Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 26226–26245.
DOI: 10.3934/math.20261051
Received: 01 July 2026
Revised: 25 July 2026
Accepted: 13 August 2026
Published: 24 August 2026

Research article

Coupled fixed point results for Burton-type large contractions in partially ordered metric spaces

Mouataz Billah Mesmouli[1,*], Doha A. Abulhamil[2] and Ovidiu Popescu[3]

[1] Department of Mathematics, College of Science, University of Hail, Hail 55473, Saudi Arabia

[2] Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 21493, Saudi Arabia

[3] Department of Mathematics and Computer Sciences, Transilvania University of Braşov, Braşov 500091, Romania

* **Correspondence:** Email: m.mesmouli@uoh.edu.sa.

Abstract: In this paper, we introduce a class of coupled large contractions for mixed monotone mappings in partially ordered metric spaces. The proposed contractive condition extends the classical coupled contraction of Bhaskar and Lakshmikantham by replacing the fixed Lipschitz constant with a Burton-type nonuniform contractive condition. Under suitable monotonicity and order assumptions, we establish the existence of coupled fixed points for mappings $F : X \times X \to X$. An example is presented to show that the obtained result is not covered by the classical theorem of Bhaskar and Lakshmikantham. Additionally, we derive a uniqueness criterion and obtain a fixed point result for the associated single-valued mapping $f : X \to X$ defined by $f(x) = F(x, x)$.

Keywords: coupled fixed point; mixed monotone mapping; partially ordered metric space; large contraction; Burton contraction
Mathematics Subject Classification: 06F30, 34A12, 47H10, 54H25

1. Introduction

The classical starting point of the metric fixed point theory is the Banach contraction principle [1], which guarantees the existence and uniqueness of a fixed point for a contraction mapping on a complete metric space. Since Banach's pioneering work, many authors have generalized the contraction principle in different directions. Khan, Swaleh, and Sessa introduced fixed point results involving altering distance functions [2], while several systematic treatments of generalized contractions and Lipschitzian-type mappings can be found in the monographs of Smart [3], Rus [4], Granas and

Dugundji [5], and Agarwal, O'Regan, and Sahu [6]. These works demonstrate the flexibility of fixed point methods and their importance in both pure and applied mathematics.

A significant extension of the Banach contraction principle was introduced by Burton [7] through the concept of large contractions. Unlike the classical contraction condition, which requires the existence of a fixed constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \tag{1.1}$$

a large contraction allows the contractive coefficient to depend on the size of the distance between the points. More precisely, a large contraction is strictly contractive for distinct points and, for every positive lower bound on the distance, there exists a contraction coefficient less than one. This nonuniform behavior makes large contractions particularly useful in problems where the standard Banach condition is too restrictive.

The theory of large contractions was further developed in several recent works. Burton and Purnaras studied necessary and sufficient conditions for large contractions in the fixed point theory [8]. Dehici, Mesmouli, and Karapinar investigated fixed point results for large Kannan contraction mappings and their applications [9]. More recently, Mesmouli et al. established fixed point theorems for large contraction mappings with applications to delay fractional differential equations [10]. Further developments include large triangle–perimeter contractions in metric spaces [11]. These contributions show that Burton-type contractions provide a useful method to extend the classical fixed point theory.

Another important line of research concerns the fixed point theory in partially ordered metric spaces. In many nonlinear problems, the underlying space is endowed not only with a metric structure but also with an order structure. This is particularly useful when lower and upper solutions or monotone iterative techniques are involved. Bhaskar and Lakshmikantham introduced coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces and applied their results to differential equations [12]. Since then, this direction has been widely developed. Aghajani, Abbas, and Pourhadi Kallehbasti obtained coupled fixed point theorems in partially ordered metric spaces with applications [13]. Berinde established generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces [14]. Ćirić and Lakshmikantham studied coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces [15]. Further extensions were obtained by Hassan, Ishtiaq, and Ray in partially ordered M-metric spaces [16], by Paknazar, Eshaghi, and Vaezpour in partially ordered complex-valued metric spaces [17], and by Seshagiri Rao and Kalyani for generalized contractions related to coupled fixed point theorems in partially ordered metric spaces [18].

Motivated by the above developments, the main purpose of this paper is to combine the coupled fixed point theory of Bhaskar and Lakshmikantham with Burton's theory of large contractions. More precisely, we introduce a class of coupled large contractions for mixed monotone mappings

$$F : X \times X \to X$$

defined on partially ordered metric spaces. In this setting, the classical Banach-type coupled contraction is replaced by a Burton-type nonuniform contractive condition depending on the quantity

$$d(x, u) + d(y, v).$$

The main contributions of this paper are the following:

- We introduce a coupled version of large contractions for mappings $F : X \times X \to X$;
- We establish coupled fixed point theorems for mixed monotone mappings in partially ordered complete metric spaces; and
- We derive fixed point results for mappings $f : X \to X$ satisfying

$$f(x) = F(x, x).$$

Thus, our results extend the coupled fixed point theorem of Bhaskar and Lakshmikantham by replacing the classical Banach-type contraction with a Burton-type large contraction, and extend Burton's theorem-type to the coupled fixed point setting in partially ordered metric spaces.

The remainder of the paper is organized as follows: Section 2, we recall some basic notions and preliminary results concerning partially ordered metric spaces, mixed monotone mappings, coupled fixed points, and Burton-type large contractions; Section 3, we introduce the notion of coupled large contractions and prove our main coupled fixed point theorem for mixed monotone mappings in partially ordered complete metric spaces, we present an example showing that our result is not covered by the classical coupled fixed point theorem of Bhaskar and Lakshmikantham, and discuss the uniqueness and derive a fixed point result for the associated single-valued mapping; and Section 4 contains the conclusion and some possible directions for future research.

2. Preliminaries

In this section, we recall some basic notions and results that will be used throughout the paper. We begin with standard definitions concerning partially ordered metric spaces, product order, mixed monotone mappings, coupled fixed points, and large contractions.

Definition 2.1. [12] Let X be a nonempty set. A relation $\leq$ on X is called a partial order if it is reflexive, antisymmetric, and transitive. The pair $(X, \leq)$ is called a partially ordered set.

Definition 2.2. [12] Let $(X, \leq)$ be a partially ordered set. Two elements $x, y \in X$ are said to be comparable if either

$$x \leq y$$

or

$$y \leq x.$$

Definition 2.3. [12] Let $(X, \leq)$ be a partially ordered set, and let d be a metric on X. Then, $(X, \leq, d)$ is called a partially ordered metric space. If (X, d) is complete, then $(X, \leq, d)$ is called a partially ordered complete metric space.

For a partially ordered set $(X, \leq)$, the product set $X \times X$ is endowed with the partial order $\leq$ defined by

$$(u, v) \leq (x, y) \iff u \leq x, \qquad v \geq y. \tag{2.1}$$

This order is natural in the study of mixed monotone mappings, since the first component is ordered in the usual direction while the second component is ordered in the reverse direction.

Definition 2.4. [12] Let $(X, \leq)$ be a partially ordered set, and let $F : X \times X \to X$ be a mapping. We say that F has the mixed monotone property if F is monotone nondecreasing in its first variable and monotone nonincreasing in its second variable, that is, for all $x_1, x_2, y \in X$,

$$x_1 \leq x_2 \quad \Longrightarrow \quad F(x_1, y) \leq F(x_2, y), \tag{2.2}$$

and for all $x, y_1, y_2 \in X$,

$$y_1 \leq y_2 \quad \Longrightarrow \quad F(x, y_1) \geq F(x, y_2). \tag{2.3}$$

The product order $\preceq$ is compatible with this property: increasing the first variable and decreasing the second variable correspond to moving upward with respect to $\preceq$.

Definition 2.5. [12] Let X be a nonempty set, and let $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is called a coupled fixed point of F if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{2.4}$$

Definition 2.6. [7] Let (X, d) be a metric space, and let $T : X \to X$ be a mapping. We say that T is a large contraction if

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$, and for every $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0, 1)$ such that

$$d(x, y) \geq \varepsilon \quad \Longrightarrow \quad d(Tx, Ty) \leq \delta(\varepsilon) d(x, y). \tag{2.5}$$

The previous definition was introduced by Burton as a nonuniform extension of the Banach contraction principle. The following theorem is the main fixed point theorem for large contractions.

Theorem 2.7 (Burton's fixed point theorem [7])**.** *Let (X, d) be a complete metric space, and let $T : X \to X$ be a large contraction mapping. Suppose that there exist $x_0 \in X$ and $L > 0$ such that*

$$d(x_0, T^n x_0) \leq L, \qquad \forall n \geq 1.$$

Then, T has a unique fixed point in X.

Additionnaly, we recall the classical coupled fixed point theorem of Bhaskar and Lakshmikantham in partially ordered metric spaces.

Theorem 2.8 (Bhaskar–Lakshmikantham coupled fixed point theorem [12])**.** *Let $(X, \leq)$ be a partially ordered set, and let d be a metric on X such that (X, d) is complete. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property.*

Assume that there exists $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \tag{2.6}$$

for all $x, y, u, v \in X$ with

$$x \geq u, \qquad y \leq v.$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \qquad y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point, that is, there exist $x, y \in X$ such that

$$F(x, y) = x, \qquad F(y, x) = y.$$

3. Main results

In this section, we establish coupled fixed point results for mixed monotone mappings satisfying a Burton-type large contraction condition in partially ordered metric spaces. Throughout this section, $(X, \leq)$ denotes a partially ordered set and d is a metric on X such that (X, d) is complete.

We begin by introducing the main ordered contractive condition used in the paper.

Definition 3.1. Let $(X, \leq)$ be a partially ordered set, and let d be a metric on X. Let $F : X \times X \to X$ be a mapping. We say that F is a *coupled large contraction* if, for all $x, y, u, v \in X$ with

$$x \geq u, \qquad y \leq v, \qquad (x, y) \neq (u, v),$$

we have

$$d(F(x, y), F(u, v)) < d(x, u) + d(y, v), \tag{3.1}$$

and for every $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0, 1)$ such that

$$d(x, u) + d(y, v) \geq \varepsilon$$

implies

$$d(F(x, y), F(u, v)) \leq \frac{1}{2}\delta(\varepsilon)[d(x, u) + d(y, v)], \tag{3.2}$$

whenever

$$x \geq u, \qquad y \leq v.$$

Lemma 3.2. *Let $(X, \leq)$ be a partially ordered set and let $F : X \times X \to X$ have the mixed monotone property. Suppose that there exist $x_0, y_0 \in X$ such that*

$$x_0 \leq F(x_0, y_0), \qquad y_0 \geq F(y_0, x_0).$$

Define sequences $\{x_n\}$ and $\{y_n\}$ by the following:

$$x_{n+1} = F(x_n, y_n), \qquad y_{n+1} = F(y_n, x_n), \qquad n \geq 0.$$

Then,

$$x_n \leq x_{n+1}, \qquad y_n \geq y_{n+1}, \qquad \forall n \geq 0.$$

Equivalently,

$$(x_n, y_n) \leq (x_{n+1}, y_{n+1}), \qquad \forall n \geq 0,$$

where $\leq$ denotes the product order on $X \times X$.

Proof. By assumption,

$$x_0 \leq F(x_0, y_0) = x_1, \qquad y_0 \geq F(y_0, x_0) = y_1.$$

Thus,

$$x_0 \leq x_1, \qquad y_0 \geq y_1.$$

Assume that, for some $n \geq 0$,

$$x_n \leq x_{n+1}, \qquad y_n \geq y_{n+1}.$$

Since F has the mixed monotone property, we have

$$F(x_n, y_n) \leq F(x_{n+1}, y_n)$$

because $x_n \leq x_{n+1}$. Additionally, since $y_{n+1} \leq y_n$ and F is nonincreasing in its second variable, we get the following:

$$F(x_{n+1}, y_n) \leq F(x_{n+1}, y_{n+1}).$$

Therefore,

$$x_{n+1} = F(x_n, y_n) \leq F(x_{n+1}, y_{n+1}) = x_{n+2}.$$

Similarly, since $y_{n+1} \leq y_n$ and F is nondecreasing in its first variable, we have the following:

$$F(y_{n+1}, x_{n+1}) \leq F(y_n, x_{n+1}).$$

Moreover, since $x_n \leq x_{n+1}$ and F is nonincreasing in its second variable, we get the following:

$$F(y_n, x_{n+1}) \leq F(y_n, x_n).$$

Hence,

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_n) = y_{n+1}.$$

Thus,

$$x_{n+1} \leq x_{n+2}, \qquad y_{n+1} \geq y_{n+2}.$$

By induction,

$$x_n \leq x_{n+1}, \qquad y_n \geq y_{n+1}, \qquad \forall n \geq 0.$$

Equivalently,

$$(x_n, y_n) \leq (x_{n+1}, y_{n+1}), \qquad \forall n \geq 0.$$

□

Theorem 3.3. *Let $(X, \leq)$ be a partially ordered set, and let d be a metric on X such that (X, d) is complete. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property.*

Assume that F is a coupled large contraction. Suppose that there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \qquad y_0 \geq F(y_0, x_0).$$

Then, F has a coupled fixed point, that is, there exist $x, y \in X$ such that

$$F(x, y) = x, \qquad F(y, x) = y.$$

Proof. Define the sequences $\{x_n\}$ and $\{y_n\}$ by the following:

$$x_{n+1} = F(x_n, y_n), \qquad y_{n+1} = F(y_n, x_n), \qquad n \geq 0.$$

By the preceding lemma, we have the following:

$$x_n \leq x_{n+1}, \qquad y_n \geq y_{n+1}, \qquad \forall n \geq 0.$$

For simplicity, put

$$\Delta_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n), \qquad n \geq 0.$$

If there exists $n_0 \geq 0$ such that

$$\Delta_{n_0} = 0,$$

then

$$d(x_{n_0+1}, x_{n_0}) = 0 \quad \text{and} \quad d(y_{n_0+1}, y_{n_0}) = 0.$$

Hence,

$$x_{n_0+1} = x_{n_0}, \qquad y_{n_0+1} = y_{n_0}.$$

Therefore,

$$F(x_{n_0}, y_{n_0}) = x_{n_0}, \qquad F(y_{n_0}, x_{n_0}) = y_{n_0},$$

and so (x_{n_0}, y_{n_0}) is a coupled fixed point of F.

Now assume that

$$\Delta_n > 0, \qquad \forall n \geq 0.$$

Since

$$x_{n+1} \geq x_n \quad \text{and} \quad y_{n+1} \leq y_n,$$

we may apply the coupled large contraction condition to the pairs (x_{n+1}, y_{n+1}) and (x_n, y_n). Taking

$$\varepsilon = \Delta_n > 0,$$

we obtain

$$d(x_{n+2}, x_{n+1}) = d(F(x_{n+1}, y_{n+1}), F(x_n, y_n)) \leq \frac{1}{2}\delta(\Delta_n)\Delta_n.$$

Similarly, since

$$y_n \geq y_{n+1} \quad \text{and} \quad x_n \leq x_{n+1},$$

we apply the same condition to the pairs (y_n, x_n) and (y_{n+1}, x_{n+1}) and get

$$d(y_{n+2}, y_{n+1}) = d(F(y_{n+1}, x_{n+1}), F(y_n, x_n)) \leq \frac{1}{2}\delta(\Delta_n)\Delta_n.$$

Consequently,

$$\Delta_{n+1} = d(x_{n+2}, x_{n+1}) + d(y_{n+2}, y_{n+1}) \leq \delta(\Delta_n)\Delta_n < \Delta_n.$$

Thus, $\{\Delta_n\}$ is decreasing and bounded below by 0. Hence, there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} \Delta_n = \gamma.$$

We claim that

$$\gamma = 0.$$

Contrarily, suppose that $\gamma > 0$. Since

$$\Delta_n \geq \gamma, \qquad \forall n \geq 0,$$

the coupled large contraction condition gives a number $\delta(\gamma) \in (0, 1)$ such that

$$\Delta_{n+1} \leq \delta(\gamma)\Delta_n, \qquad \forall n \geq 0.$$

Therefore, by induction,

$$\Delta_n \leq [\delta(\gamma)]^n \Delta_0.$$

Letting $n \to \infty$, we obtain

$$\Delta_n \to 0,$$

which contradicts $\Delta_n \to \gamma > 0$. Hence,

$$\gamma = 0.$$

Therefore,

$$\lim_{n\to\infty} \Delta_n = 0.$$

Next, we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Contrarily, suppose that this is not true. Then, there exists $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there exist integers $m > n \geq N$ satisfying

$$d(x_m, x_n) + d(y_m, y_n) \geq \varepsilon_0.$$

Thus, we may choose sequences of integers $\{m_k\}$ and $\{n_k\}$ such that

$$m_k > n_k \geq k,$$

and

$$d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k}) \geq \varepsilon_0.$$

Moreover, we choose m_k to be the smallest integer greater than n_k with this property. Hence,

$$d(x_{m_k-1}, x_{n_k}) + d(y_{m_k-1}, y_{n_k}) < \varepsilon_0.$$

Using the triangle inequality, we get the following:

$$\begin{aligned}
&d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k}) \\
&\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) + d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k}) \\
&= \Delta_{m_k-1} + d(x_{m_k-1}, x_{n_k}) + d(y_{m_k-1}, y_{n_k}) \\
&< \Delta_{m_k-1} + \varepsilon_0.
\end{aligned}$$

Since $\Delta_n \to 0$, it follows that

$$\lim_{k\to\infty} [d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k})] = \varepsilon_0.$$

Additionally, by the triangle inequality,

$$\begin{aligned}
&\left|[d(x_{m_k-1}, x_{n_k-1}) + d(y_{m_k-1}, y_{n_k-1})] - [d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k})]\right| \\
&\leq d(x_{m_k}, x_{m_k-1}) + d(y_{m_k}, y_{m_k-1}) + d(x_{n_k}, x_{n_k-1}) + d(y_{n_k}, y_{n_k-1}) \\
&= \Delta_{m_k-1} + \Delta_{n_k-1}.
\end{aligned}$$

Since $\Delta_n \to 0$, we obtain

$$\lim_{k\to\infty}[d(x_{m_k-1}, x_{n_k-1}) + d(y_{m_k-1}, y_{n_k-1})] = \varepsilon_0.$$

For k sufficiently large, we have

$$d(x_{m_k-1}, x_{n_k-1}) + d(y_{m_k-1}, y_{n_k-1}) \geq \frac{\varepsilon_0}{2}.$$

Since $m_k - 1 > n_k - 1$, the monotonicity of the sequences gives

$$x_{m_k-1} \geq x_{n_k-1}, \qquad y_{m_k-1} \leq y_{n_k-1}.$$

Therefore, by the coupled large contraction condition,

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &= d(F(x_{m_k-1}, y_{m_k-1}), F(x_{n_k-1}, y_{n_k-1})) \\ &\leq \frac{1}{2}\delta\left(\frac{\varepsilon_0}{2}\right)[d(x_{m_k-1}, x_{n_k-1}) + d(y_{m_k-1}, y_{n_k-1})]. \end{aligned}$$

Similarly, since

$$y_{n_k-1} \geq y_{m_k-1}, \qquad x_{n_k-1} \leq x_{m_k-1},$$

we apply the coupled large contraction condition to the pairs (y_{n_k-1}, x_{n_k-1}) and (y_{m_k-1}, x_{m_k-1}). Hence, using the symmetry of the metric,

$$\begin{aligned} d(y_{m_k}, y_{n_k}) &= d(F(y_{m_k-1}, x_{m_k-1}), F(y_{n_k-1}, x_{n_k-1})) \\ &= d(F(y_{n_k-1}, x_{n_k-1}), F(y_{m_k-1}, x_{m_k-1})) \\ &\leq \frac{1}{2}\delta\left(\frac{\varepsilon_0}{2}\right)[d(y_{n_k-1}, y_{m_k-1}) + d(x_{n_k-1}, x_{m_k-1})] \\ &= \frac{1}{2}\delta\left(\frac{\varepsilon_0}{2}\right)[d(y_{m_k-1}, y_{n_k-1}) + d(x_{m_k-1}, x_{n_k-1})]. \end{aligned}$$

Adding the last two inequalities, we obtain

$$\begin{aligned} &d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k}) \\ &\leq \delta\left(\frac{\varepsilon_0}{2}\right)[d(x_{m_k-1}, x_{n_k-1}) + d(y_{m_k-1}, y_{n_k-1})]. \end{aligned}$$

Letting $k \to \infty$, we get

$$\varepsilon_0 \leq \delta\left(\frac{\varepsilon_0}{2}\right)\varepsilon_0.$$

Since

$$0 < \delta\left(\frac{\varepsilon_0}{2}\right) < 1,$$

this is impossible. Hence, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Since (X, d) is complete, there exist $x, y \in X$ such that

$$x_n \to x, \qquad y_n \to y.$$

Finally, since F is continuous, we obtain

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y),$$

and similarly,

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x).$$

Therefore,

$$F(x, y) = x, \qquad F(y, x) = y.$$

Thus, (x, y) is a coupled fixed point of F. □

Remark 3.4. We observe that, under the present definition of coupled large contraction, the boundedness assumption

$$d(x_0, x_n) + d(y_0, y_n) \leq L, \qquad n \geq 1,$$

is not needed in the proof. Indeed, if

$$\Delta_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n),$$

then applying the contractive condition to the two components of the coupled iteration gives

$$\Delta_{n+1} \leq \delta(\Delta_n)\Delta_n < \Delta_n$$

whenever $\Delta_n > 0$. This estimate implies that the successive differences tend to zero. Moreover, the Cauchy property of the iterative sequences follows from the same nonuniform contraction argument used in the proof. Hence, the boundedness condition may be omitted from the statement of the theorem.

Example 3.5. Let $X = [0, 1]$ be endowed with the usual metric

$$d(x, y) = |x - y|,$$

and the usual order. Then, (X, d) is a complete metric space. Define $F : X \times X \to X$ by the following:

$$F(x, y) = \frac{x}{2} - \frac{x^2}{4} + \frac{1 - y}{8}.$$

Clearly, $F(X \times X) \subset X$. Indeed, for $x, y \in [0, 1]$, we have

$$0 \leq \frac{x}{2} - \frac{x^2}{4} \leq \frac{1}{4}$$

and

$$0 \leq \frac{1 - y}{8} \leq \frac{1}{8}.$$

Therefore,

$$0 \leq F(x, y) \leq \frac{1}{4} + \frac{1}{8} = \frac{3}{8} < 1.$$

The mapping F is continuous. Moreover, F has the mixed monotone property. Indeed, let

$$\varphi(x) = \frac{x}{2} - \frac{x^2}{4}.$$

Then,

$$F(x, y) = \varphi(x) + \frac{1 - y}{8}.$$

Since

$$\varphi'(x) = \frac{1}{2} - \frac{x}{2} \geq 0, \qquad x \in [0, 1],$$

the function F is nondecreasing in its first variable. Additionally, since

$$\frac{1-y}{8}$$

is nonincreasing in y, the function F is nonincreasing in its second variable. Hence, F has the mixed monotone property.

Now, we prove that F is a coupled large contraction. Let $x, y, u, v \in X$ with

$$(x, y) \neq (u, v).$$

Then,

$$\begin{aligned} d(F(x,y), F(u,v)) &= \left| \left(\frac{x}{2} - \frac{x^2}{4}\right) - \left(\frac{u}{2} - \frac{u^2}{4}\right) + \frac{v-y}{8} \right| \\ &\leq \left| \left(\frac{x}{2} - \frac{x^2}{4}\right) - \left(\frac{u}{2} - \frac{u^2}{4}\right) \right| + \frac{1}{8}|v - y|. \end{aligned}$$

By the mean value theorem,

$$\left| \left(\frac{x}{2} - \frac{x^2}{4}\right) - \left(\frac{u}{2} - \frac{u^2}{4}\right) \right| \leq \frac{1}{2}|x - u|.$$

Therefore,

$$d(F(x,y), F(u,v)) \leq \frac{1}{2} d(x,u) + \frac{1}{8} d(y,v).$$

Since $(x, y) \neq (u, v)$, we have the following:

$$d(x,u) + d(y,v) > 0.$$

Hence,

$$\frac{1}{2} d(x,u) + \frac{1}{8} d(y,v) < d(x,u) + d(y,v).$$

Thus,

$$d(F(x,y), F(u,v)) < d(x,u) + d(y,v)$$

for all, $(x, y) \neq (u, v)$.

Now, let $x, y, u, v \in X$ satisfy the following:

$$x \geq u, \qquad y \leq v.$$

Put

$$h = x - u \geq 0, \qquad r = v - y \geq 0.$$

Then,

$$d(x,u) + d(y,v) = h + r.$$

Moreover, since F is nondecreasing in the first variable and nonincreasing in the second variable, we have

$$F(x,y) \geq F(u,v).$$

Therefore,

$$\begin{aligned} d(F(x,y),F(u,v)) &= F(x,y) - F(u,v) \\ &= \left(\frac{x}{2} - \frac{x^2}{4}\right) - \left(\frac{u}{2} - \frac{u^2}{4}\right) + \frac{v-y}{8} \\ &= h\left(\frac{1}{2} - \frac{x+u}{4}\right) + \frac{r}{8}. \end{aligned}$$

Since $x = u + h$, we have

$$x + u = 2u + h \geq h.$$

Thus,

$$\frac{1}{2} - \frac{x+u}{4} \leq \frac{1}{2} - \frac{h}{4}.$$

Consequently,

$$d(F(x,y),F(u,v)) \leq h\left(\frac{1}{2} - \frac{h}{4}\right) + \frac{r}{8}.$$

Hence,

$$2d(F(x,y),F(u,v)) \leq h - \frac{h^2}{2} + \frac{r}{4}.$$

Let $\varepsilon > 0$ and assume that

$$d(x,u) + d(y,v) = h + r \geq \varepsilon.$$

Since $x, y, u, v \in [0,1]$, we have

$$0 \leq h + r \leq 2.$$

Define

$$\delta(\varepsilon) = \begin{cases} 1 - \dfrac{\varepsilon}{16}, & 0 < \varepsilon \leq 2, \\ \dfrac{1}{2}, & \varepsilon > 2. \end{cases}$$

Then,

$$\delta(\varepsilon) \in (0,1)$$

for every $\varepsilon > 0$.

First, assume that $0 < \varepsilon \leq 2$. We claim that

$$h - \frac{h^2}{2} + \frac{r}{4} \leq \left(1 - \frac{\varepsilon}{16}\right)(h + r).$$

Indeed,

$$(h + r) - \left(h - \frac{h^2}{2} + \frac{r}{4}\right) = \frac{h^2}{2} + \frac{3r}{4}.$$

Additionally,

$$\frac{h^2}{2} + \frac{3r}{4} \geq \frac{(h+r)^2}{16}.$$

To see this, observe that

$$\begin{aligned}16\left(\frac{h^2}{2}+\frac{3r}{4}\right)-(h+r)^2 &= 8h^2+12r-h^2-2hr-r^2\\ &= 7h^2-2hr+12r-r^2.\end{aligned}$$

Since $0 \le h, r \le 1$, we have

$$-2hr \ge -2r$$

and

$$-r^2 \ge -r.$$

Thus,

$$7h^2-2hr+12r-r^2 \ge 7h^2-2r+12r-r = 7h^2+9r \ge 0.$$

Therefore,

$$\frac{h^2}{2}+\frac{3r}{4} \ge \frac{(h+r)^2}{16}.$$

It follows that

$$\begin{aligned}h-\frac{h^2}{2}+\frac{r}{4} &= (h+r)-\left(\frac{h^2}{2}+\frac{3r}{4}\right)\\ &\le (h+r)-\frac{(h+r)^2}{16}\\ &= \left(1-\frac{h+r}{16}\right)(h+r).\end{aligned}$$

Since $h+r \ge \varepsilon$, we get

$$1-\frac{h+r}{16} \le 1-\frac{\varepsilon}{16}.$$

Hence,

$$h-\frac{h^2}{2}+\frac{r}{4} \le \left(1-\frac{\varepsilon}{16}\right)(h+r).$$

Therefore,

$$2d(F(x,y),F(u,v)) \le \left(1-\frac{\varepsilon}{16}\right)(h+r),$$

that is,

$$d(F(x,y),F(u,v)) \le \frac{1}{2}\delta(\varepsilon)[d(x,u)+d(y,v)].$$

If $\varepsilon > 2$, then the condition

$$d(x,u)+d(y,v) \ge \varepsilon$$

cannot occur, because

$$d(x,u)+d(y,v) \le 2.$$

Thus, the implication in the definition is vacuously true.

Hence, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0,1)$ such that

$$d(F(x,y),F(u,v)) \le \frac{1}{2}\delta(\varepsilon)[d(x,u)+d(y,v)]$$

whenever

$$d(x,u)+d(y,v)\geq\varepsilon,\qquad x\geq u,\qquad y\leq v.$$

Therefore, F is a coupled large contraction.

Now choose

$$x_0=0,\qquad y_0=1.$$

Then,

$$F(x_0,y_0)=F(0,1)=0,$$

and hence

$$x_0=0\leq F(x_0,y_0)=0.$$

Additionally,

$$F(y_0,x_0)=F(1,0)=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}=\frac{3}{8}.$$

Thus,

$$y_0=1\geq F(y_0,x_0)=\frac{3}{8}.$$

Therefore, all assumptions of the main theorem are satisfied. Consequently, F has a coupled fixed point.

Now, we compute the coupled fixed point. Let $(x,y)\in X\times X$ satisfy

$$F(x,y)=x,\qquad F(y,x)=y.$$

Then,

$$x=\frac{x}{2}-\frac{x^2}{4}+\frac{1-y}{8},$$

and

$$y=\frac{y}{2}-\frac{y^2}{4}+\frac{1-x}{8}.$$

Subtracting the second equation from the first, we obtain

$$x-y=\frac{x-y}{2}-\frac{x^2-y^2}{4}+\frac{x-y}{8}.$$

Hence

$$x-y=(x-y)\left(\frac{5}{8}-\frac{x+y}{4}\right).$$

Therefore,

$$(x-y)\left(1-\frac{5}{8}+\frac{x+y}{4}\right)=0,$$

that is,

$$(x-y)\left(\frac{3}{8}+\frac{x+y}{4}\right)=0.$$

Since

$$\frac{3}{8}+\frac{x+y}{4}>0,$$

we get

$$x = y.$$

Let

$$x = y = t.$$

Then,

$$t = \frac{t}{2} - \frac{t^2}{4} + \frac{1-t}{8}.$$

Multiplying by 8, we obtain

$$8t = 4t - 2t^2 + 1 - t.$$

Thus,

$$2t^2 + 5t - 1 = 0.$$

Solving this quadratic equation gives

$$t = \frac{-5 + \sqrt{33}}{4},$$

which is the unique root belonging to $[0, 1]$. Consequently,

$$\left(\frac{-5 + \sqrt{33}}{4}, \frac{-5 + \sqrt{33}}{4} \right)$$

is the coupled fixed point of F.

Finally, we show that this example is not covered by the classical theorem of Bhaskar and Lakshmikantham. Contrarily, suppose that there exists $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ satisfying

$$x \geq u, \qquad y \leq v.$$

Take

$$u = 0, \qquad v = y, \qquad x > 0.$$

Then,

$$d(y, v) = 0, \qquad d(x, u) = x.$$

Moreover,

$$F(x, y) - F(0, y) = \frac{x}{2} - \frac{x^2}{4}.$$

Hence,

$$\frac{x}{2} - \frac{x^2}{4} \leq \frac{k}{2}x.$$

Dividing by $x > 0$, we obtain

$$\frac{1}{2} - \frac{x}{4} \leq \frac{k}{2}.$$

Letting $x \to 0^+$ gives

$$\frac{1}{2} \le \frac{k}{2}.$$

Therefore,

$$k \ge 1,$$

which contradicts $k \in [0, 1)$. Thus, no constant $k \in [0, 1)$ satisfies the classical coupled contraction condition of Bhaskar and Lakshmikantham, while the previous verification shows that F satisfies the coupled large contraction condition of the present paper.

The previous theorem guarantees the existence of a coupled fixed point. To obtain uniqueness, an additional comparability assumption is needed. This is natural because the contractive condition is only imposed on ordered pairs in the product order.

Corollary 3.6. *Assume the hypotheses of the previous theorem. Suppose that any two coupled fixed points of F are comparable with respect to the product order $\preceq$ on $X \times X$, that is, if (x, y) and (u, v) are coupled fixed points of F, then either*

$$(u, v) \preceq (x, y),$$

or

$$(x, y) \preceq (u, v).$$

Equivalently, either

$$x \ge u, \qquad y \le v,$$

or

$$u \ge x, \qquad v \le y.$$

Then, the coupled fixed point of F is unique.

Proof. Let (x, y) and (u, v) be two coupled fixed points of F. Then,

$$F(x, y) = x, \qquad F(y, x) = y,$$

and

$$F(u, v) = u, \qquad F(v, u) = v.$$

Assume that

$$(x, y) \neq (u, v).$$

By the hypothesis, the two coupled fixed points are comparable in the product order. Without loss of generality, suppose that

$$(u, v) \preceq (x, y),$$

that is,

$$x \ge u, \qquad y \le v.$$

Put

$$S = d(x, u) + d(y, v).$$

Since $(x, y) \neq (u, v)$, we have

$$S > 0.$$

Using the coupled large contraction condition, we obtain

$$\begin{aligned} d(x,u) &= d(F(x,y),F(u,v)) \\ &\le \frac{1}{2}\delta(S)[d(x,u)+d(y,v)] \\ &= \frac{1}{2}\delta(S)S. \end{aligned}$$

Additionally, since

$$v \ge y, \qquad u \le x,$$

we can apply the same condition to the ordered pairs (v,u) and (y,x). Thus, using the symmetry of the metric,

$$\begin{aligned} d(y,v) &= d(F(y,x),F(v,u)) \\ &= d(F(v,u),F(y,x)) \\ &\le \frac{1}{2}\delta(S)[d(y,v)+d(x,u)] \\ &= \frac{1}{2}\delta(S)S. \end{aligned}$$

Adding the last two inequalities gives

$$S = d(x,u)+d(y,v) \le \delta(S)S.$$

Since

$$0 < \delta(S) < 1$$

and

$$S > 0,$$

this is impossible. Therefore, our assumption

$$(x,y) \neq (u,v)$$

is false. Hence,

$$(x,y) = (u,v).$$

Thus, the coupled fixed point of F is unique. □

Now, we derive a fixed point result for a single-valued mapping associated with F. If F has a coupled fixed point whose two components coincide, then this point immediately yields an ordinary fixed point for the mapping $f : X \to X$ defined by $f(x) = F(x,x)$.

Corollary 3.7. *Assume the hypotheses of the main theorem. Suppose that F has a coupled fixed point (x,y) such that $x = y$. Define $f : X \to X$ by $f(x) = F(x,x)$. Then, f has a fixed point.*

Proof. Since (x,y) is a coupled fixed point of F, we have

$$F(x,y) = x, \qquad F(y,x) = y.$$

Since $x = y$, we obtain $F(x,x) = x$. By the definition of f,

$$f(x) = F(x,x) = x.$$

Therefore, x is a fixed point of f. □

We close this section with a comparison between the present results and the classical coupled fixed point theorem of Bhaskar and Lakshmikantham. This comparison clarifies how the present theorem extends the known Banach-type coupled contraction to a Burton-type nonuniform setting.

Remark 3.8. The preceding results generalize the coupled fixed point theorem of Bhaskar and Lakshmikantham. Indeed, the classical condition

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)], \qquad k \in [0,1),$$

for $x \geq u$ and $y \leq v$, is a uniform contractive condition. In contrast, the coupled large contraction condition used in this paper is nonuniform: the coefficient depends on the positive lower bound of $d(x,u)+d(y,v)$. Thus, the present result extends the Banach-type coupled contraction framework to a Burton-type large contraction setting.

4. Conclusions

In this paper, we studied coupled fixed point results for mixed monotone mappings in partially ordered metric spaces under a Burton-type large contraction condition. The main novelty of our approach is that the contractive coefficient is not fixed globally, as in the classical Bhaskar–Lakshmikantham theorem, but depends on a positive lower bound of the quantity $d(x,u)+d(y,v)$. This gives a nonuniform coupled contraction that properly extends the usual Banach-type coupled contraction setting.

We proved the existence of coupled fixed points for mappings $F : X \times X \to X$ satisfying the mixed monotone property and the proposed coupled large contraction condition. Additionaly, we established a uniqueness criterion under an additional comparability assumption and derived a fixed point result for the associated single-valued mapping $f(x) = F(x,x)$. Furthermore, we provided an example showing that our result is not a direct consequence of the classical coupled fixed point theorem of Bhaskar and Lakshmikantham.

Future work may extend the present coupled large contraction results to generalized metric spaces, multivalued mappings, weaker order structures, and applications to iterative analytical methods. In particular, the proposed nonuniform coupled contraction condition may be useful in the convergence analysis of computational techniques such as the Variational Iteration Method, especially in situations where the associated iterative operator does not satisfy a classical global Lipschitz contraction. Another possible direction is to study asymmetric coupled large contraction conditions for bivariate mappings which do not necessarily satisfy the mixed monotone property. Such extensions would require additional order-regularity or alternative comparison conditions ensuring that the limits of the iterative sequences preserve the structure needed to pass to the coupled fixed point relations.

Author contributions

Mouataz Billah Mesmouli: Writing–review and editing, methodology, investigation; Doha A. Abulhamil: Writing–review and editing, visualization; Ovidiu Popescu: Writing–review and editing, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

AI tools used: ChatGPT (developed by OpenAI).

ChatGPT was used exclusively for English language editing, including grammar correction, improvement of readability, and refinement of sentence structure and academic language. The author has carefully reviewed and verified all AI-assisted content and takes full responsibility for the accuracy, integrity, and originality of the final manuscript.

The AI-assisted language editing was applied throughout the manuscript, particularly to improve the clarity, readability, and linguistic quality of the text.

Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. Math.*, **3** (1922), 133–181. http://dx.doi.org/10.4064/fm-3-1-133-181
2. M. S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Austral. Math. Soc.*, **30** (1984), 1–9. http://dx.doi.org/10.1017/S0004972700001659
3. D. R. Smart, *Fixed point theorems*, Cambridge: Cambridge University Press, 1980.
4. I. A. Rus, *Generalized contractions and applications*, Cluj-Napoca: Cluj University Press, 2001.
5. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2003. https://doi.org/10.1007/978-0-387-21593-8
6. R. P. Agarwal, D. O'Regan, D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, New York: Springer, 2009. https://doi.org/10.1007/978-0-387-75818-3
7. T. A. Burton, Integral equations, implicit functions, and fixed points, *Proc. Amer. Math. Soc.*, **124** (1996), 2383–2390. http://dx.doi.org/10.1090/S0002-9939-96-03533-2
8. T. A. Burton, I. Purnaras, Necessary and sufficient conditions for large contractions in fixed point theory, *Electron. J. Qual. Theory Differ. Equ.*, **2019** (2019), 1–24. http://dx.doi.org/10.14232/ejqtde.2019.1.94
9. A. Dehici, M. B. Mesmouli, E. Karapınar, On the fixed points of large-Kannan contraction mappings and applications, *Appl. Math. E-Notes*, **19** (2019), 535–551.
10. M. B. Mesmouli, E. Akın, L. F. Iambor, O. Tunç, T. S. Hassan, On the fixed point theorem for large contraction mappings with applications to delay fractional differential equations, *Fractal Fract.*, **8** (2024), 703. http://dx.doi.org/10.3390/fractalfract8120703
11. M. B. Mesmouli, L. F. Iambor, T. S. Hassan, Fixed-point results in metric spaces for large triangle–perimeter contractions, *Mathematics*, **14** (2026), 457. http://dx.doi.org/10.3390/math14030457
12. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.*, **65** (2006), 1379–1393. https://doi.org/10.1016/j.na.2005.10.017

13. A. Aghajani, M. Abbas, E. Pourhadi Kallehbasti, Coupled fixed point theorems in partially ordered metric spaces and application, *Math. Commun.*, **17** (2012), 497–509.
14. V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, *Nonlinear Anal.*, **74** (2011), 7347–7355. https://doi.org/10.1016/j.na.2011.07.053
15. L. Ćirić, V. Lakshmikantham, Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Stoch. Anal. Appl.*, **27** (2009), 1246–1259. https://doi.org/10.1080/07362990903259967
16. A. Hassan, U. Ishtiaq, S. Ray, Certain new type of fixed point results in partially ordered M-metric space, *Int. J. Nonlinear Anal. Appl.*, **14** (2023), 193–205. https://doi.org/10.22075/ijnaa.2023.29834.4271
17. M. Paknazar, M. Eshaghi, S. M. Vaezpour, Coupled fixed point theorems in partially ordered complex valued metric spaces with application, *Int. J. Nonlinear Anal. Appl.*, **13** (2022), 13–20. https://doi.org/10.22075/ijnaa.2017.449.1065
18. N. Seshagiri Rao, K. Kalyani, Generalized contractions to coupled fixed point theorems in partially ordered metric spaces, *J. Sib. Fed. Univ. Math. Phys.*, **13** (2020), 492–502. http://dx.doi.org/10.17516/1997-1397-2020-13-4-492-502

AIMS Press