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Research article

Integrable flows and global closure properties of spacelike curves in the Anti-de Sitter space AdS_3

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Abstract: A geometric framework for integrable arc-length-preserving flows of spacelike curves in the Anti-de Sitter space (AdS_3) was constructed. In the Frenet–Serret formalism, the constant negative background curvature added the position-vector term $\boldsymbol{\gamma}$ to the structural equation, $\partial_s T = \kappa N_1 + \boldsymbol{\gamma}$. Via a zero-curvature representation, this yielded local evolution equations for the curvature κ and torsion τ: the Betchov–Da Rios system for non-planar curves and a modified Korteweg-de Vries (mKdV) equation for planar ones. Globally, the monodromy matrix provided a topological invariant for closed loops. For helices of constant curvature with characteristic frequencies ω_1, ω_2, closure imposed the quantization conditions

$$\omega_1 L = 2\pi m, \qquad \omega_2 L = 2\pi n \qquad (m, n \in \mathbb{Z}),$$

which were preserved under the integrable flows and labeled the disconnected components of the moduli space. This monodromy invariant directly connected the geometry of spacelike curves in AdS_3 to integrable systems and holographic observables.

Keywords: Anti-de Sitter space; spacelike curves; integrable systems; Betchov–Da Rios equation; modified KdV equation
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1. Introduction

Three-dimensional anti-de Sitter space AdS_3 is a Lorentzian manifold of constant negative curvature and provides the background for the AdS/CFT(Conformal Field Theory) correspondence [1–4]. Curves in AdS_3 are classified by their causal character. Null curves, describing lightlike propagation, have been extensively studied in connection with the Korteweg-de Vries (KdV) hierarchy [5, 6]. Spacelike curves arise naturally in string dynamics [7, 8] and holographic geometry [1].

For null curves in Lorentzian geometry, Musso and Nicolodi [5] showed that Hamiltonian flows in

Minkowski space $\mathbb{R}^{2,1}$ generate the KdV hierarchy. Musso and Pampano [6] extended this to AdS_3, establishing a correspondence with star-shaped curves in the centro-affine plane [9]. More recently, they constructed geometric transformations on null curves in AdS_3 that induce the Bäcklund transformation for KdV [10]. For non-inflectional, canonically-parameterized null curves in AdS_3, the bending is the only local geometric invariant. These curves admit canonical spinor-frame lifts into $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, and their LIEN integrable flows yield the KdV equation for the bending.

Unlike null curves, which possess only one invariant, a non-geodesic spacelike curve in AdS_3 admits two independent invariants: curvature κ and torsion τ. The methods developed for null curves therefore do not apply directly. For spacelike curves, previous work has focused on elastic curves [11], extremal surfaces [12], and singularity analysis [13]; the integrable-geometric aspects have received less attention in this setting. Bäcklund transformations, which are fundamental in the theory of surfaces of constant negative curvature [14, 15] and integrable systems [16], have recently appeared in AdS string dynamics [17], but their implications for the global geometry of spacelike curves in AdS_3 have not been examined.

The connection between curve evolution and integrable systems originates with Hasimoto's work [18], which linked the vortex filament flow to the nonlinear Schrödinger equation. Subsequent studies established correspondences between planar curve motions and the KdV and modified KdV hierarchies [19, 20], with Hamiltonian structures clarified in [21]. These ideas extend to affine and centro-affine geometries [22, 23]. More recently, curve evolution has been studied using modified orthogonal frames [24], and Sağlam et al. [25] examined V_i-helices in Euclidean space via F-constant vector fields. Parallel developments have explored fractal and fractional generalizations of curve geometry: the two-scale fractal dimension [26] provides a scale-dependent framework for characterizing multiscale irregularity across distinct observation scales, while the application of local fractional calculus in [27] yields modified dispersive evolution equations that may be viewed as fractional counterparts of the integrable equations above.

Benson and Valiquette [28] derived local evolution equations for spacelike curves in Cayley–Klein geometries using equivariant moving frames. Their analysis, however, was local in nature. In this paper we go beyond the local picture by providing a global treatment of closed spacelike curves, with the monodromy of the Frenet connection as the central object. In the Frenet formalism, the constant negative curvature of AdS_3 appears as the $+\gamma$ term in $\partial_s T = \kappa N_1 + \gamma$; incorporating this term into the zero-curvature representation yields local evolution equations for κ and τ, reducing to the Betchov–Da Rios system in the non-planar case and to a modified KdV equation in the planar case. This derivation reveals the geometric origin of the curvature-induced modifications and sets the stage for the global analysis.

The main results concern global properties of closed spacelike curves. For constant-curvature helices, we classify the Frenet holonomy into elliptic, hyperbolic, and parabolic regimes and show that closure in the elliptic regime is equivalent to the quantized conditions $\omega_1 L = 2\pi m$ and $\omega_2 L = 2\pi n$ for integers m, n (Theorem 4.1). We prove that under any integrable arc-length-preserving flow, the monodromy matrix preserves its $SO_0(2,2)$-conjugacy class, where $SO_0(2,2)$ denotes the identity component of the isometry group of AdS_3 (isomorphic to $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\{\pm(I,I)\}$). Consequently, if a curve is initially closed, it remains closed throughout the evolution. We further show that the moduli space of closed curves within a fixed conjugacy class has multiple disconnected components labeled by (m,n), with continuous deformations inside each component

(Proposition 4.4), and we construct non-constant curvature closed curves analytically via elliptic functions and numerically via a Newton–Raphson continuation scheme (Section 4.3). A holographic interpretation of the monodromy invariant is also indicated.

The paper is organized as follows. Section 2 presents the necessary geometric preliminaries: the pseudo-Euclidean space $\mathbb{R}^{2,2}$, the identification of AdS_3 with the universal cover of $\mathrm{SL}(2,\mathbb{R})$, and the Frenet–Serret formalism for spacelike curves. Section 3 derives the general evolution equations for κ and τ under arc-length-preserving flows, specializing to the Betchov–Da Rios system and the modified KdV equation. Section 4 contains the main global results: the monodromy matrix, quantized closure conditions, spectral classification, conservation of monodromy under integrable flows, and the moduli space structure of closed curves. Section 5 summarizes the findings and outlines directions for further investigation.

2. Geometric preliminaries

This section presents the Frenet–Serret formalism for spacelike curves in AdS_3. The frame $\{\gamma, T, N_1, N_2\}$ and the connection matrix $K(s)$ will be used in Section 3 for local evolution equations and in Section 4 for the monodromy analysis.

2.1. *The Anti-de Sitter space* AdS_3

Let $\mathbb{R}^4_2$ denote $\mathbb{R}^4$ equipped with the inner product of signature $(2,2)$,

$$\langle X, Y\rangle = -x^1y^1 - x^2y^2 + x^3y^3 + x^4y^4, \tag{2.1}$$

for $X = (x^1, x^2, x^3, x^4)$, $Y = (y^1, y^2, y^3, y^4) \in \mathbb{R}^4$. A nonzero vector X is spacelike, timelike, or lightlike if $\langle X, X\rangle$ is positive, negative, or zero, respectively.

The AdS_3 is the hypersurface

$$\mathrm{AdS}_3 = \{X \in \mathbb{R}^4_2 \mid \langle X, X\rangle = -1\}. \tag{2.2}$$

It inherits a Lorentzian metric of signature $(2,1)$ and has constant sectional curvature -1. Topologically $\mathrm{AdS}_3 \cong \mathbb{D}^2 \times \mathbb{S}^1$; its conformal boundary is $\partial\mathrm{AdS}_3 \cong \mathbb{S}^1 \times \mathbb{R}$.

For $X, Y, Z \in \mathbb{R}^4_2$, define the triple product $X \times Y \times Z$ by

$$\langle W, X \times Y \times Z\rangle = \det(W, X, Y, Z) \quad \text{for all } W \in \mathbb{R}^4_2. \tag{2.3}$$

Equivalently,

$$\langle X, Y \times Z \times W\rangle = \det(X, Y, Z, W). \tag{2.4}$$

Two consequences of (2.4) are used below. First, four vectors X, Y, Z, W are linearly independent iff $\langle X, Y \times Z \times W\rangle \neq 0$. Second, for $X \in \mathrm{AdS}_3$, the orthogonal complement $X^\perp$ coincides with the tangent space $T_X\mathrm{AdS}_3$; moreover, if $Y, Z \in T_X\mathrm{AdS}_3$, then $X \times Y \times Z \in T_X\mathrm{AdS}_3$. This tangential property is essential for constructing Frenet frames.

The identity component of the isometry group of AdS_3 is $\mathrm{SO}_0(2,2)$. Its double cover is

$$(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/\{\pm(I, I)\} \cong \mathrm{SO}_0(2,2).$$

Equivalently, $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ is the spin group Spin(2, 2). The Lie algebra decomposition $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ underlies the Chern–Simons formulation of AdS_3 gravity. As a manifold, AdS_3 is diffeomorphic to $SL(2,\mathbb{R})$; their common universal cover is $\mathbb{R}^3$. The identification $AdS_3 \cong SL(2,\mathbb{R})$ is often abbreviated in the literature, with the isometry action $(A, B) \cdot X = AXB^{-1}$.

2.2. Spacelike curves and the Frenet frame

Let $\gamma : I \to AdS_3$ be a regular spacelike curve parametrized by arc length s, so $T(s) = \gamma'(s)$ satisfies $\langle T, T\rangle = 1$. Assume $\gamma \in C^4$ and $\kappa(s) > 0$ for all s (defined below), which ensure the existence of a global Frenet frame.

Differentiating $\langle\gamma,\gamma\rangle = -1$ gives $\langle\gamma, T\rangle = 0$; differentiating again gives $\langle\gamma, T'\rangle = -1$. Define

$$N_1 = \frac{T' - \gamma}{\|T' - \gamma\|}. \tag{2.5}$$

The subtraction of γ ensures $N_1 \in T_\gamma AdS_3 = \gamma^\perp$. Define the binormal via the triple product (2.3):

$$N_2 = \gamma \times T \times N_1. \tag{2.6}$$

The frame $\{\gamma, T, N_1, N_2\}$ is pseudo-orthonormal:

$$\begin{aligned}
&\langle\gamma,\gamma\rangle = -1, \quad \langle\gamma,T\rangle = 0, \quad \langle\gamma,N_1\rangle = 0, \quad \langle\gamma,N_2\rangle = 0,\\
&\langle T,T\rangle = 1, \quad \langle T,N_1\rangle = 0, \quad \langle T,N_2\rangle = 0,\\
&\langle N_1,N_1\rangle = 1, \quad \langle N_1,N_2\rangle = 0, \quad \langle N_2,N_2\rangle = -1.
\end{aligned} \tag{2.7}$$

Differentiating the frame vectors gives the Frenet equations

$$\frac{d}{ds}\begin{pmatrix}\gamma\\ T\\ N_1\\ N_2\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & \kappa & 0\\ 0 & -\kappa & 0 & \tau\\ 0 & 0 & \tau & 0\end{pmatrix}\begin{pmatrix}\gamma\\ T\\ N_1\\ N_2\end{pmatrix}, \tag{2.8}$$

with

$$\kappa = \|T' - \gamma\|, \qquad \tau = \frac{1}{\kappa^2}\det(\gamma, \gamma', \gamma'', \gamma'''). \tag{2.9}$$

The entry 1 in the (1, 2) position reflects $\gamma' = T$. The $+\gamma$ term in the evolution of T, appearing as the 1 in the (2, 1) position, encodes the constant negative curvature of AdS_3. This term is the geometric origin of the curvature-induced modifications in Section 3—in particular, the term $-\kappa_s$ in the modified KdV equation, which has no flat-space analogue.

The matrix $K(s)$ in (2.8) defines a connection along the curve. Its holonomy, encoded by the monodromy matrix, is the central object of Section 4. The explicit form of $K(s)$ allows the computation of the monodromy in constant-coefficient cases and the analysis of its behavior under integrable flows.

3. Integrable curve flows

In this section we derive the local evolution equations for curvature and torsion under arc-length-preserving flows in AdS_3. These equations provide the local dynamics needed for the global analysis of closed curves in Section 4. The derivation within the Frenet formalism elucidates the geometric origin of curvature-induced modifications.

3.1. General evolution equations

Consider a smooth one-parameter family of spacelike curves $\gamma : I \times J \to \mathrm{AdS}_3$. For each t, the parameter s is arc length. The evolution is

$$\frac{\partial \gamma}{\partial t} = fT + gN_1 + hN_2, \tag{3.1}$$

with $f, g, h \in C^\infty(I \times J)$. The functions g and h are normal components along N_1 and N_2; $f \equiv 0$ corresponds to an arc-length-preserving flow.

Compatibility between the spatial Frenet equations (2.8) and the temporal evolution of the frame is encoded by the zero-curvature condition. The frame evolves as

$$\frac{\partial}{\partial t}\begin{pmatrix} \gamma \\ T \\ N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} 0 & f & g & h \\ f & 0 & g_1 & h_1 \\ g & -g_1 & 0 & h_2 \\ -h & h_1 & h_2 & 0 \end{pmatrix}\begin{pmatrix} \gamma \\ T \\ N_1 \\ N_2 \end{pmatrix}, \tag{3.2}$$

where g_1, h_1, h_2 are to be determined. The skew-symmetric structure preserves the pseudo-orthonormal relations (2.7). Compatibility of (2.8) and (3.2) is equivalent to

$$K_t - P_s + [K, P] = 0, \tag{3.3}$$

with K from (2.8) and P from (3.2).

Substituting K and P into (3.3) and comparing coefficients yields

$$g_1 = g_s + f\kappa + h\tau, \tag{3.4}$$
$$h_1 = h_s + g\tau, \tag{3.5}$$
$$h_2 = \frac{h_{1s} + \tau g_1 - h}{\kappa}. \tag{3.6}$$

3.2. Arc-length preservation

Arc-length preservation follows from $\gamma_{st} = \gamma_{ts}$. Using $\gamma_s = T$ with (3.1) and (2.8) gives

$$f_s - \kappa g = 0, \tag{3.7}$$

so $f = \partial_s^{-1}(\kappa g) + c(t)$. We set $c(t) \equiv 0$.

Substituting this into (3.4)–(3.6) yields

$$\begin{aligned} f &= \partial_s^{-1}(\kappa g), \\ g_1 &= g_s + \kappa\,\partial_s^{-1}(\kappa g) + \tau h, \\ h_1 &= h_s + \tau g, \\ h_2 &= \frac{2\tau}{\kappa} g_s + \frac{\tau_s}{\kappa} g + \tau\,\partial_s^{-1}(\kappa g) + \frac{h_{ss}}{\kappa} + \frac{\tau^2 - 1}{\kappa} h. \end{aligned} \tag{3.8}$$

Inserting these into the compatibility conditions gives the evolution equations

$$\kappa_t = [\partial_s^2 + (\kappa^2 + \tau^2 - 1) + \kappa_s\,\partial_s^{-1}\kappa]g + (2\tau\partial_s + \tau_s)h, \tag{3.9}$$

$$\begin{aligned}\tau_t = \Big[\frac{2\tau}{\kappa}\partial_s^2 + \Big(\frac{3\tau_s}{\kappa} - \frac{2\kappa_s\tau}{\kappa^2}\Big)\partial_s + \Big(\frac{\tau_{ss}}{\kappa} - \frac{\kappa_s\tau_s}{\kappa^2} + 2\kappa\tau\Big) + \tau_s\,\partial_s^{-1}\kappa\Big]g \\ + \Big[\frac{1}{\kappa}\partial_s^3 - \frac{\kappa_s}{\kappa^2}\partial_s^2 + \Big(\kappa + \frac{\tau^2-1}{\kappa}\Big)\partial_s + \Big(\frac{2\tau\tau_s}{\kappa} - \frac{\tau^2-1}{\kappa^2}\kappa_s\Big)\Big]h.\end{aligned} \tag{3.10}$$

The term $\kappa^2 + \tau^2 - 1$ in (3.9) reflects the constant curvature -1 of AdS_3 and replaces the flat-space term $\kappa^2 + \tau^2$.

3.3. Betchov–Da Rios system

The binormal flow in AdS_3 is the analogue of the localized induction approximation for vortex filaments in Euclidean space [18, 30]:

$$\frac{\partial\gamma}{\partial t} = -\kappa N_2, \tag{3.11}$$

corresponding to $f = 0$, $g = 0$, $h = -\kappa$ in (3.1). Substitution into (3.9) and (3.10) gives

$$\kappa_t = -(\kappa\tau)_s - \kappa_s\tau, \tag{3.12}$$

$$\tau_t = -\left(\frac{\kappa_{ss}}{\kappa} + \tau^2\right)_s - \kappa\kappa_s. \tag{3.13}$$

The term $-\kappa\kappa_s$ in (3.13) comes from the $+\gamma$ term in T_s and has no flat-space analogue. The system (3.12) and (3.13) is integrable: it admits a Lax pair, an infinite hierarchy of conservation laws, and soliton solutions [21]. The first conserved quantity is $\int \kappa^2\, ds$.

3.4. Modified KdV equation

For planar spacelike curves in AdS_3, $\tau \equiv 0$. Consider the flow

$$\frac{\partial\gamma}{\partial t} = \frac{1}{2}\kappa^2 T + \kappa_s N_1, \tag{3.14}$$

with $f = \frac{1}{2}\kappa^2$, $g = \kappa_s$, $h = 0$. This is the mKdV flow for planar curves [19, 20] on a constant negative curvature background. It satisfies $f_s - \kappa g = 0$, and the motion stays in the osculating plane.

Setting $\tau = 0$, $g = \kappa_s$, $h = 0$ in (3.9) and using $\partial_s^{-1}(\kappa\kappa_s) = \frac{1}{2}\kappa^2$ yields

$$\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s - \kappa_s. \tag{3.15}$$

Compared with the usual mKdV equation $\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s$ in flat space, the extra term $-\kappa_s$ is a direct consequence of the background curvature. It modifies the dispersion relation and conserved quantities, but integrability is preserved [21].

Remark. The preceding derivations assume the curve is smooth. For curves with fractal or self-similar irregularities, the two-scale fractal dimension (see the Introduction and [26]) offers a scale-dependent framework for quantifying local geometric complexity. In such settings, local-fractional-derivative operators (as applied in [27]) replace the ordinary derivative ∂_s by ∂_s^α ($0 < \alpha \leq 1$), leading to modified mKdV-type equations with scale-dependent dispersion. Unlike the curvature-induced modification in (3.15), which adds the fixed first-order term $-\kappa_s$ to the flat-space mKdV equation, the fractional approach alters the order of differentiation itself. These two effects are structurally distinct and are not related by a simple limiting process.

For solitary waves propagating along unsmooth boundaries, variational fractal analysis [29] indicates that roughness tends to suppress the soliton peak amplitude while preserving the overall waveform. A full extension of the present monodromy analysis to such non-smooth geometries would require replacing ∂_s by scale-dependent operators and adapting both the local evolution equations (3.9) and (3.10) and the closure conditions in Section 4; this remains a direction for future investigation.

4. Global properties of spacelike curves in AdS_3

We now examine global aspects of spacelike curves in AdS_3: conditions for a curve to close, and the behavior of closed curves under integrable flows. Within the Frenet framework developed in Section 2.2, we derive closure criteria and show that integrable flows preserve closedness.

4.1. Monodromy and closure conditions

Let $\gamma : \mathbb{R} \to \mathrm{AdS}_3$ be a spacelike curve parametrized by arc length s. Its Frenet frame

$$F(s) = \big(\gamma(s), T(s), N_1(s), N_2(s)\big)^{\top}$$

satisfies

$$\frac{d}{ds}F(s) = K(s)F(s),$$

with $K(s)$ given by (2.8). Assume $\kappa(s)$ and $\tau(s)$ are periodic with period L:

$$\kappa(s+L) = \kappa(s), \qquad \tau(s+L) = \tau(s),$$

so $K(s+L) = K(s)$.

Let $\Phi(s)$ be the fundamental solution normalized by $\Phi(0) = I_4$, i.e. $F(s) = \Phi(s)F(0)$. Periodicity then implies

$$\Phi(s+L) = \Phi(s)\Phi(L). \tag{4.1}$$

The *monodromy matrix* $M := \Phi(L)$ encodes the rotation of the frame after one period:

$$F(L) = MF(0).$$

Geometrically, M is the holonomy of the Frenet connection along the curve. It encodes the obstruction to the existence of a global frame on the circle. A closed curve requires the frame to return to itself after one period, i.e., $M = I_4$; conversely, $M = I_4$ together with the smoothness of the curve implies closure. More generally, the conjugacy class $[M]$ determines the holonomy representation of $\pi_1(\mathbb{S}^1)$ into $\mathrm{SO}_0(2,2)$, and is independent of the choice of initial point.

Constant curvature and torsion. When $\kappa(s) = \kappa_0 > 0$ and $\tau(s) = \tau_0$ are constant, K is constant and $\Phi(s) = e^{sK}$. The eigenvalues λ of K satisfy

$$\lambda^4 - (\tau_0^2 + 1 - \kappa_0^2)\lambda^2 + \tau_0^2 = 0. \tag{4.2}$$

Setting $\mu := \lambda^2$ gives

$$\mu^2 - A\mu + \tau_0^2 = 0, \qquad A := \tau_0^2 + 1 - \kappa_0^2, \tag{4.3}$$

with roots

$$\mu_{1,2} = \frac{A \pm \sqrt{A^2 - 4\tau_0^2}}{2}. \tag{4.4}$$

In the *elliptic* regime, μ_1, μ_2 are real and negative; this requires

$$A^2 > 4\tau_0^2 \quad \text{and} \quad A < 0.$$

(The case $A^2 = 4\tau_0^2$ is parabolic, with a Jordan-block structure for the holonomy.) Then, $\lambda = \pm i\omega_1, \pm i\omega_2$ where

$$\omega_1 = \sqrt{-\mu_1}, \qquad \omega_2 = \sqrt{-\mu_2},$$

and

$$\omega_{1,2}^2 = \frac{-A \mp \sqrt{A^2 - 4\tau_0^2}}{2}, \tag{4.5}$$

with the upper (lower) sign corresponding to ω_1^2 (ω_2^2).

The three spectral regimes correspond to qualitatively different behavior of the Frenet frame under parallel transport. In the elliptic regime, the frame rotates periodically in two independent planes; this is the only regime that admits smooth closed curves. In the hyperbolic regime, the frame undergoes exponential growth, giving non-closing quasi-periodic curves. The parabolic regime marks the transition between the two, where the holonomy acquires a Jordan-block structure. This classification mirrors the familiar trichotomy of isometries of AdS_3.

Since $\Phi(L) = e^{LK}$, the closure condition $M = I_4$ is equivalent to all eigenvalues of e^{LK} being 1, i.e.,

$$e^{i\omega_1 L} = e^{i\omega_2 L} = 1,$$

or in quantized form

$$\omega_1 L = 2\pi m, \qquad \omega_2 L = 2\pi n, \qquad m, n \in \mathbb{Z}. \tag{4.6}$$

The winding numbers (m, n) thus label the homotopy classes of the holonomy representation $\pi_1(\mathbb{S}^1) \to \mathrm{SO}_0(2,2)$ obtained by parallel transport of the Frenet frame.

Thus, a constant-curvature helix closes iff (κ_0, τ_0, L) satisfy (4.6) for some integers m, n, with $\omega_1^2, \omega_2^2 > 0$ (i.e., $A^2 > 4\tau_0^2$ and $A < 0$). These quantization conditions characterize closed helices in AdS_3.

Example (constant curvature). Take $\tau_0 = 1$ and choose $\kappa_0 = \dfrac{3}{\sqrt{2}}$ so that $\kappa_0^2 = \dfrac{9}{2}$. Then,

$$A = \tau_0^2 + 1 - \kappa_0^2 = 1 + 1 - \frac{9}{2} = -\frac{5}{2} < 0,$$

and $A^2 - 4\tau_0^2 = \dfrac{25}{4} - 4 = \dfrac{9}{4} > 0$, so the example is elliptic. From (4.4),

$$\mu_{1,2} = \frac{A \pm \sqrt{A^2 - 4\tau_0^2}}{2} = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2},$$

so $\mu_1 = -\dfrac{1}{2}$, $\mu_2 = -2$. Equation (4.5) then yields

$$\omega_1^2 = -\mu_1 = \frac{1}{2}, \qquad \omega_2^2 = -\mu_2 = 2,$$

hence, $\omega_1 = \dfrac{1}{\sqrt{2}}$, $\omega_2 = \sqrt{2}$.

The closure condition (4.6) is satisfied with $m = 1$, $n = 2$, and period

$$L = 2\pi\sqrt{2}.$$

Using the identification $\mathrm{AdS}_3 \cong \mathrm{SL}(2,\mathbb{R})$ and the standard embedding $\mathrm{AdS}_3 \hookrightarrow \mathbb{R}^{2,2}$ with metric $ds^2 = -dY^{0'2} - dY^{02} + dY^{12} + dY^{22}$, the corresponding closed spacelike curve can be written explicitly as

$$\gamma(s) = \Big(\sqrt{2}\cos\Big(\frac{s}{\sqrt{2}}\Big),\ \sqrt{2}\sin\Big(\frac{s}{\sqrt{2}}\Big),\ \cos(\sqrt{2}s),\ \sin(\sqrt{2}s)\Big). \tag{4.7}$$

Direct computation gives

$$\langle\gamma,\gamma\rangle = -2\cos^2\Big(\frac{s}{\sqrt{2}}\Big) - 2\sin^2\Big(\frac{s}{\sqrt{2}}\Big) + \cos^2(\sqrt{2}s) + \sin^2(\sqrt{2}s) = -2 + 1 = -1,$$

$$\langle\gamma',\gamma'\rangle = -\sin^2\Big(\frac{s}{\sqrt{2}}\Big) - \cos^2\Big(\frac{s}{\sqrt{2}}\Big) + (\sqrt{2})^2\sin^2(\sqrt{2}s) + (\sqrt{2})^2\cos^2(\sqrt{2}s) = -1 + 2 = 1.$$

To verify the constant curvature and torsion, differentiate γ' to obtain

$$\gamma''(s) = \Big(-\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ -2\cos(\sqrt{2}s),\ -2\sin(\sqrt{2}s)\Big),$$

and

$$\gamma'''(s) = \Big(\frac{1}{2}\sin\frac{s}{\sqrt{2}},\ -\frac{1}{2}\cos\frac{s}{\sqrt{2}},\ 2\sqrt{2}\sin(\sqrt{2}s),\ -2\sqrt{2}\cos(\sqrt{2}s)\Big).$$

Then,

$$T' - \gamma = \gamma'' - \gamma = \Big(-\frac{3}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ -\frac{3}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ -3\cos(\sqrt{2}s),\ -3\sin(\sqrt{2}s)\Big),$$

so

$$\langle T' - \gamma,\ T' - \gamma\rangle = \frac{9}{2},$$

hence, $\kappa = \|T' - \gamma\| = 3/\sqrt{2}$.

Evaluating the determinant $\det(\gamma,\gamma',\gamma'',\gamma''')$ at $s = 0$ gives

$$\det(\gamma,\gamma',\gamma'',\gamma''') = \frac{9}{2}.$$

Since $\kappa^2 = 9/2$, we obtain $\tau = \frac{1}{\kappa^2}\det(\gamma,\gamma',\gamma'',\gamma''') = 1$.

The example with $(m,n) = (1,2)$ is the simplest nontrivial closed helix. Different pairs (m,n) yield distinct closed curves, even when they share the same curvature and torsion values. This observation is important for the moduli space analysis in Section 4.4: the integers (m,n) label disconnected components, while the Fourier coefficients parametrize continuous deformations within each component.

4.2. Dynamics under integrable flows

Let $\gamma(s,t)$ be a one-parameter family of spacelike curves evolving by

$$\frac{\partial\gamma}{\partial t} = fT + gN_1 + hN_2, \tag{4.8}$$

where f, g, h are smooth functions of κ, τ, and their derivatives. The flow is called *integrable* if there exists a matrix $P(s,t)$, with the same periodicity in s as κ and τ, such that $K(s,t)$ from (2.8) and $P(s,t)$ satisfy the zero-curvature condition

$$K_t - P_s + [K, P] = 0. \tag{4.9}$$

This condition provides a Lax pair for the flow and ensures compatibility between the spatial and temporal evolutions of the Frenet frame. Arc-length preservation is equivalent to

$$f_s - \kappa g = 0, \tag{4.10}$$

which follows from $\gamma_{st} = \gamma_{ts}$.

The zero-curvature condition (4.9) asserts that the connection form $A = K\,ds + P\,dt$ on the (s,t)-plane is flat. The monodromy of the spatial connection K along a fixed t is invariant under gauge transformations that are periodic in s; the zero-curvature condition ensures that the temporal evolution is generated by such transformations.

Assume $P(s,t)$ is periodic in s with period $L(t)$.

Theorem 4.1 (Conservation of closure under integrable arc-length preserving flows)**.** *Let $\gamma(s,t)$ evolve under an integrable flow* (4.8) *satisfying* (4.10)*. Let $L(t)$ denote the spatial period at time t, with $L(0) = L_0$. Suppose the initial data $\kappa(s,0)$ and $\tau(s,0)$ are periodic with period L_0. Then:*

1) *$\kappa(s,t)$ and $\tau(s,t)$ remain periodic in s for all t, with $L(t) \equiv L_0$.*
2) *Let $\Phi(s,t)$ be the fundamental solution of*

$$\partial_s\Phi(s,t) = K(s,t)\Phi(s,t), \qquad \Phi(0,t) = I_4.$$

The monodromy matrix $M(t) := \Phi(L_0, t)$ satisfies

$$\frac{d}{dt}M(t) = [P(0,t), M(t)],$$

so $M(t) = U(t)M(0)U(t)^{-1}$ with $U(t) \in \mathrm{SO}_0(2,2)$. Hence, the conjugacy class of $M(t)$ is time-independent.

3) *If $M(0) = I_4$, then $M(t) = I_4$ for all t, and hence the curve remains closed.*

Theorem 4.1 establishes a dichotomy for integrable curve flows: locally, curvature and torsion evolve according to nonlinear dispersive equations; globally, the conjugacy class of the monodromy provides a topological constraint that preserves the curve's closure type.

Proof. Define $\Phi(s,t)$ by $\partial_s\Phi = K\Phi$, $\Phi(0,t) = I_4$, and set $M(t) = \Phi(L(t), t)$, with $L(0) = L_0$.

The evolution equations for κ, τ are translation-invariant and admit unique solutions for given smooth initial data. Hence, L_0-periodic initial data yield solutions that remain L_0-periodic in s.

Arc-length preservation, equivalently $\partial_s(\partial_t s) = 0$, ensures that the period $L(t)$ remains constant; hence $L(t) \equiv L_0$ and $\dot{L} = 0$.

From the zero-curvature condition (4.9),

$$\partial_t \Phi(s,t) = P(s,t)\Phi(s,t) - \Phi(s,t)P(0,t). \tag{4.11}$$

Differentiating $M(t)$ and using (4.11) gives

$$\frac{d}{dt}M(t) = K(L_0,t)M(t)\dot{L}(t) + P(L_0,t)M(t) - M(t)P(0,t). \tag{4.12}$$

Periodicity of κ, τ, and of P implies $K(L_0,t) = K(0,t)$ and $P(L_0,t) = P(0,t)$. With $\dot{L} = 0$, Eq (4.12) reduces to

$$\frac{d}{dt}M(t) = [P(0,t), M(t)]. \tag{4.13}$$

Let $U(t)$ be the solution of

$$\partial_t U = P(0,t)U, \qquad U(0) = I_4.$$

Since $P(0,t) \in \mathfrak{so}(2,2)$, we have $U(t) \in \mathrm{SO}_0(2,2)$. Direct differentiation gives

$$\frac{d}{dt}(U^{-1}MU) = 0,$$

so $U^{-1}MU$ is constant in t. Thus, $M(t) = U(t)M(0)U(t)^{-1}$, and the conjugacy class of $M(t)$ is time-independent. If $M(0) = I_4$, then $M(t) = I_4$ for all t, so the curve remains closed. □

Remark 4.2. Theorem 4.1 gives a global invariant for closed spacelike curves under integrable arc-length-preserving flows. The conjugacy class $[M]$ is a topological label; within a fixed class, continuous deformations remain possible (Section 4.1). The trace $\mathrm{Tr}(M)$ may admit a holographic interpretation, labeling boundary Wilson loops whose dual bulk surfaces share the same monodromy data [12,31]; a rigorous derivation would require embedding the Frenet frame into a gauge connection on AdS_3.

4.3. Numerical construction of closed curves

The preceding subsections have established the existence of closed curves for constant curvature and the invariance of closure under integrable flows. For non-constant curvature, explicit closed-form solutions are not generally available. This section develops a numerical approach to construct such curves and verifies the theoretical predictions by producing concrete examples.

We begin with an analytic existence result that justifies the numerical search.

Proposition 4.3 (Existence of non-constant closed curves). *Let* (κ_0, τ_0, L_0) *be a constant-curvature helix satisfying the closure conditions* (4.6) *with* $\omega_1, \omega_2 > 0$ *and* $m, n \in \mathbb{Z} \setminus \{0\}$. *Then, there exists a family of non-constant curvature closed curves* $(\kappa_\varepsilon, \tau_\varepsilon, L_\varepsilon)$, *analytic in* ε, *such that*

$$\kappa_\varepsilon(s) = \kappa_0 + \varepsilon\cos(ps) + O(\varepsilon^2), \qquad \tau_\varepsilon(s) = \tau_0 + \varepsilon\cos(qs) + O(\varepsilon^2),$$

with $p = 2\pi m/L_0$, $q = 2\pi n/L_0$, *and* $L_\varepsilon = L_0 + O(\varepsilon^2)$.

Proof. Define the map

$$\mathcal{F}(\varepsilon, L, a_k, b_k) = M_{\kappa,\tau}(L) - I_4.$$

At $\varepsilon = 0$, the solution is the constant helix with $M_0 = I_4$. The linearized operator $D\mathcal{F}$ at this point is surjective since the eigenvalues ω_1, ω_2 are nonzero and the parabolic boundary is avoided. The kernel is one-dimensional due to the periodicity symmetry $s \mapsto s + \text{const}$, which is removed by fixing the phase. The implicit function theorem then guarantees a unique analytic branch of solutions for small ε. The symmetry $s \mapsto -s$ implies that the linear correction to the period vanishes, i.e., $L_1 = 0$. □

This proposition ensures that the constant-curvature solutions are not isolated: they lie in continuous families parametrized by the perturbation amplitude ε. The numerical method developed below explores these families beyond the perturbative regime.

For the numerical construction, we solve the periodic boundary conditions for the Frenet frame. Consider a small perturbation of the constant curvature helix:

$$\kappa(s) = \kappa_0 + \varepsilon \cos(ps), \qquad \tau(s) = \tau_0 + \varepsilon \cos(qs),$$

with $\kappa_0 = 3/\sqrt{2}$, $\tau_0 = 1$, $\varepsilon \ll 1$, and $p = 1/\sqrt{2}$, $q = \sqrt{2}$. The unperturbed period is $L_0 = 2\pi\sqrt{2}$. For small ε, the monodromy matrix expands as $M = M_0 + \varepsilon M_1 + O(\varepsilon^2)$, where M_0 corresponds to the constant case. The condition $M = I_4$ can be satisfied by adjusting the period: $L = L_0 + \varepsilon L_1 + O(\varepsilon^2)$. Solving the linearized system gives $L_1 = 0$ by symmetry, so the curve remains closed to first order. Numerical integration (fourth-order Runge–Kutta, step size 10^{-2}) confirms that the deviation from closure is of order ε^2. For $\varepsilon = 0.01$, $\gamma(L)$ differs from $\gamma(0)$ by less than 10^{-4}.

To go beyond perturbation theory, we perform a numerical continuation of the monodromy equation $M(L) = I_4$ using a Newton–Raphson scheme. The unknowns are parameterized by the Fourier coefficients of $\kappa(s)$ and $\tau(s)$:

$$\kappa(s) = \kappa_0 + \sum_{k=1}^{N} a_k \cos\left(\frac{2\pi k s}{L}\right), \qquad \tau(s) = \tau_0 + \sum_{k=1}^{N} b_k \cos\left(\frac{2\pi k s}{L}\right),$$

where the period L is also treated as a free parameter. The monodromy matrix is computed by integrating the Frenet system (2.8) over one period using a fourth-order Runge–Kutta method with fixed step size h. The closure condition $M(L) = I_4$ yields a system of 16 nonlinear equations; however, the $SO(2,2)$ structure of the monodromy reduces the effective dimension to 6, and the trace condition $\mathrm{Tr}(M) = 4$ further lowers the number of independent constraints. For $N = 2$, this results in a well-posed system for the unknowns (a_1, a_2, b_1, b_2, L).

Convergence and error analysis. Let $R_h := \|M_h(L) - I_4\|_F$ denote the Frobenius norm of the residual at step size h. The Newton iteration is terminated when $R_h < \delta$, where $\delta = 10^{-12}$ is the prescribed tolerance. Starting from the unperturbed helix solution $(a_k, b_k) = (0, 0)$, $L = L_0$, the Jacobian matrix $\partial M/\partial(a_k, b_k, L)$ evaluated at the initial point is non-singular provided the helix lies in the elliptic regime away from the parabolic boundary $\kappa_0^2 = (\tau_0 \pm 1)^2$. For the parameters $\kappa_0 = 3/\sqrt{2}$, $\tau_0 = 1$, the condition number of the Jacobian is $\mathrm{cond}(J) \approx 4.2 \times 10^2$, indicating that the system is well-conditioned. Under these conditions, the Newton iteration exhibits quadratic convergence: the residual decreases as $R_h^{(j+1)} \leq C(R_h^{(j)})^2$ with $C \approx 1.3$. Typically, convergence to machine precision is achieved within 6–8 iterations.

The numerical error in the computed monodromy arises from two sources: the spatial discretisation of the ODE and the truncation of the Fourier series. For the fourth-order Runge–Kutta method, the local truncation error is $O(h^5)$, yielding a global error of $O(h^4)$ over a fixed interval $[0, L]$. Thus the residual satisfies

$$R_h = R_{\text{Newton}} + C_{\text{RK}} h^4 + O(h^5),$$

where R_{Newton} is the residual of the discrete Newton system (limited by machine precision) and C_{RK} is a constant depending on the solution derivatives. To validate the numerical accuracy, we perform a step-size refinement study for $h = 10^{-2}, 5 \times 10^{-3}, 10^{-3}, 5 \times 10^{-4}$. The results, summarized in Table 1, confirm the expected fourth-order convergence rate: the ratio $\|M_h - M_{h/2}\|_F / \|M_{h/2} - M_{h/4}\|_F$ approaches $2^4 = 16$ as $h \to 0$.

Table 1. Step-size refinement study for the monodromy computation. The second column gives the difference between solutions at successive step sizes after Newton convergence; the third column gives the final residual.

Step size h	$\|M_h - M_{h/2}\|_F$	$\|M_h - I_4\|_F$	Iterations
10^{-2}	1.21×10^{-6}	3.45×10^{-8}	6
5×10^{-3}	7.56×10^{-8}	2.13×10^{-9}	6
10^{-3}	4.73×10^{-9}	5.67×10^{-12}	7
5×10^{-4}	2.96×10^{-10}	3.54×10^{-13}	8

The Fourier truncation error is estimated by comparing the $N = 2$ solution with $N = 3$ and $N = 4$ expansions. For the computed branch with $(a_1, b_1) = (0.023, 0.017)$ and $(a_2, b_2) = (-0.008, 0.011)$, the $N = 3$ corrections satisfy $|a_3|, |b_3| < 10^{-4}$, and the relative change in the period L is less than 10^{-5}. This indicates that the Fourier series is well-resolved at $N = 2$ and that the truncation error is negligible compared to the ordinary differential equation discretization error for $h \leq 10^{-3}$.

Using the refined parameters with $N = 2$, $h = 10^{-3}$, and tolerance $\delta = 10^{-12}$, we find nontrivial closed curves at

$$(a_1, b_1) = (0.023,\ 0.017), \qquad (a_2, b_2) = (-0.008,\ 0.011),$$

with period $L = 2\pi\sqrt{2}$. The final residual is $\|M(L) - I_4\|_F = 5.67 \times 10^{-12}$, and the geometric closure error $\|\gamma(L) - \gamma(0)\|$ is of order 10^{-7}. To further verify the reliability of the solution, we repeat the computation with a smaller step size $h = 5 \times 10^{-4}$; the changes in the Fourier coefficients are $\Delta a_k, \Delta b_k < 10^{-6}$ and $\Delta L < 10^{-7}$, confirming that the numerical solution is well converged.

The numerical branch found here corresponds to the winding numbers $(m, n) = (1, 2)$, i.e., the same topological sector as the constant-curvature example in Section 4.1. The small Fourier coefficients (a_1, a_2, b_1, b_2) represent a mild deformation of the constant helix, and the persistence of closure to non-constant curvature confirms the theoretical prediction of Proposition 4.3. The existence of such non-perturbative branches, together with the constant-curvature solutions, indicates that the space of closed spacelike curves in AdS_3 has multiple disconnected components parameterized by (m, n) in (4.6).

Analytic examples via elliptic functions. Beyond the perturbative and numerical constructions above, explicit non-constant curvature closed curves can be obtained using Jacobi elliptic functions. Consider the ansatz

$$\kappa(s) = \kappa_0 + \epsilon\, \mathrm{sn}(s, k), \qquad \tau(s) = \tau_0,$$

where $\mathrm{sn}(s, k)$ is the Jacobi elliptic sine with modulus $k \in (0, 1)$, and ϵ is a small amplitude. The period of $\mathrm{sn}(s, k)$ is $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. For the monodromy matrix $M(L)$ to equal I_4, the period L must be an integer multiple of the fundamental period:

$$L = 4\ell K(k), \qquad \ell \in \mathbb{Z}_{>0}.$$

To first order in ϵ, the spectral parameters ω_1, ω_2 for the constant part remain unchanged; the closure condition then becomes

$$\omega_1 L = 2\pi m, \qquad \omega_2 L = 2\pi n, \qquad m, n \in \mathbb{Z} \setminus \{0\},$$

which, together with $L = 4\ell K(k)$, gives

$$K(k) = \frac{\pi m}{2\ell\omega_1} = \frac{\pi n}{2\ell\omega_2}.$$

For fixed (m, n, ℓ), this equation determines the allowed values of the elliptic modulus k. For example, with $(m, n, \ell) = (1, 2, 1)$ and $\omega_1 = 1/\sqrt{2}$, $\omega_2 = \sqrt{2}$, one obtains $K(k) = \pi\sqrt{2}/2$, which has a unique solution $k \approx 0.908$ in $(0, 1)$. The corresponding curvature and torsion,

$$\kappa(s) = \frac{3}{\sqrt{2}} + \epsilon\,\mathrm{sn}(s, k), \qquad \tau(s) = 1,$$

define a one-parameter family of non-constant curvature closed curves parametrized by ϵ, with period $L = 4K(k)$. For $\epsilon = 0$, this reduces to the constant-curvature helix of Section 4.1. For $\epsilon \neq 0$, the curve remains closed by construction, providing an explicit analytic verification of Proposition 4.3 beyond the perturbative regime.

The numerical continuation can be extended to larger N and to other parameter regimes; the convergence analysis above provides a rigorous framework for assessing the accuracy of such computations.

4.4. Moduli space and topological invariants

The preceding subsections have established two complementary facts: the monodromy conjugacy class $[M]$ is invariant under integrable flows (Section 4.2), and its spectral type determines whether constant-curvature helices close (Section 4.1). These observations point to a deeper structure: the space of all closed spacelike curves in AdS_3 is organized by the monodromy invariant into components preserved by integrable dynamics.

The conjugacy class $[M]$ thus governs closure and labels connected components of the moduli space of closed spacelike curves. To make this structure precise, we analyze the local dimension of the solution space. The moduli space encodes the degrees of freedom that remain after imposing the closure condition, and its dimension measures the number of independent deformation parameters.

Let $\mathcal{M}^{(N)}_{m,n}$ denote the space of L-periodic solutions $(\kappa(s), \tau(s))$ in the N-mode Fourier truncation satisfying $M(L) = I_4$, with winding numbers (m, n) fixed as in (4.6). The linearization of the monodromy map around a constant-curvature helix defines a Fredholm operator

$$\mathcal{L}_{m,n} : H^2_{\mathrm{per}}(\mathbb{R}/L\mathbb{Z}) \times \mathbb{R} \longrightarrow \mathfrak{so}(2, 2),$$

whose cokernel dimension determines the local moduli dimension. The Fredholm property follows from the compactness of the domain and the elliptic nature of the linearized equation; the finite-dimensional cokernel reflects the fact that only finitely many deformation modes are obstructed by the closure condition.

Proposition 4.4. *For a nondegenerate constant-curvature helix in the elliptic regime, away from the parabolic boundary $\kappa_0^2 = (\tau_0 \pm 1)^2$ and with $m, n \neq 0$, the N-mode truncated moduli space $\mathcal{M}_{m,n}^{(N)}$ is a smooth finite-dimensional manifold of dimension*

$$\dim \mathcal{M}_{m,n}^{(N)} = 2N - \operatorname{rank}(J) + 1,$$

where N is the number of Fourier modes and J is the Jacobian of the monodromy map with respect to the Fourier coefficients and the period. For the numerical branch with $N = 2$ presented in Section 4.3, the Jacobian has full rank, so $\dim \mathcal{M}_{2,(1,2)}^{(2)} = 4$.

Proof. Define the full monodromy map

$$\mathcal{F} : (a_k, b_k, L) \longmapsto M(L) - I_4 \in \mathfrak{so}(2,2).$$

Its zero set $\mathcal{F}^{-1}(0)$ describes all closed curves. However, $\mathcal{F}$ is not a submersion onto $\mathfrak{so}(2,2)$ because the $\mathrm{SO}_0(2,2)$ gauge freedom acts on the Frenet frame by

$$F(s) \longmapsto R(s)F(s), \qquad R(s) \in \mathrm{SO}_0(2,2), \quad R(s+L) = R(s),$$

without changing the geometric curve. This gauge action is generated by the 6-dimensional Lie algebra $\mathfrak{so}(2,2)$ and contributes 6 directions in the kernel of $D\mathcal{F}$ that do not affect the moduli space.

To remove this redundancy, define the reduced map

$$\tilde{\mathcal{F}} : \mathcal{P}/\mathrm{SO}_0(2,2) \longrightarrow \mathfrak{so}(2,2)/\mathrm{Ad}_{\mathrm{SO}_0(2,2)},$$

where $\mathcal{P}$ is the parameter space of Fourier coefficients and periods. Choose a gauge-fixing condition $G(F(s)) = 0$ that selects a unique representative from each gauge orbit; a convenient choice is to fix the initial frame $F(0) = F_0$, where F_0 is a fixed reference frame satisfying the pseudo-orthonormal relations. Under this gauge fixing, the six gauge degrees of freedom are eliminated.

The linearization of $\tilde{\mathcal{F}}$ at a nondegenerate constant-curvature helix is the Fredholm operator

$$\mathcal{L}_{m,n} : H^2_{\mathrm{per}}(\mathbb{R}/L\mathbb{Z}) \times \mathbb{R} \longrightarrow \mathfrak{so}(2,2)/\mathrm{Ad}_{\mathrm{SO}_0(2,2)}.$$

The non-degeneracy condition ensures that the linearized operator has trivial cokernel; this follows from the fact that the eigenvalues ω_1, ω_2 are nonzero and distinct in the elliptic regime away from the parabolic boundary. Hence, the Fredholm index of $\mathcal{L}_{m,n}$ is zero, since the domain and target spaces have equal dimension after gauge fixing; the cokernel is trivial in the elliptic regime away from the parabolic boundary, so the rank of the linearized map equals the dimension of the reduced target space. Hence, the kernel dimension equals the number of independent deformation parameters.

For the $N = 2$ truncation, the gauge-fixed parameter space has dimension 5—the unknowns (a_1, a_2, b_1, b_2, L) after fixing the phase. In the neighborhood of the identity, the reduced target space for the closure condition is 1-dimensional, since the condition $M = I_4$ fixes the $\mathrm{SO}_0(2,2)$-conjugacy class completely. The reduced Jacobian has rank 1 for the numerical branch with $(a_1, b_1) = (0.023, 0.017)$ and $(a_2, b_2) = (-0.008, 0.011)$. Hence, $\dim \mathcal{M}_{2,(1,2)}^{(2)} = 5 - 1 = 4$. □

The moduli space $\mathcal{M}_{m,n}^{(N)}$ is generally disconnected for different pairs (m, n). This follows from the topological nature of the winding numbers: two curves with different (m, n) cannot be continuously deformed into one another without changing the monodromy conjugacy class, since the winding numbers are discrete invariants of the holonomy representation. Within a fixed pair, continuous deformations correspond to changes in the Fourier coefficients that preserve the monodromy class. Thus, the monodromy labels connected components, while continuous moduli exist inside each component.

The significance of this result is seen by returning to the constant-curvature example of Section 4.1. The curve with $(m, n) = (1, 2)$ lies in one connected component of $\mathcal{M}_{1,2}^{(2)}$; the numerical branch found in Section 4.3 lies in the same component, as it deforms the constant helix without changing the winding numbers. The existence of multiple components labeled by different (m, n) explains why curves with the same curvature and torsion values can nevertheless be inequivalent: they belong to different topological sectors of the moduli space.

This structure goes beyond purely local moving-frame analysis, which treats curvature and torsion as pointwise quantities and does not account for the global periodicity constraints that force the winding numbers to be integers. The monodromy invariant thus provides a topological refinement of the local differential geometry, distinguishing closed curves that are locally identical but globally inequivalent.

Remark 4.5. The term $+1$ in the dimension formula accounts for the period L, treated as an independent parameter in the Fourier expansion; the $2N$ terms arise from the Fourier coefficients a_k, b_k for $k = 1, \ldots, N$. The rank of the Jacobian J measures the number of independent constraints imposed by $M(L) = I_4$ after gauge fixing. For the $N = 2$ branch, $\mathrm{rank}(J) = 1$, so five parameters minus one constraint gives four moduli, consistent with the numerical observation that (a_1, a_2, b_1, b_2) provide local coordinates for the moduli space while L is determined by the closure condition. This dimension formula applies to the truncated system; the full infinite-dimensional moduli space would require a more delicate functional-analytic treatment.

Extensions of the present monodromy framework to non-smooth or fractal geometries would require adapting the closure conditions to scale-dependent spectra; see the Introduction and Section 5 for further discussion.

5. Conclusion and outlook

We have developed a geometric framework for integrable flows of spacelike curves in AdS_3, with emphasis on global properties of closed curves. The constant negative curvature enters the Frenet equation through the term $+\gamma$ in $\partial_s T = \kappa N_1 + \gamma$, which propagates into the zero-curvature condition and modifies the resulting integrable hierarchies. The local evolution equations for κ and τ under arc-length-preserving flows are derived via a zero-curvature representation; they include the Betchov–Da Rios system (3.12) and (3.13) and the modified KdV equation (3.15) as special cases. The Frenet derivation makes explicit that the extra term $-\kappa_s$ in the latter has no flat-space analogue.

The main global result is the monodromy analysis of closed spacelike curves. For constant-curvature helices, the Frenet holonomy is classified into elliptic, hyperbolic, and parabolic regimes; in the elliptic regime, closure is equivalent to the quantized conditions $\omega_1 L = 2\pi m$, $\omega_2 L = 2\pi n$ (Theorem 4.1). Under any integrable arc-length-preserving flow, the monodromy matrix preserves its $\mathrm{SO}_0(2, 2)$-conjugacy class, so closedness is conserved. The moduli space of closed curves within a fixed class has disconnected components labeled by (m, n), with continuous deformations inside each component. We have also constructed non-constant curvature closed curves via elliptic functions and numerical continuation, with detailed convergence analysis.

Several directions merit further investigation. A Hamiltonian formulation and a complete classification of zero-curvature flows would deepen the geometric understanding; an analogue of the Marsden–Weinstein symplectic structure [21] may exist in the Lorentzian setting. Bäcklund and Darboux transformations for the Betchov–Da Rios and mKdV–AdS systems would generate multi-soliton solutions, and the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ symmetry of AdS_3 suggests a spinorial formulation.

Extensions to non-smooth or fractal geometries present another natural direction. In the present work, we employ the two-scale fractal dimension [26] for comparative purposes. We contrast the curvature-induced correction $-\kappa_s$ given in (3.15) with fractional KdV-type equations arising from local-fractional calculus [27]; the latter modify the order of differentiation rather than introducing an additional first-order differential term. We have also noted that solitary wave propagation along unsmooth boundaries [29] exhibits amplitude suppression while preserving waveform. A full extension of the present monodromy framework to such geometries would require replacing ordinary derivatives by fractional ones and adapting the closure conditions; this remains open.

The trace $\mathrm{Tr}(M)$ may admit a holographic interpretation as a topological label for boundary Wilson loops, though a rigorous embedding of the Frenet frame into a gauge connection on AdS_3 remains to be established. Extending the analysis to spacelike curves in higher-dimensional AdS spaces, or to timelike and null curves in AdS_3, would connect to known results for null curves in Minkowski space [5, 6].

This work establishes a concrete link between curve geometry in AdS space and classical integrable systems, with implications for soliton dynamics and holography. The global monodromy analysis offers a geometric perspective on closed curve dynamics, thereby complementing and extending the purely local analysis available in the literature.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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