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Research article

On some topological properties in single-valued neutrosophic soft ideal spaces

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Abstract: In this paper, we introduce and investigate several new concepts in the framework of single-valued neutrosophic soft ideal topological spaces, including **t**-single valued neutrosophic soft $\delta\mathcal{I}$-cluster points and $\theta\mathcal{I}$-cluster points, together with their corresponding $\delta\mathcal{I}$-closure and $\theta\mathcal{I}$-closure operators. We further study **t**-single valued neutrosophic soft $\delta\mathcal{I}$-closed and $\theta\mathcal{I}$-closed sets and examine their fundamental properties. In addition, various types of continuity are introduced and analyzed, such as single-valued neutrosophic soft $\delta\mathcal{I}$-continuity, $\theta\mathcal{I}$-continuity, and almost $\mathcal{I}$-continuity, along with stronger forms including strongly θ-ideal continuity and almost ideal continuity. Moreover, the notions of single-valued neutrosophic soft $\mathcal{I}$-regular and almost $\mathcal{I}$-regular spaces are established within this setting, and several relationships among these concepts are derived, providing a deeper understanding of their structural properties and potential applications.

Keywords: svns-ideal topology; $\delta\mathcal{I}$-closure operator; $\theta\mathcal{I}$-closure operator; $\delta\mathcal{I}$-continuity; $\theta\mathcal{I}$-continuity; almost $\mathcal{I}$-continuity; $\mathcal{I}$-regular space; almost $\mathcal{I}$-regular space
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1. Introduction

Uncertainty modeling is an area in mathematics, as well as applied sciences, that has developed considerably within the last several decades. Classical set theory and conventional topological techniques are frequently not sufficient to handle imprecise or ambiguous data. In order to overcome this shortcoming, Zadeh in 1965 proposed the concept of fuzzy set in which elements of a set are free to be members with different membership degrees (see, e.g., [1]) as opposed to being limited to binary values (0 or 1).

Moreover, Molodtsov [2] introduced the theory of soft sets as a novel mathematical framework for handling uncertainty. Since its introduction, soft set theory has been extensively developed and successfully applied in various fields, including medicine, economics, mathematics, and engineering. Next, abstract research of soft-set operators was taken into account by Maji et al. [3], uses in decision-making problems. In addition, Shabir and Naz [4], proposed the concept of soft topological spaces (st-spaces) determined on an initial universe with a known set of parameters. Majumdar and Samanta [5], studied mappings on soft sets and how they can be applied in medical diagnosis. The concept of soft mappings was presented by Kharal and Ahmed [6], and the concept of soft continuity was later elaborated by Aygunoglu and Aygun. [7]. Since then, many studies have been devoted to soft continuity which are recorded in [8–10]. The generalization of soft open sets (S-open sets) plays an effective role in a soft topology through their ability to improve on many results, or to open the door to explore and discuss several soft topological notions such as soft continuity [11–13].

The notion of fuzzy soft sets (fs- sets) was first defined in 2001 by Maji et al. [14], which combines the f-set [1] and s-set [2]. The concept of fst-spaces was introduced and many of its properties such as fs-continuity, fs-closure operators, fs-interior operators, and fs-subspaces were studied [15, 16] based on fuzzy topologies in the sense of $\check{S}$ostak [17]. Also, a novel approach to discussing fs-regularity axioms and fs-separation axioms using fs-sets was explored by Taha [18, 19]. The notions of r-FS-regularly-open sets, r-FS-pre-open sets, r-FS-semi-open sets, r-FS-α-open sets, and r-FS-β- open sets were introduced by the authors [20–23]. Furthermore, Alshammari et al. [24] defined and investigated the concept of fs-δ-continuous functions based on fuzzy topologies in the sense of Sostak [17].

Another theory that came out to be an improvement upon fuzzy and intuitionistic fuzzy sets is neutrosophic set theory, introduced by Smarandache [25], where you introduce indeterminacy as well as truth and falsity, and three separate degrees of membership are used. Single-valued neutrosophic sets have been the most frequently used type because they are easier to implement in code and easier to apply in practice (see [26,27]). In recent times, Saber et al. [28,29] familiarized the concepts of single-valued neutrosophic closure space, single-valued neutrosophic ideal, r-single-valued neutrosophic connected sets, and single-valued neutrosophic ideal open local functions based on single-valued neutrosophic topological space.

Saber et al. [30] proposed the concept of the novel concept of single-valued neutrosophic soft topological spaces and single-valued neutrosophic soft closure spaces. The notions of singlevalued neutrosophic soft uniformities, single-valued neutrosophic soft topologies, and single-valued neutrosophic soft interior operators were introduced by Saber et al. [31]. Furthermore, Alsharari et al. [32] defined and investigated the concepts of stratified single-valued neutrosophic soft topogenous (stratified svns-topogenous), stratified single-valued neutrosophic soft filter (stratified svns-filter), stratified single-valued neutrosophic soft quasi uniformity (stratified svnsq-uniformity), and stratified single-valued neutrosophic soft quasi proximity based on single-valued neutrosophic soft ideal topological spaces [28].

In this paper, we aim to introduce and investigate several new concepts within the framework of single-valued neutrosophic soft ideal topological spaces. In particular, we will define **t** single-valued neutrosophic soft $\delta\mathcal{I}$-cluster points and $\theta\mathcal{I}$-cluster points, along with their corresponding $\delta\mathcal{I}$-closure and $\theta\mathcal{I}$-closure operators. We will also study **t**-single valued neutrosophic soft $\delta\mathcal{I}$-closed and $\theta\mathcal{I}$-closed sets and examine their fundamental properties. Furthermore, we will introduce and analyze various types of continuity, including single-valued neutrosophic soft $\delta\mathcal{I}$-continuity, $\theta\mathcal{I}$-continuity, and almost $\mathcal{I}$-continuity, as well as stronger forms such as strongly θ-ideal continuity and almost ideal continuity. In addition, we will establish the notions of single-valued neutrosophic soft $\mathcal{I}$-regular and almost $\mathcal{I}$-regular spaces and explore the relationships among these concepts.

2. Preliminaries

In this study, the symbols Φ and $\mathcal{K}$ are used to denote nonempty sets. Let $\hbar$ denote the set of all parameters associated with Φ, and let $M \subseteq \hbar$. The family of all single-valued neutrosophic sets on Φ is denoted by $\mathbf{I}^{\Phi}$, where $\mathbf{I}_{\circ} = (0, 1]$ and $\mathbf{I} = [0, 1]$. For any $\gamma \in \mathbf{I}$, the constant single-valued neutrosophic set $\overline{\gamma}$ is defined by $\overline{\gamma}(\varkappa) = \gamma$ for every $\varkappa \in \Phi$. Table 1 lists the symbols used throughout the paper.

Table 1. List of symbols used throughout the paper.

Symbol	Description
Φ	Universe of discourse (nonempty set)
$\mathcal{K}$	A nonempty set
$\hbar$	Set of all parameters associated with Φ.
$\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}$	Single-valued neutrosophic soft topology
$Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$	single-valued neutrosophic soft open $Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}$-neighborhood of $\varkappa_{\gamma}^{(m,n,r)}$
$\mathcal{H}_{\hbar}^{\varrho\sigma\varphi}$	Single-valued neutrosophic soft ideal on Φ
$C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}$	Single-valued neutrosophic soft closure operator
$\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}$	Single-valued neutrosophic soft interior operator
$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}$	$\delta\mathcal{I}$-closure operator
SVNS-$\delta\mathcal{I}$-continuous	δ-ideal continuity in single-valued neutrosophic soft topological spaces
SVNS-$\theta\mathcal{I}$-continuous	θ-ideal continuity in single-valued neutrosophic soft topological spaces.
SVNS-$\mathcal{I}$-regular	Single valued neutrosophic softideal regular
SVNS-almost-$\mathcal{I}$-regular	Single valued neutrosophic soft almost ideal regular

Definition 1. *[25] Let Φ be a universal set. A neutrosophic set (briefly, n-set) Ξ on Φ is defined as*

$$\Xi = \{(\varkappa, \langle \varrho_{\Xi}(\varkappa), \sigma_{\Xi}(\varkappa), \varphi_{\Xi}(\varkappa)\rangle) \mid \varkappa \in \Phi,\ \varrho_{\Xi}(\varkappa), \sigma_{\Xi}(\varkappa), \varphi_{\Xi}(\varkappa) \in]^{-}0, 1^{+}[\},$$

where $\varrho_{\Xi}(\varkappa)$, $\sigma_{\Xi}(\varkappa)$, and $\varphi_{\Xi}(\varkappa)$ denote the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively.

Definition 2. *[33] Let Φ be a universe of objects with a generic element $\varkappa \in \Phi$. A single-valued neutrosophic set (briefly, svn-set) Ξ in Φ is defined as $\Xi = \{(\varkappa, \langle \varrho_{\Xi}(\varkappa), \sigma_{\Xi}(\varkappa), \varphi_{\Xi}(\varkappa)\rangle)\}$, where $\varrho_{\Xi}(\varkappa), \sigma_{\Xi}(\varkappa), \varphi_{\Xi}(\varkappa) : \Phi \to \mathbf{I}$. Here, $\varrho_{\Xi}(\varkappa)$, $\sigma_{\Xi}(\varkappa)$, and $\varphi_{\Xi}(\varkappa)$ represent the truth-membership, indeterminacy-membership, and falsity-membership functions of H, respectively.*

Definition 3. *[30] Let Ξ_A be a single-valued neutrosophic soft set (briefly, svns-set) on Φ, where $\Xi : \hbar \to \mathbf{I}^{\Phi}$. In other words, for each $\varsigma \in A$, the mapping $\Xi(\varsigma)$ (denoted by Ξ_{ς}) represents a single-valued neutrosophic set (svn-set) on Φ. For parameters $\varsigma \notin A$, it is defined that $\Xi(\varsigma) = \langle 0, 1, 1\rangle$.*

Each svn-set $\Xi(\varsigma)$ is regarded as an element of the svnfs Ξ_A. Consequently, a svns-set $\Xi_\hbar$ on Φ can be expressed as

$$(\Xi, \hbar) = \{(\varsigma, \Xi(\varsigma)) \mid \varsigma \in \hbar,\ \Xi(\varsigma) \in \mathbf{I}^\Phi\} = \{(\varsigma, \langle \varrho_\Xi(\varsigma), \sigma_\Xi(\varsigma), \varphi_\Xi(\varsigma)\rangle) \mid \varsigma \in \hbar,\ \Xi(\varsigma) \in \mathbf{I}^\Phi\}.$$

Here, $\varrho_\Xi : \hbar \to \mathbf{I}$ denotes the membership function, $\sigma_\Xi : \hbar \to \mathbf{I}$ represents the indeterminacy function, and $\varphi_\Xi : \hbar \to \mathbf{I}$ denotes the non-membership function of the svns-set.

The pair $\widetilde{(\Phi, \hbar)}$ represents the family of all svns-sets defined on Φ and is referred to as the universe of svns-sets.

A svns-sets $\Xi_\hbar$ on Φ is called a null-svns-set (briefly, $\widetilde{\Phi}$) if $\varrho_\Xi(\varsigma) = 0,\ \sigma_\Xi(\varsigma) = 1,\ \varphi_\Xi(\varsigma) = 1$, for every $\varsigma \in \hbar$.

A svns-sets $\Xi_\hbar$ on Φ is called a absolute-svns-set (briefly, $\widetilde{\hbar}$) if $\varrho_\Xi(\varsigma) = 1,\ \sigma_\Xi(\varsigma) = 0,\ \varphi_\Xi(\varsigma) = 0$, for every $\varsigma \in \hbar$.

Furthermore, a svns-sets $\Xi_\hbar$ on Φ is said to be a $\mathbf{s}$-absolute svns-set (briefly, $\widetilde{\hbar}_\mathbf{s}$) if $\varrho_\Xi(\varsigma) = \mathbf{s},\ \sigma_\Xi(\varsigma) = 0,\ \varphi_\Xi(\varsigma) = 0$, for all $\varsigma \in \hbar$, where $\mathbf{s} \in \mathbf{I}$.

Definition 4. *[30] Let $\Xi_A, \Upsilon_B \in \widetilde{(\Phi, \hbar)}$ be an svns-sets on Φ. Then,*

(i) Inclusion of two sets (for short, $\Xi_A \sqsubseteq \Upsilon_B$) defined as

$$\varrho_\Xi(\varsigma) \leq \varrho_\Upsilon(\varsigma), \quad \sigma_\Xi(\varsigma) \geq \sigma_\Upsilon(\varsigma), \quad \varphi_\Xi(\varsigma) \geq \varphi_\Upsilon(\varsigma);$$

(ii) The complement of the set Ξ_A denoted by (for short, $(\Xi_A)^c$) is defined as

$$(\Xi_A)^c = \{(\varsigma, \langle \varphi_\Xi(\varsigma), \sigma_{(\Xi)^c}(\varsigma), \varrho_\Xi(\varsigma)\rangle) \mid \varsigma \in \hbar, \Xi(\varsigma) \in \mathbf{I}^\Phi\}.$$

Definition 5. *[34] Let Φ be a universal set and let $\hbar$ denote a collection of parameters. A single-valued neutrosophic soft point, written as $\varkappa_\gamma^{(m,n,r)}$ on Φ, is defined as a mapping from $\hbar$ into the class of single-valued neutrosophic soft points of Φ satisfying*

$$\varkappa_\gamma^{(m,n,r)}(\overline{\gamma})(\mathbf{y}) = \begin{cases} (m, n, r), & \text{if } \overline{\gamma} = \gamma, \text{ and } \mathbf{y} = \varkappa, \\ \langle 0, 1, 1\rangle, & \text{otherwise}, \end{cases}$$

where γ is a single-valued neutrosophic point on Φ. Moreover, we say that $\varkappa_\gamma^{(m,n,r)}$ belongs to Ξ_A ($\varkappa_\gamma^{(m,n,r)} \widetilde{\in} \Xi_A$) if $m \leq \varrho_{\Xi_A}(\varkappa)(\gamma)$, $n \geq \sigma_{\Xi_A}(\varkappa)(\gamma)$ and $r \geq \varphi_{\Xi_A}(\varkappa)(\gamma)$. On Φ, $P_{\widetilde{(m,n,r)}}(\Phi)$ is the family of all svns-points.

Definition 6. *[34] Let $\Phi \neq 0$ and let $\Xi_A \in \widetilde{(\Phi, \hbar)}$. For a point $\varkappa_\gamma^{(m,n,r)} \in P_{\widetilde{(m,n,r)}}(\Phi)$, the point $\varkappa_\gamma^{(m,n,r)}$ is said to be single-valued neutrosophic soft quasi-coincident (for short, svnsqc-coincident) with Ξ_A, denoted by $\varkappa_\gamma^{(m,n,r)}\ \Lambda\ \Xi_A$, if there exists $\gamma \in \Phi$ such that*

$$m + \varrho_{\Xi_A}(\varkappa)(\gamma) > 1, \qquad m + \sigma_{\Xi_A}(\varkappa)(\gamma) \leq 1, \qquad r + \varphi_{\Xi_A}(\varkappa)(\gamma) \leq 1.$$

Furthermore, for any $\Xi_A, \Upsilon_B \in \widetilde{(\Phi, \hbar)}$, we write $\Xi_A\ \Lambda\ \Upsilon_B$ whenever there exists an element $\gamma \in \Phi$ satisfying

$$\varrho_{\Xi_A}(\varkappa)(\gamma) + \varrho_{\Upsilon_B}(\varkappa)(\gamma) > 1,\ \sigma_{\Xi_A}(\varkappa)(\gamma) + \sigma_{\Upsilon_B}(\varkappa)(\gamma) \leq 1,\ \ \varphi_{\Xi_A}(\varkappa)(\gamma) + \varphi_{\Upsilon_B}(\varkappa)(\gamma) \leq 1.$$

Definition 7. *[30] We say that* $\mathcal{G}^{\varrho}_{\varsigma}, \mathcal{G}^{\sigma}_{\varsigma}, \mathcal{G}^{\varphi}_{\varsigma} : \hbar \to \mathbf{I}^{\widetilde{(\Phi,\hbar)}}$ *forms a single-valued neutrosophic soft topology (svnst) on* Φ, *if it satisfies the subsequent axioms for every* $\varsigma \in \hbar$:

($\mathcal{G}_1$) $\mathcal{G}^{\varrho}_{\varsigma}(\phi) = 1, \mathcal{G}^{\sigma}_{\varsigma}(\phi) = 0, \mathcal{G}^{\varphi}_{\varsigma}(\phi) = 0$ *and* $\mathcal{G}^{\varrho}_{\varsigma}(\widetilde{\hbar}) = 1, \mathcal{G}^{\sigma}_{\varsigma}(\widetilde{\hbar}) = 0, \mathcal{G}^{\varphi}_{\varsigma}(\widetilde{\hbar}) = 0$;
($\mathcal{G}_2$) $\mathcal{G}^{\varrho}_{\varsigma}(\Xi_A \sqcap \Upsilon_B) \geq \mathcal{G}^{\varrho}_{\varsigma}(\Xi_A) \wedge \mathcal{G}^{\varrho}_{\varsigma}(\Upsilon_B)$, $\mathcal{G}^{\sigma}_{\varsigma}(\Xi_A \sqcap \Upsilon_B) \leq \mathcal{G}^{\sigma}_{\varsigma}(\Xi_A) \wedge \mathcal{G}^{\sigma}_{\varsigma}(\Upsilon_B)$, $\mathcal{G}^{\varphi}_{\varsigma}(\Xi_A \sqcap \Upsilon_B) \leq \mathcal{G}^{\varphi}_{\varsigma}(\Xi_A) \wedge \mathcal{G}^{\varphi}_{\varsigma}(\Upsilon_B)$, $\forall\ \Xi_A, \Upsilon_B \in \widetilde{(\Phi,\hbar)}$;
($\mathcal{G}_3$) $\mathcal{G}^{\varrho}_{\varsigma}(\bigsqcup_{i\in\dagger}(\Xi_A)_i) \geq \bigwedge_{i\in\dagger}(\mathcal{G}^{\varrho}_{\varsigma}((\Xi_A)_i)$, $\mathcal{G}^{\sigma}_{\varsigma}(\bigsqcup_{i\in\dagger}(\Xi_A)_i) \leq \bigvee_{i\in\dagger}(\mathcal{G}^{\sigma}_{\varsigma}((\Xi_A)_i)$, $\mathcal{G}^{\varphi}_{\varsigma}(\bigsqcup_{i\in\dagger}(\Xi_A)_i) \leq \bigvee_{i\in\dagger}(\mathcal{G}^{\varphi}_{\varsigma}((\Xi_A)_i)$, $\forall\ \Xi_A \in \widetilde{(\Phi,\hbar)}$.

A quadruple $(\Phi, \mathcal{G}^{\varrho}_{\varsigma}, \mathcal{G}^{\sigma}_{\varsigma}, \mathcal{G}^{\varphi}_{\varsigma})$ *is said to form a single-valued neutrosophic soft topological space (svnst-space). When no ambiguity occurs, we simply denote* $(\mathcal{G}^{\varrho}_{\varsigma}, \mathcal{G}^{\sigma}_{\varsigma}, \mathcal{G}^{\varphi}_{\varsigma})$ *by* $\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}$.

Definition 8. *[30] Let* $(\Phi, \mathcal{G}^{\varrho}_{\hbar}, \mathcal{G}^{\sigma}_{\hbar}, \mathcal{G}^{\varphi}_{\hbar})$ *be an svnst-space. For* $\Xi_A \in \widetilde{(\Phi,\hbar)}$, $\varsigma \in \hbar$, *and* $\mathbf{t} \in \mathbf{I}_\circ$, *the operators* $C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}$ *and* $\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}$ *are mappings from* $\hbar \times \widetilde{(\Phi,\hbar)} \times \mathbf{I}_\circ$ *into* $\widetilde{(\Phi,\hbar)}$ *defined as follows:*

$$C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \prod \left\{ \Upsilon_B \in \widetilde{(\Phi,\hbar)} \;\middle|\; \Xi_A \sqsubseteq \Upsilon_B, \quad \mathcal{G}^{\varrho}_{\varrho}(\Upsilon^c_B) \geq \alpha, \quad \mathcal{G}^{\sigma}_{\varsigma}(\Upsilon^c_B) \leq 1 - \mathbf{t}, \mathcal{G}^{\varphi}_{\varsigma}(\Upsilon^c_B) \leq 1 - \mathbf{t} \right\},$$

$$\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \coprod \left\{ \Upsilon_B \in \widetilde{(\Phi,\hbar)} \;\middle|\; \Upsilon_B \sqsubseteq \Xi_A \quad \mathcal{G}^{\varrho}_{\varsigma}(\Upsilon_B) \geq \mathbf{t}, \quad \mathcal{G}^{\sigma}_{\varrho}(\Upsilon_B) \leq 1 - \mathbf{t}, \mathcal{G}^{\varphi}_{\varsigma}(\Upsilon_B) \leq 1 - \mathbf{t} \right\}.$$

Definition 9. *[34] Let* $(\Phi, \mathcal{G}^{\varrho}_{\varsigma}, \mathcal{G}^{\sigma}_{\varsigma}, \mathcal{G}^{\varphi}_{\varsigma})$ *be an svnst-space. For* $\mathbf{t} \in \mathbf{I}_\circ$, $\Xi_A \in \widetilde{(\Phi,\hbar)}$, *and* $\varkappa^{(m,n,r)}_{\gamma} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$, *a single-valued neutrosophic soft open* $Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}$*-neighborhood of* $\varkappa^{(m,n,r)}_{\gamma}$ *is defined by*

$$Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa^{(m,n,r)}_{\gamma}, \mathbf{t}) = \left\{ \Xi_A \;\middle|\; \varkappa^{(m,n,r)}_{\gamma} \Lambda\ \Xi_A, \;\; \mathcal{G}^{\varrho}_{\varsigma}(\Xi_A) \geq \mathbf{t},\; \mathcal{G}^{\sigma}_{\varsigma}(\Xi_A) \leq 1 - \mathbf{t},\; \mathcal{G}^{\varphi}_{\varsigma}(\Xi_A) \leq 1 - \mathbf{t} \right\}.$$

Definition 10. *[28] A mapping* $\mathcal{H}^{\varrho}_{\varsigma}, \mathcal{H}^{\sigma}_{\varsigma}, \mathcal{H}^{\varphi}_{\varsigma} : \mathbf{I}_\circ \to \mathbf{I}^{\Phi}$ *is said to be a single-valued neutrosophic soft ideal on* Φ *if it satisfies the following properties:*

($\mathcal{H}_1$) $\mathcal{H}^{\varrho}_{\varsigma}(0) = 1$ *and* $\mathcal{H}^{\sigma}_{\varsigma}(0) = \mathcal{H}^{\varphi}_{\varsigma}(0) = 0$.
($\mathcal{H}_2$) *If* $\Xi_A \sqsubseteq \Upsilon_B$, *then* $\mathcal{H}^{\varrho}_{\varsigma}(\Xi_A) \leq \mathcal{H}^{\varrho}_{\varsigma}(\Upsilon_b)$, $\mathcal{H}^{\sigma}_{\varsigma}(\Xi_A) \geq \mathcal{H}^{\sigma}_{\varsigma}(\Upsilon_B)$, $\mathcal{H}^{\varphi}_{\varsigma}(\Xi_A) \geq \mathcal{H}^{\varphi}_{\varsigma}(\Upsilon_B)$, *for every svn-sets* $\Xi_A, \Upsilon_B \in \widetilde{(\Phi,\hbar)}$.
($\mathcal{H}_3$) *For anysvn-sets* $\Xi_A, \Upsilon_B \in \widetilde{(\Phi,\hbar)}$, $\mathcal{H}^{\varrho}_{\varsigma}(\Xi_A \sqcup \Upsilon_B) \geq \mathcal{H}^{\varrho}_{\varsigma}(\Xi_A) \cap \mathcal{H}^{\varrho}_{\varsigma}(\Upsilon_B)$, $\mathcal{H}^{\sigma}_{\varsigma}(\Xi_A \sqcup \Upsilon_B) \leq \mathcal{H}^{\sigma}_{\varsigma}(\Xi_A) \cup \mathcal{H}^{\sigma}_{\varsigma}(\Upsilon_B)$, $\mathcal{H}^{\varphi}_{\varsigma}(\Xi_A \sqcup \Upsilon_B) \leq \mathcal{H}^{\varphi}_{\varsigma}(\Xi_A) \cup \mathcal{H}^{\varphi}_{\varsigma}(\Upsilon_B)$.

Remark 1. *The introduction of ideals in single-valued neutrosophic soft topological spaces provides a useful mechanism for describing negligible or insignificant components within the considered structure. In this framework, ideals play an essential role in defining and analyzing generalized operators by controlling the subsets that can be ignored in the topological structure. Consequently, the presence of ideals allows us to obtain new closure and interior operators and establish more general results than those obtained in the absence of ideals. Thus, ideals are not merely additional assumptions, but they represent a fundamental component that contributes to the development of the proposed structure and its related properties.*

Definition 11. *[28] Let* $(\Phi, \mathcal{G}^{\varrho}_{\varsigma}, \mathcal{G}^{\sigma}_{\varsigma}, \mathcal{G}^{\varphi}_{\varsigma})$ *be an svnst-space and let* $\Xi_A \in \widetilde{(\Phi,\hbar)}$. *The single-valued neutrosophic soft ideal open local function of* Ξ_A, *denoted by* $\Xi^{\star}_A(\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$, *is defined as the union of all single-valued neutrosophic soft points* $\varkappa^{(m,n,r)}_{\gamma}$ *satisfying the following condition: If*

$\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ *and* $\mathcal{H}_{\varsigma}^{\varrho}(\Xi_A) \geq \mathbf{t}$, $\mathcal{H}_{\varsigma}^{\sigma}(\Xi_A) \leq 1 - \mathbf{t}$, $\mathcal{H}_{\varsigma}^{\varphi}(\Xi_A) \leq 1 - \mathbf{t}$, *then there exists an element* $\gamma \in \Phi$ *such that*

$$\begin{aligned}
&\varrho_{\Xi_A}(\varkappa) + \varrho_{\Upsilon_B}(\varkappa) - 1 > \varrho_{\mathcal{R}_C}(\varkappa),\\
&\sigma_{\Xi_A}(\varkappa) + \sigma_{\Upsilon_B}(\varkappa) - 1 \leq \sigma_{\mathcal{R}_C}(\varkappa),\\
&\varphi_{\Xi_A}(\varkappa) + \varphi_{\Upsilon_B}(\varkappa) - 1 \leq \varphi_{\mathcal{R}_C}(\varkappa).
\end{aligned}$$

Remark 2. *[28] Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *be a single-valued neutrosophic soft ideal topological spaces (svnsit-spaces) and let* $\Xi_A \in \widetilde{(\Phi, \hbar)}$*. Then, we define*

$$C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \Xi_A \sqcup \Xi_A^{\star}, \qquad \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \Xi_A \sqcap [(\Xi_A^{c})^{\star}]^{c}.$$

Definition 12. *Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *be a svnsit-spaces and let* $\mathbf{t} \in \mathbf{I}_{\circ}$, $\varsigma \in \hbar$*. Then,* $\Xi_A \in \widetilde{(\Phi, \hbar)}$ *is said to be:*

(i) An $\mathbf{t}$*-single-valued neutrosophic soft pre-ideal open set (abbreviated as* $\mathbf{t}$*-SVNSPIO) iff* $\Xi_A \leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$*;*

(ii) An $\mathbf{t}$*-single-valued neutrosophic soft regular ideal open set (abbreviated as* $\mathbf{t}$*-SVNSRIO) iff* $\Xi_A = \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$*;*

(iii) An $\mathbf{t}$*-single-valued neutrosophic soft strongly* β*-ideal open set (abbreviated as* $\mathbf{t}$*-SVNSS*β*IO) iff* $\Xi_A \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}), \mathbf{t})$*;*

(iv) An $\mathbf{t}$*-single-valued neutrosophic soft* δ*-ideal open set (abbreviated as* $\mathbf{t}$*-SVNS*δ*IO) iff* $\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}) \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$*;*

(v) An $\mathbf{t}$*-single-valued neutrosophic soft* $\mathcal{IR}_{\hbar}^{\mathcal{G}^{\varrho\sigma\varphi}}$*-neighborhood of* $\varkappa_{\gamma}^{(m,n,r)}$ *if* $\varkappa_{\gamma}^{(m,n,r)}\ \Lambda\ \Xi_A$ *and* Ξ_A *is* $\mathbf{t}$*-SVNSRIO. We denote*

$$\mathcal{IR}_{\hbar}^{\mathcal{G}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t}) = \left\{ \Xi_A \in \widetilde{(\Phi, \hbar)} \;\middle|\; \varkappa_{\gamma}^{(m,n,r)}\ \Lambda\ \Xi_A,\ \ \Xi_A \text{ is } \mathbf{t} - SVNSRIO \right\}.$$

3. t-single-valued neutrosophic soft $\theta\mathcal{I}$-open and $\delta\mathcal{I}$-open sets

In this section, we introduce and study the concept of **t**-single-valued neutrosophic soft $\delta\mathcal{I}$-cluster points. Based on this concept, we define the $\delta\mathcal{I}$-closure and $\theta\mathcal{I}$-closure operators as mappings

$$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}} : \hbar \times \widetilde{(\Phi, \hbar)} \times \mathbf{I}_{\circ} \to \widetilde{(\Phi, \hbar)}, \quad \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}} : \hbar \times \widetilde{(\Phi, \hbar)} \times \mathbf{I}_{\circ} \to \widetilde{(\Phi, \hbar)},$$

and investigate their fundamental properties.

Furthermore, we introduce and analyze the notions of **t**-svn-$\delta\mathcal{I}$-closed and **t**-svn-$\theta\mathcal{I}$-closed sets, examining their interrelationships within the framework of single-valued neutrosophic soft topological spaces.

Definition 13. *Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *be svnsit-spaces. For each* $\mathbf{t} \in \mathbf{I}_{\circ}$, $\varsigma \in \hbar$, $\Xi_A \in \widetilde{(\Phi, \hbar)}$*, and* $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$*. Then:*

(i) $\varkappa_{\gamma}^{(m,n,r)}$ *is called* $\mathbf{t}$*-single-valued neutrosophic soft* $\delta\mathcal{I}$*-cluster point (*$\mathbf{t}$*-*$\delta\mathcal{I}$*-cluster point) of* Ξ_A *if for every* $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ *we have* $\Xi_A\ \Lambda\ \mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})$*;*

(ii) $\delta\mathcal{I}$-closure operator is a mapping $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}} : \hbar \times \widetilde{(\Phi, \hbar)} \times \mathbf{I}_{\circ} \to \widetilde{(\Phi, \hbar)}$ *defined as*

$$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \bigsqcup \left\{ \varkappa_{\gamma}^{(m,n,r)} \,\widetilde{\in}\, P_{\widetilde{(m,n,r)}}(\Phi) \,\middle|\, \varkappa_{\gamma}^{(m,n,r)} \text{ is } \mathbf{t} - \delta\mathcal{I} - \text{cluster point of } \ \Xi_A \right\}.$$

Theorem 1. *Let $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ be svnsit-spaces, for each $\mathbf{t} \in \mathbf{I}_{\circ}$, $\varsigma \in \hbar$ and $\Xi_A, \Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)}$. Then, the following properties hold:*

(1) $\Xi_A \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.
(2) If $\Xi_A \leq \Upsilon_B$, *then* $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})$.
(3) $\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$ *is* $\mathbf{t}$*-SVNSRIO.*
(4) $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \sqcap\{\Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)} | \Xi_A \leq \Upsilon_B, \ \Upsilon_B$ *is* $\mathbf{t}$*-SVNSRIC*$\}$.
(5) $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.

Proof. (1) and (2) are easily proved from Definition 13.

(3) Let $\Upsilon_B \,\widetilde{\in}\, \widetilde{(\Phi, \hbar)}$ and $\Upsilon_B = \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$. Then, we have

$$\begin{aligned}
\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}) &= \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}), \mathbf{t}), \mathbf{t}) \\
&\leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}), \mathbf{t}) \\
&= \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}) = \Upsilon_B.
\end{aligned}$$

Since

$$\Upsilon_B = \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}),$$

hence

$$\Upsilon_B = \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}).$$

(4) Setting $\mathcal{R} = \sqcap\{\Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)} | \Xi_A \leq \Upsilon_B, \ \Upsilon_B$ is $\mathbf{t}$-SVNSRIC$\}$, and let $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \ngeq \mathcal{R}$. Then there exist $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_{\circ}$ such that

$$\begin{cases}
\varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) < m < \varrho_{\mathcal{R}}(\varkappa), \\
\sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \geq n \geq \sigma_{\mathcal{R}}(\varkappa), \\
\varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \geq r \geq \varphi_{\mathcal{R}}(\varkappa).
\end{cases} \tag{3.1}$$

Then, $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A. So, there exists

$$\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t}), \ \Xi_A \leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c.$$

Thus,

$$\Xi_A \leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c = C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t}).$$

Since $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t})$ is $\mathbf{t}$-SVNSRIC,

$$\varrho_{\mathcal{R}}(\varkappa) \leq \varrho_{C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t})}(\varkappa) < m,$$

$$\sigma_{\mathcal{R}}(\varkappa) \geq \sigma_{C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t})}(\varkappa) \geq n,$$

$$\varphi_{\mathcal{R}}(\varkappa) \geq \varphi_{C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t})}(\varkappa) \geq r.$$

This contradicts Eq (3.1). Thus $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \geq \mathcal{R}$.

Suppose $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \nleq \mathcal{R}$. Then, there exists $\mathbf{t}$-$\delta\mathcal{I}$-cluster point $y_{\gamma}^{(\acute{m},\acute{n},\acute{r})} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$ of Ξ_A, such that

$$\begin{cases} \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, \Xi_A, \mathbf{t})}(y) > \acute{m} > \varrho_{\mathcal{R}}(y), \\ \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma, \Xi_A, \mathbf{t})}(y) \leq \acute{n} \leq \sigma_{\mathcal{R}}(y), \\ \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma, \Xi_A, \mathbf{t})}(y) \leq \acute{r} \leq \varphi_{\mathcal{R}}(y). \end{cases} \tag{3.2}$$

By the definition of $\mathcal{R}$, there exists $\mathbf{t}$-SVNSRIC $\Upsilon_B \in \widetilde{(\Phi, \hbar)}$ with $\Xi_A \leq \Upsilon_B$ such that

$$\varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, \Xi_A, \mathbf{t})}(y) > \acute{m} > \varrho_{\Upsilon_B}(y) \geq \varrho_R(y),$$

$$\sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma, \Xi_A, \mathbf{t})}(y) \leq \acute{n} \leq \sigma_{\Upsilon_B}(y) \leq \sigma_R(y),$$

$$\varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma, \Xi_A, \mathbf{t})}(y) \leq \acute{r} \leq \varphi_{\Upsilon_B}(y) \leq \varphi_R(y).$$

Then, $\Upsilon_B^c \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, y_{\gamma}^{(\acute{m},\acute{n},\acute{r})}, \mathbf{t})$. So,

$$\Xi_A \leq \Upsilon_B = \left(\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t})\right)^c.$$

Therefore, $\Xi_A \;\overline{\Lambda}\; \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t})$. Hence, $y_{\gamma}^{(\acute{m},\acute{n},\acute{r})}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A, that is,

$$\varrho_{\delta C^{\star}_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, \Xi_A, \mathbf{t})} < \acute{m}, \qquad \sigma_{\delta C^{\star}_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma, \Xi_A, \mathbf{t})} \geq \acute{n}, \qquad \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma, \Xi_A, \mathbf{t})} \geq \acute{r}.$$

This contradicts Eq (3.2). Thus, $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \mathcal{R}$.

(5) Suppose that $\Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)}$ and $\mathbf{t} \in \mathbf{I}_{\circ}$ such that

$$C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \nleq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}).$$

Then, there exist $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_{\circ}$ such that

$$\varrho_{C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) > m > \varrho_{\delta C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa),$$

$$\sigma_{C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \leq n \leq \sigma_{\delta C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa),$$

$$\varphi_{C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \leq r \leq \varphi_{\delta C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa).$$

Since $\varrho_{\delta C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m$, $\sigma_{\delta C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n$, and $\varphi_{\delta C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r$, $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A. So, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$, and

$$\Xi_A \leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c.$$

Hence,

$$\varrho_{C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq \varrho_{(\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Upsilon_B,\mathbf{t}),\mathbf{t}))^c}(\varkappa) < m,$$

$$\sigma_{C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \sigma_{(\mathcal{I}_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Upsilon_B,\mathbf{t}),\mathbf{t}))^c}(\varkappa) \geq n,$$

$$\varphi_{C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \varphi_{(\mathcal{I}_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Upsilon_B,\mathbf{t}),\mathbf{t}))^c}(\varkappa) \geq r.$$

This is a contradiction. Thus,

$$C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}).$$

□

Lemma 1. *Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *be svnsit-spaces, and* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *a* $\mathbf{t}$-*SVNSβIO set. Then,* $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.

Proof. We establish the following inequality:

$$C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \geq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}).$$

Assume, to the contrary, that

$$C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \not\geq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}).$$

Then, there exist $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_{\circ}$ such that

$$\begin{cases} \varrho_{C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m < \varrho_{\delta C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa), \\ \sigma_{C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa), \\ \varphi_{C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r \geq \varphi_{\delta C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa). \end{cases} \tag{3.3}$$

According to the definition of $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}$, there exists $\Upsilon_B \in \widetilde{(\Phi, \hbar)}$ such that

$$\mathcal{G}_{\varrho}^{\varrho}(\Upsilon_B^c) \geq \alpha, \quad \mathcal{G}_{\varsigma}^{\sigma}(\Upsilon_B^c) \leq 1 - \mathbf{t}, \quad \mathcal{G}_{\varsigma}^{\varphi}(\Upsilon_B^c) \leq 1 - \mathbf{t}, \quad \text{and} \quad \Xi_A \sqsubseteq \Upsilon_B,$$

such that

$$\varrho_{C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq \varrho_{\Upsilon_B}(\varkappa) < m < \varrho_{\delta C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa),$$

$$\sigma_{C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \sigma_{\Upsilon_B}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa),$$

$$\varphi_{C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \varphi_{\Upsilon_B}(\varkappa) \geq r \geq \varphi_{\delta C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa).$$

Then, $\Upsilon_B^c \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$, and

$$\Xi_A^C \geq \Upsilon_B^c \Rightarrow \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A^c, \mathbf{t}) \geq \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}) = (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c = (C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c = \Upsilon_B^c.$$

Then, $C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})\ \overline{\Lambda}\ \Upsilon^c_B$ implies that

$$\begin{aligned}\Upsilon^c_B &\leq (C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t})^c \Rightarrow C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon^c_B,\mathbf{t}),\mathbf{t}) \leq (\mathcal{I}^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t}))^c\\ &\Rightarrow \mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon^c_B,\mathbf{t}),\mathbf{t}) \leq (C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\mathcal{I}^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t}),\mathbf{t}))^c\\ &\Rightarrow \mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon^c_B,\mathbf{t}),\mathbf{t}) \leq (C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t}),\mathbf{t}))^c.\end{aligned}$$

Thus,

$$\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon^c_B,\mathbf{t}),\mathbf{t})\ \overline{\Lambda}\ C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t}),\mathbf{t}).$$

Since Ξ_A is $\mathbf{t}$-SVNSβIO, $\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon^c_B,\mathbf{t}),\mathbf{t})\ \overline{\Lambda}\ \Xi_A$. So, $\varkappa^{(m,n,r)}_{\gamma}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A. Thus,

$$\begin{gathered}\varrho_{\delta C_{\mathcal{G}^{\varrho}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m,\\ \sigma_{\delta C_{\mathcal{G}^{\sigma}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n,\\ \varphi_{\delta C_{\mathcal{G}^{\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r.\end{gathered}$$

This contradicts Eq (3.3). Hence,

$$C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}) \geq \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}).$$

By Theorem 1(5),

$$C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}) = \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}).$$

□

Definition 14. *Let $(\Phi,\mathcal{G}^{\varrho\sigma\varphi}_{\hbar},\mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ be svnsit-spaces. For $\mathbf{t}\in\mathbf{I}_\circ$, $\varsigma\in\hbar$, $\Xi_A\in\widetilde{(\Phi,\hbar)}$ and $\varkappa^{(m,n,r)}_{\gamma}\ \widetilde{\in}\ P_{\widetilde{(m,n,r)}}(\Phi)$. Then*

(i) Ξ_A is called $\mathbf{t}$-single valued neutrosophic soft $\delta\mathcal{I}$-cluster point ($\mathbf{t}$-$\theta\mathcal{I}$-cluster point) of Ξ_A if for every $\Upsilon_B\ \widetilde{\in}\ Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\varkappa^{(m,n,r)}_{\gamma},\mathbf{t})$ we have $\Xi_A\ \Lambda\ C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon_B,\mathbf{t})$;

(ii) The $\theta\mathcal{I}$-closure operator is a mapping $\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}:\hbar\times\widetilde{(\Phi,\hbar)}\times\mathbf{I}_\circ\to\widetilde{(\Phi,\hbar)}$ defined as

$$\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t}) = \sqcup\{\varkappa^{(m,n,r)}_{\gamma}\ \widetilde{\in}\ P_{\widetilde{(m,n,r)}}(\Phi)\ \Big|\ \varkappa^{(m,n,r)}_{\gamma}\ is\ \mathbf{t}-\theta I-cluster\ point\ of\ \ \Xi_A\}.$$

Theorem 2. *Let $(\Phi,\mathcal{G}^{\varrho\sigma\varphi}_{\hbar},\mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ be svnsit-spaces, for each $\Xi_A,\Upsilon_B\widetilde{\in}\widetilde{(\Phi,\hbar)}$, $\varsigma\in\hbar$ and $\mathbf{t}\in\mathbf{I}_\circ$. Then, the following properties hold:*

(1) $\Xi_A\leq\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})$.
(2) If $\Xi_A\leq\Upsilon_B$, then $\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})\leq\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon_B,\mathbf{t})$.
(3) $C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})\leq\sqcup\{\varkappa^{(m,n,r)}_{\gamma}\ \widetilde{\in}\ P_{\widetilde{(m,n,r)}}(\Phi)\ \Big|\ \varkappa^{(m,n,r)}_{\gamma}\ is\ \mathbf{t}-\delta\mathcal{I}-cluster\ point\ of\ \ \Xi_A\}$.
(4) $\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})=\sqcap\{\Upsilon_B\widetilde{\in}\widetilde{(\Phi,\hbar)}\ \Big|\ \Xi_A\leq\mathcal{I}^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Upsilon_B,\mathbf{t}),\ \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}(\Upsilon^c_B)\geq\mathbf{t}\}$.
(5) $\delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})=\sqcap\{\Upsilon_B\widetilde{\in}\widetilde{(\Phi,\hbar)}\ \Big|\ \Xi_A\leq\Upsilon_B$ is $\mathbf{t}$-SVNSδIC$\}$.
(6) $\varkappa^{(m,n,r)}_{\gamma}$ is $\mathbf{t}-\theta I$-cluster point of Ξ_A iff $\varkappa^{(m,n,r)}_{\gamma}\ \widetilde{\in}\ \theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})$.
(7) $\varkappa^{(m,n,r)}_{\gamma}$ is $\mathbf{t}-\delta I$-cluster point of Ξ_A iff $\varkappa^{(m,n,r)}_{\gamma}\ \widetilde{\in}\ \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma,\Xi_A,\mathbf{t})$.

(8) If $\Xi_A = C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$*, then* $\Xi_A = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$

(9) $\Xi_A \leq C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.

(10) $\mathcal{W}(\varsigma, \Xi_A \sqcup \Upsilon_B, \mathbf{t}) = \mathcal{W}(\varsigma, \Xi_A, \mathbf{t}) \cup \mathcal{W}((\varsigma, \Upsilon_B, \mathbf{t}))$ *for each* $\mathcal{W} = \{\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}, \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}\}$.

(11) $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$.

Proof. From Definition 14, (1) and (2) are easily proved.

(3) Setting $\mathcal{R} = \sqcup\{\varkappa_{\gamma}^{(m,n,r)} \,\widetilde{\in}\, P_{\widetilde{(m,n,r)}}(\Phi) \;\big|\; \varkappa_{\gamma}^{(m,n,r)},\ is\ \mathbf{t} - \delta\mathcal{I} - cluster\ point\ of\ \ \Xi_A\}$.

Suppose that $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \nleq \mathcal{R}$. Then, there exists $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_{\circ}$ such that

$$\begin{cases} \varrho_{C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) > m > \varrho_{\mathcal{R}}(\varkappa), \\ \sigma_{C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq n \leq \sigma_{\mathcal{R}}(\varkappa), \\ \varphi_{C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq r \leq \varphi_{\mathcal{R}}(\varkappa). \end{cases} \tag{3.4}$$

Then, $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t} - \delta\mathcal{I} - cluster\ point\ of\ \ \Xi_A$. So, there exists $\Upsilon_B \,\widetilde{\in}\, Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$,

$$\Xi_A \leq [\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})]^c \leq \Upsilon_B^c.$$

By the definition of $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}$,

$$\begin{gathered} \varrho_{C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq \varrho_{\Upsilon_B^c}(\varkappa) < m, \\ \sigma_{C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \sigma_{\Upsilon_B^c}(\varkappa) \geq n, \\ \varphi_{C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \varphi_{\Upsilon_B^c}(\varkappa) \geq r. \end{gathered}$$

This contradicts Eq (3.4). Thus, $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \mathcal{R}$,

(4) $\mathcal{S} = \sqcap\{\Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)} \;\big|\; \Xi_A \leq \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}),\ \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}(\Upsilon_B^c) \geq \mathbf{t}\}$.

Suppose that $\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \ngeq \mathcal{S}$. Then, there exists $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_{\circ}$ such that

$$\begin{cases} \varrho_{\theta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m < \varrho_{\mathcal{S}}(\varkappa), \\ \sigma_{\theta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n \geq \varrho_{\mathcal{S}}(\varkappa), \\ \varphi_{\theta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r \geq \varphi_{\mathcal{S}}(\varkappa). \end{cases} \tag{3.5}$$

Then, $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t} - \theta\mathcal{I} - cluster\ point\ of\ \ \Xi_A$. So, there exists $\Upsilon_B \,\widetilde{\in}\, Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ and $\Xi_A \leq (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c$. Hence, $\Xi_A \leq (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c = \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t})$, $\mathcal{G}_{\varsigma}^{\varrho}(\Upsilon_B) \geq \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\sigma}(\Upsilon_B) \leq 1 - \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\varphi}(\Upsilon_B) \leq 1 - \mathbf{t}$. Thus,

$$\varrho_{\mathcal{S}}(\varkappa) \leq \varrho_{\Upsilon_B^c}(\varkappa) < m, \quad \sigma_{\mathcal{S}}(\varkappa) \geq \sigma_{\Upsilon_B^c}(\varkappa) \geq n, \quad \varphi_{\mathcal{S}}(\varkappa) \geq \varphi_{\Upsilon_B^c}(\varkappa) \geq r.$$

This contradicts Eq (3.5). Thus, $\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \geq \mathcal{S}$.

Suppose that $\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \nleq \mathcal{S}$. Then, there exists $\mathbf{t}$-$\theta\mathcal{I}$-cluster point $y_{\gamma}^{(\acute{m},\acute{n},\acute{r})} \,\widetilde{\in}\, P_{\widetilde{(m,n,r)}}(\Phi)$ of Ξ_A, such that

$$\begin{cases} \varrho_{\theta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(y) > \acute{m} > \varrho_{\mathcal{S}}(y), \\ \sigma_{\theta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(y) \leq \acute{n} \leq \sigma_{\mathcal{S}}(y), \\ \varphi_{\theta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(y) \leq \acute{r} \leq \varphi_{\mathcal{S}}(y). \end{cases} \tag{3.6}$$

By the definition of $\mathcal{S}$, there exists $\Upsilon_B \,\widetilde{\in}\, \widetilde{(\Phi,\hbar)}$ with $\Xi_A \leq \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})$, $\mathcal{G}^{\varrho}_{\varsigma}(\Upsilon^c_B) \geq \mathbf{t}$, $\mathcal{G}^{\sigma}_{\varsigma}(\Upsilon^c_B) \leq 1 - \mathbf{t}$, $\mathcal{G}^{\varphi}_{\varsigma}(\Upsilon^c_B) \leq 1 - \mathbf{t}$ such that

$$\varrho_{\theta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(y) > \acute{m} > \varrho_{\Upsilon_B}(y) \geq \varrho_S(y),$$

$$\sigma_{\theta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(y) \leq \acute{n} \leq \sigma_{\Upsilon_B}(y) \leq \sigma_S(y),$$

$$\varphi_{\theta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(y) \leq \acute{r} \leq \varphi_{\Upsilon_B}(y) \leq \varphi_S(y).$$

Then, $\Upsilon^c_B \,\widetilde{\in}\, Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, y_{\gamma}^{(\acute{m},\acute{n},\acute{r})}, \mathbf{t})$. Furthermore, $\Xi_A \leq \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) = (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon^c_B, \mathbf{t}))^c$ implies $\Xi_A \,\overline{\Lambda}\, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon^c_B, \mathbf{t})$. Hence, $y_{\gamma}^{(\acute{m},\acute{n},\acute{r})}$ is not a $\mathbf{t}$-$\theta\mathcal{I}$-cluster point of Ξ_A, that is,

$$\varrho_{\theta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < \acute{m}, \qquad \sigma_{\theta C^{\star}_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \acute{n}, \qquad \varphi_{\theta C^{\star}_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \acute{r}.$$

This contradicts Eq (3.6). Thus, $\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \mathcal{S}$.

(5) It is also proved as in (3) and (4).

(6) ($\Rightarrow$) It is trivial.

($\Leftarrow$) Suppose that $\varkappa_{\gamma}^{(m,n,r)}$ is not $\mathbf{t} - \theta\mathcal{I}$-cluster point of Ξ_A. Then, there exists $\Upsilon_B \,\widetilde{\in}\, Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Xi^c_B$. Thus, $\Xi_A \leq (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c = \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon^c_B, \mathbf{t})$. By (4),

$$\varrho_{\theta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq \varrho_{\Upsilon^c_B}(\varkappa) < m,$$

$$\sigma_{\theta C^{\star}_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \sigma_{\Upsilon^c_B}(\varkappa) \geq n,$$

$$\varphi_{\theta C^{\star}_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})} \geq \varphi_{\Upsilon^c_B}(\varkappa) \geq r.$$

Hence, $\varkappa_{\gamma}^{(m,n,r)} \,\widetilde{\notin}\, \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.

(7) It can be proved in the same manner as (6).

(8) It is an immediate consequence of Theorem 1(4).

(9) From Theorem 1(5), we show that only $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$. Let $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \nleq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$. Then, there exists $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_{\circ}$ such that

$$\begin{cases} \varrho_{\delta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) > m > \varrho_{\theta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa), \\ \sigma_{\delta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq n \leq \sigma_{\theta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa), \\ \varphi_{\delta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq r \leq \varphi_{\theta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa). \end{cases} \tag{3.7}$$

Since $\varrho_{\theta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m$, $\sigma_{\theta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n$, and $\varphi_{\theta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r$, it follows that $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\theta\mathcal{I}$-cluster point of Ξ_A. Therefore, there exists $\Upsilon_B \,\widetilde{\in}\, Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$, $\Xi_A \leq (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c$. This implies that $\Xi_A \,\overline{\Lambda}\, \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})$. Hence, $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A. By (7), we have

$$\varrho_{\delta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m, \quad \sigma_{\delta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n, \quad \varphi_{\delta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r,$$

which contradicts (3.7). Thus, $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.

(10) Let $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \sqcup \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \ngeq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A \sqcup \Upsilon_B, \mathbf{t})$. Then, there exists $\varkappa \in \Phi$ and $m, n, r \in (0, 1)$ such that

$$\begin{cases} \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \sqcup \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Upsilon_B,\mathbf{t})}(\varkappa) < m < \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A\sqcup\Upsilon_B,\mathbf{t})}(\varkappa), \\ \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \sqcup \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Upsilon_B,\mathbf{t})}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A\sqcup\Upsilon_B,\mathbf{t})}(\varkappa), \\ \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \sqcup \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Upsilon_B,\mathbf{t})}(\varkappa) \geq r \geq \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A\sqcup\Upsilon_B,\mathbf{t})}(\varkappa). \end{cases} \tag{3.8}$$

Since

$$\varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m,$$
$$\sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n,$$
$$\varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r,$$

and

$$\varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Upsilon_B,\mathbf{t})}(\varkappa) < m,$$
$$\sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Upsilon_B,\mathbf{t})}(\varkappa) \geq n,$$
$$\varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Upsilon_B,\mathbf{t})}(\varkappa) \geq r,$$

it follows that $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A and Υ_B. Therefore, there exists

$$(\Xi_A)_1(\Upsilon_B)_1 \,\widetilde{\in}\, Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t}),$$

$$\Xi_A \leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Xi_A)_1, \mathbf{t}), \mathbf{t}))^c,$$

and

$$\Upsilon_B \leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Upsilon_B)_1, \mathbf{t}), \mathbf{t}))^c.$$

Thus, $(\Xi_A)_1 \sqcap (\Upsilon_B)_1 \,\widetilde{\in}\, Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ and

$$\begin{aligned} \Xi_A \sqcup \Upsilon_B &\leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Xi_A)_1, \mathbf{t}), \mathbf{t}) \sqcap \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Upsilon_B)_1, \mathbf{t}), \mathbf{t}))^c \\ &= (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Xi_A)_1, \mathbf{t}) \sqcap C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Upsilon_B)_1, \mathbf{t}), \mathbf{t}), \mathbf{t}))^c \\ &\leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Xi_A)_1 \sqcap (\Upsilon_B)_1, \mathbf{t}), \mathbf{t}))^c. \end{aligned}$$

This implies that $\Xi_A \sqcup \Upsilon_B \;\overline{\Lambda}\; \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Xi_A)_1 \sqcap (\Upsilon_B)_1, \mathbf{t}), \mathbf{t})$. Hence, $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of $\Xi_A \sqcup \Upsilon_B$. By (7), we have

$$\varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A\sqcup\Upsilon_B,\mathbf{t})}(\varkappa) < m, \quad \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A\sqcup\Upsilon_B,\mathbf{t})}(\varkappa) \geq n, \quad \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A\sqcup\Upsilon_B,\mathbf{t})}(\varkappa) \geq r.$$

This contradicts Eq (3.8). Thus,

$$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \sqcup \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \geq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A \sqcup \Upsilon_B, \mathbf{t}).$$

On the other hand, $\Xi_A, \Upsilon_B \geq \Xi_A \sqcup \Upsilon_B$. Hence,

$$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \sqcup \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A \sqcup \Upsilon_B, \mathbf{t}).$$

Thus,

$$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \sqcup \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A \sqcup \Upsilon_B, \mathbf{t}).$$

(11) Since $\Xi_A \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$, it follows that

$$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}).$$

On the other hand, suppose that $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \not\geq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$. Then, there exists $\varkappa \in \Phi$ and $m, n, r \in \mathbb{I}_\circ$ such that

$$\begin{cases} \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m < \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t})}(\varkappa), \\ \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t})}(\varkappa), \\ \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t})}(\varkappa). \end{cases} \tag{3.9}$$

Since $\varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m$, $\sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n$, and $\varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r$, it follows that $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A. Therefore, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that

$$\Xi_A \leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Upsilon_B^c, \mathbf{t}), \mathbf{t}))^c = C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}).$$

Since $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})$ is $\mathbf{t}$-SVNSRIC and

$$\Xi_A \leq C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}).$$

Then, by Theorem 1(5), $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})$. Again,

$$\begin{aligned} \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}) &\leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}), \mathbf{t}) \\ &= C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}). \end{aligned}$$

Hence,

$$\varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t})}(\varkappa) < m,$$

$$\sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t})}(\varkappa) \geq n,$$

$$\varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t}),\mathbf{t})}(\varkappa) \geq r.$$

This contradicts Eq (3.9). Hence, $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$. □

Example 1. *Let $\Phi = \{\varkappa_1, \varkappa_2\}$, $\hbar = \{\varsigma\}\}$ and define $\Xi_{\hbar}, \Upsilon_{\hbar}, \Omega_{\hbar}, \widetilde{\in}\widetilde{(\Phi, \hbar)}$, as follows:*

$$\Xi_{\hbar} = \{(\varsigma, \{\langle\varkappa_1, 0.6, 0.4, 0.6\rangle, \langle\varkappa_2, 0.6, 0.4, 0.6\rangle\})\},$$

$$\Upsilon_{\hbar} = \{(\varsigma, \{\langle\varkappa_1, 0.6, 0.6, 0.6\rangle, \langle\varkappa_2, 0.6, 0.6, 0.6\rangle\})\},$$

$$\Omega_{\hbar} = \{(\varsigma, \{\langle \varkappa_1, 0.5, 0.5, 0.5\rangle, \langle \varkappa_2, 0.5, 0.5, 0.5\rangle\})\}.$$

Define $\mathcal{G}_{\varsigma}^{\varrho}, \mathcal{G}_{\varsigma}^{\sigma}, \mathcal{G}_{\varsigma}^{\varphi}, \mathcal{H}_{\varsigma}^{\varrho}, \mathcal{H}_{\varsigma}^{\sigma}, \mathcal{H}_{\varsigma}^{\varphi} : \hbar \longrightarrow \mathbf{I}^{\widetilde{(\Phi,\hbar)}}$:

$$\mathcal{G}_{\varsigma}^{\varrho}(\mathcal{Z}_{\hbar}) = \begin{cases} 1, & if \;\; \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{2}, & if \;\; \mathcal{Z}_{\hbar} = \Xi_{\hbar}, \\ 0, & if \quad otherwise. \end{cases} \qquad \mathcal{H}_{\varsigma}^{\varrho}(\mathcal{Z}_{\hbar}) = \begin{cases} 1, & if \;\; \mathcal{Z} = \phi, \\ \frac{2}{3}, & if \;\; \langle \mathcal{Z}_{\hbar} = \Omega_{\hbar}, \\ \frac{1}{2}, & if \;\; \langle 0,1,1\rangle < \mathcal{Z}_{\hbar} < \Omega_{\hbar}, \\ 0, & if \quad otherwise. \end{cases}$$

$$\mathcal{G}_{\varsigma}^{\sigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & if \;\; \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{2}, & if \;\; \mathcal{Z}_{\hbar} = \Xi_{\hbar}, \\ 1, & if \quad otherwise. \end{cases} \qquad \mathcal{H}_{\varsigma}^{\sigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & if \;\; \mathcal{Z} = \phi, \\ \frac{1}{3}, & if \;\; \langle \mathcal{Z}_{\hbar} = \Omega_{\hbar}, \\ \frac{1}{2}, & if \;\; \langle 0,1,1\rangle < \mathcal{Z}_{\hbar} < \Omega_{\hbar}, \\ 1, & if \quad otherwise. \end{cases}$$

$$\mathcal{G}_{\varsigma}^{\varphi}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & if \;\; \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{2}, & if \;\; \mathcal{Z}_{\hbar} = \Xi_{\hbar}, \\ 1, & if \quad otherwise. \end{cases} \qquad \mathcal{H}_{\varsigma}^{\varphi}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & if \;\; \mathcal{Z} = \phi, \\ \frac{1}{3}, & if \;\; \langle \mathcal{Z}_{\hbar} = \Omega_{\hbar}, \\ \frac{1}{2}, & if \;\; \langle 0,1,1\rangle < \mathcal{Z}_{\hbar} < \Omega_{\hbar}, \\ 1, & if \quad otherwise. \end{cases}$$

From Theorem 1(4), we obtain $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}} : \hbar \times \widetilde{(\Phi, \hbar)} \times \mathbf{I}_{\circ} \to \widetilde{(\Phi, \hbar)}$ *as follows:*

$$\delta C_{\mathcal{G}_{\hbar}^{\star\varrho\sigma\varphi}}(\varsigma, \mathcal{Z}_{\hbar}, \mathbf{t}) = \begin{cases} \langle 0,1,1\rangle, & if \;\; \mathcal{Z}_{\hbar} = \langle 0,1,1\rangle, \\ \Upsilon_{\hbar}, & if \;\; \langle 0,1,1\rangle \neq \mathcal{Z} \leq \Upsilon_{\hbar}, \;\; 0 < r \leq \frac{1}{2}, \\ \langle 1,0,0\rangle, & \qquad otherwise. \end{cases}$$

This example shows that the converse of Theorem 2(8) does not hold in general. Indeed,

$$\Upsilon_{\hbar} = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_{\hbar}, \frac{1}{2}),$$

whereas

$$\Upsilon_{\hbar} \neq C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_{\hbar}, \frac{1}{2}), \frac{1}{2}).$$

Hence, the converse implication of Theorem 2(8) is not valid in general.

Theorem 3. *Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *be svnsit-spaces. For each* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, *and* $\mathbf{t} \in \mathbf{I}_{\circ}$, Ξ_A *is a* **t**-*SVNSPIC if and only if*

$$C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}).$$

Proof. Let Ξ_A is a **t**-SVNSPIC. Then, $\Xi_A \leq C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$ and by Theorem 1(3)(4), we have

$$\begin{aligned} \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) &\leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}), \mathbf{t}) = C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}) \\ &\leq C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}). \end{aligned}$$

Conversely, suppose that there exist $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, $\mathbf{t} \in \mathbf{I}_\circ$, and $m, , n, r \in \mathbf{I}_\circ$ such that

$$\varrho_{\delta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) > m > \varrho_{C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa),$$
$$\sigma_{\delta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \le n \le \sigma_{C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa),$$
$$\varphi_{\delta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \le r \le \varphi_{C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa).$$

Then, $\varkappa_\gamma^{(m,n,r)}$ is not a $\mathbf{t}-\delta\mathcal{I}-cluster\ point\ of\ \ \Xi_A$. So, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ with $\Xi_A \le \Upsilon_B^c$. Since $\varkappa_\gamma^{(m,n,r)}$ is a $\mathbf{t} - \delta\mathcal{I} - cluster\ point\ of\ \ \Xi_A$, for $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, we have

$$\Xi_A \wedge \mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^\star_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}).$$

Since,

$$\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^\star_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}) \le \mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^\star_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A^c, \mathbf{t}), \mathbf{t}),$$

it follows that

$$\begin{aligned} \Xi_A \ge (\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^\star_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}))^c \ &\ge (\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^\star_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A^c, \mathbf{t}), \mathbf{t}))^c \\ &= C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \mathcal{I}^\star_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}). \end{aligned}$$

Hence, Ξ_A is not $\mathbf{t}$-SVNSPIC. □

Definition 15. *Let $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ be svnsit-spaces, for $\varsigma \in \hbar$, $\Xi_A, \Upsilon_B \in \widetilde{(\Phi, \hbar)}$ and $\mathbf{t} \in \mathbf{I}_\circ$. Then:*

(i) $\mathcal{A}$ is called is $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed (resp. $\mathbf{t}$-svn-$\theta\mathcal{I}$-closed) iff $\delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \Xi_A$ (resp. $\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \Xi_A$). We define

$$\Delta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \left\{\Upsilon_B \in \widetilde{(\Phi, \hbar)} \,\middle|\, \Xi_A \le \Upsilon_B,\ \Upsilon_B = \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t})\right\},$$

$$\Theta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \left\{\Upsilon_B \in \widetilde{(\Phi, \hbar)} \,\middle|\, \Xi_A \le \Upsilon_B,\ \Upsilon_B = \theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t})\right\}.$$

(ii) The complement of $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed (resp. $\mathbf{t}$-svn-$\theta\mathcal{I}$-closed) set is called $\mathbf{t}$-svn-$\delta\mathcal{I}$-open (resp. $\mathbf{t}$-svn-$\theta\mathcal{I}$-open).

Theorem 4. *Let $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ be svnsit-spaces, for each $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$ and $\mathbf{t} \in \mathbf{I}_\circ$. Then, the following properties are satisfied:*

(1) $\Delta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t})$;
(2) $\Delta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t})$ is $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed;
(3) $\Theta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Theta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$;
(4) $\Theta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t})$ is $\mathbf{t}$-svn-θI-closed;
(5) $\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) \le \Theta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t})$.

Proof. (1) From Theorem 2(9)(11), $\Xi_A \le \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$ implies $\Delta_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) \le \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t})$.

Suppose that $\Delta_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \ngeq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$. Then, there exist $\varsigma \in \hbar$, $\mathbf{t} \in \mathbf{I}_\circ$ and $m, , n, r \in \mathbf{I}_\circ$ such that

$$\varrho_{\Delta_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m < \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa),$$
$$\sigma_{\Delta_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa),$$
$$\varphi_{\Delta_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r \geq \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa).$$

From the definition of $\Delta_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, \Xi_A, \mathbf{t})$. There exist $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$ and $\Xi_A \leq \Upsilon_B = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})$ such that

$$\varrho_{\Delta_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq \varrho_{\Upsilon_B}(\varsigma) < m < \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa),$$
$$\sigma_{\Delta_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \sigma_{\Upsilon_B}(\varsigma) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa),$$
$$\varphi_{\Delta_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \varphi_{\Upsilon_B}(\varsigma) \geq r \geq \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa).$$

On the other hand, $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) = \Upsilon_B$. This is a contradiction. Hence,

$$\Delta_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \geq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}).$$

(2) From Theorem 2(11), it is trivial.

(3) Let $\Xi_A \leq (\Upsilon_B)_i = \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Upsilon_B)_i, \mathbf{t})$, for each $i \in j$. Then,

$$\sqcap_{i\in j}(\Upsilon_B)_i \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \sqcap_{i\in j}(\Upsilon_B)_i, \mathbf{t}) \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\Upsilon_B)_i, \mathbf{t}) = (\Upsilon_B)_i.$$

So, $\sqcap_{i\in j}(\Upsilon_B)_i \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \sqcap_{i\in j}(\Upsilon_B)_i, \mathbf{t})$. Hence, $\Theta_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Theta_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$.

(4) From(3), it is trivial.

(5) Since $\Xi_A \leq \Theta_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$, by (3),

$$\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Theta_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}) = \Theta_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}).$$

□

Lemma 2. *If* $\Xi_A, \Upsilon_B \;\widetilde{\in}\; \widetilde{(\Phi, \hbar)}$, $\mathbf{t} \in \mathbf{I}_\circ$ *such that* $\Xi_A \;\overline{\Lambda}\; \Upsilon_B$ *where* Υ_B *is* $\mathbf{t}$*-svn-*$\delta\mathcal{I}$*-open, then* $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \;\overline{\Lambda}\; \Upsilon_B$.

Proof. Let $\Xi_A \;\overline{\Lambda}\; \Upsilon_B$ where Υ_B is $\mathbf{t}$-svn-$\delta\mathcal{I}$-open. Then, $\Xi_A \leq \Upsilon_B^c = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t})$, and by Theorem 2(11),

$$\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}) = \Upsilon_B^c.$$

Hence, $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \;\overline{\Lambda}\; \Upsilon_B$. □

Lemma 3. *Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *be svnsit-spaces and* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *a* $\mathbf{t}$*-svn-*$\delta\mathcal{I}$*-open set iff for every* $\varkappa_\gamma^{(m,n,r)} \;\widetilde{\in}\; \widetilde{P_{(m,n,r)}}(\Phi)$ *with* $\varkappa_\gamma^{(m,n,r)} \;\Lambda\; \Xi_A$*, there exists* $\mathbf{t}$*-SVNSRIO set* $\Upsilon_B \;\widetilde{\in}\; \widetilde{(\Phi, \hbar)}$ *such that* $\varkappa_\gamma^{(m,n,r)} \;\Lambda\; \Upsilon_B \leq \Xi_A$.

Proof. Let $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} \widetilde{P_{(m,n,r)}}(\Phi)$ with $\varkappa_{\gamma}^{(m,n,r)} \wedge \Xi_A$. Then, $\varkappa_{\gamma}^{(m,n,r)} \notin \Xi_A^c$. Since Ξ_A is a $\mathbf{t}$-svn-δI-open set, $\varkappa_{\gamma}^{(m,n,r)} \notin \Xi_A^c = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})$. Thus, $\varkappa_{\gamma}^{(m,n,r)}$ is not $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A^c. So, there exists $\mathcal{D}_C \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that $\Xi_A^c \overline{\wedge}\ \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}), \mathbf{t})$. Setting $\Upsilon_B = \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}), \mathbf{t})$, then Υ_B is $\mathbf{t}$-SVNSRIO set with $\varkappa_{\gamma}^{(m,n,r)} \wedge \Upsilon_B \leq \Xi_A$.

Conversely, suppose $\Xi_A^c \neq \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})$. Then, there exist $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_{\circ}$ such that

$$\varrho_{\Xi_A^c}(\varkappa) < m < \varrho_{\delta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa),$$
$$\sigma_{\Xi_A^c}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa),$$
$$\varphi_{\Xi_A^c}(\varkappa) \geq r \geq \varphi_{\delta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa).$$

Since $\varkappa_{\gamma}^{(m,n,r)} \wedge \Xi_A$, there exists $\mathbf{t}$-SVNSRIO set Υ_B such that $\varkappa_{\gamma}^{(m,n,r)} \wedge \Upsilon_B \leq \Xi_A$. It implies $\Xi_A^c \leq \Upsilon_B^c = C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t})$. By Theorem 1(4),

$$\varrho_{C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa) \leq \varrho_{\Upsilon_B^c}(\varkappa) < m$$
$$\sigma_{C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa) \geq \sigma_{\Upsilon_B^c}(\varkappa) \geq n$$
$$\varphi_{C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa) \geq \varphi_{\Upsilon_B^c}(\varkappa) \geq r.$$

This is a contradiction. Hence, $\Xi_A^c = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})$ i.e., Ξ_A is $\mathbf{t}$-svn-$\delta\mathcal{I}$-open set. □

Lemma 4. *If $\mathcal{G}_{\varsigma}^{\varrho}(\Xi_A) \geq \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\sigma}(\Xi_A) \leq 1 - \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\varphi}(\Xi_A) \leq 1 - \mathbf{t}$, then $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.*

Proof. It an immediate consequence of Theorem 2(8). □

4. On $\delta\mathcal{I}$-continuity and $\theta\mathcal{I}$-continuity in single-valued neutrosophic soft topological spaces

In this section, we introduce and investigate the concepts of δ-ideal continuity and θ-ideal continuity in single-valued neutrosophic soft topological spaces. Furthermore, we study related notions such as strongly θ-ideal continuity and almost ideal continuity, supported by illustrative examples.

Definition 16. *Let $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ and $(\mathcal{K}, \mathcal{T}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{F}_{E}^{\varrho\sigma\varphi})$ be svnsit-spaces, $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$, and $\mathbf{t} \in \mathbf{I}_{\circ}$. An svns-function $\psi_{\varepsilon} : \widetilde{(\Phi, \hbar)} \to \widetilde{(\mathcal{K}, E)}$ is called:*

(i) SVNS-$\delta\mathcal{I}$-continuous if and only if, for each $\Xi_A \widetilde{\in} Q_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_{\varepsilon}(\varkappa_{\gamma}^{(m,n,r)}), \mathbf{t})$, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that

$$\psi_{\varepsilon}\Big(\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})\Big) \leq \mathcal{I}_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, C^{\star}_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t});$$

(ii) SVNS-$\theta\mathcal{I}$-continuous if and only if, for each $\Xi_A \widetilde{\in} Q_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_{\varepsilon}(\varkappa_{\gamma}^{(m,n,r)}), \mathbf{t})$, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that $\psi_{\varepsilon}\Big(C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})\Big) \leq C^{\star}_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t})$;

(iii) SVNSS-$\theta\mathcal{I}$-continuous, if and only if, for each $\Xi_A \widetilde{\in} Q_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_{\varepsilon}(\varkappa_{\gamma}^{(m,n,r)}), \mathbf{t})$, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that $\psi_{\varepsilon}\Big(C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})\Big) \leq \Xi_A$;

(iv) *SVNSA-$\mathcal{I}$-continuous if and only if, for each* $\Xi_A \widetilde{\in} Q_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$*, there exists* $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ *such that* $\psi_\varepsilon(\Upsilon_B) \leq \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})$.

Remark 3. *From the above definition, we obtain the following diagram:*

$$\text{SVNSS-}\theta\mathcal{I}\text{-continuous} \Longrightarrow \text{SVNS} - \delta\mathcal{I}\text{-continuous} \Longrightarrow \text{SVNSA-}\mathcal{I}\text{-continuous}.$$

Theorem 5. *Let* $(\Phi, \mathcal{G}_\hbar^{\varrho\sigma\varphi}, \mathcal{H}_\hbar^{\varrho\sigma\varphi})$ *and* $(\mathcal{K}, \mathcal{T}_\hbar^{\varrho\sigma\varphi}, \mathcal{F}_E^{\varrho\sigma\varphi})$ *be svnsit-spaces and* $\psi_\varepsilon : \widetilde{(\Phi, \hbar)} \to \widetilde{(\mathcal{K}, E)}$ *be an SVNS-function. Then, the following statements are equivalent for every* $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$ *and* $\mathbf{t} \in \mathbf{I}_\circ$.

(1) ψ_ε *is SVNS-$\delta\mathcal{I}$-continuous.*
(2) *For each* $\Xi_A \in \mathcal{IR}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$*, there exists* $\Upsilon_B \in \mathcal{IR}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$*, such that* $\psi_\varepsilon(\Upsilon_B) \leq \Xi_A$.
(3) $\psi_\varepsilon(\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})) \leq \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})$*, for each* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$.
(4) $\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \psi_\varepsilon^{-1}(\delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}))$*, for each* $\Upsilon_B \widetilde{\in} \widetilde{(\mathcal{K}, E)}$.
(5) *For each* $\mathbf{t}$*-svn-δI-closed (resp.* $\mathbf{t}$*-svn-δI-open) set* $\Upsilon_B \widetilde{\in} \widetilde{(\mathcal{K}, E)}$, $\psi_\varepsilon^{-1}(\Upsilon_B)$ *is a* $\mathbf{t}$*-svn-δI-closed (resp.* $\mathbf{t}$*-svn-δI-open) set in* Φ.
(6) *For each* $\mathbf{t}$*-SVNSRIO (resp.* $\mathbf{t}$*-SVNSRIC) set* $\mathcal{D}_C \widetilde{\in} \widetilde{(\mathcal{K}, E)}$, $\psi_\varepsilon^{-1}(\mathcal{D}_C)$ *is a* $\mathbf{t}$*-svn-δI-open (resp.* $\mathbf{t}$*-svn-δI-closed) set in* Φ.

Proof. (1) ⇒ (2): The result follows directly from Definition 16.
(2)⇒(3): Suppose there exist $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$ and $\mathbf{t} \in \mathbf{I}_\circ$ such that

$$\psi_\varepsilon(\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})) \nleq \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\Xi_A), \mathbf{t}).$$

Then, there exists $y \in \mathcal{K}$, $m, n, r \in \mathbf{I}_\circ$, $\varsigma \in \hbar$, and $e \in E$ such that

$$\begin{aligned}
\varrho_{\psi_\varepsilon(\delta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t}))}(y) &> m > \varrho_{\delta C_{\mathcal{T}_e^\varrho}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(y),\\
\sigma_{\psi_\varepsilon(\delta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t}))}(y) &\leq n \leq \sigma_{\delta C_{\mathcal{T}_e^\sigma}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(y),\\
\varphi_{\psi_\varepsilon(\delta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t}))}(y) &\leq r \leq \varphi_{\delta C_{\mathcal{T}_e^\varphi}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(y).
\end{aligned}$$

If $\psi_\varepsilon^{-1}(\{y\}) = \emptyset$, then, by the definition of the inverse image $\psi_\varepsilon(\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}))(y) = \langle 0, 1, 1 \rangle$.
If $\psi_\varepsilon^{-1}(\{y\}) \neq \emptyset$, there exists $\varkappa \in \psi_\varepsilon^{-1}(\{y\})$ such that

$$\begin{aligned}
\varrho_{\psi_\varepsilon(\delta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t}))}(y) &\geq \varrho_{\delta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) > m > \varrho_{\delta C_{\mathcal{T}_e^\varrho}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)),\\
\sigma_{\psi_\varepsilon(\delta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t}))}(y) &\leq \sigma_{\delta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \leq n \leq \sigma_{\delta C_{\mathcal{T}_e^\sigma}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)),\\
\varphi_{\psi_\varepsilon(\delta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t}))}(y) &\leq \varphi_{\delta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \leq r \leq \varphi_{\delta C_{\mathcal{T}_e^\varphi}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)).
\end{aligned} \tag{4.1}$$

Since

$$\begin{aligned}
\varrho_{\delta C_{\mathcal{T}_e^\varrho}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)) &< m,\\
\sigma_{\delta C_{\mathcal{T}_e^\sigma}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)) &\geq n,\\
\varphi_{\delta C_{\mathcal{T}_e^\varphi}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)) &\geq r,
\end{aligned}$$

by Theorem 2(7), $\psi_\varepsilon(\varkappa_\gamma^{(m,n,r)})$ is not a $\mathbf{t}$-δI-cluster point of $\psi_\varepsilon(\Xi_A)$. So, there exists $\mathcal{D}_C \mathrel{\widetilde{\in}} \mathcal{Q}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$, such that $\psi_\varepsilon(\Xi_A) \leq (\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \mathcal{D}_C, \mathbf{t}), \mathbf{t}))^c$. Since $\mathcal{D}_C \mathrel{\widetilde{\in}} \mathcal{Q}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$, $\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}), \mathbf{t}) \mathrel{\widetilde{\in}} \mathcal{IR}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$. By (2), there exists $\Upsilon_B \in \mathcal{IR}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, such that $\psi_\varepsilon(\Upsilon_B) \leq \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \mathcal{D}_C, \mathbf{t}), \mathbf{t})$. Hence, $\Upsilon_B \mathrel{\widetilde{\in}} \mathcal{Q}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ and

$$\psi_\varepsilon(\Xi_A) \leq (\psi_\varepsilon(\Upsilon_B))^c = (\psi_\varepsilon(\mathcal{I}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c$$

imply that $\Xi_A \leq (\mathcal{I}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c$. Thus, $\varkappa_\gamma^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A, and by Theorem 2(7),

$$\varrho_{\delta C_{\mathcal{G}_\varsigma^{\varrho}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m,$$

$$\sigma_{\delta C_{\mathcal{G}_\varsigma^{\sigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n,$$

$$\varphi_{\delta C_{\mathcal{G}_\varsigma^{\varphi}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r.$$

This contradicts Eq (4.1).

(3)⇒(4): For all $\Upsilon_B \mathrel{\widetilde{\in}} (\widetilde{\mathcal{K}, E})$ and $\mathbf{t} \in \mathbf{I}_\circ$. Set $\Xi_A = \psi_\varepsilon^{-1}(\Upsilon_B)$ from (3). Then,

$$\psi_\varepsilon(\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\psi_\varepsilon^{-1}(\Upsilon_B)), \mathbf{t}) \leq \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}).$$

This implies

$$\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}) \leq \psi_\varepsilon^{-1}(\psi_\varepsilon(\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}))) \leq \psi_\varepsilon^{-1}(\delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t})).$$

(4)⇒(5): Let $\Upsilon_B \mathrel{\widetilde{\in}} (\widetilde{\mathcal{K}, E})$ be $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed. By (4), we have

$$\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \psi_\varepsilon^{-1}(\delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t})) = \psi_\varepsilon^{-1}(\Upsilon_B),$$

and $\psi_\varepsilon^{-1}(\Upsilon_B) \leq \delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})$ implies $\psi_\varepsilon^{-1}(\Upsilon_B) = \delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})$. Hence, $\psi_\varepsilon^{-1}(\Upsilon_B)$ is a $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed set. The other is similarly proved.

(5)⇒(6): The result follows directly from Theorem 2(8).

(6)⇒(1): Let $\Xi_A \mathrel{\widetilde{\in}} \mathcal{Q}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$. Then,

$$\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}) \mathrel{\widetilde{\in}} \mathcal{IR}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t}).$$

By (6),

$$(\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})))^c = \delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, (\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})))^c, \mathbf{t})).$$

Since $\psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}) \wedge \Xi_A \leq \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})$, it follows that

$$\varkappa_\gamma^{(m,n,r)} \wedge \psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})).$$

That is, $\varkappa_\gamma^{(m,n,r)}$ is not $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of $(\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})))^c$. Then, there exists $\Upsilon_B \mathrel{\widetilde{\in}} \mathcal{Q}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ such that

$$(\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})))^c \leq (\mathcal{I}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c.$$

Hence,

$$\psi_\varepsilon(\mathcal{I}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})) \le \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(\varsigma, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}).$$

Therefore, ψ_ε is SVNS-$\delta\mathcal{I}$-continuous. □

Theorem 6. *Let $(\Phi, \mathcal{G}_\hbar^{\varrho\sigma\varphi}, \mathcal{H}_\hbar^{\varrho\sigma\varphi})$ and $(\mathcal{K}, \mathcal{T}_\hbar^{\varrho\sigma\varphi}, \mathcal{F}_E^{\varrho\sigma\varphi})$ be svnsit-spaces and $\psi_\varepsilon : \widetilde{(\Phi, \hbar)} \to \widetilde{(\mathcal{K}, E)}$ be an SVNS-function. Then, the following statements are equivalent for every $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$ and $\mathbf{t} \in \mathbf{I}_\circ$.*

(1) ψ_ε is SVNS-$\theta\mathcal{I}$-continuous.
(2) For each $\Xi_A \in \mathcal{IR}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$, there exists $\Upsilon_B \in Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, such that $\psi_\varepsilon(C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})) \le \Xi_A$.
(3) $\psi_\varepsilon(\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})) \le \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})$, for each $\Xi_A \,\widetilde{\in}\, \widetilde{(\Phi, \hbar)}$.
(4) $\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \le \psi_\varepsilon^{-1}(\delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}))$, for each $\Upsilon_B \,\widetilde{\in}\, \widetilde{(\mathcal{K}, E)}$.
(5) For each $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed (resp. $\mathbf{t}$-svn-$\delta\mathcal{I}$-open) set $\Upsilon_B \,\widetilde{\in}\, \widetilde{(\mathcal{K}, E)}$, $\psi_\varepsilon^{-1}(\Upsilon_B)$ is a $\mathbf{t}$-svn-$\theta\mathcal{I}$-closed (resp. $\mathbf{t}$-svn-$\theta\mathcal{I}$-open) set in Φ.

Proof. (1)⇒(2): The result follows directly from Definition 16.

(2)⇒(3): Suppose there exist $\Xi_A \,\widetilde{\in}\, \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$, and $\mathbf{t} \in \mathbf{I}_\circ$ such that

$$\psi_\varepsilon(\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})) \not\le \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\Xi_A), \mathbf{t}).$$

Then, there exists $y \in \mathcal{K}$, $m, n, r \in \mathbf{I}_\circ$, $\varsigma \in \hbar$, and $e \in E$ such that

$$\varrho_{\psi_\varepsilon(\theta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t}))}(y) > m > \varrho_{\delta C_{\mathcal{T}_e^\varrho}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(y),$$
$$\sigma_{\psi_\varepsilon(\theta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t}))}(y) \le n \le \sigma_{\delta C_{\mathcal{T}_e^\sigma}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(y),$$
$$\varphi_{\psi_\varepsilon(\theta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t}))}(y) \le r \le \varphi_{\delta C_{\mathcal{T}_e^\varphi}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(y).$$

If $\psi_\varepsilon^{-1}(\{y\}) = \emptyset$, then, by the definition of the inverse image, $\psi_\varepsilon(\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}))(y) = \langle 0, 1, 1 \rangle$.

If $\psi_\varepsilon^{-1}(\{y\}) \ne \emptyset$, there exists $\varkappa \in \psi_\varepsilon^{-1}(\{y\})$ such that

$$\begin{aligned}
&\varrho_{\psi_\varepsilon(\theta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t}))}(y) \ge \varrho_{\theta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) > m > \varrho_{\delta C_{\mathcal{T}_e^\varrho}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)),\\
&\sigma_{\psi_\varepsilon(\theta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t}))}(y) \le \sigma_{\theta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \le n \le \sigma_{\delta C_{\mathcal{T}_e^\sigma}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)),\\
&\varphi_{\psi_\varepsilon(\theta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t}))}(y) \le \varphi_{\theta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \le r \le \varphi_{\delta C_{\mathcal{T}_e^\varphi}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)).
\end{aligned} \tag{4.2}$$

Since

$$\varrho_{\delta C_{\mathcal{T}_e^\varrho}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)) < m,$$
$$\sigma_{\delta C_{\mathcal{T}_e^\sigma}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)) \ge n,$$
$$\varphi_{\delta C_{\mathcal{T}_e^\varphi}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})}(\psi_\varepsilon(\varkappa)) \ge r,$$

it follows that $\psi_\varepsilon(\varkappa_\gamma^{(m,n,r)})$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of $\psi_\varepsilon(\Xi_A)$. So, there exists $\mathcal{D}_C \,\widetilde{\in}\, Q_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$, such that $\psi_\varepsilon(\Xi_A) \le (\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \mathcal{D}_C, \mathbf{t}), \mathbf{t}))^c$. Then

$$\mathcal{D}_C \,\widetilde{\in}\, Q_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t}), \quad \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}), \mathbf{t}) \,\widetilde{\in}\, \mathcal{IR}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t}).$$

By (2), there exists $\Upsilon_B \in Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, such that

$$\psi_\varepsilon(C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})) \leq \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \mathcal{D}_C, \mathbf{t}), \mathbf{t}).$$

Hence, $\psi_\varepsilon(\Xi_A) \leq (\psi_\varepsilon(C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c$ imply that $\Xi_A \leq (C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c$. Thus, $\varkappa_\gamma^{(m,n,r)}$ is not a $\mathbf{t}$-$\theta\mathcal{I}$-cluster point of Ξ_A, and by Theorem 2(6),

$$\varrho_{\theta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) < m,$$

$$\sigma_{\theta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \geq n,$$

$$\varphi_{\theta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \geq r.$$

This contradicts Eq (4.2).

(3)⇒(4): For all $\Upsilon_B \,\widetilde{\in}\, \widetilde{(\mathcal{K}, E)}$ and $\mathbf{t} \in \mathbf{I}_\circ$. set $\Xi_A = \psi_\varepsilon^{-1}(\Upsilon_B)$ from (3). Then,

$$\psi_\varepsilon(\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\psi_\varepsilon^{-1}(\Upsilon_B)), \mathbf{t}) \leq \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}).$$

This implies that

$$\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}) \leq \psi_\varepsilon^{-1}(\psi_\varepsilon(\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}))) \leq \psi_\varepsilon^{-1}(\delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}))$$

(4)⇒(5): Let $\Upsilon_B \,\widetilde{\in}\, \widetilde{(\mathcal{K}, E)}$ be $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed. By (4), we have

$$\theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \psi_\varepsilon^{-1}(\delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t})) = \psi_\varepsilon^{-1}(\Upsilon_B),$$

and since $\psi_\varepsilon^{-1}(\Upsilon_B) \leq \theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})$, $\psi_\varepsilon^{-1}(\Upsilon_B) = \theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})$. Hence, $\psi_\varepsilon^{-1}(\Upsilon_B)$ is a $\mathbf{t}$-svn-$\theta\mathcal{I}$-closed set. The other case is proved similarly.

(5)⇒(1): Let $\Xi_A \,\widetilde{\in}\, Q_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$. Then,

$$\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}) \,\widetilde{\in}\, \mathcal{I}\mathcal{R}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t}),$$

and by Theorem 2(8),

$$(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}))^c = \delta C_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, (\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}))^c, \mathbf{t}).$$

Thus, $\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})$ is a $\mathbf{t}$-svn-$\delta\mathcal{I}$-open set. By (5),

$$(\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}))^c) = \theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, (\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}))^c), \mathbf{t}).$$

Since, $\psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}) \wedge \Xi_A \leq \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})$, it follows that

$$\varkappa_\gamma^{(m,n,r)} \wedge \psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})).$$

Thus, $\varkappa_\gamma^{(m,n,r)}$ is not a $\mathbf{t}$-$\theta\mathcal{I}$-cluster point of $(\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})))^c$. Then, there exists $\Upsilon_B \,\widetilde{\in}\, Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ such that

$$(\psi_\varepsilon^{-1}(\mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t})))^c \leq (C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c.$$

Hence,

$$\psi_\varepsilon(C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})) \leq \mathcal{I}_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}), \mathbf{t}) \leq C^\star_{\mathcal{T}_E^{\varrho\sigma\varphi}}(e, \Xi_A, \mathbf{t}).$$

Therefore, ψ_ε is SVNS-θI-continuous. □

Theorem 7. *Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *and* $(\mathcal{K}, \mathcal{T}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{F}_{E}^{\varrho\sigma\varphi})$ *be svnsit-spaces and* $\psi_\varepsilon : \widetilde{(\Phi, \hbar)} \to \widetilde{(\mathcal{K}, E)}$ *be an SVNS-function. Then the following statements are equivalent for all* $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$ *and* $\mathbf{t} \in \mathbf{I}_\circ$.

(1) ψ_ε *is SVNS-*$\theta\mathcal{I}$*-continuous.*

(2) $\psi_\varepsilon(\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})) \leq \theta C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})$*, for every* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$.

(3) $\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \psi_\varepsilon^{-1}(\theta C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}))$*, for every* $\Upsilon_B \widetilde{\in} \widetilde{(\mathcal{K}, E)}$.

Proof. The proof is analogous to that of Theorem 6, with minor modifications, and is therefore omitted. □

Theorem 8. *Let* $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ *and* $(\mathcal{K}, \mathcal{T}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{F}_{E}^{\varrho\sigma\varphi})$ *be svnsit-spaces and* $\psi_\varepsilon : \widetilde{(\Phi, \hbar)} \to \widetilde{(\mathcal{K}, E)}$ *be an SVNS-function. Then, the following statements are equivalent for all* $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$ *and* $\mathbf{t} \in \mathbf{I}_\circ$.

(1) ψ_ε *is SVNSS-*$\theta\mathcal{I}$*-continuous.*

(2) $\psi_\varepsilon(\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})) \leq C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\Xi_A), \mathbf{t})$*, for every* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$.

(3) $\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \psi_\varepsilon^{-1}(C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}))$*, for every* $\Upsilon_B \widetilde{\in} \widetilde{(\mathcal{K}, E)}$.

(4) For each $\mathcal{T}_e^{\varrho}(\Upsilon_B^c) \geq \mathbf{t}$, $\mathcal{T}_e^{\sigma}(\Upsilon_B^c) \leq 1 - \mathbf{t}$, $\mathcal{T}_e^{\varphi}(\Upsilon_B^c) \leq 1 - \mathbf{t}$ *(resp.* $\mathcal{T}_e^{\varrho}(\Upsilon_B) \geq \mathbf{t}$, $\mathcal{T}_e^{\sigma}(\Upsilon_B) \leq 1 - \mathbf{t}$*, and* $\mathcal{T}_e^{\varphi}(\Upsilon_B) \leq 1 - \mathbf{t}$*),* $\psi_\varepsilon^{-1}(\Upsilon_B)$ *is a* $\mathbf{t}$*-svn-*$\theta\mathcal{I}$*-closed (resp.* $\mathbf{t}$*-svn-*$\theta\mathcal{I}$*-open) set in* Φ.

Proof. (1)⇒(2): The proof is analogous to that of Theorem 6(3).

(2)⇒(3): For all $\Upsilon_B \widetilde{\in} \widetilde{(\mathcal{K}, E)}$ and $\mathbf{t} \in \mathbf{I}_\circ$, set $\Xi_A = \psi_\varepsilon^{-1}(\Upsilon_B)$ from (2). Then,

$$\psi_\varepsilon(\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\psi_\varepsilon^{-1}(\Upsilon_B)), \mathbf{t}) \leq C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}).$$

This implies that

$$\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}) \leq \psi_\varepsilon^{-1}(\psi_\varepsilon(\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}))) \leq \psi_\varepsilon^{-1}(C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t})).$$

(3)⇒(4): Let $\mathcal{T}_e^{\varrho}(\Upsilon_B^c) \geq \mathbf{t}$, $\mathcal{T}_e^{\sigma}(\Upsilon_B^c) \leq 1 - \mathbf{t}$, $\mathcal{T}_e^{\varphi}(\Upsilon_B^c) \leq 1 - \mathbf{t}$. Then $\Upsilon_B = C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t})$. By (3)

$$\theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t})) \leq \psi_\varepsilon^{-1}(C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t})) = \psi_\varepsilon^{-1}(\Upsilon_B),$$

and since

$$\psi_\varepsilon^{-1}(\Upsilon_B) \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}),$$
$$\psi_\varepsilon^{-1}(\Upsilon_B) = \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_\varepsilon^{-1}(\Upsilon_B), \mathbf{t}).$$

Hence, $\psi_\varepsilon^{-1}(\Upsilon_B)$ is a $\mathbf{t}$-svn-$\theta\mathcal{I}$-closed set. The other case is similarly proved.

(4)⇒(1): Let $\Xi_A \widetilde{\in} Q_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_\varepsilon(\varkappa_\gamma^{(m,n,r)}), \mathbf{t})$. Then, $\mathcal{T}_e^{\varrho}(\Xi_A) \geq \mathbf{t}$, $\mathcal{T}_e^{\sigma}(\Xi_A) \leq 1 - \mathbf{t}$, and $\mathcal{T}_e^{\varphi}(\Xi_A) \leq 1 - \mathbf{t}$. By (4), $(\psi_\varepsilon^{-1}(\Xi_A))^C \leq \theta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, (\psi_\varepsilon^{-1}(\Xi_A))^C, \mathbf{t}))$. Since $\psi_\varepsilon(\varkappa_\gamma^{(m,n,r)})\ \Lambda\ \Xi_A$, it follows that $\varkappa_\gamma^{(m,n,r)}\ \Lambda\ \psi_\varepsilon^{-1}(\Xi_A)$, that is,

$$\varrho_{\theta C_{\mathcal{G}_{\varsigma}^{\varrho}}(\varsigma, (\psi_\varepsilon^{-1}(\Xi_A))^C, \mathbf{t}))}(\varkappa) = \varrho_{(\psi_\varepsilon^{-1}(\Xi_A))^C}(\varkappa) < m,$$
$$\sigma_{\theta C_{\mathcal{G}_{\varsigma}^{\sigma}}(\varsigma, (\psi_\varepsilon^{-1}(\Xi_A))^C, \mathbf{t}))}(\varkappa) = \sigma_{(\psi_\varepsilon^{-1}(\Xi_A))^C}(\varkappa) \geq n,$$
$$\varphi_{\theta C_{\mathcal{G}_{\varsigma}^{\varphi}}(\varsigma, (\psi_\varepsilon^{-1}(\Xi_A))^C, \mathbf{t}))}(\varkappa) = \varphi_{(\psi_\varepsilon^{-1}(\Xi_A))^C}(\varkappa) \geq r.$$

Thus, $\varkappa_\gamma^{(m,n,r)}$ is not a $\mathbf{t}$-$\theta\mathcal{I}$-cluster point of $(\psi_\varepsilon^{-1}(\Xi_A))^c$. Then, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ such that

$$(\psi_\varepsilon^{-1}(\Xi_A))^c \leq (C^\star_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}))^c.$$

Hence, $\psi_\varepsilon(C^\star_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t})) \leq \Xi_A$. Therefore, ψ_ε is SVNSS-$\theta\mathcal{I}$-continuous. □

Theorem 9. *Let $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ and $(\mathcal{K}, \mathcal{T}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{F}_{E}^{\varrho\sigma\varphi})$ be svnsit-spaces, and $\psi_{\varepsilon} : \widetilde{(\Phi, \hbar)} \to \widetilde{(\mathcal{K}, E)}$ be an SVNS-function. Then, the following statements are equivalent for all $\varsigma \in \hbar$, $(e = \varepsilon(\varsigma) \in E)$ and* $\mathbf{t} \in \mathbf{I}_{\circ}$.

(1) ψ_{ε} *is SVNSA-*$\mathcal{I}$*-continuous.*
(2) $\psi_{\varepsilon}(C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})) \leq \delta C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \psi_{\varepsilon}(\Xi_A), \mathbf{t})$*, for all* $\Xi_A \mathrel{\widetilde{\in}} \widetilde{(\Phi, \hbar)}$.
(3) $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_{\varepsilon}^{-1}(\Upsilon_B), \mathbf{t})) \leq \psi_{\varepsilon}^{-1}(\delta C_{\mathcal{T}_{E}^{\varrho\sigma\varphi}}(e, \Upsilon_B, \mathbf{t}))$*, for all* $\Upsilon_B \mathrel{\widetilde{\in}} \widetilde{(\mathcal{K}, E)}$.
(4) For each $\mathbf{t}$*-svn-*$\delta\mathcal{I}$*-closed (resp.* $\mathbf{t}$*-svn-*$\delta\mathcal{I}$*-open) set* $\Upsilon_B \mathrel{\widetilde{\in}} \widetilde{(\mathcal{K}, E)}$, $\mathcal{G}_{\varsigma}^{\varrho}((\psi_{\varepsilon}^{-1}(\Xi_A))^c) \geq \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\sigma}((\psi_{\varepsilon}^{-1}(\Xi_A))^c) \leq 1 - \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\varphi}((\psi_{\varepsilon}^{-1}(\Xi_A))^c) \leq 1 - \mathbf{t}$ *(resp.* $\mathcal{G}_{\varsigma}^{\varrho}(\psi_{\varepsilon}^{-1}(\Xi_A)) \geq \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\sigma}(\psi_{\varepsilon}^{-1}(\Xi_A)) \leq 1 - \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\varphi}(\psi_{\varepsilon}^{-1}(\Xi_A)) \leq 1 - \mathbf{t}$*).*
(5) For each $\mathbf{t}$*-SVNSRIO (resp.* $\mathbf{t}$*-SVNSRIC) set* $\Upsilon_B \mathrel{\widetilde{\in}} \widetilde{(\mathcal{K}, E)}$, $\mathcal{G}_{\varsigma}^{\varrho}((\psi_{\varepsilon}^{-1}(\Xi_A))^c) \geq \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\sigma}((\psi_{\varepsilon}^{-1}(\Xi_A))^c) \leq 1 - \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\varphi}((\psi_{\varepsilon}^{-1}(\Xi_A))^c) \leq 1 - \mathbf{t}$ *(resp.* $\mathcal{G}_{\varsigma}^{\varrho}(\psi_{\varepsilon}^{-1}(\Xi_A)) \geq \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\sigma}(\psi_{\varepsilon}^{-1}(\Xi_A)) \leq 1 - \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\varphi}(\psi_{\varepsilon}^{-1}(\Xi_A)) \leq 1 - \mathbf{t}$*).*

Proof. The proof is analogous to that of Theorem 5. □

Example 2. *Let* $\Phi = \{\varkappa_1, \varkappa_2\}$, $\hbar = \{\varsigma_1, \varsigma_2\}$*, and* $E = \{e_1, e_2\}$*, and define* $\Xi_{\hbar}, \Upsilon_{\hbar}, \Omega_{\hbar}, \widetilde{\in}\widetilde{(\Phi, \hbar)}$*, as follows:*

$$\Xi_{\hbar} = \{(\varsigma_1, \{\langle \varkappa_1, 0.6, 0.4, 0.6\rangle, \langle \varkappa_2, 0.6, 0.4, 0.6\rangle\}), (\varsigma_2, \{\langle \varkappa_1, 0.6, 0.4, 0.6\rangle, \langle \varkappa_2, 0.6, 0.4, 0.6\rangle\})\},$$

$$\Upsilon_{\hbar} = \{(\varsigma_1, \{\langle \varkappa_1, 0.3, 0.7, 0.3\rangle, \langle \varkappa_2, 0.3, 0.7, 0.3\rangle\}), (\varsigma_2, \{\langle \varkappa_1, 0.3, 0.7, 0.3\rangle, \langle \varkappa_2, 0.3, 0.7, 0.3\rangle\})\},$$

$$\Omega_{\hbar} = \{(\varsigma_1, \{\langle \varkappa_1, 0.8, 0.2, 0.8\rangle, \langle \varkappa_2, 0.8, 0.2, 0.8\rangle\}), (\varsigma_2, \{\langle \varkappa_1, 0.8, 0.2, 0.8\rangle, \langle \varkappa_2, 0.8, 0.2, 0.8\rangle\})\}.$$

Define $\mathcal{G}_{\varsigma}^{\varrho}, \mathcal{G}_{\varsigma}^{\sigma}, \mathcal{G}_{\varsigma}^{\varphi}, \mathcal{G}_{\varsigma}^{\star\varrho}, \mathcal{G}_{\varsigma}^{\star\sigma}, \mathcal{G}_{\varsigma}^{\star\varphi} : \hbar \longrightarrow \mathbf{I}^{\widetilde{(\Phi,\hbar)}}$*:*

$$\mathcal{G}_{\varsigma}^{\varrho}(\mathcal{Z}_{\hbar}) = \begin{cases} 1, & \text{if } \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{2}, & \text{if } \mathcal{Z}_{\hbar} = \Xi_{\hbar}, \\ 0, & \text{if } \quad \text{otherwise}. \end{cases} \qquad \mathcal{G}_{\varsigma}^{\star\varrho}(\mathcal{Z}_{\hbar}) = \begin{cases} 1, & \text{if } \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{2}{3}, & \text{if } \mathcal{Z}_{\hbar} = \{\Xi_{\hbar}, \Upsilon_{\hbar}\}, \\ 0, & \text{if } \quad \text{otherwise}. \end{cases}$$

$$\mathcal{G}_{\varsigma}^{\sigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & \text{if } \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{2}, & \text{if } \mathcal{Z}_{\hbar} = \Xi_{\hbar}, \\ 1, & \text{if } \quad \text{otherwise}. \end{cases} \qquad \mathcal{G}_{\varsigma}^{\star\sigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & \text{if } \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{3}, & \text{if } \mathcal{Z}_{\hbar} = \{\Xi_{\hbar}, \Upsilon_{\hbar}\}, \\ 1, & \text{if } \quad \text{otherwise}. \end{cases}$$

$$\mathcal{G}_{\varsigma}^{\varphi}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & \text{if } \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{2}, & \text{if } \mathcal{Z}_{\hbar} = \Xi_{\hbar}, \\ 1, & \text{if } \quad \text{otherwise}. \end{cases} \qquad \mathcal{G}_{\varsigma}^{\star\varphi}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & \text{if } \mathcal{Z} = \{\phi, \widetilde{\hbar}\}, \\ \frac{1}{3}, & \text{if } \mathcal{Z}_{\hbar} = \{\Xi_{\hbar}, \Upsilon_{\hbar}\}, \\ 1, & \text{if } \quad \text{otherwise}. \end{cases}$$

Define $\mathcal{H}_{\varsigma}^{\varrho}, \mathcal{H}_{\varsigma}^{\sigma}, \mathcal{H}_{\varsigma}^{\varphi}, \mathcal{H}_{\varsigma}^{\star\varrho}, \mathcal{H}_{\varsigma}^{\star\sigma}, \mathcal{H}_{\varsigma}^{\star\varphi} : \hbar \longrightarrow \mathbf{I}^{\widetilde{(\Phi,\hbar)}}$*:*

$$\mathcal{H}_{\varsigma}^{\varrho}(\mathcal{Z}_{\hbar}) = \begin{cases} 1, & \text{if } \mathcal{Z} = \phi, \\ \frac{2}{3}, & \text{if } \langle 0, 1, 1\rangle \leq \mathcal{Z}_{\hbar} \leq \Omega_{\hbar}, \\ 0, & \text{if } \quad \text{otherwise}. \end{cases} \qquad \mathcal{H}_{\varsigma}^{\star\varrho}(\mathcal{Z}_{\hbar}) = \begin{cases} 1, & \text{if } \mathcal{Z} = \phi, \\ \frac{2}{3}, & \text{if } \langle 0, 1, 1\rangle \leq \mathcal{Z}_{\hbar} \leq \Xi_{\hbar}, \\ 0, & \text{if } \quad \text{otherwise}. \end{cases}$$

$$\mathcal{H}_{\varsigma}^{\sigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & \text{if } \mathcal{Z} = \phi, \\ \frac{1}{3}, & \text{if } \langle 0, 1, 1\rangle \leq \mathcal{Z}_{\hbar} \leq \Omega_{\hbar}, \\ 1, & \text{if } \quad \text{otherwise}. \end{cases} \qquad \mathcal{H}_{\varsigma}^{\star\sigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & \text{if } \mathcal{Z} = \phi, \\ \frac{1}{3}, & \text{if } \langle 0, 1, 1\rangle \leq \mathcal{Z}_{\hbar} \leq \Xi_{\hbar}, \\ 1, & \text{if } \quad \text{otherwise}. \end{cases}$$

$$\mathcal{H}^{\varphi}_{\varsigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & if\ \mathcal{Z} = \phi, \\ \frac{1}{3}, & if\ \langle 0,1,1\rangle \leq \mathcal{Z}_{\hbar} \leq \Omega_{\hbar}, \\ 1, & if\quad otherwise. \end{cases} \qquad \mathcal{H}^{\star\varphi}_{\varsigma}(\mathcal{Z}_{\hbar}) = \begin{cases} 0, & if\ \mathcal{Z} = \phi, \\ \frac{1}{3}, & if\ \langle 0,1,1\rangle \leq \mathcal{Z}_{\hbar} \leq \Xi_{\hbar}, \\ 1, & if\quad otherwise. \end{cases}$$

From Theorems 1(4) and 2(4), we obtain $\delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}, \theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}} : \hbar \times \widetilde{(\Phi, \hbar)} \times \mathbf{I}_{\circ} \to \widetilde{(\Phi, \hbar)}$ as follows:

$$C_{\mathcal{G}^{\star\varrho\sigma\varphi}_{\hbar}}(\varsigma, \mathcal{Z}_{\hbar}, \mathbf{t}) = \begin{cases} \langle 0,1,1\rangle, & \text{if } \mathcal{Z}_{\hbar} = \langle 0,1,1\rangle, \\ \Upsilon^{c}_{\hbar}, & \text{if } \langle 0,1,1\rangle \neq \mathcal{Z} \leq \Upsilon^{c}_{\hbar},\ \ 0 < r \leq \frac{1}{2}, \\ \Xi^{c}_{\hbar}, & \text{if } \Upsilon^{c}_{\hbar} \neq \mathcal{Z} \leq \Xi^{c}_{\hbar},\ \ 0 < r \leq \frac{1}{2}, \\ \langle 1,0,0\rangle, & \text{otherwise.} \end{cases}$$

$$(\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}} = \delta C_{\mathcal{G}^{\star\varrho\sigma\varphi}_{\hbar}} = C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}})(\varsigma, \mathcal{Z}_{\hbar}, \mathbf{t}) = \begin{cases} \langle 0,1,1\rangle, & \text{if } \mathcal{Z}_{\hbar} = \langle 0,1,1\rangle, \\ \Xi^{c}_{\hbar}, & \text{if } \langle 0,1,1\rangle \neq \mathcal{Z} \leq \Xi^{c}_{\hbar},\ \ 0 < r \leq \frac{1}{2}, \\ \langle 1,0,0\rangle, & \text{otherwise.} \end{cases}$$

By Theorem 5(3), the identity mapping $id_X : (\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar}) \to (\Phi, \mathcal{G}^{\star\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\star\varrho\sigma\varphi}_{\hbar})$ is SVNS-$\delta\mathcal{I}$-continuous, but it is not SVNSS-$\theta\mathcal{I}$-continuous because, by Theorem 8(2),

$$\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \mathcal{Z}_{\hbar}, \mathbf{t}) \nleq C_{\mathcal{G}^{\star\varrho\sigma\varphi}_{\hbar}}(\varsigma, \mathcal{Z}_{\hbar}, \mathbf{t}).$$

5. Single-valued neutrosophic soft $\mathcal{I}$-regular and almost $\mathcal{I}$-regular spaces

In this section, we develop the notions ofsingle-valued neutrosophic soft $\mathcal{I}$-regular (SVNS-$\mathcal{I}$-regular) and single-valued neutrosophic soft almost $\mathcal{I}$-regular (SVNS-almost-$\mathcal{I}$-regular) spaces within the setting of single-valued neutrosophic soft deal topology. We first recall the necessary preliminaries and introduce the formal definitions of these concepts. Then, we investigate their fundamental properties and establish several characterizations that illustrate their structural behavior. In addition, we examine the relationships between these classes and other related forms of single-valued neutrosophic regularity, with particular emphasis on the role of the ideal in controlling separation properties. The results obtained in this section contribute to a clearer understanding of these notions and highlight their significance in the broader context of single-valued neutrosophic soft ideal topological structures.

Definition 17. *Let $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ be svnsit-spaces, for $\varsigma \in \hbar$, $\Xi_A, \Upsilon_B \in \widetilde{(\Phi, \hbar)}$ and $\mathbf{t} \in \mathbf{I}_{\circ}$. Then, Φ is called:*

(i) SVNS-$\mathcal{I}$-regular if for each $\Xi_A \widetilde{\in} Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(e, \varkappa^{(m,n,r)}_{\gamma}, \mathbf{t})$, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa^{(m,n,r)}_{\gamma}, \mathbf{t})$ such that $C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Xi_A$.

(ii) SVNS-almost-$\mathcal{I}$-regular if for each $\Xi_A \widetilde{\in} \mathcal{IR}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa^{(m,n,r)}_{\gamma}, \mathbf{t})$, there exists $\Upsilon_B \widetilde{\in} \mathcal{IR}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa^{(m,n,r)}_{\gamma}, \mathbf{t})$ such that $C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Xi_A$.

Theorem 10. *Let $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ be svnsit-spaces, for each $\Xi_A, \Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, and $\mathbf{t} \in \mathbf{I}_{\circ}$. Then, the following statements are equivalent:*

(1) $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ is called SVNS-almost-$\mathcal{I}$-regular.

(2) For each $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$ *and for each* $\Xi_A \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$*, there exists* $\Upsilon_B \widetilde{\in} \mathcal{IR}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ *such that* $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$.

(3) For each $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$ *and for each* $\Xi_A \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$*, there exists* $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ *such that* $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t})$.

(4) For each $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$ *and* $\mathbf{t}$*-SVNSRIC* $\mathcal{D}_C \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *with* $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\notin} \mathcal{D}_C$*, there exists* $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ *and* Ξ_A *is* $\mathbf{t}$*-svns-*$\star$*-open set such that* $\mathcal{D}_C \leq \Xi_A$ *and* $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \overline{\Lambda} C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.

(5) For each $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$ *and* $\mathbf{t}$*-SVNSRIC* $\mathcal{D}_C \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *with* $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\notin} \mathcal{D}_C$*, there exists* $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ *and* Ξ_A *is a* $\mathbf{t}$*-svns-*$\star$*-open set such that* $\mathcal{D}_C \leq \Xi_A$ *and* $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \overline{\Lambda} \Xi_A$.

(6) For each $\mathbf{t}$*-SVNSRIO* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *with* $\mathcal{D}_C \Lambda \Xi_A$*, there exists* $\mathbf{t}$*-SVNSRIO* $\Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *such that* $\mathcal{D}_C \Lambda \Upsilon_B \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Xi_A$.

(7) For each $\mathbf{t}$*-SVNSRIC* $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *with* $\mathcal{D}_C \not\leq \Xi_A$*, there exists* $\mathbf{t}$*-SVNSRIO* $\Upsilon_B \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *and is a* $\mathbf{t}$*-svns-*$\star$*-open set* $\Omega_D \widetilde{\in} \widetilde{(\Phi, \hbar)}$ *such that* $\mathcal{D}_C \Lambda \Upsilon_B$*,* $\Xi_A \leq \Omega_D$*, and* $\Upsilon_B \overline{\Lambda} \Omega_D$.

Proof. The proofs of (1)⇒(2) and (2)⇒(3) are immediate.

(3)⇒(1): Let $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$ and $\Xi_A \widetilde{\in} \mathcal{IR}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$. Then, by (3), there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}), \mathbf{t}) = \Xi_A$. Since $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$, it follows that

$$\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}) \widetilde{\in} \mathcal{IR}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t}).$$

Also, since $\mathcal{D}_C = \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}) \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})$, we have $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}) \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})$, and hence $\varkappa_{\gamma}^{(m,n,r)} \Lambda \mathcal{D}_C \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}) \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Xi_A$ where $\mathcal{D}_C \widetilde{\in} \mathcal{IR}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_{\varepsilon}(\varkappa_{\gamma}^{(m,n,r)}), \mathbf{t})$.

(3)⇒(4): Let $\mathcal{D}_C$ be a $\mathbf{t}$-SVNSRIC set in Φ and $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\in} P_{\widetilde{(m,n,r)}}(\Phi)$ with $\varkappa_{\gamma}^{(m,n,r)} \widetilde{\notin} \mathcal{D}_C$. Then, $\varkappa_{\gamma}^{(m,n,r)} \Lambda \mathcal{D}_C^c$ and $\mathcal{D}_C^c \widetilde{\in} \mathcal{IR}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \psi_{\varepsilon}(\varkappa_{\gamma}^{(m,n,r)}), \mathbf{t}) \sqsubseteq Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$. By (3), there exists $\Omega_D \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that

$$C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Omega_D, \mathbf{t}) \leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C^c, \mathbf{t}), \mathbf{t}) = \mathcal{D}_C^c.$$

Now, $\varkappa_{\gamma}^{(m,n,r)} \Lambda \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Omega_D, \mathbf{t}), \mathbf{t})$. Then,

$$\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Omega_D, \mathbf{t}), \mathbf{t}) \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t}),$$

and hence, by the hypothesis, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that $C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Omega_D, \mathbf{t}), \mathbf{t})$. Then, $\mathcal{D}_C \leq (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Omega_D, \mathbf{t}))^c$. Set $\Xi_A = (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Omega_D, \mathbf{t}))^c$. Then, Ξ_A is a $\mathbf{t}$-svns-$\star$-open set. Hence,

$$C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \leq (\mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Omega_D, \mathbf{t}), \mathbf{t}))^c \leq (C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c.$$

Hence, $Cl^{\star}(\Upsilon_B, r) \overline{\Lambda} C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$

The implication (4)⇒(5) is straightforward.

(5)⇒(6): Suppose that Ξ_A is a $\mathbf{t}$-SVNSRIO set with $\mathcal{D}_C \wedge \Xi_A$. Then, $\mathcal{D}_C \not\leq \Xi_A^c$. Therefore, there exists $\varkappa_\gamma^{(m,n,r)} \widetilde{\in} \widetilde{P_{(m,n,r)}}(\Phi)$ such that $\varkappa_\gamma^{(m,n,r)} \in \mathcal{D}_C$ and $\mathcal{D}_C \not\leq \Xi_A^c$ where Ξ_A^c is a $\mathbf{t}$-SVNSRIC set. By (5), there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, and Ω_D is a $\mathbf{t}$-svns-$\star$-open set such that $\Xi_A^c \leq \Omega_D$ and $C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \overline{\wedge} \Omega_D$. From $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, we have $\varkappa_\gamma^{(m,n,r)} \wedge \Upsilon_B \leq \mathcal{I}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})$. Setting $(\Upsilon_B)_1 = \mathcal{I}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})$, we have $\mathcal{D}_C \wedge (\Upsilon_B)_1$, and $(\Upsilon_B)_1$ is a $\mathbf{t}$-SVNSRIO set such that

$$\mathcal{D}_C \wedge (\Upsilon_B)_1 \leq C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, (\Upsilon_B)_1, \mathbf{t}) \leq C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Omega_D^c \leq \Xi_A.$$

(6)⇒(7): Let Ξ_A be a $\mathbf{t}$-SVNSRIC set with $\mathcal{D}_C \not\leq \Xi_A$. Then, $\mathcal{D}_C \wedge \Xi_A^c$, and hence by (6) there exists a $\mathbf{t}$-SVNSRIO set $\Upsilon_B \in \widetilde{(\Phi, \hbar)}$ such that $\mathcal{D}_C \wedge \Upsilon_B \leq C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Xi_A^c$. Then, Υ_B is a $\mathbf{t}$-SVNSRIO set and $(C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c$ is a $\mathbf{t}$-svns-$\star$-open such that $\mathcal{D}_C \wedge \Upsilon_B$, $\Xi_A \leq (C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c$ and $\Upsilon_B \overline{\wedge} (C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}))^c$.

(7)⇒(1): Let $\Xi_A \widetilde{\in} \mathcal{IR}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$. Then, $\varkappa_\gamma^{(m,n,r)} \notin \Xi_A^c$ and Ξ_A^c is $\mathbf{t}$-SVNSRIC set. By (7), there exist a $\mathbf{t}$-SVNSRIO set $\Upsilon_B \in \widetilde{(\Phi, \hbar)}$ and a $\mathbf{t}$-svns-$\star$-open set $\Omega_D \in \widetilde{(\Phi, \hbar)}$ such that $\varkappa_\gamma^{(m,n,r)} \wedge \Upsilon_B$, $\Xi_A^c \leq \Omega_D$, and $\Upsilon_B \overline{\wedge} \Omega_D$. Then, $\Upsilon_B \in \mathcal{IR}_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$. Since Ω_D is a $\mathbf{t}$-svns-$\star$-open set, we have $C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \overline{\wedge} \Omega_D$. Thus, $\varkappa_\gamma^{(m,n,r)} \wedge \Upsilon_B \leq C^\star_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Omega_D^c \leq \Xi_A$. Hence, $(\Phi, \mathcal{G}_\hbar^{\varrho\sigma\varphi}, \mathcal{H}_\hbar^{\varrho\sigma\varphi})$ is called SVNS-almost-$\mathcal{I}$-regular. □

Theorem 11. *Let $(\Phi, \mathcal{G}_\hbar^{\varrho\sigma\varphi}, \mathcal{H}_\hbar^{\varrho\sigma\varphi})$ be svnsit-spaces, for each $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, and $\mathbf{t} \in \mathbf{I}_\circ$. Then, the following statements are equivalent:*

(1) $(\Phi, \mathcal{G}_\hbar^{\varrho\sigma\varphi}, \mathcal{H}_\hbar^{\varrho\sigma\varphi})$ is called SVNS-$\mathcal{I}$-regular.

(2) For each $\varkappa_\gamma^{(m,n,r)} \widetilde{\in} \widetilde{P_{(m,n,r)}}(\Phi)$ and for each $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, with $\mathcal{G}_\varsigma^\varrho(\Xi_A^c) \geq \mathbf{t}$, $\mathcal{G}_\varsigma^\sigma(\Xi_A^c) \leq 1 - \mathbf{t}$, $\mathcal{G}_\varsigma^\varphi(\Xi_A^c) \leq 1 - \mathbf{t}$ and $\varkappa_\gamma^{(m,n,r)} \notin \Xi_A$, there exists $\Upsilon_B \in \widetilde{(\Phi, \hbar)}$ where Υ_B is a $\mathbf{t}$-svns-$\star$-open such that $\varkappa_\gamma^{(m,n,r)} \notin C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t})$ and $\Xi_A \leq \Upsilon_B$.

(3) For each $\varkappa_\gamma^{(m,n,r)} \widetilde{\in} \widetilde{P_{(m,n,r)}}(\Phi)$ and for each $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, with $\mathcal{G}_\varsigma^\varrho(\Xi_A^c) \geq \mathbf{t}$, $\mathcal{G}_\varsigma^\sigma(\Xi_A^c) \leq 1 - \mathbf{t}$, $\mathcal{G}_\varsigma^\varphi(\Xi_A^c) \leq 1 - \mathbf{t}$, and $\varkappa_\gamma^{(m,n,r)} \notin \Xi_A$, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ and $\Omega_D \widetilde{\in} \widetilde{(\Phi, \hbar)}$ where Ω_D is a $\mathbf{t}$-svns-$\star$-open such that $\Xi_A \leq \Upsilon_B$ and $\Upsilon_B \overline{\wedge} \Omega_D$.

(4) For each $\mathcal{D}_C, \Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$ with $\mathcal{G}_\varsigma^\varrho(\Xi_A^c) \geq \mathbf{t}$, $\mathcal{G}_\varsigma^\sigma(\Xi_A^c) \leq 1 - \mathbf{t}$, $\mathcal{G}_\varsigma^\varphi(\Xi_A^c) \leq 1 - \mathbf{t}$, and $\mathcal{D}_C \not\leq \Xi_A$, there exists $\Upsilon_B \in Q_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, and $\Upsilon_B, \Omega_D \widetilde{\in} \widetilde{(\Phi, \hbar)}$ with $\mathcal{G}_\varsigma^\varrho(\Upsilon_B) \geq \mathbf{t}$, $\mathcal{G}_\varsigma^\sigma(\Upsilon_B) \leq 1 - \mathbf{t}$, $\mathcal{G}_\varsigma^\varphi(\Upsilon_B) \leq 1 - \mathbf{t}$ and Ω_D is $\mathbf{t}$-svns-$\star$-open such that $\mathcal{D}_C \wedge \Upsilon_B$, $\Xi_A \leq \Omega_D$ and $\Upsilon_B \overline{\wedge} \Omega_D$.

Proof. The proof proceeds in a manner similar to that of Theorem 10, and is therefore omitted. □

Theorem 12. *An svnsit-space $(\Phi, \mathcal{G}_\hbar^{\varrho\sigma\varphi}, \mathcal{H}_\hbar^{\varrho\sigma\varphi})$ is SVNS-almost-$\mathcal{I}$-regular iff for all $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, and $\mathbf{t} \in \mathbf{I}_\circ$, $\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.*

Proof. From Theorem 2(9), we only show that $\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \geq \theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$. Let $\delta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) \not\geq \theta C_{\mathcal{G}_\hbar^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$. Then, there exists $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_\circ$ such that

$$\begin{cases} \varrho_{\delta C_{\mathcal{G}_\varsigma^\varrho}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) < m < \varrho_{\theta C_{\mathcal{G}_\hbar^\varrho}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa), \\ \sigma_{\delta C_{\mathcal{G}_\varsigma^\sigma}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \geq n \geq \sigma_{\theta C_{\mathcal{G}_\hbar^\sigma}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa), \\ \varphi_{\delta C_{\mathcal{G}_\varsigma^\varphi}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa) \geq r \geq \varphi_{\theta C_{\mathcal{G}_\hbar^\varphi}(\varsigma, \Xi_A, \mathbf{t})}(\varkappa). \end{cases} \tag{5.1}$$

Since

$$\varrho_{\delta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) < m, \quad \sigma_{\delta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq n, \quad \varphi_{\delta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq r,$$

it follows that $\varkappa_\gamma^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A. So, there exists $\Upsilon_B \,\widetilde{\in}\, Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ with $\Xi_A \leq (\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c$. Since $\Upsilon_B \,\widetilde{\in}\, Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$, we have

$$\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}) \,\widetilde{\in}\, \mathcal{IR}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t}).$$

By the SVNS-almost-$\mathcal{I}$-regularity of Φ, there exists $\mathcal{D}_C \,\widetilde{\in}\, \mathcal{IR}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ such that $C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \mathcal{D}_C, \mathbf{t})) \leq \mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t})$. Thus,

$$\Xi_A \leq (\mathcal{I}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}), \mathbf{t}))^c \leq (C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \mathcal{D}_C, \mathbf{t}))^c = \mathcal{I}^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \mathcal{D}^c_C, \mathbf{t}),$$

and $\mathcal{G}^{\varrho}_{\varsigma}(\mathcal{D}_C) \geq \mathbf{t}$, $\mathcal{G}^{\sigma}_{\varsigma}(\mathcal{D}_C) \leq 1 - \mathbf{t}$, $\mathcal{G}^{\varphi}_{\varsigma}(\mathcal{D}_C) \leq 1 - \mathbf{t}$. By Theorem 2(4),

$$\varrho_{\theta C_{\mathcal{G}^{\varrho}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \leq \varrho_{\mathcal{D}^c_C}(\varkappa) < m,$$

$$\sigma_{\theta C_{\mathcal{G}^{\sigma}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \sigma_{\mathcal{D}^c_C}(\varkappa) \geq n,$$

$$\varphi_{\theta C_{\mathcal{G}^{\varphi}_{\varsigma}}(\varsigma,\Xi_A,\mathbf{t})}(\varkappa) \geq \varphi_{\mathcal{D}^c_C}(\varkappa) \geq r.$$

This contradicts Eq (5.1).

Similarly, the proof of the converse direction is straightforward. □

Theorem 13. *An svnsit-space $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ is SVNS-almost-$\mathcal{I}$-regular iff for all $\mathbf{t}$-SVNSRIC sets $\Xi_A \,\widetilde{\in}\, \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, and $\mathbf{t} \in \mathbf{I}_\circ$, $\theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) = \Xi_A$.*

Proof. The necessary part is a direct consequence of Theorem 12 and the fact that the $\mathbf{t}$-SVNSRIC set is $\mathbf{t}$-svn-$\delta\mathcal{I}$-closed.

Conversely, let Ξ_A be any $\mathbf{t}$-SVNSRIC set with $\varkappa_\gamma^{(m,n,r)} \notin \Xi_A$. Then, $\varkappa_\gamma^{(m,n,r)} \notin \theta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t})$, and hence, $\varkappa_\gamma^{(m,n,r)}$ is not a $\mathbf{t}$-δI-cluster point of Ξ_A. Thus, there exists $\Upsilon_B \widetilde{\in} Q_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \varkappa_\gamma^{(m,n,r)}, \mathbf{t})$ such that $\Xi_A \;\overline{\wedge}\; C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t})$. Thus, that $\Xi_A \leq (C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t}))^c = \mathcal{D}_C$ and $\mathcal{D}_C$ is a $\mathbf{t}$-svns-$\star$-open set implies $\mathcal{D}_C \;\overline{\wedge}\; C^{\star}_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t})$. Hence, by Theorem 11(5), $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ is SVNS-almost-I-regular. □

Lemma 5. *If $\Xi_A, \Upsilon_B \,\widetilde{\in}\, \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, and $\mathbf{t} \in \mathbf{I}_\circ$ such that $\Xi_A \;\overline{\wedge}\; \Upsilon_B$ where Υ_B is $\mathbf{t}$-svn-$\delta\mathcal{I}$-open, then $\delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) \;\overline{\wedge}\; \Upsilon_B$.*

Proof. Let $\Xi_A \;\overline{\wedge}\; \Upsilon_B$ where Υ_B is $\mathbf{t}$-svn-$\delta\mathcal{I}$-open. Then, $\Xi_A \leq \Upsilon^c_B = \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon_B, \mathbf{t})$, and by Theorem 2(11),

$$\delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) \leq \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon^c_B, \mathbf{t}), \mathbf{t}) = \delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Upsilon^c_B, \mathbf{t}) = \Upsilon^c_B.$$

Hence, $\delta C_{\mathcal{G}^{\varrho\sigma\varphi}_{\hbar}}(\varsigma, \Xi_A, \mathbf{t}) \;\overline{\wedge}\; \Upsilon_B$. □

Lemma 6. *Let $(\Phi, \mathcal{G}^{\varrho\sigma\varphi}_{\hbar}, \mathcal{H}^{\varrho\sigma\varphi}_{\hbar})$ be svnsit-spaces, and $\Xi_A \,\widetilde{\in}\, \widetilde{(\Phi, \hbar)}$ be a $\mathbf{t}$-svn-δI-open set iff for every $\varkappa_\gamma^{(m,n,r)} \,\widetilde{\in}\, \widetilde{P_{(m,n,r)}}(\Phi)$ with $\varkappa_\gamma^{(m,n,r)} \wedge \Xi_A$, there exists $\mathbf{t}$-SVNSRIO set $\Upsilon_B \widetilde{\in}\, \widetilde{(\Phi, \hbar)}$ such that $\varkappa_\gamma^{(m,n,r)} \wedge \Upsilon_B \leq \Xi_A$.*

Proof. Let $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\widetilde{\in}} \widetilde{P_{(m,n,r)}}(\Phi)$ with $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Xi_A$. Then, $\varkappa_{\gamma}^{(m,n,r)} \notin \Xi_A^c$. Since Ξ_A is a $\mathbf{t}$-svn-$\delta\mathcal{I}$-open set, $\varkappa_{\gamma}^{(m,n,r)} \notin \Xi_A^c = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})$. Thus, $\varkappa_{\gamma}^{(m,n,r)}$ is not a $\mathbf{t}$-$\delta\mathcal{I}$-cluster point of Ξ_A^c. Thus, there exists $\mathcal{D}_C \mathrel{\widetilde{\in}} Q_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \varkappa_{\gamma}^{(m,n,r)}, \mathbf{t})$ such that $\Xi_A^c \mathrel{\overline{\Lambda}} \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}), \mathbf{t})$. Setting $\Upsilon_B = \mathcal{I}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{D}_C, \mathbf{t}), \mathbf{t})$, then, Υ_B is a $\mathbf{t}$-SVNSRIO set with $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Upsilon_B \leq \Xi_A$.

Conversely, suppose $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A^c, \mathbf{t}) \neq \Xi_A^c$. Then, there exist $\varkappa \in \Phi$ and $m, n, r \in \mathbf{I}_\circ$ such that

$$\varrho_{\Xi_A^c}(\varkappa) < m < \varrho_{\delta C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa),$$
$$\sigma_{\Xi_A^c}(\varkappa) \geq n \geq \sigma_{\delta C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa),$$
$$\varphi_{\Xi_A^c}(\varkappa) \geq n \geq \varphi_{\delta C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa).$$

Since $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Xi_A$, there exists a $\mathbf{t}$-SVNSRIO set Υ_B such that $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Upsilon_B \leq \Xi_A$. This implies

$$\Xi_A^c \leq \Upsilon_B^c = C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \mathcal{I}^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B^c, \mathbf{t}), \mathbf{t}).$$

By Theorem 1(4),

$$\varrho_{\delta C_{\mathcal{G}_{\hbar}^{\varrho}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa) \leq \varrho_{\Upsilon_B^c}(\varkappa) < m$$
$$\sigma_{\delta C_{\mathcal{G}_{\hbar}^{\sigma}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa) \geq \sigma_{\Upsilon_B^c}(\varkappa) \geq n,$$
$$\varphi_{\delta C_{\mathcal{G}_{\hbar}^{\varphi}}(\varsigma, \Xi_A^c, \mathbf{t})}(\varkappa) \geq \varphi_{\Upsilon_B^c}(\varkappa) \geq r.$$

This is a contradiction. Hence, $\delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A^c, \mathbf{t}) = \Xi_A^c$, i.e., Ξ_A is a $\mathbf{t}$-svn-$\delta\mathcal{I}$-open set. □

Lemma 7. *If $\mathcal{G}_{\varsigma}^{\varrho}(\Xi_A) \geq \mathbf{t}$, $\mathcal{G}_{\varsigma}^{\sigma}(\Xi_A) \leq 1 - \mathbf{t}$, and $\mathcal{G}_{\varsigma}^{\varphi}(\Xi_A) \leq 1 - \mathbf{t}$, then $C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t}) = \delta C_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Xi_A, \mathbf{t})$.*

Proof. The proof of this lemma follows directly from Theorem 2(8). □

Theorem 14. *Let $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ be svnsit-spaces, for each $\Xi_A \widetilde{\in} \widetilde{(\Phi, \hbar)}$, $\varsigma \in \hbar$, and $\mathbf{t} \in \mathbf{I}_\circ$. Then, the following statements are equivalent:*

(1) $(\Phi, \mathcal{G}_{\hbar}^{\varrho\sigma\varphi}, \mathcal{H}_{\hbar}^{\varrho\sigma\varphi})$ is called SVNS-almost-$\mathcal{I}$-regular.
(2) For each $\mathbf{t}$-svn-$\delta\mathcal{I}$-open set $\Xi_A \mathrel{\widetilde{\in}} \widetilde{(\Phi, \hbar)}$ and each $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\widetilde{\in}} \widetilde{P_{(m,n,r)}}(\Phi)$ with $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Xi_A$, there exists a $\mathbf{t}$-svn-δI-open set $\Upsilon_B \mathrel{\widetilde{\in}} \widetilde{(\Phi, \hbar)}$, such that $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Upsilon_B \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Xi_A$.

Proof. (1)⇒(2): Let Ξ_A be a $\mathbf{t}$-svn-$\delta\mathcal{I}$-open set such each $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Xi_A$. Then, by Lemma 6, there exists a $\mathbf{t}$-SVNSRIO set $\Omega_D \in \widetilde{(\Phi, \hbar)}$ such that $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Omega_D \leq \Xi_A$. By the SVNS-almost-$\mathcal{I}$-regularity of Φ, there exists a $\mathbf{t}$-SVNSRIO set Υ_B (which is also a $\mathbf{t}$-svn-$\delta\mathcal{I}$-open set) such that $\varkappa_{\gamma}^{(m,n,r)} \mathrel{\Lambda} \Upsilon_B \leq C^{\star}_{\mathcal{G}_{\hbar}^{\varrho\sigma\varphi}}(\varsigma, \Upsilon_B, \mathbf{t}) \leq \Omega_D \leq \Xi_A$.

(2)⇒(1): It is obvious. □

6. Conclusions

In this paper, we have introduced and investigated several new concepts within the framework of single-valued neutrosophic soft ideal topological spaces, including $\mathbf{t}$-single-valued neutrosophic soft $\delta\mathcal{I}$-cluster points and θI-cluster points, along with their corresponding $\delta\mathcal{I}$-closure and $\theta\mathcal{I}$-closure operators. We examined $\mathbf{t}$-single-valued neutrosophic soft δI-closed and θI-closed sets and established their fundamental properties. We also explored various types of continuity, such as single-valued neutrosophic soft $\delta\mathcal{I}$-continuity, $\theta\mathcal{I}$-continuity, and almost $\mathcal{I}$-continuity, in addition to stronger forms including strongly θ-ideal continuity and almost ideal continuity. Moreover, the notions of single-valued neutrosophic soft $\mathcal{I}$-regular and almost $\mathcal{I}$-regular spaces were developed, and several relationships among these concepts were derived, providing deeper insight into their structural properties and suggesting directions for future research.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools have been used in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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