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Research article

On classes of analytic functions with respect to symmetric points defined using the Mittag-Leffler function of Le Roy type

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Abstract: In this article, we will define an operator using the Hadamard product of an analytic function involving the Mittag-Leffler function of the Le Roy type. The defined operator extends and unifies the well-known operators that were defined to study various subclasses of univalent functions. Using the defined operator, we introduce a new family of analytic functions with respect to symmetric points. The analytic characterization of the defined function class was mainly motivated by the so-called α-exponentially convex functions. Further, the defined function class involves (r, ℓ)-symmetrical function which is a generalization of the notions such as even, odd, and ℓ-symmetrical functions. Estimates of the initial bounds, Fekete-Szegő inequality, and the integral representation of the defined function class are the main results of the paper. Applications of the main results are presented as inclusion relations and corollaries.

Keywords: starlike function; convex function; exponential convex function; Mittag-Leffler function of the Le Roy type; Fekete-Szegő inequality, subordination
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

The primary objective of this study is to introduce a family of analytic functions using a function which unifies the notions of even, odd, and ℓ-symmetrical functions. Several studies dealing with the theory's special functions can be found in the literature, but the studies involving an exponential functions explicitly in the characterization are very rare. To fill this gap, here, we introduce a family whose characterization is mainly motivated by the so-called α-exponentially convex functions and their related studies (see [1–4]). If extension, generalization, and unification of the existing studies is our goal, then naturally, we will have to define the function class using a differential operator. For this purpose, here, we will define a presumably new operator involving a Mittag-Leffler function of the Le Roy type using Hadamard product of analytic functions.

1.1. A short introduction to Mittag-Leffler functions of the Le Roy type

Let $\mathcal{U}_r = \{\zeta : |\zeta| < r\}$, $\mathcal{U} = \mathcal{U}_1 = \{\zeta : |\zeta| < 1\}$, and $\mathcal{H}(a, n)$ denote the class of analytic functions J defined in $\mathcal{U}$ having a power series of the form

$$J(\zeta) = a + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \cdots .$$

Let $\mathcal{A} = \mathcal{H}(1, 1)$ with $a_1 = 1$. The special function $\mathcal{L}^{\gamma}(\zeta)$ (see [5]), given by

$$\mathcal{L}^{\gamma}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{[n!]^{\gamma}} = \sum_{n=0}^{\infty} \frac{\zeta^n}{[\Gamma(n+1)]^{\gamma}}, \qquad (\zeta \in \mathbb{C}; \gamma > 0), \tag{1.1}$$

is known as the Le Roy function. Another popular special function, which is regarded as a leading function of fractional calculus, is the well-known Mittag-Leffler function, which is defined by

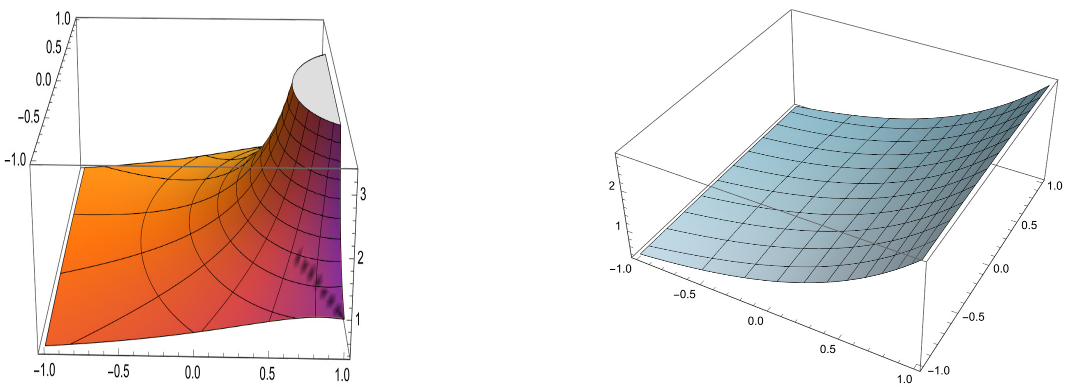
$$E_{\theta}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\theta n + 1)}, \quad (\zeta \in \mathbb{C}, \theta \in \mathbb{R}, \theta > 0),$$

(see Figures 1 and 2 for its graphical illustration) and its two-parameter version $E_{\theta,\vartheta}(\zeta)$ is defined by following power series, convergent in the whole complex plane:

$$E_{\theta,\vartheta}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\theta n + \vartheta)} \quad (\zeta \in \mathbb{C}, \vartheta, \in \mathbb{C}, \theta > 0). \tag{1.2}$$

Combining (1.1) and (1.2), Gerhold [6] (also [7]) introduced the so-called Mittag-Leffler function of the Le Roy type, which is an extension of both the Le Roy and Mittag-Leffler functions that is given by

$$\mathcal{L}^{\gamma}_{\theta,\vartheta}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{[\Gamma(\theta n + \vartheta)]^{\gamma}}, \qquad (\zeta \in \mathbb{C}; \vartheta, \in \mathbb{C}; \theta, \gamma > 0). \tag{1.3}$$



(a) $E_{0.2}(\zeta)$ **(b)** $E_{0.95}(\zeta)$

Figure 1. Complex three-dimensional plot of Mittag-Leffler function $E_{\theta}(\zeta)$ if the domain is a unit disc.

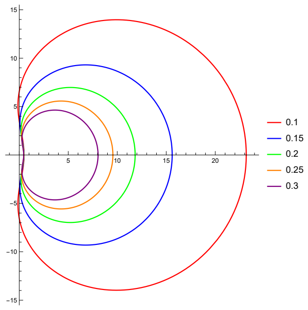


Figure 2. The image of the unit disc under the transformation $E_\theta(\zeta)$ for various values of θ.

Analogous to the Mittag-Leffler-Prabhakar function [8, Eq (1.3)] (also see Srivastava and Tomovski's extension of the Mittag-Leffler-Prabhakar function [9]), Tomovski and Mehrez in [10] introduced and studied the Mittag-Leffler-Prabhakar function of the Le Roy type which is given by

$$\mathcal{L}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)=\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)\zeta^n}{n!\Gamma(\delta)\left[\Gamma(\theta n+\vartheta)\right]^{\gamma}},\qquad (\zeta\in\mathbb{C};\vartheta,\in\mathbb{C};\theta,\ \gamma>0).\tag{1.4}$$

Using $\mathcal{L}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)$, we intend to define an operator on the class $\mathcal{A}$ which would unify various studies pertaining to univalent functions. For this purpose, we need to modify $\mathcal{L}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)$ so that it satisfies the required normalization, which can be achieved by multiplying the term $\zeta\,[\vartheta]^{\gamma}$ to it. That is

$$\begin{aligned}\mathcal{G}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)&=\zeta\left[\Gamma(\vartheta)\right]^{\gamma}\mathcal{L}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)=\zeta\left[\Gamma(\vartheta)\right]^{\gamma}\left[\frac{\zeta}{\left[\Gamma(\vartheta)\right]^{\gamma}}+\sum_{n=1}^{\infty}\frac{\Gamma(\delta+n)\zeta^n}{n!\Gamma(\delta)\left[\Gamma(\theta n+\vartheta)\right]^{\gamma}}\right]\\&=\zeta\left[\Gamma(\vartheta)\right]^{\gamma}\left[\frac{\zeta}{\left[\Gamma(\vartheta)\right]^{\gamma}}+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n-1)\zeta^{n-1}}{(n-1)!\Gamma(\delta)\left[\Gamma(\theta(n-1)+\vartheta)\right]^{\gamma}}\right].\end{aligned}$$

Precisely, we have

$$\mathcal{G}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)=\zeta\left[\Gamma(\vartheta)\right]^{\gamma}\mathcal{L}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n-1)\left[\Gamma(\vartheta)\right]^{\gamma}}{(n-1)!\Gamma(\delta)\left[\Gamma(\theta n+\vartheta-\theta)\right]^{\gamma}}\zeta^n.\tag{1.5}$$

1.2. Generalized differential operator

Motivated by [11, 12], we now define the following operator $\Re_\lambda^m(\theta,\ \vartheta,\ \gamma)J(\zeta)\ :\ \mathcal{U}\longrightarrow\mathcal{U}$ as:

$$\Re_\lambda^0(\theta,\ \vartheta,\ \gamma)J(\zeta)=J(\zeta)*\mathcal{G}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)=\zeta+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n-1)\left[\Gamma(\vartheta)\right]^{\gamma}}{(n-1)!\Gamma(\delta)\left[\Gamma(\theta n+\vartheta-\theta)\right]^{\gamma}}a_n\zeta^n,$$

$$\begin{aligned}\Re_\lambda^1(\theta,\ \vartheta,\ \gamma)J(\zeta)&=(1-\lambda)\left(J(\zeta)*\mathcal{G}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)\right)+\lambda\,\zeta\left(J(\zeta)*\mathcal{G}_{\theta,\vartheta}^{\gamma,\delta}(\zeta)\right)'\\&=(1-\lambda)\zeta+\sum_{n=2}^{\infty}(1-\lambda)\ \frac{\Gamma(\delta+n-1)\left[\Gamma(\vartheta)\right]^{\gamma}}{(n-1)!\Gamma(\delta)\left[\Gamma(\theta n+\vartheta-\theta)\right]^{\gamma}}a_n\zeta^n\end{aligned}$$

$$+ \lambda\zeta + \sum_{n=2}^{\infty} \lambda n \frac{\Gamma(\delta+n-1)\,[\Gamma(\vartheta)]^{\gamma}}{(n-1)!\Gamma(\delta)\,[\Gamma(\theta n+\vartheta-\theta)]^{\gamma}} a_n \zeta^n$$

$$= \zeta + \sum_{n=2}^{\infty} [1-\lambda+\lambda n] \frac{\Gamma(\delta+n-1)\,[\Gamma(\vartheta)]^{\gamma}}{(n-1)!\Gamma(\delta)\,[\Gamma(\theta n+\vartheta-\theta)]^{\gamma}} a_n \zeta^n. \tag{1.6}$$

Similarly, we have

$$\Re_{\lambda}^{2}(\theta,\,\vartheta,\,\gamma)J(\zeta) = \Re_{\lambda}^{1}\left(\Re_{\lambda}^{1}(\theta,\,\vartheta,\,\gamma)J(\zeta)\right). \tag{1.7}$$

Hence, applying the steps detailed as in (1.6) recursively $m-1$ times, we get the following:

$$\Re_{\lambda}^{m}(\theta,\,\vartheta,\,\gamma)J(\zeta) = \zeta + \sum_{n=2}^{\infty} [1-\lambda+\lambda n]^{m} \frac{\Gamma(\delta+n-1)\,[\Gamma(\vartheta)]^{\gamma}}{(n-1)!\Gamma(\delta)\,[\Gamma(\theta n+\vartheta-\theta)]^{\gamma}} a_n \zeta^n. \tag{1.8}$$

Letting $m = 0$ in (1.8), we get the operator recently studied by Aktaş and Görgülü in [13]. Morever, letting $\delta = 1$ and $\theta = 0$ in (1.8), we get the differential operator introduced by Al-Oboudi [14]. Furthermore, for $\delta = 1 = \lambda$ and $\theta = 0$, we get the operator introduced by Sălăgean [15]. Many (well known and new) integral and differential operators can be obtained by specializing the parameters.

1.3. A short overview of the existing literature on geometric function theory

We let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are one to one. Let $\mathcal{P}$ denote the family consisting of the functions $\mathcal{K}$ that is analytic on $\mathcal{U}$ with $\mathcal{K}(0) = 1$ and $\mathrm{Re}[\mathcal{K}(\zeta)] > 0$. Geometrically defined function classes of starlike and convex functions (see [16,17]) which we will denote here by $\mathcal{S}^*$ and $\mathcal{C}$, satisfy the following respective differential inclusions:

$$\mathcal{A}_s(\zeta) = \frac{\zeta J'(\zeta)}{J(\zeta)} \in \mathcal{P} \quad \text{and} \quad \mathcal{A}_c(\zeta) = 1 + \frac{\zeta J''(\zeta)}{J'(\zeta)} \in \mathcal{P}.$$

The well-known classes, namely close-to-convex ($\mathcal{C}^*$) (see [18]) and quasi-convex ($\mathcal{QC}$), is known to satisfy the condition

$$\frac{J'(\zeta)}{g'(\zeta)} \in \mathcal{P} \quad \text{and} \quad \frac{(\zeta J'(\zeta))'}{g'(\zeta)} \in \mathcal{P}, \qquad (g \in \mathcal{C};\ \zeta \in \mathcal{U}).$$

Ma and Minda [19] obtained the coefficient inequalities of the classes $\mathcal{S}^*(\mathfrak{K})$ and $\mathcal{C}(\mathfrak{K})$ which satisfy the following conditions, respectively

$$\mathcal{A}_s(\zeta) \prec \mathfrak{K}(\zeta) \quad \text{and} \quad \mathcal{A}_c(\zeta) \prec \mathfrak{K}(\zeta), \tag{1.9}$$

where $\mathfrak{K}(\zeta) \in \mathcal{P}$ satisfies a few geometrical conditions and is of the form

$$\mathfrak{K}(\zeta) = 1 + \kappa_1\zeta + \kappa_2\zeta^2 + \kappa_3\zeta^3 + \cdots, \qquad (\kappa_1 \neq 0;\ \zeta \in \mathcal{U}). \tag{1.10}$$

Motivated by Ma and Minda [19], several authors restudied the subclasses of $\mathcal{S}$ by restricting the differential inequalities given in (1.9) to a conic region. By expressing a certain analytic characterization listed in (1.9), for example $\mathcal{A}_s(\zeta) \prec \mathfrak{K}(\zeta)$ ($\prec$ denotes subordination), and choosing $\mathfrak{K}(\zeta)$ to map $\mathcal{U}$ on to conic regions such as cardioid [20,21], petal-shaped [22], crescent-shaped [23,24], and leaf-like regions on the right-half of the complex plane [25], various interesting families of functions have been introduced and studied by several authors.

Arango et al. [26] introduced the so-called α-exponentially convex functions that consist of a functions $J \in \mathcal{A}$ such that $e^{\alpha J(\zeta)}$ is convex univalent in $\mathbb{U}$. Precisely for $\alpha \in \mathbb{C} \setminus \{0\}$, a function $J \in \mathcal{A}$ is in $\varepsilon(\alpha)$ if and only if

$$\operatorname{Re}\left(1 + \frac{\zeta J''(\zeta)}{J'(\zeta)} + \alpha \zeta J'(\zeta)\right) > 0, \quad (\zeta \in \mathbb{U}). \tag{1.11}$$

In this direction, Karthikeyan et al. [27] obtained the coefficient inequalities of functions $J \in \mathcal{R}(\Re)$ which satisies the condition

$$1 + \left[\frac{\zeta^2 J''(\zeta)}{J(\zeta)} + \frac{\zeta J'(\zeta)}{J(\zeta)} - \left(\frac{\zeta J'(\zeta)}{J(\zeta)}\right)^2\right] e^{\frac{\zeta J'(\zeta)}{J(\zeta)} - 1} \prec \Re(\zeta),$$

where $\Re \in \mathcal{P}$ is defined as in (1.10). Refer to [2] for details on the studies on geometric function theory which involves an exponential function in its analytic characterization.

For $J \in \mathcal{A}$, Sakaguchi [28, Theorem 3] proved that all functions $J \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left\{\frac{\zeta J'(\zeta)}{\frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \frac{J(\varepsilon^\sigma \zeta)}{\varepsilon^\sigma}}\right\} > 0, \quad (\zeta \in \mathbb{U}),$$

are close-to-convex and therefore are univalent in $\mathbb{U}$ for all $\ell \in \mathbb{N}$ and $\varepsilon = \exp\left(\frac{2\pi i}{\ell}\right)$. Wang et al. [29] (also see [30]) introduced the classes of starlike and convex functions with respect to ℓ-symmetric points which are defined by subjecting $J \in \mathcal{A}$ to satisfy the conditions

$$\frac{\zeta J'(\zeta)}{J_\ell(\zeta)} \prec \Re(\zeta) \quad \text{and} \quad \frac{(\zeta J'(\zeta))'}{J'_\ell(\zeta)} \prec \Re(\zeta), \quad (\ell \in \mathbb{N};\ \Re \in \mathcal{R}), \tag{1.12}$$

where $\prec$ denotes subordination and $J_\ell(\zeta) = \frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \frac{J(\varepsilon^\sigma \zeta)}{\varepsilon^\sigma}$. Here, we let $\Upsilon_s^\ell(\Re)$ and $C_s^\ell(\Re)$ denote the classes of starlike and convex functions with respect to ℓ-symmetric points. Liczberski and Połubiński [31] introduced the following extension:

$$J_{r,\ell}(\zeta) = \frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \frac{J(\varepsilon^\sigma \zeta)}{\varepsilon^{\sigma r}}, \quad (J \in \mathcal{A}), \tag{1.13}$$

where $\varepsilon = \exp\left(\frac{2\pi i}{\ell}\right)$, $\ell \geq 1$ is a fixed integer and $r = 0, 1, 2, \ldots, \ell - 1$. Karthikeyan [32] further extended $\Upsilon_s^\ell(\Re)$ and $C_s^\ell(\Re)$ by replacing $J_\ell(\zeta)$ with $J_{r,\ell}(\zeta)$ in (1.12).

For $J(\zeta) = \zeta + \sum_{n=2}^\infty a_n \zeta^n$, we can get

$$J_{r,\ell}(\zeta) = \sum_{n=1}^\infty a_n \Upsilon_{n,r}^\ell \zeta^n, \quad (a_1 = 1), \quad \Upsilon_{n,r}^\ell = \frac{1}{\ell}\sum_{\sigma=0}^{\ell-1}\left[\exp\left(\frac{2\pi i}{\ell}\right)\right]^{(n-r)\sigma}.$$

If σ is an integer, then from (1.13), we have

$$\begin{aligned}
J_{r,\ell}(\varepsilon^\sigma \zeta) &= \varepsilon^{\sigma r} J_{r,\ell}(\zeta), \\
J'_{r,\ell}(\varepsilon^\sigma \zeta) &= \varepsilon^{\sigma r - \sigma} J'_{r,\ell}(\zeta) = \frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \frac{J'(\varepsilon^\sigma \zeta)}{\varepsilon^{\sigma r - \sigma}}, \\
J''_{r,\ell}(\varepsilon^\sigma \zeta) &= \varepsilon^{\sigma r - 2\sigma} J''_{r,\ell}(\zeta) = \frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \frac{J''(\varepsilon^\sigma \zeta)}{\varepsilon^{\sigma r - 2\sigma}}.
\end{aligned} \tag{1.14}$$

For developments related to the functions with respect to $(r,\ \ell)$-symmetric points, refer to [33, 34].

Throughout this paper, we assume that $p,\ \ell \in \mathbb{N}$, $\varepsilon = \exp(2\pi i/\ell)$, and

$$J_{r,\ell}^{m,\lambda}(\theta,\ \vartheta,\ \gamma;\ \zeta) = \frac{1}{\ell}\sum_{\sigma=0}^{\ell-1}\varepsilon^{-\sigma r}\left(\Re_{\lambda}^{m}(\theta,\ \vartheta,\ \gamma)J(\varepsilon^{\sigma}\zeta)\right) = \zeta + \cdots, \tag{1.15}$$

$$(J \in \mathcal{A};\ \ell = 1,\ 2,\ \ldots;\ r = 0,\ 1,\ 2,\ \ldots(\ell-1)).$$

Clearly, for $r = \ell = 1$, we have

$$J_{1,1}^{m,\lambda}(\theta,\ \vartheta,\ \gamma;,\ \zeta) = \Re_{\lambda}^{m}(\theta,\ \vartheta,\ \gamma)J(\zeta).$$

From (1.15), we can get

$$J_{r,\ell}^{m,\lambda}(\theta,\ \vartheta,\ \gamma;\ \zeta) = \sum_{n=1}^{\infty}a_n\mathcal{M}_n\Upsilon_{n,r}^{\ell}\zeta^n, \quad (a_1 = \mathcal{M}_1 = 1), \quad \Upsilon_{n,r}^{\ell} = \frac{1}{\ell}\sum_{\sigma=0}^{\ell-1}\varepsilon^{(n-r)\sigma}, \tag{1.16}$$

where

$$\mathcal{M}_n = [1-\lambda+\lambda n]^m\ \frac{\Gamma(\delta+n-1)\,[\Gamma(\vartheta)]^{\gamma}}{(n-1)!\Gamma(\delta)\,[\Gamma(\theta n+\vartheta-\theta)]^{\gamma}}. \tag{1.17}$$

1.4. New classes of non-univalent functions with respect to $(r,\ \ell)$ symmetric points

Motivated by the class studied by Arango et al. [26] and Karthikeyan et al. [27], we now define the following. A function $J \in \mathcal{A}$ belongs to the class $\mathfrak{E}(\Re)$ if it satisfies

$$1 + \left[\frac{\zeta^2 J''(\zeta)}{J_{r,\ell}(\zeta)} + \frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)} - \frac{\zeta^2 J'_{r,\ell}(\zeta)J'(\zeta)}{(J_{r,\ell}(\zeta))^2}\right]e^{\frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)}-1} \prec \Re(\zeta), \tag{1.18}$$

where $\Re(\zeta) \in \mathcal{P}$ has a power series representation of the form (1.10).

Remark 1.1. *It is well-known that if $J \in \Upsilon_s^{\ell}(\Re)$, then $J_\ell(\zeta) \in \mathcal{S}^*$. Now the question arises if $J \in \mathfrak{E}(\Re)$ imply $J_{r,\ell}(\zeta) \in \mathcal{R}(\Re)$. Unfortunately, such an inclusion cannot be established, as the definition of $\mathfrak{E}(\Re)$ has an exponential term.*

Here, we provide the details of how the exponential term prevents us from obtaining the inclusion. From the definition of $\mathfrak{E}(\Re)$, we have

$$\mathrm{Re}\left(1 + \left[\frac{\zeta^2 J''(\zeta)}{J_{r,\ell}(\zeta)} + \frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)} - \frac{\zeta^2 J'_{r,\ell}(\zeta)J'(\zeta)}{(J_{r,\ell}(\zeta))^2}\right]e^{\frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)}-1}\right) > 0, \tag{1.19}$$

since Re$\{\psi(\zeta)\} > 0$. If we replace ζ by $\varepsilon^{\sigma}\zeta$ in (1.19), then (1.19) will be of the form

$$\mathrm{Re}\left\{1 + \left[\frac{\varepsilon^{2\sigma}\zeta^2 J''(\varepsilon^{\sigma}\zeta)}{J_{r,\ell}(\varepsilon^{\sigma}\zeta)} + \frac{\varepsilon^{\sigma}\zeta J'(\varepsilon^{\sigma}\zeta)}{J_{r,\ell}(\varepsilon^{\sigma}\zeta)} - \frac{\varepsilon^{2\sigma}\zeta^2 J'_{r,\ell}(\varepsilon^{\sigma}\zeta)J'(\varepsilon^{\sigma}\zeta)}{(J_{r,\ell}(\varepsilon^{\sigma}\zeta))^2}\right]e^{\frac{\varepsilon^{\sigma}\zeta J'(\varepsilon^{\sigma}\zeta)}{J_{r,\ell}(\varepsilon^{\sigma}\zeta)}-1}\right\} > 0, \tag{1.20}$$

where $z \in \mathcal{U}$; $\sigma = 0,\ 1,\ 2,\ \ldots,\ \eta-1$. Using (1.14) in (1.20), we get

$$\mathrm{Re}\left\{1 + \left[\frac{\varepsilon^{2\sigma}\zeta^2 J''(\varepsilon^{\sigma}\zeta)}{\varepsilon^{\sigma r}J_{r,\ell}(\zeta)} + \frac{\varepsilon^{\sigma}\zeta J'(\varepsilon^{\sigma}\zeta)}{\varepsilon^{\sigma r}J_{r,\ell}(\zeta)} - \frac{\varepsilon^{2\sigma}\zeta^2\,\varepsilon^{\sigma r-\sigma}J'_{r,\ell}(\zeta)J'(\varepsilon^{\sigma}\zeta)}{\varepsilon^{2\sigma r}\,(J_{r,\ell}(\zeta))^2}\right]e^{\frac{\varepsilon^{\sigma}\zeta J'(\varepsilon^{\sigma}\zeta)}{\varepsilon^{\sigma r}J_{r,\ell}(\zeta)}-1}\right\} > 0. \tag{1.21}$$

Since $\sigma = 0, 1, 2, \ldots, \eta - 1$ in (1.21), summing them in a way that obtains our result is not possible because of the exponential term in our definition. Precisely, $e^{x_1} + e^{x_2} + \cdots + e^{x_n}$ is not equal to $e^{x_1+x_2+\cdots+x_n}$, which we would have needed to obtain the inclusion.

To highlight the deviation of this present study from the class that was introduced and studied in [27], we consider the following example.

Example 1.2. *In [27], it was shown that $J(\zeta) = \frac{5\zeta}{5-\zeta}$ belongs to $\mathcal{R}(\mathfrak{K})$. Now for $J(\zeta) = \frac{5\zeta}{5-\zeta}$ with its corresponding function $J_{1,2}(\zeta) = \frac{10\zeta}{10-\zeta^2}$, on computation,* (1.18) *reduces to*

$$\Xi(\zeta) = 1 + \left(\frac{5\zeta\left(25-\zeta^2\right)}{(5-\zeta)^3} + \frac{5\left(25-\zeta^2\right)}{2(5-\zeta)^2} - \left(\frac{5\left(25-\zeta^2\right)}{2(5-\zeta)^2} \right) \left(\frac{25+\zeta^2}{25-\zeta^2} \right) \right) e^{\frac{5\left(25-\zeta^2\right)}{2(5-\zeta)^2} - 1}.$$

The function $\Xi(\zeta)$ does not map the unit disc entirely onto a right half-plane (see Figure 3); hence the function $J(\zeta) = \frac{5\zeta}{5-\zeta}$ does not belong to $\mathfrak{E}(\mathfrak{K})$ for the choice of $r = 1$ and $\ell = 2$.

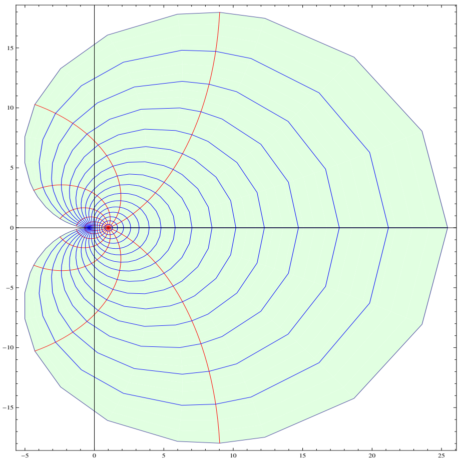
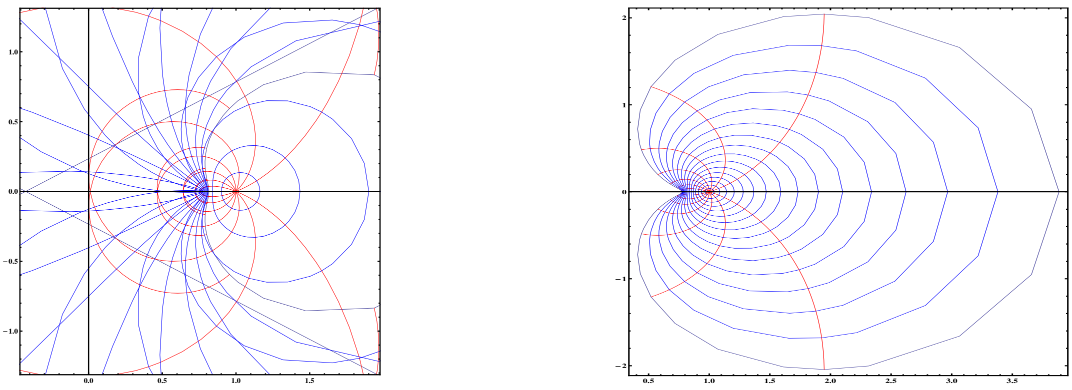


Figure 3. The image of $\Xi(\zeta)$ if the domain is a unit disc.

Example 1.3. *Let $J(\zeta) = \frac{\zeta}{(1-\zeta)^2}$, which is a well-known extremal function of class $\mathcal{S}$. On replacing $J(\zeta)$ with a Koebe function in* (1.18)*, we have*

$$\mathfrak{B}(\zeta) = 1 + \left(\frac{2\zeta(\zeta+2)}{(1-\zeta)^2} + \frac{1+z}{1-z} - \left(\frac{(1+z)}{(1-z)} \right)^2 \right) e^{\frac{1+\zeta}{1-\zeta} - 1}.$$

We can see that it does not belong to $\mathfrak{E}(\mathfrak{K})$ for the choice of $r = 1$ and $\ell = 1$ (see Figure 4(a)). However, for a disc of a smaller radius $J(\zeta) = \frac{\zeta}{(1-\zeta)^2}$ belongs to $\mathfrak{E}(\mathfrak{K})$ for the choice of $r = 1$ and $\ell = 1$ (see Figure 4(b)).



(a) $|\zeta| < 1$ **(b)** $|\zeta| < 0.3$

Figure 4. Images of the unit disc under $\mathfrak{B}(\zeta)$ if (a) $|\zeta| < 1$; (b) $|\zeta| < 0.3$.

Now we will focus our discussion on what motivated the definition of the function class $\mathfrak{E}(\mathfrak{K})$. Let us suppose that a function $p(\zeta) \in \mathcal{P}$ satisfies $p(\zeta) \prec \mathfrak{K}(\zeta)$ and also suppose that $J \in \mathfrak{E}(\mathfrak{K})$. Now (1.18) can be rewritten as

$$\left[\frac{\zeta^2 J''(\zeta)}{J_{r,\ell}(\zeta)} + \frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)} - \frac{\zeta^2 J'_{r,\ell}(\zeta) J'(\zeta)}{(J_{r,\ell}(\zeta))^2}\right] e^{\frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)} - 1} = p(\zeta) - 1. \tag{1.22}$$

Equivalently, (1.22) can be rewritten as

$$\frac{d}{d\zeta}\left[e^{\frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)} - 1}\right] = \frac{p(\zeta) - 1}{\zeta}, \tag{1.23}$$

which, upon integrating, would lead to

$$\frac{\zeta J'(\zeta)}{J_{r,\ell}(\zeta)} - 1 = \log\left\{1 + \int_0^{\zeta} \frac{p(t) - 1}{t} dt\right\}, \quad (\zeta \in \mathcal{U};\ \log 1 = 0). \tag{1.24}$$

On replacing ζ by $\varepsilon^{\sigma}\zeta$ in (1.24), we have

$$\frac{\varepsilon^{\sigma}\zeta J'(\varepsilon^{\sigma}\zeta)}{J_{r,\ell}(\varepsilon^{\sigma}\zeta)} - 1 = \log\left\{1 + \int_0^{\varepsilon^{\sigma}\zeta} \frac{p(t) - 1}{t} dt\right\}, \quad (\zeta \in \mathcal{U};\ \log 1 = 0). \tag{1.25}$$

From (1.14), Eq (1.25) reduces to

$$\frac{\varepsilon^{\sigma - \sigma r}\zeta J'(\varepsilon^{\sigma}\zeta)}{J_{r,\ell}(\zeta)} - 1 = \log\left\{1 + \int_0^{\varepsilon^{\sigma}\zeta} \frac{p(t) - 1}{t} dt\right\}, \quad (\zeta \in \mathcal{U};\ \log 1 = 0). \tag{1.26}$$

If we let $\sigma = 0, 1, 2, \ldots, \ell - 1$ in (1.26) and sum them, we get

$$\frac{J'_{r,\ell}(\zeta)}{J_{r,\ell}(\zeta)} - \frac{1}{\zeta} = \frac{1}{\zeta}\left(\frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \log\left\{1 + \int_0^{\varepsilon^{\sigma}\zeta} \frac{p(t) - 1}{t} dt\right\}\right), \quad (\zeta \in \mathcal{U};\ \log 1 = 0). \tag{1.27}$$

If we integrate the expression (1.27), on simplification, we have

$$J_{r,\ell}(\zeta) = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{\eta}\left(\frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \log\left\{1 + \int_0^{\varepsilon^{\sigma}\eta} \frac{p(t) - 1}{t} dt\right\}\right) d\eta\right\}. \tag{1.28}$$

Upon summarizing the above mentioned deliberation, we get the following result.

Theorem 1.4. *A function $J \in \mathfrak{E}(\mathfrak{R})$ if and only if a function $p \in \mathcal{P}$ exists such that $p \prec \mathfrak{R}$ and*

$$J_{r,\ell}(\zeta) = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{\eta}\left(\frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \log\left\{1 + \int_0^{\varepsilon^{\sigma}\eta} \frac{p(t)-1}{t}dt\right\}\right)d\eta\right\}, \quad (\zeta \in \mathcal{U};\ \log 1 = 0). \tag{1.29}$$

To make this study more versatile and broad, we study the class $\mathfrak{E}(\mathfrak{R})$ involving the operator $\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J$. The primary motive to redefine the class $\mathfrak{E}(\mathfrak{R})$ involving the operator is to generalize and unify various new and well-known studies pertaining to univalent function theory. The presence of the parameter ϑ and δ in $\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)$ will help us to reduce the analytic characterization related to starlikeness and convexity, whereas λ will helps us to reduce the analytic characterization related to the well-known Mocanu class.

Definition 1.5. *A function $J \in \mathcal{A}$ belongs to the class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$ if it satisfies*

$$1 + \left[\frac{\zeta\left[\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} - \frac{\zeta^2\left[J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)\right]'\left[\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{\left[J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)\right]^2}\right.$$
$$\left. + \frac{\zeta^2\left[\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]''}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)}\right] e^{\frac{\zeta\left[\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} - 1} \prec \mathfrak{R}(\zeta), \tag{1.30}$$

where $\mathfrak{R}(\zeta) \in \mathcal{P}$ has a power series representation of the form (1.10).

Remark 1.6. *Note that $J \in \mathcal{A}$ belongs to the class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$ if and only if $\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J \in \mathfrak{E}(\mathfrak{R})$. If we let $r = 1$ and $\ell = 2$ in the class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$,* (1.30) *will reduce to*

$$1 + \zeta\left[\frac{2\zeta\left[\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta) - \mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(-\zeta)} - 1\right]' e^{\frac{2\zeta\left[\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{\mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta) - \mathfrak{N}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(-\zeta)} - 1} \prec \mathfrak{R}(\zeta).$$

We use $\mathfrak{S}\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$ to denote the class of functions in $\mathcal{A}$ satisfying the abovementioned condition.

For $m = \theta = 0$, $r = 0$, and $\delta = \ell = 1$, the class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$ reduces to the class recently studied by Karthikeyan et al. in [27]. Upon varying the parameters involved in the class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$, we get the class $\mathfrak{E}(\mathfrak{R})$.

Remark 1.7. *Louis de Branges [35] proved the Bieberbach conjecture which states that the coefficients of the univalent functions $J \in \mathcal{A}$ are majorized by those of the Koebe function which maps the unit disc on to a radially slit plane. Hence from the coefficients' estimates that will presented in the next section, we can establish that the functions $J \in \mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$ are not univalent.*

Using arguments similar to those detailed in Theorem 1.4, we can prove the following integral representation.

Theorem 1.8. *A function $J \in \mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{R})$ if and only if a function $p \in \mathcal{P}$ exists such that $p \prec \mathfrak{R}$ and*

$$J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta) = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{\eta}\left(\frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \log\left\{1 + \int_0^{\varepsilon^{\sigma}\eta} \frac{p(t)-1}{t}dt\right\}\right)d\eta\right\}, \tag{1.31}$$

where $\zeta \in \mathcal{U}$ and $\log(.)$ *denotes the single valued branch of the complex logarithm with* $\log 1 = 0$.

Proof. Let us suppose that a function $p(\zeta) \in \mathcal{P}$ satisfies $p(\zeta) \prec \mathfrak{K}(\zeta)$ and also suppose that $J \in \mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{K})$. Now (1.30) can be rewritten as

$$\left[\frac{\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} - \frac{\zeta^2\left[J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)\right]'\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{\left[J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)\right]^2}\right.$$
$$\left.+\frac{\zeta^2\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]''}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)}\right] e^{\frac{\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta,\vartheta,\gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)}-1} = p(\zeta) - 1. \tag{1.32}$$

Equivalently, (1.32) can be rewritten as

$$\frac{d}{d\zeta}\left[e^{\frac{\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta,\vartheta,\gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)}-1}\right] = \frac{p(\zeta)-1}{\zeta}, \tag{1.33}$$

which, upon integrating, would lead to

$$e^{\frac{\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta,\vartheta,\gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)}-1} = 1 + \int_0^{\zeta} \frac{p(t)-1}{t}dt, \quad (\zeta \in \mathcal{U}). \tag{1.34}$$

Further simplifying (1.34), we have

$$\frac{\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} - 1 = \log\left[1 + \int_0^{\zeta} \frac{p(t)-1}{t}dt\right], \quad (\zeta \in \mathcal{U}), \tag{1.35}$$

where log(.) in (1.35) denotes the single valued branch of the complex logarithm with $\log 1 = 0$. On replacing ζ by $\varepsilon^{\sigma}\zeta$ in (1.35), we have

$$\frac{\varepsilon^{\sigma}\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\varepsilon^{\sigma}\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \varepsilon^{\sigma}\zeta)} - 1 = \log\left[1 + \int_0^{\varepsilon^{\sigma}\zeta} \frac{p(t)-1}{t}dt\right], \quad (\zeta \in \mathcal{U}). \tag{1.36}$$

From (1.14), Eq (1.36) reduces to

$$\frac{\varepsilon^{\sigma-\sigma r}\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\varepsilon^{\sigma}\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \varepsilon\zeta)} - 1 = \log\left\{1 + \int_0^{\varepsilon^{\sigma}\zeta} \frac{p(t)-1}{t}dt\right\}, \quad (\zeta \in \mathcal{U};\ \log 1 = 0). \tag{1.37}$$

Let $\sigma = 0, 1, 2, \ldots, \ell - 1$ in (1.37). By summing them, we get

$$\frac{\left[J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} - \frac{1}{\zeta} = \frac{1}{\zeta}\left(\frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \log\left\{1 + \int_0^{\varepsilon^{\sigma}\zeta} \frac{p(t)-1}{t}dt\right\}\right), \quad (\zeta \in \mathcal{U};\ \log 1 = 0). \tag{1.38}$$

Integrating the expression (1.38) and by simplifying, we have

$$J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta) = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{\eta}\left(\frac{1}{\ell}\sum_{\sigma=0}^{\ell-1} \log\left\{1 + \int_0^{\varepsilon^{\sigma}\eta} \frac{p(t)-1}{t}dt\right\}\right)d\eta\right\}. \tag{1.39}$$

The difficulty of integrating (1.38) with the presence of the first-order pole at the origin has been avoided by integrating from ζ_0 to ζ and then letting $\zeta_0 \to 0$.

□

Theorem 1.9. *A function $J \in \mathfrak{S}\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \Re)$ if and only if a function $p \in \mathcal{P}$ exists such that $p \prec \Re$ and*

$$\frac{\Re_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta) - \Re_{\lambda}^{m}(\theta, \vartheta, \gamma)J(-\zeta)}{2} = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{\eta}\left(\frac{1}{2}\sum_{\sigma=0}^{1}\log\left\{1 + \int_0^{\varepsilon^{\sigma}\eta} \frac{p(t)-1}{t}dt\right\}\right)d\eta\right\},$$

where $\zeta \in \mathcal{U}$ and $\log(.)$ *denotes the single-valued branch of the complex logarithm with* $\log 1 = 0$.

2. Initial coefficient bounds and other main results:

Lemma 2.1. *[36, Theorem 1] If $L(\zeta) = 1 + \sum\limits_{r=1}^{\infty} l_r\zeta^r \in \mathcal{P}$ and $\rho \in \mathbb{C}$, then*

$$|l_{\varepsilon} - \rho l_r l_{\varepsilon-r}| \leq 2\max\{1; |2\rho - 1|\},$$

for all $1 \leq r \leq \varepsilon - 1$.

We will now proceed to find the upper bound of the coefficient estimates in $J \in \mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \Re)$.

Theorem 2.2. *If we let $J(\zeta) \in \mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \Re)$, then we have*

$$|a_2| \leq \frac{\kappa_1}{\left|\mathcal{M}_2\left(3 - 2\Upsilon_{2,r}^{\ell}\right)\right|}, \tag{2.1}$$

$$|a_3| \leq \frac{\kappa_1}{\left|\mathcal{M}_3\left(7 - 3\Upsilon_{3,r}^{\ell}\right)\right|}\max\left\{1; \left|\frac{\kappa_2}{\kappa_1} - \frac{\kappa_1\left(7 - 12\Upsilon_{2,r}^{\ell} + 4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3 - 2\Upsilon_{2,r}^{\ell}\right)^2}\right|\right\} \tag{2.2}$$

and for all $\rho \in \mathbb{C}$, we have

$$\left|a_3 - \rho a_2^2\right| \leq \frac{\kappa_1}{\left|\mathcal{M}_3\left(7 - 3\Upsilon_{3,r}^{\ell}\right)\right|}\max\left\{1; \left|\frac{\kappa_2}{\kappa_1} - \frac{\kappa_1\left(7 - 12\Upsilon_{2,r}^{\ell} + 4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3 - 2\Upsilon_{2,r}^{\ell}\right)^2} - \frac{\rho\kappa_1\mathcal{M}_3\left(7 - 3\Upsilon_{3,r}^{\ell}\right)}{\mathcal{M}_2^2\left(3 - 2\Upsilon_{2,r}^{\ell}\right)^2}\right|\right\}, \tag{2.3}$$

where $\Upsilon_{n,r}^{\ell}$ and $\mathcal{M}_n$ are defined as shown in (1.16) *and* (1.17)*, respectively. The inequalities are sharp.*

Proof. As $J(\zeta) \in \mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \Re)$, by (1.30), we have

$$1 + \left[\frac{\zeta\left[\Re_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} - \frac{\zeta^2\left[J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)\right]'\left[\Re_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{\left[J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)\right]^2} + \frac{\zeta^2\left[\Re_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]''}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)}\right] e^{\frac{\zeta\left[\Re_{\lambda}^{m}(\theta,\vartheta,\gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)} - 1} = \Re\left[w(\zeta)\right]. \tag{2.4}$$

Let $\vartheta(\zeta) \in \mathcal{P}$ be of the form $\vartheta(\zeta) = 1 + \sum_{n=1}^{\infty}\vartheta_n\zeta^n$, we consider

$$\vartheta(\zeta) = \frac{1 + w(\zeta)}{1 - w(\zeta)},$$

where $w(\zeta)$ is such that $w(0)=0$ and $|w(\zeta)|<1$. On simple computation, we have

$$\begin{aligned} w(\zeta) &= \frac{\vartheta(\zeta)-1}{\vartheta(\zeta)+1} \\ &= \frac{\vartheta_1\zeta+\vartheta_2\zeta^2+\vartheta_3\zeta^3+\cdots}{2+\vartheta_1\zeta+\vartheta_2\zeta^2+\vartheta_3\zeta^3+\cdots} \\ &= \frac{1}{2}\vartheta_1\zeta+\frac{1}{2}\left(\vartheta_2-\frac{1}{2}\vartheta_1^2\right)\zeta^2+\frac{1}{2}\left(\vartheta_3-\vartheta_1\vartheta_2+\frac{1}{4}\vartheta_1^3\right)\zeta^3+\cdots. \end{aligned} \tag{2.5}$$

Using (2.5) in (1.10), we have

$$\begin{aligned} \Re\left[w(\zeta)\right] &= 1+\kappa_1 w(\zeta)+\kappa_2[w(\zeta)]^2+\kappa_3[w(\zeta)]^3+\cdots \\ &= 1+\frac{\vartheta_1\kappa_1}{2}\zeta+\frac{\kappa_1}{2}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{\kappa_2}{\kappa_1}\right)\right]\zeta^2+\cdots. \end{aligned} \tag{2.6}$$

The left-hand side of (2.4) will be of the form

$$\begin{aligned} &1+\left[\frac{\zeta\left[\Re_\lambda^m(\theta,\vartheta,\gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)}-\frac{\zeta^2\left[J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)\right]'\left[\Re_\lambda^m(\theta,\vartheta,\gamma)J(\zeta)\right]'}{\left[J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)\right]^2}\right. \\ &\left.+\frac{\zeta^2\left[\Re_\lambda^m(\theta,\vartheta,\gamma)J(\zeta)\right]''}{J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)}\right]e^{\frac{\zeta\left[\Re_\lambda^m(\theta,\vartheta,\gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta,\vartheta,\gamma;\zeta)}-1} = 1+a_2\mathcal{M}_2\left(3-2\Upsilon_{2,r}^\ell\right)\zeta+ \\ &\left[a_3\mathcal{M}_3\left(7-3\Upsilon_{3,r}^\ell\right)+a_2^2\mathcal{M}_2^2\left(7-12\Upsilon_{2,r}^\ell+4\left(\Upsilon_{2,r}^\ell\right)^2\right)\right]\zeta^2+\cdots. \end{aligned} \tag{2.7}$$

From (2.7) and (2.6), we obtain

$$a_2=\frac{\vartheta_1\kappa_1}{2\mathcal{M}_2\left(3-2\Upsilon_{2,r}^\ell\right)} \tag{2.8}$$

and

$$a_3=\frac{\kappa_1}{2\mathcal{M}_3\left(7-3\Upsilon_{3,r}^\ell\right)}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{\kappa_2}{\kappa_1}+\frac{\kappa_1\left(7-12\Upsilon_{2,r}^\ell+4\left(\Upsilon_{2,r}^\ell\right)^2\right)}{\left(3-2\Upsilon_{2,r}^\ell\right)^2}\right)\right]. \tag{2.9}$$

Equation (2.1) can be obtained by applying $|\vartheta_1|\le 2$ ([37, p. 41]) in (2.8). Taking the modulus and using Lemma 2.1 in the equality of a_3, we get the bounds for a_3.

Now to prove (2.3), we consider

$$\begin{aligned} \left|a_3-\rho a_2^2\right| &= \left|\frac{\kappa_1}{2\mathcal{M}_3\left(7-3\Upsilon_{3,r}^\ell\right)}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{\kappa_2}{\kappa_1}+\frac{\kappa_1\left(7-12\Upsilon_{2,r}^\ell+4\left(\Upsilon_{2,r}^\ell\right)^2\right)}{\left(3-2\Upsilon_{2,r}^\ell\right)^2}\right)\right]-\frac{\rho\vartheta_1^2\kappa_1^2}{4\mathcal{M}_2^2\left(3-2\Upsilon_{2,r}^\ell\right)^2}\right| \\ &= \left|\frac{\kappa_1}{2\mathcal{M}_3\left(7-3\Upsilon_{3,r}^\ell\right)}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{\kappa_2}{\kappa_1}+\frac{\kappa_1\left(7-12\Upsilon_{2,r}^\ell+4\left(\Upsilon_{2,r}^\ell\right)^2\right)}{\left(3-2\Upsilon_{2,r}^\ell\right)^2}+\frac{\rho\kappa_1\mathcal{M}_3\left(7-3\Upsilon_{3,r}^\ell\right)}{\mathcal{M}_2^2\left(3-2\Upsilon_{2,r}^\ell\right)^2}\right)\right]\right| \end{aligned}$$

$$\leq \frac{\kappa_1}{2\left|\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)\right|}\left[2+\frac{|\vartheta_1|^2}{2}\left(\left|\frac{\kappa_2}{\kappa_1}-\frac{\kappa_1\left(7-12\Upsilon_{2,r}^{\ell}+4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}-\frac{\rho\kappa_1\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)}{\mathcal{M}_2^2\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}\right|-1\right)\right]. \tag{2.10}$$

Taking

$$\mathfrak{B} := \left|\frac{\kappa_2}{\kappa_1}-\frac{\kappa_1\left(7-12\Upsilon_{2,r}^{\ell}+4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}-\frac{\rho\kappa_1\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)}{\mathcal{M}_2^2\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}\right|,$$

if $\mathfrak{B} \leq 1$, from (2.10), we obtain

$$\left|a_3-\rho a_2^2\right| \leq \frac{\kappa_1}{\left|\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)\right|}. \tag{2.11}$$

Further, if $\mathfrak{B} \geq 1$ from (2.10), we deduce

$$\left|a_3-\rho a_2^2\right| \leq \frac{\kappa_1}{\left|\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)\right|}\left|\frac{\kappa_2}{\kappa_1}-\frac{\kappa_1\left(7-12\Upsilon_{2,r}^{\ell}+4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}-\frac{\rho\kappa_1\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)}{\mathcal{M}_2^2\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}\right|. \tag{2.12}$$

An examination of the proof shows that the equality for (2.11) holds if $\vartheta_1 = 0$ and $\vartheta_2 = 2$. Equivalently, by Lemma 2.1, we have $\mathfrak{K}(\zeta^2) = \mathfrak{K}_2(\zeta) = \dfrac{1+\zeta^2}{1-\zeta^2}$. Therefore, the extremal function of the class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{K})$ is given by

$$\frac{\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} = 1+\log\left[\int_0^{\zeta}\left(\frac{1+t^2}{1-t^2}-1\right)\frac{dt}{t}\right].$$

Similarly, the equality for (2.12) holds if $\vartheta_2 = 2$. Equivalently, by Lemma 2.1, we have $\mathfrak{K}(\zeta) = \mathfrak{K}_1(\zeta) = \dfrac{1+\zeta}{1-\zeta}$. Therefore, the extremal function in $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{K})$ is given by

$$\frac{\zeta\left[\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)\right]'}{J_{r,\ell}^{m,\lambda}(\theta, \vartheta, \gamma; \zeta)} = 1+\log\left[\int_0^{\zeta}\left(\frac{1+t}{1-t}-1\right)\frac{dt}{t}\right].$$

The proof of the theorem is complete. □

Corollary 2.3. *If we let $J(\zeta) \in \mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma;\ 1+\zeta e^{\zeta})$, then we have*

$$|a_2| \leq \frac{1}{\left|\mathcal{M}_2\left(3-2\Upsilon_{2,r}^{\ell}\right)\right|},$$

$$|a_3| \leq \frac{1}{\left|\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)\right|}\max\left\{1;\ \left|1-\frac{\left(7-12\Upsilon_{2,r}^{\ell}+4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}\right|\right\}$$

and for all $\rho \in \mathbb{C}$, we have

$$\left|a_3-\rho a_2^2\right| \leq \frac{1}{\left|\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)\right|}\max\left\{1;\left|1-\frac{\left(7-12\Upsilon_{2,r}^{\ell}+4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}-\frac{\rho\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)}{\mathcal{M}_2^2\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}\right|\right\},$$

where $\Upsilon_{n,r}^{\ell}$ and $\mathcal{M}_n$ are defined as in (1.16) *and* (1.17)*, respectively. The inequalities are sharp.*

Proof. The function $\Re(\zeta) = 1+\zeta e^{\zeta} \in \mathcal{P}$ (see [38]) maps the unit disc on to a cardioid. On expanding, we can find that coefficient of ζ and ζ^2 is 1. Replacing $\kappa_1 = \kappa_2 = 1$ in Theorem 2.2, we get the proof of the theorem. □

Corollary 2.4. *If we let $J(\zeta) \in \mathfrak{E}_{\lambda}^{m}\left(\theta,\, \vartheta,\, \gamma;\, \frac{2\sqrt{1+\zeta}}{1+e^{-\zeta}}\right)$, then we have*

$$|a_2| \leq \frac{1}{\left|\mathcal{M}_2\left(3-2\Upsilon_{2,r}^{\ell}\right)\right|},$$

$$|a_3| \leq \frac{1}{\left|\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)\right|}\max\left\{1;\left|\frac{1}{8}-\frac{\left(7-12\Upsilon_{2,r}^{\ell}+4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}\right|\right\}$$

and for all $\rho \in \mathbb{C}$, we have

$$\left|a_3-\rho a_2^2\right| \leq \frac{1}{\left|\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)\right|}\max\left\{1;\left|\frac{1}{8}-\frac{\left(7-12\Upsilon_{2,r}^{\ell}+4\left(\Upsilon_{2,r}^{\ell}\right)^2\right)}{\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}-\frac{\rho\mathcal{M}_3\left(7-3\Upsilon_{3,r}^{\ell}\right)}{\mathcal{M}_2^2\left(3-2\Upsilon_{2,r}^{\ell}\right)^2}\right|\right\},$$

where $\Upsilon_{n,r}^{\ell}$ and $\mathcal{M}_n$ are defined as in (1.16) *and* (1.17)*, respectively. The inequalities are sharp.*

Proof. The function $\Re(\zeta) = \frac{2\sqrt{1+\zeta}}{1+e^{-\zeta}} \in \mathcal{P}$ (see [39]) maps the unit disc on to a balloon-shaped region. On expanding, we can find that the coefficients of ζ and ζ^2 are 1 and $\frac{1}{8}$, respectively. Replacing $\kappa_1 = \kappa_2 = \frac{1}{8}$ in Theorem 2.2, we get the proof of the theorem. □

Letting $\ell = 2$ and $r = 1$ in Theorem 2.2, we get the following:

Corollary 2.5. *If we let $J(\zeta) \in \mathfrak{S}\mathfrak{E}_{\lambda}^{m}(\theta,\, \vartheta,\, \gamma;\, \Re)$ (see Remark 1.1), then we have*

$$|a_2| \leq \frac{\kappa_1}{3\,|\mathcal{M}_2|}, \quad |a_3| \leq \frac{\kappa_1}{4\,|\mathcal{M}_3|}\max\left\{1;\left|\frac{\kappa_2}{\kappa_1}-\frac{7\kappa_1}{9}\right|\right\}$$

and for all $\rho \in \mathbb{C}$, we have

$$\left|a_3-\rho a_2^2\right| \leq \frac{\kappa_1}{4\,|\mathcal{M}_3|}\max\left\{1;\left|\frac{\kappa_2}{\kappa_1}-\frac{7\kappa_1}{9}-\frac{4\rho\kappa_1\mathcal{M}_3}{9\mathcal{M}_2^2}\right|\right\},$$

where $\Upsilon_{n,r}^{\ell}$ and $\mathcal{M}_n$ are defined as in (1.16) *and* (1.17)*, respectively. The inequalities are sharp.*

Letting $\ell = 2$ and $r = 1$ in Theorem 2.2, we get the following:

Corollary 2.6. *Let $J(\zeta) \in \mathcal{A}$ satisfy the condition*

$$1 + \left[\frac{2\zeta^2 J''(\zeta)}{[J(\zeta) - J(-\zeta)]} + \frac{2\zeta J'(\zeta)}{[J(\zeta) - J(-\zeta)]} - \frac{2\zeta^2 \left[J(\zeta) - J(-\zeta)\right]' J'(\zeta)}{\left([J(\zeta) - J(-\zeta)]\right)^2} \right] e^{\frac{2\zeta J'(\zeta)}{[J(\zeta)-J(-\zeta)]} - 1} \prec \Re(\zeta).$$

We then have

$$|a_2| \leq \frac{\kappa_1}{3}, \quad |a_3| \leq \frac{\kappa_1}{4} \max\left\{1; \left|\frac{\kappa_2}{\kappa_1} - \frac{7\kappa_1}{9}\right|\right\}$$

and for all $\rho \in \mathbb{C}$, we have

$$\left|a_3 - \rho a_2^2\right| \leq \frac{\kappa_1}{4} \max\left\{1; \left|\frac{\kappa_2}{\kappa_1} - \frac{(7 + 4\rho)\kappa_1}{9}\right|\right\}.$$

The inequalities are sharp.

Letting $m = \theta = 0$, $r = 0$, and $\delta = \ell = 1$ in Theorem 2.2, we get the following result.

Corollary 2.7. *[27] If we let $J(\zeta) \in \mathcal{E}(\Re(\zeta))$, then we have*

$$|a_2| \leq \kappa_1 \quad \text{and} \quad |a_3| \leq \frac{\kappa_1}{4} \max\left\{1; \left|\frac{\kappa_2}{\kappa_1} + \kappa_1\right|\right\}$$

and for all $\rho \in \mathbb{C}$, we have

$$\left|a_3 - \rho a_2^2\right| \leq \frac{\kappa_1}{4} \max\left\{1; \left|\frac{\kappa_2}{\kappa_1} + (1 - 4\rho)\kappa_1\right|\right\}.$$

The inequalities are sharp.

Letting $\Psi(\zeta) = \frac{1+\zeta}{1-\zeta}$ in Theorem 2.2, we get the following result.

Corollary 2.8. *[27] If we let $J(\zeta) \in \mathcal{E}(1 + \zeta/1 - \zeta)$, then we have*

$$|a_2| \leq 2 \quad \text{and} \quad |a_3| \leq \frac{3}{2}$$

and for all $\rho \in \mathbb{C}$, we have

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{2} \max\left\{1; |3 - 8\rho|\right\}.$$

The inequalities are sharp.

3. Conclusions

The function class $\mathfrak{E}(\mathfrak{K})$ has been defined, whose characterization involved the exponential function and a function which unifies the notion of odd, even, and ℓ-symmetrical functions. Further, we defined the class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{K})$ by replacing $J(\zeta)$ with an operator $\mathfrak{R}_{\lambda}^{m}(\theta, \vartheta, \gamma)J(\zeta)$ in the class $\mathfrak{E}(\mathfrak{K})$. The purpose of studying a family of analytic function in tandem with the Le Roy type Mittag-Leffler function and exponential functions is to unify and generalize the recent studies of various other researchers. It is well-known that family of starlike functions preserves the symmetric property, which is the reason we introduced a family of analytic functions with respect to symmetric points. In Remark 1.1, we established that the important inclusion relation analogous to $J \in \Upsilon_{s}^{\ell}(\mathfrak{K}) \Longrightarrow J_{\ell}(\zeta) \in \mathcal{S}^{*}$ does not hold for the family $\mathfrak{E}(\mathfrak{K})$.

The defined function class $\mathfrak{E}_{\lambda}^{m}(\theta, \vartheta, \gamma; \mathfrak{K})$ is not univalent in $\mathcal{U}$ but it consists of functions which can be univalent, non-univalent, starlike and convex functions, which we illustrated in the introduction section. We obtained several interesting properties of the defined function class, for example, the inclusion relationship, initial coefficient bounds, and the Fekete-Szegő inequalities.

As a concluding remark, we will now enumerate further extensions of this paper.

1. Radius problems which determine the largest disk $|\zeta| < r$ for which the functions from $\mathfrak{E}(\mathfrak{K})$ satisfy geometric and algebraic properties like symmetry, convexity, starlikeness, and univalence.
2. On replacing the classical derivative with a quantum derivative, the class $\mathfrak{E}(\mathfrak{K})$ can be redefined and studied. But it should be noted that replacing $e^{[\ln f(z)]'-1}$ in $\mathfrak{E}(\mathfrak{K})$ with $e_q^{D_q[\ln f(z)]-1}$ (where D_q denotes the quantum derivative) would be incorrect, as ln and e_q are not inverse to each other. So defining such a class would require many adaptations (see [40]).

Appendix

Here, we provide the Mathematica code that was used to calculate the coefficients' estimates and some graphs.

Figure A5 is the screenshot of the input and output that was obtained on entering the following code to obtain the coefficient estimate:

```
Series[1 +(((2*a*z + 6*b*z^2 + 12*c*z^3)/
(1 + R*z + S*z^2 + T*z^3) + (1 + 2*a*z + 3*b*z^2 + 4*c*z^3)/
(1 + R*z + S*z^2 +T*z^3) - (((1 + 2*a*z + 3*b*z^2 + 4*c*z^3)
*(1 + 2*R*z +3*S*z^2 + 4*T*z^3))/((1 + a*z + b*z^2 + c*z^3)
*(1 + R*z +S*z^2 + T*z^3))))*(E^((1 + 2*a*z + 3*b*z^2 + 4*c*z^3)
/(1 + R*z + S*z^2 + T*z^3) - 1))), {z, 0, 3}].
```

Similarly, the following code was used to compute the Figure 3:

```
ParametricPlot[
 With[{z = r*E^(I*t)}, {Re[1 + ((5*z*(25 - z^2))/
 (5 - z)^3 + (5*(25 - z^2))/(2*(5 - z)^2) - ((5*(25 - z^2))/
 (2*(5 - z)^2)) ((25 + z^2)/(25 - z^2)))
 E^((5*(25 - z^2))/(2*(5 - z)^2) - 1)],
 Im[1 + ((5*z*(25 - z^2))/(5 - z)^3 + (5*(25 - z^2))/(
```

```
2∗(5 − z)^2) − ((5∗(25 − z^2))/(2∗(5 − z)^2)) ((25 + z^2)/
(25 − z^2))) E^((5∗(25 − z^2))/(2∗(5 − z)^2) − 1)]}], {t, 0,
  2 Pi}, {r, 0.0, 1}, PlotStyle −> LightGreen, PlotRange −> All,
 MeshStyle −> {Red, Blue}, ImageSize −> 600]
```

Input:

Series[

$$1+\left(\left(\frac{2*a*z+6*b*z^2+12*c*z^3}{1+R*z+S*z^2+T*z^3}+\frac{1+2*a*z+3*b*z^2+4*c*z^3}{1+R*z+S*z^2+T*z^3}-\left(\frac{(1+2*a*z+3*b*z^2+4*c*z^3)*(1+2*R*z+3*S*z^2+4*T*z^3)}{(1+a*z+b*z^2+c*z^3)*(1+R*z+S*z^2+T*z^3)}\right)\right)*\left(e^{\frac{1+2*a*z+3*b*z^2+4*c*z^3}{1+R*z+S*z^2+T*z^3}}-1\right)\right),\{z,0,3\}]$$

Output: $1+(3a-2R)\,z+\left(7a^2+7b-12aR+4R^2-3S\right)z^2+\left(7a^3+26ab+13c-26a^2R-24bR+\frac{53aR^2}{2}-7R^3-15aS+10RS-4T\right)z^3+O[z]^4$

Figure A5. Screenshot of the input and output from Mathematica Software.

Author contributions

Kadhavoor R. Karthikeyan: Conceptualization, investigation, validation, data interpretation, visualization, writing the original draft, manuscript review and editing, supervision, project administration, funding acquisition; Sakkarai Lakshmi: Conceptualization, investigation, validation, methodology, data interpretation, visualization, writing the original draft, manuscript review and editing, supervision; Dharmaraj Mohankumar: Conceptualization, investigation, methodology, experimentation, validation, data interpretation, visualization, writing the original draft, manuscript review and editing, data interpretation, resource management for the project. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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