsymbol{\alpha}'(s), \mathcal{R}_\phi(s), \mathcal{R}'_\phi(s)) = 0$, we know that the nontrivial concircular surface $F(s,u)$ is a developable surface. Since the local parametric representation of a nontrivial concircular surface is based on the curve on the totally umbilical surface with constant curvature c, we obtain the following:

(1) If $c = 0$, then $(s_0, u_0) \in I \times \mathbb{R}$ is a singular point of $F(s,u)$ if and only if $\phi \neq \pi/2$ and $u_0 = 1/\cos\phi\kappa_\alpha(s_0)$.
(2) If $c > 0$, then $(s_0, u_0) \in I \times \mathbb{R}$ is a singular point of $F(s,u)$ if and only if $\sin\phi\sqrt{c} - \cos\phi\kappa_\alpha(s_0) \neq 0$ and $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\sqrt{c})$.

In fact, when $c = 0$ and $\phi = \pi/2$, $\boldsymbol{\alpha}$ is a plane curve and $\mathcal{R}_\phi(s)$ is a constant vector. This implies that $F(s,u)$ is a cylindrical surface with no singularities. When $c > 0$ and $\sin\phi\sqrt{c} - \cos\phi\kappa_\alpha(s) = 0$, the vector $\mathcal{R}_\phi(s)$ is constant. It follows that $F(s,u)$ is a cylindrical surface without singular points. Moreover, we notice that $\mathcal{N}'(s) = [\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s)]\boldsymbol{T}_\alpha(s)$, where $\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s) \neq 0$; otherwise, $F(s,u)$ is a plane, that is, it is a trivial concircular surface.

Throughout the rest of this paper, we assume $\phi \in (0, \pi/2)$, $\sin\phi\sqrt{c} - \cos\phi\kappa_\alpha(s) \neq 0$, and $\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s) \neq 0$ to exclude trivial and non-singular cases. Under these limitations, every nontrivial concircular surface always has singular points.

3. Classification of singularities of nontrivial concircular surfaces

In this section, we use the unfolding theory of functions to classify the singularities of nontrivial concircular surfaces. We begin by defining a family of support functions. Given a curve $\boldsymbol{\alpha}\colon I \to S \subset \mathbb{R}^3$ with $\kappa_\alpha(s) \neq 0$, the support function of $\boldsymbol{\alpha}$ (relative to $\mathcal{N}(s)$) is defined as $G\colon I \times \mathbb{R}^3 \to \mathbb{R}$ by

$$G(s, \boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{\alpha}(s)) \cdot \mathcal{N}(s).$$

For each fixed $\boldsymbol{x} \in \mathbb{R}^3$, we write $g_x(s) = G(s, \boldsymbol{x})$. Then, the following result is established.

Proposition 3.1. *Let $\boldsymbol{\alpha}\colon I \to S \subset \mathbb{R}^3$ be a unit speed curve satisfying $\kappa_\alpha(s) \neq 0$.*
(1) $g_x(s) = 0$ *if and only if there exist λ, $\mu \in \mathbb{R}$ such that*

$$\boldsymbol{x} - \boldsymbol{\alpha}(s) = \lambda\boldsymbol{T}_\alpha(s) + \mu(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s)).$$

(2) $g_x(s) = g_x'(s) = 0$ *if and only if there exists $\mu \in \mathbb{R}$ such that*

$$\boldsymbol{x} - \boldsymbol{\alpha}(s) = \mu(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s)).$$

(3) $g_x(s) = g_x'(s) = g_x''(s) = 0$ *if and only if*

$$\boldsymbol{x} - \boldsymbol{\alpha}(s) = \frac{1}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s)).$$

(4) $g_x(s) = g_x'(s) = g_x''(s) = g_x'''(s) = 0$ *if and only if*

$$\boldsymbol{x} - \boldsymbol{\alpha}(s) = \frac{1}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s)),$$

and $\kappa_\alpha'(s) = 0$.
(5) $g_x(s) = g_x'(s) = g_x''(s) = g_x'''(s) = g_x^{(4)}(s) = 0$ *if and only if*

$$\boldsymbol{x} - \boldsymbol{\alpha}(s) = \frac{1}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s)),$$

and $\kappa_\alpha'(s) = \kappa_\alpha''(s) = 0$.

Proof. Since $g_x(s) = (\boldsymbol{x} - \boldsymbol{\alpha}(s))(\cos\phi\boldsymbol{v}(s) - \sin\phi\boldsymbol{N}_\alpha(s))$, a simple calculation shows that

$$\begin{aligned}
g_x &= (\boldsymbol{x} - \boldsymbol{\alpha})(\cos\phi\,\boldsymbol{v} - \sin\phi\,\boldsymbol{N}_\alpha),\\
g_x' &= (\boldsymbol{x} - \boldsymbol{\alpha})(\cos\phi\,\sqrt{c} + \sin\phi\,\kappa_\alpha)\boldsymbol{T}_\alpha,\\
g_x'' &= -(\cos\phi\,\sqrt{c} + \sin\phi\,\kappa_\alpha) + (\boldsymbol{x} - \boldsymbol{\alpha})\Big[\sin\phi\,\kappa_\alpha'\boldsymbol{T}_\alpha + (\cos\phi\,\sqrt{c} + \sin\phi\,\kappa_\alpha)(\kappa_\alpha\boldsymbol{N}_\alpha - \sqrt{c}\,\boldsymbol{v})\Big],\\
g_x''' &= -2\sin\phi\,\kappa_\alpha' + (\boldsymbol{x} - \boldsymbol{\alpha})\Big[\Big(\sin\phi\,\kappa_\alpha'' - (\cos\phi\,\sqrt{c} + \sin\phi\,\kappa_\alpha)(c + \kappa_\alpha^2)\Big)\boldsymbol{T}_\alpha - 2\sin\phi\,\sqrt{c}\,\kappa_\alpha'\,\boldsymbol{v}\\
&\qquad + (\cos\phi\,\sqrt{c} + 3\sin\phi\,\kappa_\alpha)\kappa_\alpha'\boldsymbol{N}_\alpha\Big],\\
g_x^{(4)} &= -3\sin\phi\,\kappa_\alpha'' + (\cos\phi\,\sqrt{c} + \sin\phi\,\kappa_\alpha)(c + \kappa_\alpha^2)\\
&\quad + (\boldsymbol{x} - \boldsymbol{\alpha})\Big[\Big(\sin\phi\,\kappa_\alpha''' - 3c\sin\phi\,\kappa_\alpha' - 6\sin\phi\,\kappa_\alpha'\kappa_\alpha^2 - 3\cos\phi\,\sqrt{c}\,\kappa_\alpha\kappa_\alpha'\Big)\boldsymbol{T}_\alpha\\
&\qquad - \Big(3\sin\phi\,\kappa_\alpha'' - (\cos\phi\,\sqrt{c} + \sin\phi\,\kappa_\alpha)(c + \kappa_\alpha^2)\Big)\sqrt{c}\,\boldsymbol{v}\\
&\qquad + \Big(3\sin\phi(\kappa_\alpha\kappa_\alpha'' + (\kappa_\alpha')^2) + (\cos\phi\,\sqrt{c} + \sin\phi\,\kappa_\alpha)(\kappa_\alpha'' - \kappa_\alpha(c + \kappa_\alpha^2))\Big)\boldsymbol{N}_\alpha\Big].
\end{aligned}$$

(1) Since $\boldsymbol{T}_\alpha$, $\boldsymbol{N}_\alpha$, and $\boldsymbol{v}$ are mutually orthogonal unit vectors, assertion (1) can be directly derived from the definition of $g_x(s)$.

(2) As shown in (1), we have $\boldsymbol{x} - \boldsymbol{\alpha}(s) = \lambda\boldsymbol{T}_\alpha(s) + \mu(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s))$. In this case, $g_x'(s) = 0$ if and only if $(\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s))\lambda = 0$. Since $\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s) \neq 0$, we obtain $\lambda = 0$. It follows that $g_x(s) = g_x'(s) = 0$ if and only if there exists $\mu \in \mathbb{R}$ such that $\boldsymbol{x} - \boldsymbol{\alpha}(s) = \mu(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s))$.

(3) Under the condition of assertion (2), $g_x''(s) = 0$ is equivalent to

$$\mu = \frac{1}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}.$$

Thus, we can conclude that assertion (3) is valid.

(4) When $g_x(s) = g_x'(s) = g_x''(s) = 0$, adding $g_x'''(s) = 0$ yields

$$\frac{\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s)}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}\cos\phi\kappa_\alpha'(s) = 0.$$

Since $\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s) \neq 0$ and $\phi \in (0, \pi/2)$, we obtain $\kappa_\alpha'(s) = 0$.

(5) Under the condition that $g_x(s) = g_x'(s) = g_x''(s) = g_x'''(s) = 0$, $g_x^{(4)}(s) = 0$ if and only if $\kappa_\alpha'(s) = 0$,

$$\boldsymbol{x} - \boldsymbol{\alpha}(s) = \frac{1}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{v}(s)),$$

and

$$3\cos\phi\sin\phi(\kappa_\alpha'(s))^2 + (\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s))\cos\phi\kappa_\alpha''(s) = 0.$$

From the last equation, we derive that $\kappa_\alpha''(s) = 0$. The assertion (5) holds. □

We review some basic concepts of the unfolding theory. The function germ $F\colon (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ is called an r-parameter unfolding of f, where $f(s) = F_{x_0}(s, \boldsymbol{x}_0)$. We say that $f(s)$ has A_k-singularity at s_0 if $f^{(l)}(s_0) = 0$ for all $(1 \leq l \leq k)$ and $f^{(k+1)}(s_0) \neq 0$. Let F be an r-parameter unfolding of f and suppose f has A_k-singularity $(k \geq 1)$ at s_0. For the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0, we have

$$j^{(k-1)}(\partial F/\partial x_i)(s, \boldsymbol{x}_0)(s_0) = \sum_{j=1}^{k-1}\alpha_{ji}(s - s_0)^j, \quad \text{for } i = 1, \dots, r.$$

We say that F is a versal unfolding of f if the rank of the $k \times r$ coefficient matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \leq r)$, where $\alpha_{0i} = (\partial F/\partial x_i)(s_0, \boldsymbol{x}_0)$. The discriminant set of F is represented as

$$\mathcal{D}_F = \{\boldsymbol{x} \in \mathbb{R}^r | \text{there exists } s \text{ such that } F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = 0\}.$$

The following classical classification result is given in [13].

Theorem 3.2. *Let $F\colon (\mathbb{R}\times\mathbb{R}^3, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a 3-parameter unfolding of $f(s)$ with the A_k-singularity at s_0. Assume that F is a versal unfolding of f.*
(1) If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}$.
(2) If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to SW.

Here, $C \times \mathbb{R} = \{(a_1, a_2, a_3) | a_1^2 = a_2^3\}$ is the cuspidal edge (see Figure 1), and $SW = \{(a_1, a_2, a_3) \,|\, a_1 = 3b^4 + b^2 d,\ a_2 = 4b^3 + 2bd,\ a_3 = d\}$ is the swallowtail (see Figure 2).

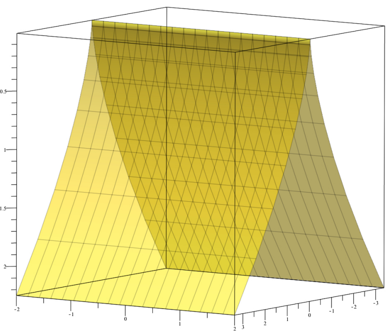

Figure 1. Cuspidal edge.

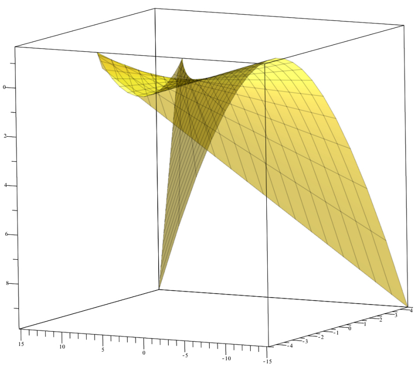

Figure 2. Swallowtail.

We obtain the following conclusion for the support function G.

Proposition 3.3. *Let $\boldsymbol{\alpha}\colon I \to S \subset \mathbb{R}^3$ be a unit speed curve with nonzero curvature $\kappa_\alpha(s)$. The support function defined on $\boldsymbol{\alpha}$ relative to the normal vector $\boldsymbol{N}$ is $G\colon I \times \mathbb{R}^3 \to \mathbb{R}$. If g_{x_0} has an A_k-singularity $(k = 2,\ 3)$ at s_0, then G is a versal unfolding of g_{x_0}.*

Proof. Let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\mathcal{N}(s) = \cos\phi \boldsymbol{v}(s) - \sin\phi \boldsymbol{N}_\alpha(s) = (\mathfrak{n}_1(s), \mathfrak{n}_2(s), \mathfrak{n}_3(s))$. We have

$$G(s, \boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{\alpha}(s))\mathcal{N}(s) = x_1\mathfrak{n}_1(s) + x_2\mathfrak{n}_2(s) + x_3\mathfrak{n}_3(s) - \boldsymbol{\alpha}(s)\mathcal{N}(s),$$

$$\frac{\partial G}{\partial x_i}(s, \boldsymbol{x}) = \mathfrak{n}_i(s), \quad \frac{\partial}{\partial s}\frac{\partial G}{\partial x_i}(s, \boldsymbol{x}) = \mathfrak{n}_i'(s), \quad \frac{\partial^2}{\partial s^2}\frac{\partial G}{\partial x_i}(s, \boldsymbol{x}) = \mathfrak{n}_i''(s), \quad (i = 1, 2, 3).$$

Then, the 2-jet can be represented as

$$\begin{aligned} j^2 \frac{\partial G}{\partial x_i}(s, \boldsymbol{x}_0)(s_0) &= \frac{\partial}{\partial s}\frac{\partial G}{\partial x_i}(s_0, \boldsymbol{x}_0)(s - s_0) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial G}{\partial x_i}(s_0, \boldsymbol{x}_0)(s - s_0)^2 \\ &= \mathfrak{n}_i'(s_0)(s - s_0) + \frac{1}{2}\mathfrak{n}_i''(s_0)(s - s_0)^2. \end{aligned}$$

Consider the following matrix

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} \mathfrak{n}_1(s_0) & \mathfrak{n}_2(s_0) & \mathfrak{n}_3(s_0) \\ \mathfrak{n}_1'(s_0) & \mathfrak{n}_2'(s_0) & \mathfrak{n}_3'(s_0) \\ \mathfrak{n}_1''(s_0) & \mathfrak{n}_2''(s_0) & \mathfrak{n}_3''(s_0) \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\, \boldsymbol{v}(s_0) - \sin\phi\, \boldsymbol{N}_\alpha(s_0) \\ (\cos\phi\, \sqrt{c} + \sin\phi\, \kappa_\alpha(s_0))\, \boldsymbol{T}_\alpha(s_0) \\ \sin\phi\, \kappa_\alpha'(s_0)\, \boldsymbol{T}_\alpha(s_0) + (\cos\phi\, \sqrt{c} + \sin\phi\, \kappa_\alpha(s_0))(-\sqrt{c}\, \boldsymbol{v}(s_0) + \kappa_\alpha(s_0)\boldsymbol{N}_\alpha(s_0)) \end{pmatrix}. \end{aligned}$$

The rank of matrix $\mathcal{A}$ is the same as the rank of matrix

$$\begin{pmatrix} 0 & -\sin\phi & \cos\phi \\ \cos\phi\, \sqrt{c} + \sin\phi\kappa_\alpha(s_0) & 0 & 0 \\ \sin\phi\kappa_\alpha'(s_0) & (\cos\phi\, \sqrt{c} + \sin\phi\kappa_\alpha(s_0))\kappa_\alpha(s_0) & -\sqrt{c}(\cos\phi\, \sqrt{c} + \sin\phi\kappa_\alpha(s_0)) \end{pmatrix},$$

because $\{\boldsymbol{T}_\alpha(s_0), \boldsymbol{N}_\alpha(s_0), \boldsymbol{v}(s_0)\}$ is an orthonormal basis for $\mathbb{R}^3$. It follows from the calculation that $\operatorname{rank}\mathcal{A} = 3$ if and only if

$$(\cos\phi\, \sqrt{c} + \sin\phi\kappa_\alpha(s_0))^2(\cos\phi\kappa_\alpha(s_0) - \sin\phi\, \sqrt{c}) \neq 0.$$

As shown in Section 2, this condition is precisely equivalent to the existence of singular points on a nontrivial concircular surface. Hence, under the assumption that the nontrivial concircular surface has singularities, this condition naturally holds.

Moreover, the rank of

$$\begin{pmatrix} \cos\phi\boldsymbol{v}(s_0) - \sin\phi\boldsymbol{N}_\alpha(s_0) \\ (\cos\phi\, \sqrt{c} + \sin\phi\kappa_\alpha(s_0))\boldsymbol{T}_\alpha(s_0) \end{pmatrix}$$

is always 2 because $\phi \in (0, \pi/2)$ and $\cos\phi\, \sqrt{c} + \sin\phi\kappa_\alpha(s_0) \neq 0$.

This finishes the proof. □

From Theorem 3.2 and Proposition 3.3, we derive the classification theorem for singularities below.

Theorem 3.4. *Let $\boldsymbol{\alpha}\colon I \to S \subset \mathbb{R}^3$ be a unit speed curve satisfying $\kappa_\alpha(s) \neq 0$.*

(1) *The image of the nontrivial concircular surface $F(s, u)$ is not singular at (s_0, u_0) if and only if $u_0 \neq 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\, \sqrt{c})$.*

(2) *The image of the nontrivial concircular surface $F(s, u)$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, u_0) if $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\, \sqrt{c})$ and $\kappa_\alpha'(s_0) \neq 0$.*

(3) *The image of the nontrivial concircular surface $F(s, u)$ is locally diffeomorphic to the swallowtail SW at (s_0, u_0) if $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\, \sqrt{c})$, $\kappa_\alpha'(s_0) = 0$ and $\kappa_\alpha''(s_0) \neq 0$.*

Proof. From $F(s,u) = \boldsymbol{\alpha}(s) + u(\cos\phi \boldsymbol{N}_\alpha(s) + \sin\phi \boldsymbol{\nu}(s))$, we can derive that

$$F_s(s,u) \times F_u(s,u) = [1 + u(\sin\phi\sqrt{c} - \cos\phi\kappa_\alpha(s))](\cos\phi\boldsymbol{\nu}(s) - \sin\phi\boldsymbol{N}_\alpha(s)).$$

Thus, (s_0, u_0) is a regular point of $F(s,u)$ if and only if $1+u(\sin\phi\sqrt{c}-\cos\phi\kappa_\alpha(s)) \neq 0$. The assertion (1) is proven.

According to assertion (2) of Proposition 3.1, the image of $F(s,u)$ is the discriminant set $\mathcal{D}_G$ of the support function G. Moreover, assertions (3)–(5) of Proposition 3.1 imply that g_{x_0} has an A_2-singularity (resp., A_3-singularity) at $s = s_0$ if and only if $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\sqrt{c})$ and $\kappa_\alpha'(s_0) \neq 0$ (resp., $u_0 = 1/(\cos\phi\kappa_\alpha(s_0)-\sin\phi\sqrt{c})$, $\kappa_\alpha'(s_0) = 0$, and $\kappa_\alpha''(s_0) \neq 0$). Applying Theorem 3.2 and Proposition 3.3 yields assertions (2) and (3). □

4. Contact

In this section, we study the geometric properties of singular points on nontrivial concircular surfaces. Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3 \times S^2$ be a curve and $M\colon \mathbb{R}^3 \times S^2 \to \mathbb{R}$ be a submersion. The curve $\boldsymbol{\gamma}$ is said to have contact of at least order k with the set $M^{-1}(0)$ at $t = t_0$ if the function $m(t) = M \circ \boldsymbol{\gamma}(t)$ satisfies $m^{(j)}(t_0) = 0$ for all $j = 0, 1, \ldots, k$. Furthermore, the contact is said to be of order k at $t = t_0$ if, in addition to the above conditions, $m^{(k+1)}(t_0) \neq 0$. For every $\boldsymbol{x} \in \mathbb{R}^3$, we consider a function $\mathfrak{m}_x\colon \mathbb{R}^3 \times S^2 \to \mathbb{R}$ defined by $\mathfrak{m}_x(\boldsymbol{\mu},\boldsymbol{\nu}) = (\boldsymbol{x}-\boldsymbol{\mu})\cdot\boldsymbol{\nu}$. The zero level set of this function is characterized by the equation $\boldsymbol{\mu}\cdot\boldsymbol{\nu} = \boldsymbol{x}\cdot\boldsymbol{\nu}$; that is, $\mathfrak{m}_x^{-1}(0) = \{(\boldsymbol{\mu},\boldsymbol{\nu}) \in \mathbb{R}^3\times S^2 | \boldsymbol{\mu}\cdot\boldsymbol{\nu} = \boldsymbol{x}\cdot\boldsymbol{\nu}\}$. Given a fixed $\boldsymbol{\nu} \in S^2$, the set $\mathfrak{m}_x^{-1}(0)|_{\mathbb{R}^3\times\{\nu\}}$ is a plane defined by $\boldsymbol{\mu}\cdot\boldsymbol{\nu} = c$(with $c = \boldsymbol{x}\cdot\boldsymbol{\nu}$), which is orthogonal to $\boldsymbol{\nu}$. Hence, it is parallel to $T_\nu S^2$. Recall that the tangent bundle of S^2 can be identified as $TS^2 = \{(\boldsymbol{\mu},\boldsymbol{\nu}) \in \mathbb{R}^3 \times S^2 | \boldsymbol{\mu}\cdot\boldsymbol{\nu} = 1\}$. Then, we consider the restriction of the canonical projection $\pi_2 : \mathbb{R}^3 \times S^2 \to S^2$ to the set $\mathfrak{m}_x^{-1}(0)$, which is denoted by $\pi_2|_{\mathfrak{m}_x^{-1}(0)}\colon \mathfrak{m}_x^{-1}(0) \to S^2$. This restriction forms a fiber bundle over S^2. Defining a map $\varphi\colon \mathfrak{m}_x^{-1}(0) \to TS^2$ by $\varphi(\boldsymbol{\mu},\boldsymbol{\nu}) = (\boldsymbol{\mu}/(\boldsymbol{x}\cdot\boldsymbol{\nu}),\boldsymbol{\nu})$ yields a bundle isomorphism. Thus, the set $TS^2(\boldsymbol{x}) = \mathfrak{m}_x^{-1}(0)$ is naturally called the *affine tangent bundle* over S^2 through $\boldsymbol{x}$.

Let $\boldsymbol{\alpha}\colon I \to S \subset \mathbb{R}^3$ be a unit speed curve satisfying $\kappa_\alpha(s) \neq 0$. Define a function

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{\nu}(s)).$$

By Theorem 3.4, $\boldsymbol{\beta}(s)$ is the singular value set of $F(s,u)$. By a direct calculation, we have

$$\boldsymbol{\beta}'(s) = -\frac{\cos\phi\kappa_\alpha'(s)}{(\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c})^2}(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{\nu}(s)).$$

It follows that $\boldsymbol{\beta}'(s) = 0$ is equal to $\kappa_\alpha'(s) \equiv 0$. Under the assumption $\kappa_\alpha'(s) \equiv 0$, the function $\boldsymbol{\beta}(s)$ reduces to a constant vector, which is denoted by $\boldsymbol{x}_0$. This implies

$$\boldsymbol{x}_0 - \boldsymbol{\alpha}(s) = \frac{1}{\cos\phi\kappa_\alpha(s) - \sin\phi\sqrt{c}}(\cos\phi\boldsymbol{N}_\alpha(s) + \sin\phi\boldsymbol{\nu}(s)),$$

and

$$\mathfrak{m}_{x_0}(\boldsymbol{\alpha}(s),\mathcal{N}(s)) = g_{x_0}(s) = (\boldsymbol{x}_0 - \boldsymbol{\alpha}(s))\cdot(\cos\phi\boldsymbol{\nu}(s) - \sin\phi\boldsymbol{N}_\alpha(s)) = 0.$$

Conversely, if there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $\mathfrak{m}_{x_0}(\boldsymbol{\alpha}(s),\mathcal{N}(s)) = 0$, then the same relation holds and we must have $\kappa_\alpha'(s) \equiv 0$. For the curve $(\boldsymbol{\alpha},\mathcal{N})\colon I \to \mathbb{R}^3 \times S^2$, based on the above analysis, we obtain the following result.

Proposition 4.1. *Let $\boldsymbol{\alpha} : I \to S \subset \mathbb{R}^3$ be a unit speed curve satisfying $\kappa_\alpha(s) \neq 0$. Then, there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $(\boldsymbol{\alpha}, \mathcal{N})(s) \subset TS^2(\boldsymbol{x}_0)$ if and only if $\kappa'_\alpha(s) \equiv 0$.*

From Proposition 3.1 (2), it follows that $(\boldsymbol{\alpha}, \mathcal{N})(s)$ is tangent to $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ if and only if there exists some $u_0 \in \mathbb{R}$ such that $\boldsymbol{x}_0 = F(s_0, u_0)$. This leads to the following conclusion.

Proposition 4.2. *Let $\boldsymbol{\alpha}\colon I \to S \subset \mathbb{R}^3$ be a unit speed curve satisfying $\kappa_\alpha(s) \neq 0$ and $\boldsymbol{x}_0 = F(s_0, u_0)$. The order of contact between $(\boldsymbol{\alpha}, \mathcal{N})$ and $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is characterized as follows:*
(1) *The contact order is 2 if and only if $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\sqrt{c})$ and $\kappa'_\alpha(s_0) \neq 0$.*
(2) *The contact order is 3 if and only if $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\sqrt{c})$, $\kappa'_\alpha(s_0) = 0$, and $\kappa''_\alpha(s_0) \neq 0$.*

Proof. By assertions (3) and (4) of Proposition 3.1, $g_x(s_0) = g'_x(s_0) = g''_x(s_0) = 0$ and $g'''_x(s_0) \neq 0$ are equivalent to $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\sqrt{c})$ and $\kappa'_\alpha(s_0) \neq 0$. Since $\mathfrak{m}_{x_0}(\boldsymbol{\alpha}, \mathcal{N}) = g_{x_0}$, these conditions indicate that the contact between $(\boldsymbol{\alpha}, \mathcal{N})$ and $TS^2(\boldsymbol{x}_0)$ is of order 2 at $s = s_0$. Assertion (2) is obtained by using assertions (4) and (5) of Proposition 3.1 in the same way. □

In addition, we provide the following geometric interpretation of the classification in Theorem 3.4.

Theorem 4.3. *Let $\boldsymbol{\alpha}\colon I \to S \subset \mathbb{R}^3$ be a unit speed curve satisfying $\kappa_\alpha(s) \neq 0$ and $\boldsymbol{x}_0 = F(s_0, u_0)$.*
(1) *The image of the nontrivial concircular surface $F(s, u)$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, u_0) if $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\sqrt{c})$ and $\kappa'_\alpha(s_0) \neq 0$. In this case, $(\boldsymbol{\alpha}, \mathcal{N})$ and $TS^2(\boldsymbol{x}_0)$ have contact of order 2 at $s = s_0$.*
(2) *The image of the nontrivial concircular surface $F(s, u)$ is locally diffeomorphic to the swallowtail SW at (s_0, u_0) if $u_0 = 1/(\cos\phi\kappa_\alpha(s_0) - \sin\phi\sqrt{c})$, $\kappa'_\alpha(s_0) = 0$, and $\kappa''_\alpha(s_0) \neq 0$. In this case, $(\boldsymbol{\alpha}, \mathcal{N})$ and $TS^2(\boldsymbol{x}_0)$ have contact of order 3 at $s = s_0$.*

5. Spherical Legendrian duality

In this section, we focus on the spherical Legendrian duality on nontrivial concircular surfaces. We begin by reviewing some basic concepts related to contact manifolds and Legendrian submanifolds. Let N be a smooth manifold of dimension $2n + 1$. A tangent hyperplane field K on N can be locally given as the kernel of a 1-form ρ. If $\rho \wedge (d\rho)^n \neq 0$ everywhere, then K is a nondegenerate hyperplane field, and (N, K) is called a contact manifold. In this case, K and ρ are termed the contact structure and the contact form, respectively. A diffeomorphism $\Phi : N \to N'$ is called a contact diffeomorphism if the contact manifolds (N, K) and (N', K') satisfy $d\Phi(K) = K'$. Two contact manifolds are said to be contact diffeomorphic if such a map exists. A submanifold $i\colon L \hookrightarrow N$ of a contact manifold (N, K) is referred to as Legendrian if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ for every $x \in L$. If the total space of a smooth fiber bundle $\pi\colon E \to M$ is endowed with a contact structure and its fibers are Legendrian submanifolds, then the fiber bundle is called a Legendrian fibration. Given such a fibration and a Legendrian submanifold $i\colon L \hookrightarrow E$, the map $\pi \circ i\colon L \to M$ is called a Legendrian map, and its image $W(L)$ is the wavefront set of i. Locally, around any $p \in E$, one can find coordinates $(x_1, \ldots, x_m, p_1, \ldots, p_m, z)$ such that the projection is given by

$$\pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z)$$

and the contact structure is defined by the 1-form $\rho = dz - \sum_{i=1}^{m} p_i dx_i$.

Next, we introduce the double fibrations associated with S^2.

$$S^2 \times S^2 \supset \Delta = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \cdot \mathbf{w} = 0\},$$

$$\pi_1 : \Delta \ni (\mathbf{v}, \mathbf{w}) \to \mathbf{v} \in S^2, \quad \pi_2 : \Delta \ni (\mathbf{v}, \mathbf{w}) \to \mathbf{w} \in S^2,$$
$$\theta_1 = d\mathbf{v} \cdot \mathbf{w}|\Delta, \quad \theta_2 = \mathbf{v} \cdot d\mathbf{w}|\Delta.$$

where $d\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{3} w_i dv_i$ and $\mathbf{v} \cdot d\mathbf{w} = \sum_{i=1}^{3} v_i dw_i$.

The tangent hyperplane fields over Δ defined by $\theta_1^{-1}(0)$ and $\theta_2^{-1}(0)$ coincide, forming a common field denoted by K. It follows that Δ is a smooth submanifold in $\mathbb{R}^3 \times \mathbb{R}^3$ and (Δ, K) is a contact manifold. Moreover, both π_1 and π_2 are Legendrian fibrations. By construction, the fibers of both π_1 and π_2 are Legendrian submanifolds in (Δ, K). For a Legendrian submanifold $L \subset \Delta$, the images $\pi_1(L)$ and $\pi_2(L)$ in S^2 are called spherical Legendrian dual to each other. In particular, for any $\mathbf{v} \in S^2$, $\pi_1^{-1}(\mathbf{v}) = \mathbf{v} \times S^2(\mathbf{v}, 0)$ and $\pi_2^{-1}(\mathbf{v}) = S^2(\mathbf{v}, 0) \times \mathbf{v}$ are Legendrian submanifolds of Δ. Hence, $\mathbf{v}$ and $S^2(\mathbf{v}, 0)$ are spherical Legendrian dual. From this, the following dual relation holds.

Theorem 5.1. *Let $\boldsymbol{\alpha} : I \to S \subset \mathbb{R}^3$ be a unit speed curve satisfying $\kappa_\alpha(s) \neq 0$, and let $F(s, u)$ be the nontrivial concircular surface of $\boldsymbol{\alpha}$. Then, the ruling direction $\mathcal{R}_\phi$ of the surface F and its unit normal vector field $\mathcal{N}$ are mutually spherical Legendrian dual.*

Proof. Define a mapping $\mathcal{L} : I \to \Delta$ by $\mathcal{L}(s) = (\mathcal{R}_\phi(s), \mathcal{N}(s))$. A direct calculation shows

$$\mathcal{R}_\phi(s) \cdot \mathcal{N}(s) = (\cos\phi \boldsymbol{N}_\alpha(s) + \sin\phi \boldsymbol{v}(s)) \cdot (\cos\phi \boldsymbol{v}(s) - \sin\phi \boldsymbol{N}_\alpha(s)) = 0,$$

which ensures that $\mathcal{L}$ is well defined. We can also easily calculate that

$$\mathcal{L}'(s) = ((-\cos\phi\kappa_\alpha(s) + \sin\phi\sqrt{c})\boldsymbol{T}_\alpha(s), (\cos\phi\sqrt{c} + \sin\phi\kappa_\alpha(s))\boldsymbol{T}_\alpha(s)),$$

and

$$\mathcal{L}^*(s)\theta_1 = (-\cos\phi\kappa_\alpha(s) + \sin\phi\sqrt{c})\boldsymbol{T}_\alpha(s) \cdot (\cos\phi \boldsymbol{v}(s) - \sin\phi \boldsymbol{N}_\alpha(s)) = 0.$$

It implies that $\mathcal{L}(s)$ is a Legendrian curve of Δ. Hence, $\mathcal{R}_\phi$ is spherical Legendrian dual to $\mathcal{N}$. □

6. Example

In this section, we present examples of concircular surfaces with singularities constructed from curves on a totally umbilical surface to illustrate the main conclusions.

Example 6.1. *We consider a plane curve $\boldsymbol{\alpha}(s) = (ln(s + \sqrt{s^2 + 1}), \sqrt{s^2 + 1}, 0)$. Then,*

$$\boldsymbol{T}_\alpha(s) = \left(\frac{1}{\sqrt{s^2+1}}, \frac{s}{\sqrt{s^2+1}}, 0\right), \quad \boldsymbol{N}_\alpha(s) = \left(\frac{-s}{\sqrt{s^2+1}}, \frac{1}{\sqrt{s^2+1}}, 0\right),$$

$\boldsymbol{v}(s) = (0, 0, 1)$. *By a direct calculation*

$$\kappa_\alpha(s) = \frac{1}{s^2+1}, \quad \kappa'_\alpha(s) = \frac{-2s}{(s^2+1)^2}, \quad \kappa''_\alpha(s) = \frac{2(3s^2-1)}{(s^2+1)^3}.$$

Therefore, the nontrivial concircular surface $F(s, u)$ generated by $\boldsymbol{\alpha}$ is

$$F(s,u) = \left(ln(s + \sqrt{s^2+1}) - \frac{us\cos\phi}{\sqrt{s^2+1}}, \sqrt{s^2+1} + \frac{u\cos\phi}{\sqrt{s^2+1}}, u\sin\phi\right).$$

When $\phi = \dfrac{\pi}{3}$*, the surface* $F(s,u)$ *is as follows*

$$F(s,u) = \left(ln(s + \sqrt{s^2+1}) - \frac{us}{2\sqrt{s^2+1}}, \sqrt{s^2+1} + \frac{u}{2\sqrt{s^2+1}}, \frac{\sqrt{3}u}{2} \right).$$

For $s \neq 0$*,* $\kappa'_\alpha(s) \neq 0$*. By Theorem 3.4 (2), the nontrivial concircular surface* $F(s,u)$ *is locally diffeomorphic to the cuspidal edge at* $(s, 2(s^2+1))$*,* $s \neq 0$*. At* $s = 0$*, we have* $\kappa'_\alpha(0) = 0$ *and* $\kappa''_\alpha(0) \neq 0$*. According to Theorem 3.4 (3), the nontrivial concircular surface* $F(s,u)$ *is locally diffeomorphic to the swallowtail at* $(0,2)$ *(see Figure 3).*

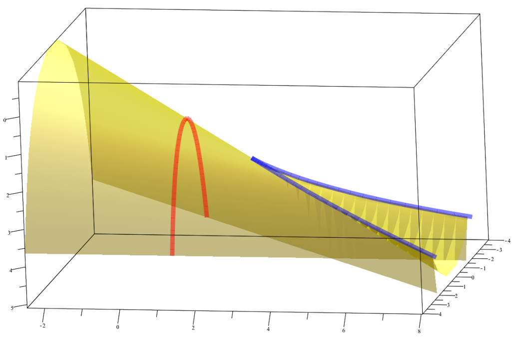

Figure 3. The red curve is $\boldsymbol{\alpha}(s)$ and the blue curve is the singularities of $F(s,u)$.

Example 6.2. *We consider a spherical curve* $\boldsymbol{\alpha}(\theta) = (\sin\theta, \sin\theta\cos\theta, \cos^2\theta)$*, where* $\theta \in [-\pi, \pi)$*. The Darboux frame along* $\boldsymbol{\alpha}(\theta)$ *is* $\{\boldsymbol{T}_\alpha(\theta), \boldsymbol{N}_\alpha(\theta), \boldsymbol{\alpha}(\theta)\}$*, where*

$$\boldsymbol{T}_\alpha(\theta) = \left(\frac{\sqrt{2}\cos\theta}{\sqrt{3+\cos 2\theta}}, \frac{\sqrt{2}(\cos^2\theta - \sin^2\theta)}{\sqrt{3+\cos 2\theta}}, \frac{-2\sqrt{2}\cos\theta\sin\theta}{\sqrt{3+\cos 2\theta}} \right),$$

$$\boldsymbol{N}_\alpha(\theta) = \left(\frac{-\sqrt{2}\cos^2\theta}{\sqrt{3+\cos 2\theta}}, \frac{\sqrt{2}(\cos^3\theta + 2\sin^2\theta\cos\theta)}{\sqrt{3+\cos 2\theta}}, \frac{-\sqrt{2}\sin^3\theta}{\sqrt{3+\cos 2\theta}} \right).$$

We can easily calculate that

$$\begin{aligned}
\kappa_\alpha(\theta) &= -\frac{\sqrt{2}\sin\theta(5+\cos 2\theta)}{(3+\cos 2\theta)^{3/2}}, \\
\kappa'_\alpha(\theta) &= \frac{\sqrt{2}[-\cos\theta(5+\cos 2\theta) + 2\sin\theta\cos\theta]}{(3+\cos 2\theta)^{3/2}} \\
&\quad - \frac{3\sqrt{2}\sin\theta\,\sin 2\theta(5+\cos 2\theta)}{(3+\cos 2\theta)^{5/2}}, \\
\kappa''_\alpha(\theta) &= \frac{\sqrt{2}[\sin\theta(5+\cos 2\theta) + 4\sin 2\theta\cos\theta + 4\sin\theta\cos 2\theta]}{(3+\cos 2\theta)^{3/2}} \\
&\quad - \frac{15\sqrt{2}\sin\theta\,\sin^2 2\theta(5+\cos 2\theta)}{(3+\cos 2\theta)^{7/2}} \\
&\quad + \frac{6\sqrt{2}[2\sin\theta\,\sin^2 2\theta - (5+\cos 2\theta)(\sin 2\theta\cos\theta + \sin\theta\cos 2\theta)]}{(3+\cos 2\theta)^{5/2}}.
\end{aligned}$$

When $\phi = \frac{\pi}{3}$, the nontrivial concircular surface $F(\theta, u)$ generated by $\boldsymbol{\alpha}(\theta)$ is

$$\begin{aligned} F(\theta, u) &= \boldsymbol{\alpha}(\theta) + u\left(\cos\frac{\pi}{3}\boldsymbol{N}_{\alpha}(\theta) + \sin\frac{\pi}{3}\boldsymbol{\alpha}(\theta)\right) \\ &= \left(\sin\theta + u\left(\frac{-\sqrt{2}\cos^2\theta}{2\sqrt{3+\cos 2\theta}} + \frac{\sqrt{3}}{2}\sin\theta\right),\right. \\ &\qquad \sin\theta\cos\theta + u\left(\frac{\sqrt{2}(\cos^3\theta + 2\sin^2\theta\cos\theta)}{2\sqrt{3+\cos 2\theta}} + \frac{\sqrt{3}}{2}\sin\theta\cos\theta\right), \\ &\qquad \left.\cos^2\theta + u\left(\frac{-\sqrt{2}\sin^3\theta}{2\sqrt{3+\cos 2\theta}} + \frac{\sqrt{3}}{2}\cos^2\theta\right)\right). \end{aligned}$$

Provided that $\theta \notin \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$, we have $\kappa'_{\alpha}(\theta) \neq 0$. By Theorem 3.4 (2), the nontrivial concircular surface $F(\theta, u)$ is locally diffeomorphic to the cuspidal edge at

$$\left(\theta, \frac{-2(3+\cos 2\theta)^{\frac{3}{2}}}{\sqrt{2}\sin\theta(5+\cos 2\theta) - \sqrt{3}(3+\cos 2\theta)^{\frac{3}{2}}}\right), \quad \textit{where } \theta \notin \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}.$$

If $\theta \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$, then $\kappa'_{\alpha}(\theta) = 0$ and $\kappa''_{\alpha}(\theta) \neq 0$. According to Theorem 3.4 (3), the nontrivial concircular surface $F(\theta, u)$ is locally diffeomorphic to the swallowtail at $(\frac{\pi}{2}, \frac{2}{2+\sqrt{3}})$ and $(-\frac{\pi}{2}, \frac{2}{2-\sqrt{3}})$ (see Figure 4).

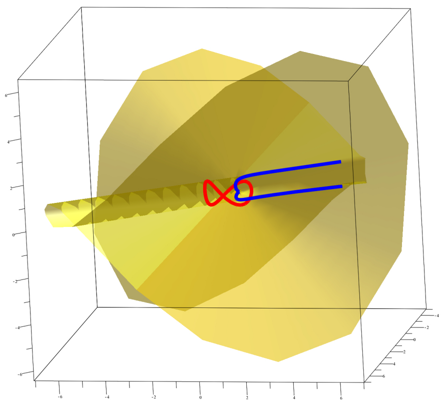

Figure 4. The red curve is $\boldsymbol{\alpha}(\theta)$ and the blue curve is the singularities of $F(\theta, u)$.

7. Conclusions

The main purpose of this paper is to investigate the singularities of concircular surfaces, which can be regarded as a special class of ruled surfaces. By applying singularity theory, we classify the generic singularities of such surfaces as cuspidal edges and swallowtails (Theorem 3.4). It is shown that the type of singularity is determined by the order of contact between the generating curve of the concircular surface and the affine tangent bundle of the unit sphere, which also provides a clear geometric interpretation of the classification (Theorem 4.3). Moreover, we establish that the normal vector field and the rulings of a concircular surface are spherical Legendrian dual to each other (Theorem 5.1). These results provide a bridge for understanding the local geometric properties and singularities of concircular surfaces. Future work may generalize the above results to the singularities of concircular submanifolds in higher-dimensional Euclidean spaces.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools for writing this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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