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Research article

Sufficient conditions for the existence of minimizers of fuzzy variational problems with generalized Hukuhara differentiable functions

Mansi Verma[1,2], **Chuei Yee Chen**[1,3,*] **and Sie Long Kek**[4]

[1] Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia; ORCID: https://orcid.org/0000-0003-1176-5968, https://orcid.org/0000-0003-1768-8668

[2] School of Computer Science and Engineering, IILM University, 1, Knowledge Center, Golf Course Rd, Sector 53, 122011, Gurugram, Haryana, India; ORCID: https://orcid.org/0000-0003-1176-5968

[3] Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia; ORCID: https://orcid.org/0000-0003-1768-8668

[4] Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, UTHM Kampus Cawangan Pagoh, Hab Pendidikan Tinggi Pagoh, KM 1, Jalan Panchor, 84600 Pagoh, Muar, Johor, Malaysia; ORCID: https://orcid.org/0000-0003-0425-4422

* **Correspondence:** Email: cychen@upm.edu.my.

Abstract: The study of fuzzy variational problems has received considerable attention in recent years due to their ability to handle the complexity and uncertainty of mathematical models in optimal control and image segmentation problems. The existing literature is focused on necessary conditions for finding minimizers, under the assumption that a minimizer always exists. This study establishes sufficient conditions for the existence of minimizers for generalized Hukuhara (gH) differentiable functions involving multivariable preinvex functions. Preinvexity, being a weaker notion than convexity, nevertheless retains the lower bound property that is sufficient for establishing the existence of minimizers.

Keywords: fuzzy variational problem; existence of minimizer; gH differentiable function; multivariable preinvex gH differentiable functions
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1. Introduction

Optimization problems serve as fundamental challenges across numerous scientific and engineering domains, aiming to find the best solution among a set of feasible options. In recent years, the integration of fuzzy variables into optimization frameworks has received attention in the fields of science and engineering such as image segmentation, vessel contouring, and other optimization problems [1–3], due to their ability to handle uncertainty and imprecision inherent in real-world scenarios. Within this landscape, fuzzy variational problems represent a distinct subset, wherein optimization objectives are subject to fuzzy constraints or involve fuzzy-valued functions. Although the theory of fuzzy partial differential equations has received significant attention [4], the research on fuzzy variational problems is still developing, with limited studies on conditions guaranteeing the existence and uniqueness of solutions [5, 6].

On the other hand, the theory of fuzzy partial differential equations (PDEs) has been well-studied, and several researchers have investigated the existence and uniqueness of fuzzy solutions to fuzzy partial hyperbolic functional differential equations under the Buckley–Feuring derivative [7, 8]. Their approach involves transforming the fuzzy PDE into a fixed-point problem for a suitable operator and establishing the Lipschitz condition of the operator that leads to the fixed point corresponding to a unique solution of the fuzzy PDE. However, it is important to note that the Buckley–Feuring derivative utilizes the classical notion of a derivative in a fuzzy sense, obtained by the associated α-level sets of the fuzzy functions, which may not always exist. Due to this, the notion of Buckley–Feuring derivative does not have a clear interpretation in the fuzzy setting, as pointed out in the work of Bede and Gal [9]. As for the generalized Hukuhara (gH) differentiability concept, it is based on the generalized Hukuhara difference, which is defined in two ways: Either the first fuzzy number contains the translation of the second, or the second includes the translation of the first. This approach helps to deal with the situation in these two ways and, as a result, increases the possibility of solving problems that allow for a more flexible approach where the derivative can be defined even in cases of imprecision or uncertainty. Therefore, the gH-differentiability concept allows for the consideration of a wider class of fuzzy functions [10].

In 2017, Long et al. [11] employed a similar approach to that of Long et al. [7], proving that the solutions to fuzzy fractional partial differential equations exist and are unique under Caputo gH differentiability. The former utilized a weaker Lipschitz condition that is under weakly contractive mapping. Subsequently, these results were used to study and establish the existence and uniqueness of solutions for fuzzy partial differential equations involving intuitionistic fuzzy sets [12].

Although some variational problems inherently involve PDEs, the literature on fuzzy variational problems remains limited [6]. It was first investigated by Farhadinia [13], who aimed to find the optimal solution for a fuzzy energy functional, where the variables are imprecise or uncertain. He derived Euler–Lagrange equations for fuzzy variational problems to calculate the extremals explicitly. For this purpose, he used Buckley–Feuring differentiability and fuzzy convexity notions in the classical sense via the α-level set approach. He provided a theoretical framework for fuzzy optimization problems by introducing the necessary conditions for optimality in fuzzy variational problems, which extend the classical Euler–Lagrange equations to the fuzzy case. However, it is important to note that most existing research does not guarantee the existence of a solution. Recent studies have employed different notions of differentiability to obtain Euler–Lagrange equations as necessary optimality conditions for local extrema in fuzzy variational problems [13, 14]. Furthermore, similar results were obtained for fuzzy fractional variational problems (see, for instance, [15, 16]). More recently, Verma et al. [5] studied the existence and uniqueness of minimizers for functions that are differentiable in the Buckley–Feuring sense.

In light of this, the present study establishes sufficient conditions for the existence and uniqueness of minimizers of fuzzy variational problems under a broader class of fuzzy differentiable functions, particularly gH differentiable functions.

The paper is organized as follows. Section 2 provides some definitions and results on fuzzy sets and functions, gH differentiable functions, invex sets, and preinvex functions, which are essential for this study. Section 3 presents the framework for studying sufficient conditions by defining multivariable preinvex gH differentiable functions and deriving the corresponding lower-bound property result. Section 4 presents the main results of this study, namely the sufficient conditions that guarantee the existence and uniqueness of minimizers. Section 5 provides an application of the main result to a fuzzy optimal control problem. The study is then concluded in Section 6.

2. Preliminaries

We now introduce the fuzzy variational problem (FVP), followed by the standard definitions and concepts used throughout the paper, including fuzzy numbers, fuzzy functions, their corresponding α-level functions, and some important concepts such as the continuity and integrability of fuzzy functions. The fuzzy variational problem is formulated as follows:

$$\begin{aligned} &\text{minimize} \qquad \tilde{\mathcal{I}}(\tilde{v}) := \int_{x_0}^{x_f} \tilde{h}(x, \tilde{v}(x), \dot{\tilde{v}}(x))\, dx, \\ &\text{subject to} \qquad \tilde{v}(x_0) \approx \tilde{v}_0, \qquad \tilde{v}(x_f) \approx \tilde{v}_f. \end{aligned} \tag{2.1}$$

Here, we wish to find the function $\tilde{v}$ that minimizes the functional $\tilde{\mathcal{I}}$ whose integrand $\tilde{h} : \Omega\times\mathcal{F}\times\mathcal{F} \to \mathcal{F}$ assigns a fuzzy number to each point $(x, \tilde{v}(x), \dot{\tilde{v}}(x))$, where $x \in \Omega = [x_0, x_f] \subseteq \mathbb{R}$, $\tilde{v}$ is the fuzzy-valued state function, $\dot{\tilde{v}}$ denotes its gH derivative, and $\mathcal{F}$ denotes the set of all fuzzy numbers on $\mathbb{R}$. Problems of the form (2.1) arise in fuzzy variational calculus when uncertainty or imprecision in the state variable is represented through fuzzy-valued functions [13]. Such problems are useful for modelling optimization problems in which the state of the system is not precisely known. In this direction, the imperfect production process model studied by Roul et al. [3] is considered in Section 5 as an application of the proposed results.

A fuzzy number is a fuzzy set $\tilde{A}$ whose membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0, 1]$ is normal, fuzzy convex and upper semicontinuous, the closure of the support of $\mu_{\tilde{A}}$ is compact [17].

Definition 2.1. (Fuzzy function [18]) Let $\Omega \subseteq \mathbb{R}$. A function $\tilde{v} : \Omega \to \mathcal{F}$ is said to be a fuzzy function if $\tilde{v}(x)$ represents a fuzzy number for every $x \in \Omega$.

The α-level set of $\tilde{v}(x)$ for $x \in \Omega$ is denoted by

$$v(x)[\alpha] = [v^l(x, \alpha), v^r(x, \alpha)],$$

where $v^l(x, \alpha) = \min_{\alpha\in[0,1]}\{v(x)[\alpha]\}$ and $v^r(x, \alpha) = \max_{\alpha\in[0,1]}\{v(x)[\alpha]\}$ are bounded functions that are monotonically increasing and decreasing functions in α, respectively. The continuity and integrability of fuzzy functions are expressed in terms of left and right endpoint functions [5, 6, 13, 19]. To proceed further, let us recall the definition of gH differentiable fuzzy functions, which is based on the concept of gH difference.

Definition 2.2. (gH difference [20]) The gH difference of two fuzzy numbers μ and ν, denoted by $\mu \ominus_{gH} \nu$, exists if there is a fuzzy number τ such that

1. $\mu = \nu + \tau$, or
2. $\nu = \mu + (-1)\tau$.

The α-level sets of τ are expressed as

$$\tau^l[\alpha] = \min\{\mu^l[\alpha] - \nu^l[\alpha], \mu^r[\alpha] - \nu^r[\alpha]\},$$

$$\tau^r[\alpha] = \max\{\mu^l[\alpha] - \nu^l[\alpha], \mu^r[\alpha] - \nu^r[\alpha]\}.$$

In general, the Hukuhara difference of μ and ν exists if μ contains a translation of ν that is essentially the part 1 of Definitions 2.2. The gH difference, on the other hand, allows for its definition to work in either direction.

Definition 2.3. (gH differentiable fuzzy function [10]) Let $\Omega \subseteq \mathbb{R}$ be an open set and $\tilde{v} : \Omega \to \mathcal{F}$ be a fuzzy function. A fuzzy function is said to be gH differentiable if

$$\dot{\tilde{v}}(x) = \lim_{h\to 0} \frac{\tilde{v}(x+h) \ominus_{gH} \tilde{v}(x)}{h}$$

exists.

In particular, if $v^l(x,\alpha)$ is an increasing function, and $v^r(x,\alpha)$ is a decreasing function of α for every $\alpha \in [0,1]$, then the gH derivative of $\tilde{v}$ expressed in its α-level set is

$$\begin{aligned} \dot{\tilde{v}}(x)[\alpha] &= \left[\lim_{h\to 0} \frac{v^l(x+h)[\alpha] - v^l(x)[\alpha]}{h}, \lim_{h\to 0} \frac{v^r(x+h)[\alpha] - v^r(x)[\alpha]}{h}\right] \\ &= \left[\dot{v}^l(x)[\alpha], \dot{v}^r(x)[\alpha]\right]. \end{aligned}$$

If $v^l(x,\alpha)$ is a decreasing function, and $v^r(x,\alpha)$ is an increasing function of α for every $\alpha \in [0,1]$, then the gH derivative of $\tilde{v}$ expressed in its α-level set is

$$\begin{aligned} \dot{\tilde{v}}(x)[\alpha] &= \left[\lim_{h\to 0} \frac{v^r(x+h)[\alpha] - v^r(x)[\alpha]}{h}, \lim_{h\to 0} \frac{v^l(x+h)[\alpha] - v^l(x)[\alpha]}{h}\right] \\ &= \left[\dot{v}^r(x)[\alpha], \dot{v}^l(x)[\alpha]\right]. \end{aligned}$$

We now recall a result by Bede and Stefanini [10] that expresses the relation of gH differentiable fuzzy functions in terms of its α-level set.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}$ be an open set. Let $\tilde{v} : \Omega \to \mathcal{F}$ be a fuzzy function such that its α-level set, $v(x)[\alpha] = [v^l(x)[\alpha], v^r(x)[\alpha]]$, where $v^l(x)[\alpha]$ and $v^r(x)[\alpha]$, are real-valued differentiable functions with respect to $x \in \Omega$ for all $\alpha \in [0,1]$. Then, the function $\tilde{v}(x)$ is gH differentiable at a fixed $x \in \Omega$ if and only if one of the following two cases holds:*

1. $\dot{v}^l(x)[\alpha]$ *is increasing, and* $\dot{v}^r(x)[\alpha]$ *is decreasing for every* $\alpha \in [0,1]$ *with* $\dot{v}^l(x)[\alpha] \leq \dot{v}^r(x)[\alpha]$.
2. $\dot{v}^r(x)[\alpha]$ *is increasing, and* $\dot{v}^l(x)[\alpha]$ *is decreasing for every* $\alpha \in [0,1]$ *with* $\dot{v}^r(x)[\alpha] \leq \dot{v}^l(x)[\alpha]$.

In particular, for every $\alpha \in [0,1]$, it is given as

$$\dot{v}_{gH}(x)[\alpha] = [\min\{\dot{v}^l(x)[\alpha], \dot{v}^r(x)[\alpha]\}, \max\{\dot{v}^l(x)[\alpha], \dot{v}^r(x)[\alpha]\}].$$

Our study is focused on fuzzy variational problems that involve minimizing functionals with integrands dependent on one independent variable x and two dependent variables, namely $\tilde{v}(x)$ and $\dot{\tilde{v}}(x)$. To proceed, we now introduce the definition of gH partial derivatives as follows:

Definition 2.4. (gH partial derivative) Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $\tilde{v} : \Omega \to \mathcal{F}$ be a fuzzy function, and $x = \{x_1, x_2, ..., x_i, .., x_n\} \in \Omega$. The function $\tilde{v}$ is said to have a gH partial derivative with respect to the ith variable if $\tilde{v}$ is gH differentiable at x_i. We denote this by $\dot{\tilde{v}}_i(x)$.

In this context, the gradient in the fuzzy setting is denoted by $\tilde{\nabla}$.

Definition 2.5. (Continuously gH differentiable fuzzy function) Given an open set $\Omega \subseteq \mathbb{R}^n$ and $x = \{x_1, x_2, ..., x_i, .., x_n\} \in \Omega$. A fuzzy function $\tilde{v} : \Omega \to \mathcal{F}$ is said to be continuously gH differentiable at x if all of the partial derivatives $\dot{\tilde{v}}_i$, for $i = 1, 2, ..., n$, exist and are continuous at x.

In particular, $\tilde{v}$ is said to be continuously gH differentiable on Ω if it is continuously gH differentiable for every $x \in \Omega$.

Another key factor to consider in the study of the existence of minimizers within variational problems is convexity. However, convexity is often too strong to be true in a fuzzy context, as evidenced by [21–24]. In contrast, preinvexity, a weaker notion than convexity, retains the lower bound property similar to that of convex functions and is sufficient for establishing the existence of minimizers. We recall the definition of preinvex functions, which are defined on invex sets.

Definition 2.6. (Invex set [25]) A set $\Omega \subseteq \mathcal{F}$ is said to be an invex set with respect to a function $\tilde{\zeta} : \Omega \times \Omega \to \mathcal{F}$ if

$$\tilde{v} + \lambda\tilde{\zeta}(\tilde{v}, \tilde{w}) \in \Omega$$

for each $\tilde{v}, \tilde{w} \in \Omega$ and $\lambda \in [0,1]$.

Definition 2.7. (Preinvex function [25]) Let $\Omega \subseteq \mathcal{F}$ be an invex set with respect to a function $\tilde{\zeta}$. A fuzzy function $\tilde{h} \colon \Omega \to \mathcal{F}$ is said to be preinvex on Ω if

$$\tilde{h}(\tilde{v} + \lambda\tilde{\zeta}(\tilde{v}, \tilde{w})) \leq \lambda\tilde{h}(\tilde{v}) + (1-\lambda)\tilde{h}(\tilde{w}) \tag{2.2}$$

for every $\tilde{v}, \tilde{v} \in \Omega$ and $\lambda \in [0,1]$. It is said to be strictly preinvex on Ω, if

$$\tilde{h}(\tilde{v} + \lambda\tilde{\zeta}(\tilde{v}, \tilde{w})) \prec \lambda\tilde{h}(\tilde{v}) + (1-\lambda)\tilde{h}(\tilde{w}) \tag{2.3}$$

holds for all $\tilde{v}, \tilde{w} \in \Omega$ and $\lambda \in [0,1]$.

Here, the relational operators $\preceq$ and $\prec$ refer to the partial ordering of the fuzzy number [18, 21]. In terms of α-level sets, (2.2) can be expressed as

$$h^l(v^l + \lambda\zeta^l(v^l, w^l), \alpha) \leq \lambda h^l(v^l, \alpha) + (1-\lambda)h^l(w^l, \alpha),$$

$$h^r(v^r + \lambda\zeta^r(v^r, w^r), \alpha) \leq \lambda h^r(v^r, \alpha) + (1-\lambda)h^r(w^r, \alpha),$$

for every $\lambda \in [0,1]$ and $\alpha \in [0,1]$.

In the fuzzy variational problem (2.1), it is assumed that $\tilde{h}$ is twice continuously gH differentiable with respect to all of its variables. Let us express the set of admissible curves as follows:

$$\begin{aligned}\tilde{V} = \{\tilde{v} \mid \tilde{v} &\text{ is twice continuously gH differentiable w.r.t } x \in [x_0, x_f],\\ &\tilde{v}(x_0) \approx \tilde{v}_0,\ \tilde{v}(x_f) \approx \tilde{v}_f\}.^*\end{aligned}$$

The corresponding fuzzy Euler–Lagrange equations derived by Fard et al. [14] for gH differentiable functions corresponding to the two cases in Definition 2.2 for the functional $\tilde{\mathcal{I}}$ in (2.1) are as follows:

Case (a)

$$\frac{\partial h^l}{\partial v^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^l}{\partial \dot{v}^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0, \tag{2.4}$$

$$\frac{\partial h^l}{\partial v^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^l}{\partial \dot{v}^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0, \tag{2.5}$$

$$\frac{\partial h^r}{\partial v^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^r}{\partial \dot{v}^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0, \tag{2.6}$$

$$\frac{\partial h^r}{\partial v^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^r}{\partial \dot{v}^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0. \tag{2.7}$$

Case (b)

$$\frac{\partial h^l}{\partial v^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^r}{\partial \dot{v}^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0, \tag{2.8}$$

$$\frac{\partial h^l}{\partial v^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^r}{\partial \dot{v}^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0, \tag{2.9}$$

$$\frac{\partial h^r}{\partial v^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^l}{\partial \dot{v}^l}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0, \tag{2.10}$$

$$\frac{\partial h^r}{\partial v^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha)) - \frac{d}{dx}\Big(\frac{\partial h^l}{\partial \dot{v}^r}(x,\alpha,v_*^l(x_\alpha),\dot{v}_*^l(x_\alpha),v_*^r(x_\alpha),\dot{v}_*^r(x_\alpha))\Big) = 0. \tag{2.11}$$

Here, for convenience, we denote $v_*^l(x,\alpha)$, $\dot{v}_*^l(x,\alpha)$, $v_*^r(x,\alpha)$, and $\dot{v}_*^r(x,\alpha)$ as $v_*^l(x_\alpha)$, $\dot{v}_*^l(x_\alpha)$, $v_*^r(x_\alpha)$, and $\dot{v}_*^r(x_\alpha)$, respectively.

3. Multivariable preinvex generalized Hukuhara differentiable function

In this section, the multivariable preinvex gH differentiable fuzzy functions are introduced and discussed. The preinvexity of the integrand in (2.1) with respect to its last two variables is defined as follows:

Let $\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \subseteq \mathcal{F} \times \mathcal{F}$ be an invex set with respect to a non-negative function $\tilde{\zeta} : (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \times (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \to \mathcal{F} \times \mathcal{F}$ such that

$$\Big((\tilde{v}_*, \dot{\tilde{v}}_*) + \lambda\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*))\Big) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}},$$

holds for each $(\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ and $\lambda \in [0, 1]$, where $\tilde{\zeta}$ is defined as

$$\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) = (\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*), \tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*)),$$

with well-defined maps $\tilde{\zeta}_1 : \Omega^1_{\mathcal{F}} \times \Omega^1_{\mathcal{F}} \to \mathcal{F}$ and $\tilde{\zeta}_2 : \Omega^2_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \to \mathcal{F}$.

In this context, $\tilde{\zeta}_2$, and $\tilde{\zeta}_1$ are selected in such a way that the gH antiderivative of $\tilde{\zeta}_2$ with respect to x is equal to $\tilde{\zeta}_1$, where $\tilde{v}, \dot{\tilde{v}}, \tilde{v}_*$, and $\dot{\tilde{v}}_*$ are functions of x. In simpler terms, the invexity of $\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ with respect to $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ is expressed as

$$(\tilde{v}_* + \lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*),\ \dot{\tilde{v}}_* + \lambda\tilde{\zeta}_2(\dot{\tilde{v}},\ \dot{\tilde{v}}_*)) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$$

for each $\lambda \in [0, 1]$. We say a fuzzy function $\tilde{v}_1 : \Omega \to \mathcal{F}$ is gH antiderivative of the fuzzy function $\tilde{v}_2 : \Omega \to \mathcal{F}$ if

$$\dot{\tilde{v}}_1 x = \tilde{v}_2 x$$

for every $x \in \Omega$, where $\Omega \subseteq \mathbb{R}$. We shall henceforth denote the fuzzy function $\tilde{v}(x)$ as $\tilde{v}$.

Invexity and preinvexity for fuzzy-valued functions have been studied by [25]. However, their use in the setting of gH differentiable fuzzy-valued integrands, particularly for fuzzy variational problems involving both the state variable and its gH derivative, needs to be formulated carefully. Motivated by this gap, we introduce the following notions.

Definition 3.1. (Multivariable preinvex gH differentiable function) Let $\Omega \subseteq \mathbb{R}$ and $\Omega^1_{\mathcal{F}}, \Omega^2_{\mathcal{F}} \subseteq \mathcal{F}$. A fuzzy function $\tilde{h} : \Omega \times \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \to \mathcal{F}$ is said to be preinvex gH differentiable in last the two variables with respect to $\tilde{\zeta} : (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \times (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \to \mathcal{F} \times \mathcal{F}$ if

$$\tilde{h}(x, \tilde{v}_* + \lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*), \dot{\tilde{v}}_* + \lambda\tilde{\zeta}_2(\dot{\tilde{v}},\ \dot{\tilde{v}}_*)) \preceq \lambda\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) + (1 - \lambda)\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*),$$

for each $(\tilde{v}, \dot{\tilde{v}})$ and $(\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ and $\lambda \in [0, 1]$, where $\tilde{\zeta}_1$ is a gH antiderivative of $\tilde{\zeta}_2$.

Definition 3.2. (Strict multivariable preinvex gH differentiable function) Let $\Omega \subseteq \mathbb{R}$ and $\Omega^1_{\mathcal{F}}, \Omega^2_{\mathcal{F}} \subseteq \mathcal{F}$. A fuzzy function $\tilde{h} : \Omega \times \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \to \mathcal{F}$ is said to be strictly preinvex gH differentiable in the last two variables with respect to $\tilde{\zeta} : (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \times (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \to \mathcal{F} \times \mathcal{F}$ if

$$\tilde{h}(x, \tilde{v}_* + \lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*), \dot{\tilde{v}}_* + \lambda\tilde{\zeta}_2(\dot{\tilde{v}},\ \dot{\tilde{v}}_*)) \prec \lambda\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) + (1 - \lambda)\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*),$$

for each $(\tilde{v}, \dot{\tilde{v}})$ and $(\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$, $\lambda \in [0, 1]$, where $\tilde{\zeta}_1$ is a gH antiderivative of $\tilde{\zeta}_2$.

Mohan and Neogy [26] proposed Condition A, which ensures that any function defined over an invex set possesses the property of preinvexity.

Let $\Omega^1_{\mathcal{F}}, \Omega^2_{\mathcal{F}} \subseteq \mathcal{F}$. A function $\tilde{\zeta} : (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \times (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \to \mathcal{F} \times \mathcal{F}$ satisfies Condition A if

$$\tilde{\zeta}\Big((\tilde{v}_*, \dot{\tilde{v}}_*), (\tilde{v}_*, \dot{\tilde{v}}_*) + \lambda\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*))\Big) = -\lambda\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)), \tag{3.1}$$

$$\tilde{\zeta}\Big((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*) + \lambda\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*))\Big) = (1 - \lambda)\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) \tag{3.2}$$

holds for each $(\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ and $\lambda \in [0, 1]$.

Alternatively, Condition A admits the following equivalent representation in terms of $\tilde{\zeta}_1, \tilde{\zeta}_2 : \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \to \mathcal{F}$ as follows:

$$\tilde{\zeta}_1(\tilde{v}_*, \tilde{v}_* + \lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*)) = -\lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*), \tag{3.3}$$
$$\tilde{\zeta}_1(\tilde{v}, \tilde{v}_* + \lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*)) = (1-\lambda)\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*), \tag{3.4}$$

and

$$\tilde{\zeta}_2(\dot{\tilde{v}}_*, \dot{\tilde{v}}_* + \lambda\tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*)) = -\lambda\tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*), \tag{3.5}$$
$$\tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_* + \lambda\tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*)) = (1-\lambda)\tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*) \tag{3.6}$$

for each $(\tilde{v}, \dot{\tilde{v}})$, $(\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ and $\lambda \in [0,1]$. We now establish a lower-bound property for multivariable preinvex gH differentiable fuzzy functions.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}$ and $\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \subseteq \mathcal{F} \times \mathcal{F}$ be an invex set with respect to a non-negative function $\tilde{\zeta} : (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \times (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \to \mathcal{F} \times \mathcal{F}$. Let $\tilde{h} : \Omega \times \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \to \mathcal{F}$ be a gH differentiable fuzzy function. Then, $\tilde{h}$ is preinvex gH differentiable in the last two variables with respect to $\tilde{\zeta}$ if and only if*

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) \succeq \tilde{\nabla}\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*)^T \tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) + \tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \tag{3.7}$$

holds for every $(\tilde{v}, \dot{\tilde{v}})$, $(\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$.

Proof. Assume that $\tilde{h}$ is preinvex gH differentiable in the last two variables with respect to $\tilde{\zeta}$ for every $(\tilde{v}, \dot{\tilde{v}})$, $(\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$. Hence,

$$h^l(x, \alpha, v^l_* + \lambda\zeta^l_1(v^l, v^l_*), \dot{v}^l_* + \lambda\zeta^l_2(\dot{v}^l,\ \dot{v}^l_*)) \le \lambda h^l(x, \alpha, v^l, \dot{v}^l) + (1-\lambda)h^l(x, \alpha, {v^l}_*, \dot{v}^l_*) \tag{3.8}$$

and

$$h^r(x, \alpha, v^r_* + \lambda\zeta^r_1(v^r, v^r_*), \dot{v}^r_* + \lambda\zeta^r_2(\dot{v}^r,\ \dot{v}^r_*)) \le \lambda h^r(x, \alpha, v^r, \dot{v}^r) + (1-\lambda)h^r(x, \alpha, {v^r}_*, \dot{v}^r_*). \tag{3.9}$$

Equations (3.8) and (3.9) can be written as

$$h^l(x, \alpha, v^l, \dot{v}^l) - h^l(x, \alpha, v^l_*, \dot{v}^l_*) \ge \frac{1}{\lambda}[h^l(x, \alpha, v^l_* + \lambda\zeta^l_1(v^l, v^l_*), \dot{v}^l_* + \lambda\zeta^l_2(\dot{v}^l,\ \dot{v}^l_*)) - h^l(x, \alpha, v^l_*, \dot{v}^l_*)] \tag{3.10}$$

and

$$h^r(x, \alpha, v^r, \dot{v}^r) - h^r(x, \alpha, v^r_*, \dot{v}^r_*) \ge \frac{1}{\lambda}[h^r(x, \alpha, v^r_* + \lambda\zeta^r_1(v^r, v^r_*), \dot{v}^r_* + \lambda\zeta^r_2(\dot{v}^r,\ \dot{v}^r_*)) - h^r(x, \alpha, v^r_*, \dot{v}^r_*)]. \tag{3.11}$$

Using Definition 2.3, we get

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) - \tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \succeq \frac{1}{\lambda}[\tilde{h}(x, \tilde{v}_* + \lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*), \dot{\tilde{v}}_* + \lambda\tilde{\zeta}_2(\dot{\tilde{v}},\ \dot{\tilde{v}}_*)) \ominus_{gH} \tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*)]. \tag{3.12}$$

Now, because $\tilde{h}$ is gH differentiable and $\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ is invex with respect to $\tilde{\zeta}$, we deduce that

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) - \tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \succeq \tilde{\nabla}\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*)^T \tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) \tag{3.13}$$

or

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) \succeq \tilde{\nabla}\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*)^T \tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) + \tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*). \tag{3.14}$$

Hence, (3.7) is obtained.

To prove the converse, suppose that (3.7) is true. Let $(\tilde{v}, \dot{\tilde{v}})$, $(\tilde{v}_*, \dot{\tilde{v}}_*) \in \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ and $(\tilde{v}^*, \dot{\tilde{v}}^*) = (\tilde{v}_* + \lambda\zeta_1(\tilde{v}, \tilde{v}_*), \dot{\tilde{v}}_* + \lambda\zeta_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*))$, where $\lambda \in [0, 1]$. Now, (3.7) can be written as

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) \geq \tilde{\nabla}\tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*)^T\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}^*, \dot{\tilde{v}}^*)) + \tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*) \tag{3.15}$$

and

$$\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \geq \tilde{\nabla}\tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*)^T\tilde{\zeta}((\tilde{v}_*, \dot{\tilde{v}}_*), (\tilde{v}^*, \dot{\tilde{v}}^*)) + \tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*). \tag{3.16}$$

Invoking Condition A on (3.15) and (3.16) leads to

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) \geq (1 - \lambda)\tilde{\nabla}\tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*)^T\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) + \tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*) \tag{3.17}$$

and

$$\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \geq (-\lambda)\tilde{\nabla}\tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*)^T\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) + \tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*). \tag{3.18}$$

Multiplying (3.17) with λ and (3.18) with $(1 - \lambda)$ and adding them gives

$$\lambda\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) + (1 - \lambda)\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \geq \tilde{h}(x, \tilde{v}^*, \dot{\tilde{v}}^*); \tag{3.19}$$

that is,

$$\lambda\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) + (1 - \lambda)\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \geq \tilde{h}((x, \tilde{v}_*, \dot{\tilde{v}}_*) + \lambda\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)). \tag{3.20}$$

Hence, $\tilde{h}$ is preinvex gH differentiable, and this completes the proof. □

Furthermore, (3.7) can be written as follows:

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) \geq \tilde{h}_{\tilde{v}}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \cdot \tilde{\zeta}_1(\tilde{v}, \tilde{v}_*) + \tilde{h}_{\dot{\tilde{v}}}(x, \tilde{v}_*, \dot{\tilde{v}}_*) \cdot \tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*) + \tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*). \tag{3.21}$$

4. Sufficient conditions for the existence and uniqueness of the minimizer

In this section, we prove sufficient conditions for the existence and uniqueness of the minimizer of the fuzzy variational problem (2.1).

Theorem 4.1. *Given the integrand $\tilde{h} \in C^2_{gH}(\Omega, \mathcal{F})$ in (2.1). Suppose that $(\tilde{v}, \dot{\tilde{v}}) \to \tilde{h}(x, \tilde{v}, \dot{\tilde{v}})$ is a preinvex gH differentiable function in the last two arguments with respect to $\tilde{\zeta}$ for every $x \in [x_0, x_f]$ with $\tilde{\zeta}((\tilde{v}(x), \dot{\tilde{v}}(x)), (\tilde{v}_*(x), \dot{\tilde{v}}_*(x))) = (\tilde{0}, \tilde{0})$ at $x = x_0$ and $x = x_f$ for all fuzzy functions $\tilde{v}(x), \tilde{v}_*(x) \in \tilde{V}$. Then, $\tilde{v}_*$ is a minimizer of the fuzzy variational problem (2.1) if it satisfies the fuzzy Euler–Lagrange equations (2.4)–(2.11).*

Proof. Let $\Omega = [x_0, x_f] \subseteq \mathbb{R}$ and $\Omega^1_{\mathcal{F}}, \Omega^2_{\mathcal{F}} \subseteq \mathcal{F}$ such that $\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}$ is a fuzzy invex set with respect to a non-negative function $\tilde{\zeta}$ satisfying Condition A, where $\tilde{\zeta} : (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \times (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \to \mathcal{F} \times \mathcal{F}$ is defined as follows:

$$\begin{aligned}\tilde{\zeta}((\tilde{v}(x), \dot{\tilde{v}}(x)), (\tilde{v}_*x, \dot{\tilde{v}}_*x)) &= (\tilde{\zeta}_1(\tilde{v}(x), \tilde{v}_*x), \tilde{\zeta}_2(\dot{\tilde{v}}(x), \dot{\tilde{v}}_*x)) \\ &= (\tilde{0}, \tilde{0})\end{aligned} \tag{4.1}$$

at $x = x_0$ and $x = x_f$ for all admissible curves. Now, because $\tilde{h} : \Omega \times \Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}} \to \mathcal{F}$ is fuzzy preinvex in the last two variables, by Definition 3.1 and Eq (3.21),

$$\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) \geq \tilde{\nabla}\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*)^T\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) + \tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*),$$

or

$$\tilde{h}(x,\tilde{v},\dot{\tilde{v}}) \geq \tilde{h}_{\tilde{v}}(x,\tilde{v}_*,\dot{\tilde{v}}_*)\cdot\tilde{\zeta}_1(\tilde{v},\tilde{v}_*) + \tilde{h}_{\dot{\tilde{v}}}(x,\tilde{v}_*,\dot{\tilde{v}}_*)\cdot\tilde{\zeta}_2(\dot{\tilde{v}},\dot{\tilde{v}}_*) + \tilde{h}(x,\tilde{v}_*.\dot{\tilde{v}}_*). \tag{4.2}$$

Because the inequality in (4.2) is a fuzzy inequality, it is interpreted through the corresponding α-level endpoint functions. Thus, for each fixed $\alpha \in [0,1]$, the fuzzy inequality gives the left and right real-valued endpoint inequalities. These endpoint inequalities can then be integrated over $[x_0, x_f]$ for every $\tilde{v},\tilde{v}_* \in \tilde{V}$. Integrating Eq (4.2) gives the following possible cases:

$$\mathcal{I}^l(v^l) \geq \mathcal{I}^l(v^l{}_*) + \int_{x_0}^{x_f}\Big[h^l_{v^l}(x,\alpha,v^l{}_*,\dot{v}^l_*)\zeta^l_1(v^l,v^l{}_*) + h^l_{\dot{v}^l}(x,\alpha,v^l_*,\dot{v}^l_*)\zeta^l_2(\dot{v}^l,\dot{v}^l_*)\Big]dx$$

and

$$\mathcal{I}^r(v^r) \geq \mathcal{I}^r(v^r{}_*) + \int_{x_0}^{x_f}\Big[h^r_{v^r}(x,\alpha,v^r{}_*,\dot{v}^r_*)\zeta^r_1(v^r,v^r{}_*) + h^r_{\dot{v}^r}(x,\alpha,v^r_*,\dot{v}^r_*)\zeta^r_2(\dot{v}^r,\dot{v}^r_*)\Big]dx.$$

Because $\tilde{\zeta}_1$ is assumed to be the gH antiderivative of $\tilde{\zeta}_2$ with respect to x, the corresponding endpoint functions satisfy

$$\frac{d}{dx}\zeta^l_1(v^l,v^l_*) = \zeta^l_2(\dot{v}^l,\dot{v}^l_*), \qquad \frac{d}{dx}\zeta^r_1(v^r,v^r_*) = \zeta^r_2(\dot{v}^r,\dot{v}^r_*).$$

Therefore, the second term in each endpoint inequality can be rewritten in terms of the derivative of ζ_1. Applying integration by parts and using the endpoint condition in (4.1), the boundary terms involving ζ_1 vanish at $x = x_0$ and $x = x_f$. Hence, we obtain

$$\mathcal{I}^l(v^l) \geq \mathcal{I}^l(v^l_*) + \int_{x_0}^{x_f}\Big[h^l_{v^l}(x,\alpha,v^l_*,\dot{v}^l_*) - \frac{d}{dx}\Big(h^l_{\dot{v}^l}(x,\alpha,v^l_*,\dot{v}^l_*)\Big)\Big]\zeta^l_1(v^l,v^l{}_*)dx,$$

$$\mathcal{I}^r(v^r) \geq \mathcal{I}^r(v^r_*) + \int_{x_0}^{x_f}\Big[h^r_{v^r}(x,\alpha,v^r_*,\dot{v}^r_*) - \frac{d}{dx}\Big(h^r_{\dot{v}^r}(x,\alpha,v^r_*,\dot{v}^r_*)\Big)\Big]\zeta^r_1(\dot{v}^r,\dot{v}^r_*)dx,$$

$$\mathcal{I}^l(v^l) \geq \mathcal{I}^l(v^l_*) + \int_{x_0}^{x_f}\Big[h^l_{v^r}(x,\alpha,v^r_*,\dot{v}^r_*) - \frac{d}{dx}\Big(h^l_{\dot{v}^r}(x,\alpha,v^r_*,\dot{v}^r_*)\Big)\Big]\zeta^l_1(\dot{v}^l,\dot{v}^l_*)dx,$$

$$\mathcal{I}^r(v^r) \geq \mathcal{I}^r(v^r_*) + \int_{x_0}^{x_f}\Big[h^r_{v^l}(x,\alpha,v^r_*,\dot{v}^r_*) - \frac{d}{dx}\Big(h^r_{\dot{v}^l}(x,\alpha,v^r_*,\dot{v}^r_*)\Big)\Big]\zeta^r_1(\dot{v}^r,\dot{v}^r_*)dx,$$

$$\mathcal{I}^l(v^l) \geq \mathcal{I}^l(v^l_*) + \int_{x_0}^{x_f}\Big[h^l_{v^l}(x,\alpha,v^r_*,\dot{v}^r_*) - \frac{d}{dx}\Big(h^r_{\dot{v}^l}(x,\alpha,v^r_*,\dot{v}^r_*)\Big)\Big]\zeta^l_1(\dot{v}^l,\dot{v}^l_*)dx,$$

$$\mathcal{I}^l(v^l) \geq \mathcal{I}^l(v^l_*) + \int_{x_0}^{x_f}\Big[h^l_{v^r}(x,\alpha,v^r_*,\dot{v}^r_*) - \frac{d}{dx}\Big(h^r_{\dot{v}^r}(x,\alpha,v^r_*,\dot{v}^r_*)\Big)\Big]\zeta^l_1(\dot{v}^l,\dot{v}^l_*)dx,$$

$$\mathcal{I}^r(v^r) \geq \mathcal{I}^r(v^r_*) + \int_{x_0}^{x_f}\Big[h^r_{v^l}(x,\alpha,v^r_*,\dot{v}^r_*) - \frac{d}{dx}\Big(h^l_{\dot{v}^l}(x,\alpha,v^r_*,\dot{v}^r_*)\Big)\Big]\zeta^r_1(\dot{v}^r,\dot{v}^r_*)dx,$$

and

$$\mathcal{I}^r(v^r) \geq \mathcal{I}^r(v^r_*) + \int_{x_0}^{x_f}\Big[h^r_{v^r}(x,\alpha,v^r_*,\dot{v}^r_*) - \frac{d}{dx}\Big(h^l_{\dot{v}^r}(x,\alpha,v^r_*,\dot{v}^r_*)\Big)\Big]\zeta^r_1(\dot{v}^r,\dot{v}^r_*)dx.$$

The above endpoint inequalities are obtained by applying the same integration by parts argument to the left and right endpoint functions and to the remaining endpoint combinations arising from the two possible cases of gH differentiability. Each bracketed term corresponds to one of the fuzzy Euler–Lagrange expressions in (2.4)–(2.11). Hence, if $\tilde{v}_*$ satisfies the fuzzy Euler–Lagrange equations (2.4)–(2.11), all such bracketed terms vanish. Consequently, we obtain

$$I^l(v^l) \geq I^l(v^l_*)$$

and

$$I^r(v^r) \geq I^r(v^r_*).$$

Therefore,

$$\tilde{I}(\tilde{v}) \succeq \tilde{I}(\tilde{v}_*).$$

Because this inequality holds for every admissible fuzzy function $\tilde{v} \in \tilde{V}$, it follows that $\tilde{v}_*$ is a minimizer of the fuzzy variational problem (2.1). This completes the proof. □

Next, we would prove the uniqueness of the minimizer of the considered fuzzy variational problem in the following theorem.

Theorem 4.2. *Given that the integrand $\tilde{h}$ in (2.1) satisfies the condition of Theorem 4.1 and $(\tilde{v}, \dot{\tilde{v}}) \to \tilde{h}(x, \tilde{v}, \dot{\tilde{v}})$ is strictly preinvex gH differentiable in the last two variables with respect to $\tilde{\zeta}$ for every $x \in [x_0, x_f]$. Then, the minimizer of the fuzzy variational problem (2.1), if it exists, is unique.*

Proof. The proof proceeds by contradiction. Assume that $\tilde{h}$ is strictly preinvex and that there exists two distinct solutions of (2.1), namely $\tilde{v}_1(x)$ and $\tilde{v}_2(x)$, corresponding to the minimum value, say $\tilde{m} \in \mathcal{F}$. Define

$$\begin{aligned}(\tilde{v}_3, \dot{\tilde{v}}_3) &= \left((\tilde{v}_2, \dot{\tilde{v}}_2) + \lambda\tilde{\zeta}((\tilde{v}_1, \dot{\tilde{v}}_1), (\tilde{v}_2, \dot{\tilde{v}}_2))\right) \\ &= (\tilde{v}_2 + \lambda\tilde{\zeta}_1(\tilde{v}_1, \tilde{v}_2), \dot{\tilde{v}}_2 + \lambda\tilde{\zeta}_2(\dot{\tilde{v}}_1,\ \dot{\tilde{v}}_2)).\end{aligned}$$

Because $\tilde{h}(x, \tilde{v}, \dot{\tilde{v}})$ is strictly preinvex gH differentiable, the following holds (see Definition 3.2):

$$\lambda\tilde{h}(x, \tilde{v}_1, \dot{\tilde{v}}_1) + (1 - \lambda)\tilde{h}(x, \tilde{v}_2, \dot{\tilde{v}}_2) \succ \tilde{h}(x, \tilde{v}_3, \dot{\tilde{v}}_3). \tag{4.3}$$

Now, integrating Eq (4.3) leads to

$$\tilde{m} \approx \lambda\tilde{\mathcal{I}}(\tilde{v}_1 x) + (1 - \lambda)\tilde{\mathcal{I}}(\tilde{v}_2 x) \succ \tilde{\mathcal{I}}(\tilde{v}_3 x) \succeq \tilde{m}.$$

Hence,

$$\int_{x_0}^{x_f} \left[\lambda\tilde{h}(x, \tilde{v}_1, \dot{\tilde{v}}_1) + (1 - \lambda)\tilde{h}(x, \tilde{v}_2, \dot{\tilde{v}}_2) - \tilde{h}(x, \tilde{v}_2 + \lambda\tilde{\zeta}_1(\tilde{v}_1, \tilde{v}_2), \dot{\tilde{v}}_2 + \tilde{\zeta}_2(\dot{\tilde{v}}_1,\ \dot{\tilde{v}}_2))\right] dx \approx \tilde{0}.$$

This leads to a contradiction, as $\tilde{h}$ is assumed to be strictly preinvex. Therefore, $\tilde{v}_1(x) = \tilde{v}_2(x)$. □

5. Application of Theorem 4.1 to a fuzzy optimal control problem

We now demonstrate how Theorem 4.1 can be utilized in a fuzzy optimal control setting. The problem under consideration involves dynamic production planning under uncertainty, where parameters such as production rate and stock levels are inherently imprecise due to factors such as fluctuating labor efficiency, variable raw material quality, and machine breakdowns. This particular fuzzy optimal control problem, which was considered in [3], aims to minimize total production costs including development costs, holding costs, and the impact of defective items. Its system dynamics and cost structure are given as follows:

$$\tilde{\mathcal{I}}_{t\in(x_0,x_f)} = \int_{x_0}^{x_f} \Big(\tilde{J}\tilde{v}(x) + \tilde{c}_1 + \Big(\frac{c_t}{1-\delta} - p\delta\Big)(\dot{\tilde{v}}(x) + d_0 - ae^{-bt}) + \frac{\beta}{(1-\delta)^2}(\dot{\tilde{v}}(x) + d_0 - ae^{-bt})^2\Big)dx. \quad (5.1)$$

Here, $\tilde{v}(x)$ represents the state variable corresponding to the fuzzy stock level. The term $\tilde{J}$ denotes the fuzzy holding cost per unit of storage, and $\tilde{c}_1$ corresponds to the fuzzy development cost, incorporating labor and technology. The parameter β accounts for wear and tear, δ indicates the fraction of defective production, and p refers to the selling price of the defective items. The constants a and b are assumed to be positive, c_t denotes the constant material cost, and d_0 specifies the constant demand at the initial stage. Moreover, the function $\tilde{v}$ is taken to be a non-negative cost function represented by a triangular fuzzy number, and all constants involved are assumed to be non-negative. The study in [3] addressed this problem using numerical methods, under the assumption that the fuzzy state variable $\tilde{v}(x)$ admits the following form as a candidate solution:

$$\tilde{v}_*x = \frac{a}{b}(1 - e^{-bx_0}) + \frac{ax_0}{bx_f}(e^{-bx_f} - 1) + \frac{(1-\delta)^2\tilde{J}x_0}{4\beta}(x_0 - x_f). \quad (5.2)$$

To analytically validate Theorem 4.1 for this fuzzy optimal control problem, we note that $\tilde{v}_*x$, as proposed in [3], was derived based on the assumption that it satisfies the fuzzy Euler–Lagrange equations. Thus, the remaining step is to verify that the integrand is preinvex gH differentiable with respect to the last two variables. Establishing this property ensures that the conditions of Theorem 4.1 are satisfied, thereby confirming that $\tilde{v}_*$ is indeed a global minimizer of the problem. Let the integrand in (5.1) be denoted by $\tilde{g}(x, \tilde{v}, \dot{\tilde{v}})$:

$$\tilde{g}(x, \tilde{v}, \dot{\tilde{v}}) = \tilde{J}\tilde{v}(x) + \tilde{c}_1 + \Big(\frac{c_t}{1-\delta} - p\delta\Big)(\dot{\tilde{v}}(x) + d_0 - ae^{-bt}) + \frac{\beta}{(1-\delta)^2}(\dot{\tilde{v}}(x) + d_0 - ae^{-bt})^2.$$

Let the function $\tilde{\zeta} : (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \times (\Omega^1_{\mathcal{F}} \times \Omega^2_{\mathcal{F}}) \to \mathcal{F} \times \mathcal{F}$ be defined as

$$\tilde{\zeta}((\tilde{v}, \dot{\tilde{v}}), (\tilde{v}_*, \dot{\tilde{v}}_*)) = (\tilde{v} - \tilde{v}_*, \dot{\tilde{v}} - \dot{\tilde{v}}_*),$$

so that $\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*) = \tilde{v} - \tilde{v}_*$ and $\tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*) = \dot{\tilde{v}} - \dot{\tilde{v}}_*$ satisfy Condition A. Now, by Definition 3.1, $\tilde{g}$ is preinvex gH differentiable if

$$\tilde{g}(x, \tilde{v}_* + \lambda\tilde{\zeta}_1(\tilde{v}, \tilde{v}_*), \dot{\tilde{v}}_* + \lambda\tilde{\zeta}_2(\dot{\tilde{v}}, \dot{\tilde{v}}_*)) \leq \lambda\tilde{h}(x, \tilde{v}, \dot{\tilde{v}}) + (1-\lambda)\tilde{h}(x, \tilde{v}_*, \dot{\tilde{v}}_*), \quad \forall \lambda \in [0, 1]. \quad (5.3)$$

Let us verify this via the α-level sets. Denote the left endpoint as $f^l(x, \alpha, v^l, \dot{v}^l)$, expressed as:

$$g^l = J^l v^l + C_1\dot{v}^l + C_2(\dot{v}^l)^2 + C_3e^{-bt} + C_4e^{-2bt} + C_5e^{-bt}\dot{v}^l + c_1^l + C_6,$$

with constants

$$
\begin{aligned}
C_1 &= \left(\frac{c_t}{1-\delta} - p\delta\right) + \frac{2\beta d_0}{(1-\delta)^2},\\
C_2 &= \frac{\beta}{(1-\delta)^2},\\
C_3 &= -a\left(\frac{c_t}{1-\delta} - p\delta\right) - \frac{2a\beta d_0}{(1-\delta)^2},\\
C_4 &= \frac{a^2\beta}{(1-\delta)^2},\\
C_5 &= \frac{-2a\beta}{(1-\delta)^2},\\
C_6 &= \left(\frac{c_t}{1-\delta} - p\delta\right)d_0 + \frac{d_0^2\beta}{(1-\delta)^2}.
\end{aligned}
$$

We evaluate

$$g^l(x, \alpha, v_*^l + \lambda(v^l - v_*^l), \dot{v}_*^l + \lambda(\dot{v}^l - \dot{v}_*^l)),$$

and aim to prove

$$g^l(x, \alpha, v_*^l + \lambda\zeta_1^l, \dot{v}_*^l + \lambda\zeta_2^l) \le \lambda g^l(x, \alpha, v^l, \dot{v}^l) + (1-\lambda)g^l(x, \alpha, v_*^l, \dot{v}_*^l). \tag{5.4}$$

Note that we have the following:

- g^l is linear in v^l, quadratic in $\dot{v}^l$, and includes convex exponential terms.
- $\zeta_1^l = v^l - v_*^l, \zeta_2^l = \dot{v}^l - \dot{v}_*^l$ are linear in arguments.
- All components of g^l preserve convexity under convex combinations.

Thus, g^l satisfies Inequality (5.4) and is preinvex. Similarly, the right endpoint function g^r also satisfies:

$$g^r(x, \alpha, v_*^r + \lambda\zeta_1^r, \dot{v}_*^r + \lambda\zeta_2^r) \le \lambda g^r(x, \alpha, v^r, \dot{v}^r) + (1-\lambda)g^r(x, \alpha, v_*^r, \dot{v}_*^r),$$

confirming that the fuzzy function $\tilde{g}$ is preinvex in the gH sense.

6. Conclusions

This paper has addressed the existence of minimizers for fuzzy variational problems with gH differentiable functions that satisfy a weaker convexity condition, namely preinvexity. We first introduced the concept of multivariable preinvex gH differentiable functions and then analyzed their lower-bound properties. Our analysis showed that preinvexity is sufficient for establishing the existence and uniqueness of minimizers for fuzzy variational problems that satisfy the fuzzy Euler-Lagrange equations. The multivariable preinvex gH differentiability framework applies to a broader class of fuzzy functions than those admissible under standard differentiability frameworks, thereby allowing the existence of minimizers to be established for a wider range of fuzzy variational problems.

Author contributions

Mansi Verma: Conceptualization, methodology, formal analysis, writing–original draft, writing–review & editing; Chuei Yee Chen: Conceptualization, methodology, validation, writing–review & editing, supervision, funding acquisition; Sie Long Kek: Validation, writing–review. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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