



Research article

Dynamic analysis and finite-time control of a memristor-based six-dimensional hyperchaotic system

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Abstract: Memristors possess memory and nonlinear characteristics. Constructing high-dimensional nonlinear circuit systems using memristors can enhance the nonlinear coupling relationships among state variables and generate more complex dynamic behaviors. However, as the system's dimensions and nonlinear coupling relationships increase, synchronization control of high-dimensional hyperchaotic systems becomes more difficult. Meanwhile, synchronization control methods for high-dimensional hyperchaotic systems often suffer from complex controller structures, model dependence, and an uncertain convergence time. Therefore, this paper designs a six-dimensional hyperchaotic circuit using memristors and circuit components to enhance the nonlinear characteristics and dynamic complexity of the system. By applying Kirchhoff's current law and Kirchhoff's voltage law, the dynamic equations of the circuit state variables are established. Furthermore, the dynamic analysis, Multisim circuit simulation, and FPGA (Field Programmable Gate Array) hardware implementation results verify that the system exhibits hyperchaotic behavior. In addition, this paper proposes a finite-time synchronization control scheme with a simple structure and an explicit convergence time estimate, which enables the synchronization of hyperchaotic systems within a finite time. Based on Lyapunov stability theory and matrix inequalities, sufficient conditions for finite-time synchronization are derived. Numerical simulations further verify the effectiveness and feasibility of the proposed scheme.

Keywords: finite-time control; synchronization; memristor; hyperchaotic system; Lyapunov stability

Mathematics Subject Classification: 37D45, 93C10, 93C27, 93D05, 93D40

1. Introduction

A memristor is a nonlinear electronic component with memory function. Its resistance depends on the history of the charge passing through it, and it can maintain its previous resistive state even after the power is cut off. Chua [1] predicted the existence of the memristor in 1971 on the basis of the completeness of circuit theory, Strukov et al. [2] physically realized the memristor in 2008 and established its corresponding mathematical model. Owing to its memory and nonlinear characteristics, the memristor has shown great application potential in many fields, including neural networks [3, 4], analog circuit design [5], chaotic systems [6], and encryption [7]. For example, in [4], Gao et al. designed a discrete sinusoidal memristive Rulkov neuron map for key-stream generation and proposed the “Encrypt a Story” method, which improves encryption efficiency by encrypting key video segments instead of all frames. In [7], Gao et al. introduced two discrete memristors into a cubic map to construct a three-dimensional memristive cubic map with higher chaotic complexity and sequence randomness, and then applied it to image encryption.

Chaos is a form of complex dynamic behavior that arises in nonlinear systems, where small variations in the initial conditions can lead to vastly different trajectories. This behavior originates from the intrinsic nonlinearity of the system and represents a deterministic form of randomness. In 1963, meteorologist Lorenz proposed a classical chaotic system [8], which played an important role in promoting the development of chaos theory. Compared with chaotic systems, hyperchaotic systems have two or more positive Lyapunov exponents, exhibit more complex dynamic behaviors, and possess greater unpredictability. Owing to these characteristics, chaotic and hyperchaotic systems have been widely applied in secure communication [9], image encryption [10, 11], and pseudorandom number generation [12]. Chaotic systems [13, 14] and hyperchaotic systems [15–20] can be constructed using different approaches. In [13], Ding et al. constructed a novel three-dimensional logistic map by introducing cross-dimensional coupling and nonlinear sine modulation, and applied it to seafood image encryption. In [14], Zhang et al. introduced a generalized hyperbolic tangent voltage-controlled memristor into the Sprott-C system to construct a novel four-dimensional memristor chaotic system with richer dynamic behaviors, and further applied it to secure color image encryption. In [15], the multisegment square function family was introduced into the Protter-C chaotic system, and a unidirectional multiturbulent hyperchaotic system with $(2N + 2)$ vortex numbers was constructed. In [16], a first-order differential equation $\dot{w} = -hx$ was introduced into the Lorenz system, and a new piecewise linear function was designed using a sign function. By replacing the state variables x and y with the piecewise linear function $g(u)$, a new hyperchaotic system was obtained. In [17], a flux-controlled memristor was introduced into the five-dimensional continuous chaotic system, and a novel six-dimensional four-wing memristive hyperchaotic system was proposed. This system is then applied to image encryption. In [18], by coupling some nonlinear terms to the existing chaotic system, a new four-dimensional hyperchaotic system was obtained, and this system was applied in encryption and decryption. Unlike conventional hyperchaotic systems, memristor-based hyperchaotic systems have equilibrium points that exhibit different dynamic behaviors at different locations [19]. In [20], a new hyperchaotic circuit was designed using three memristors, resulting in a six-dimensional hyperchaotic system. This study has significant theoretical importance and practical engineering value for advancing research on high-dimensional hyperchaotic systems based on multiple memristors.

The control of nonlinear systems has received extensive attention and has been widely applied in many fields [21–28], especially in chaotic and hyperchaotic systems. Exploring efficient and stable control methods has always been a major challenge. Researchers have proposed many simple but effective control methods, such as alternate control [29], intermittent control [30], sliding-mode control [31], impulse control [32], and synchronization [33, 34]. In [35], a robust controller with adaptive update laws was proposed, and the finite-time stability and robustness of the proposed control strategy were proved; it was then applied in secure voice communication. In [36], the finite-time synchronization of two four-dimensional hyperchaotic systems was discussed, and its application in secure communication was studied. In [37], a novel five-dimensional hyperchaotic system was constructed, and a fast terminal sliding mode control strategy was proposed to achieve finite-time synchronization with reduced chattering. In [38], an adaptive terminal sliding mode control scheme was proposed, in which a smooth fractional-order nonsingular sliding surface was designed to reduce chattering and solve the synchronization problem of uncertain fractional-order chaotic systems with disturbances. In [39], Tian et al. designed a pulse synchronization method for time-delayed hyperchaotic Chen circuits and derived the corresponding sufficient synchronization conditions.

Synchronization is different from other controls, as it can achieve synchronization or antisynchronization between two or more systems in different state spaces by designing specific controllers [40–42]. Especially for finite-time synchronization, it can achieve synchronization within a given finite time. The research in [34–39] achieved synchronization within a very short time range, but the synchronization time was not specified. For finite-time synchronization of chaotic and hyperchaotic systems, it is important not only to achieve rapid convergence within a finite time but also to consider the implementation efficiency of the control process. In many existing schemes, control inputs are applied throughout the entire synchronization process, which may lead to unnecessary control input once the synchronization error has been effectively reduced. Meanwhile, impulsive control provides instantaneous action and can rapidly suppress the synchronization error within a short time interval. However, relying solely on impulsive control may make it difficult to ensure the smooth and stable convergence of the error system in the final stage.

Based on this, this paper designs a finite-time synchronization control scheme, as shown in Figure 1. In the first stage, only impulsive control is applied to rapidly suppress the synchronization error, and the impulsive control gain decreases as the control process progresses. In the second stage, no control is applied, and the system evolves freely. In the third stage, continuous control is introduced to achieve stable convergence. Compared with traditional schemes, the proposed control scheme combines impulsive control, free evolution, and continuous control, which reduces unnecessary control input while ensuring finite-time synchronization with an explicit convergence time estimate. This staged control mechanism better reflects practical control conditions and is successfully applied to a financial hyperchaotic system to achieve finite-time synchronization.

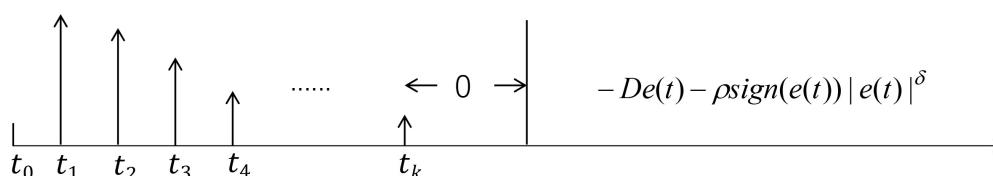


Figure 1. Schematic diagram of the proposed finite-time synchronization control scheme.

In summary, the main contributions of this paper are summarized as follows.

- A six-dimensional hyperchaotic circuit is constructed using one charge-controlled memristor, two flux-controlled memristors, two capacitors, an inductor, a linear resistor R , and a negative conductance branch $-G$. The linear resistor and the negative conductance branch modify the damping and energy compensation mechanisms of the circuit, respectively, thereby affecting the nonlinear coupling among the state variables and enhancing the hyperchaotic characteristics of the system.
- A finite-time synchronization control scheme is proposed by combining impulsive control, free evolution, and continuous control. The proposed scheme has a simpler controller structure, better flexibility, and an explicit convergence time estimate. Based on Lyapunov stability theory and matrix inequalities, sufficient synchronization conditions are derived. Simulations, comparative experiments, and an application to a financial hyperchaotic system verify the effectiveness and applicability of the proposed scheme.

The remaining parts of this paper are divided into the following sections. Section 2 presents the constructed hyperchaotic system and analyzes its dynamic characteristics. Section 3 presents the circuit simulation and FPGA implementation. Section 4 presents the proposed finite-time synchronization control scheme. Section 5 derives sufficient conditions for finite-time synchronization based on Lyapunov stability theory and matrix inequalities. Section 6 presents numerical simulations to verify the validity of the derived synchronization conditions. Section 7 summarizes the main results and concludes the paper.

2. Description of the proposed hyperchaotic system

The memristor is a nonlinear electronic component with memory characteristics. Its resistance is not fixed, but varies with the amount of charge passing through it, thereby exhibiting memory characteristics. The development and application of memristors have opened up new opportunities for the study of nonlinear circuits and hyperchaotic systems. Due to their inherent nonlinearity and memory properties, memristors are well suited for constructing high-dimensional nonlinear dynamic systems. Moreover, memristor-based hyperchaotic systems usually exhibit more complex attractor structures and richer dynamic behaviors. In this paper, the cubic smooth memristor model in [20] is adopted. The charge-controlled memristor model $\varphi(q)$ is described as follows:

$$\varphi(q) = aq + bq^3, \quad (1)$$

where a and b are two constants, q denotes the accumulated charge, $M(q)$ denotes the memristance of the charge-controlled memristor. The relationship between the memristance $M(q)$ and the flux $\varphi(q)$ is

given as follows:

$$M(q) = \frac{d\varphi(q)}{dq} = a + 3bq^2. \quad (2)$$

The voltage-current relationship of the charge-controlled memristor is given as follows:

$$v = M(q)i, \quad (3)$$

where v is the voltage across the memristor and i is the current flowing through it.

The flux-controlled memristor model adopted in this paper is a smooth cubic nonlinear model

$$q_1(\varphi_1) = c_1\varphi_1 + d_1\varphi_1^3, \quad q_2(\varphi_2) = c_2\varphi_2 + d_2\varphi_2^3, \quad (4)$$

where c_1 , d_1 , c_2 , and d_2 are constants; and φ_1 and φ_2 denote the magnetic fluxes of the two memristors; and $W_1(\varphi_1)$ and $W_2(\varphi_2)$ denote the memductances of the two flux-controlled memristors. The corresponding memductances are given as follows:

$$W_1(\varphi_1) = \frac{dq_1(\varphi_1)}{d\varphi_1} = c_1 + 3d_1\varphi_1^2, \quad W_2(\varphi_2) = \frac{dq_2(\varphi_2)}{d\varphi_2} = c_2 + 3d_2\varphi_2^2. \quad (5)$$

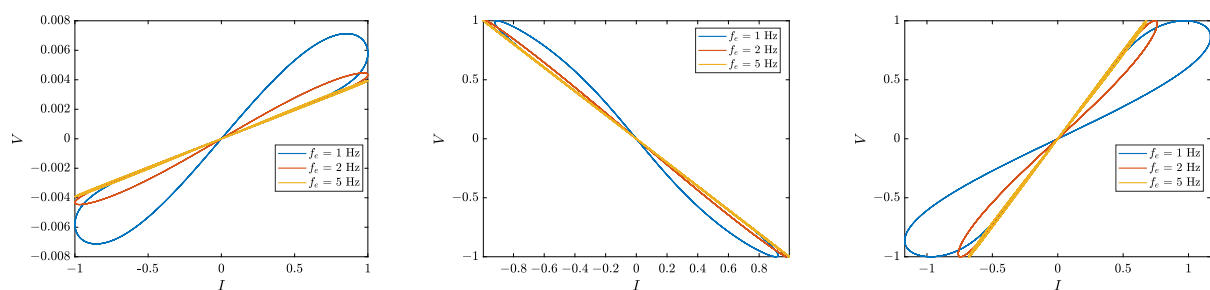
The current-voltage relationship of the flux-controlled memristor is given as follows:

$$i = W(\varphi)v, \quad (6)$$

where v and i are defined as in (3).

To verify the memristive characteristics of the adopted memristors, the volt-ampere characteristic curves are plotted under sinusoidal excitations with different frequencies. The parameters of the charge-controlled memristor in (1) are set as $a = 1/260$ and $b = 1/39$, and the sinusoidal current excitation is set as $i(t) = A \sin(2\pi f_e t)$. The parameters of the flux-controlled memristors in (4) are set as $c_1 = -1$, $d_1 = 6/5$, $c_2 = 2/3$, and $d_2 = 4$. For the flux-controlled memristors, the sinusoidal voltage excitation is set as $v(t) = A \sin(2\pi f_e t)$.

In all cases, the excitation amplitude is set as $A = 1$, and the excitation frequencies are selected as $f_e = 1$ Hz, 2 Hz, and 5 Hz. The corresponding volt-ampere characteristic curves are shown in Figure 2. As shown in Figure 2, the characteristic curves of the three memristors pass through the origin and exhibit typical pinched hysteresis loops. As the excitation frequency increases, the hysteresis loops gradually shrink, indicating the frequency-dependent behavior of the adopted memristors. These results verify that the adopted memristors possess typical memristive characteristics.



(a) Charge-controlled memristor $M(q)$. (b) Flux-controlled memristor $W_1(\varphi_1)$. (c) Flux-controlled memristor $W_2(\varphi_2)$.

Figure 2. Volt-ampere characteristic curves of different memristors.

Wang et al. [20] proposed a six-dimensional hyperchaotic circuit. On the basis of this circuit, we further extend the circuit structure and enhance its hyperchaotic behavior, as shown in Figure 3. Table 1 presents the comparison between the system in [20] and this work.

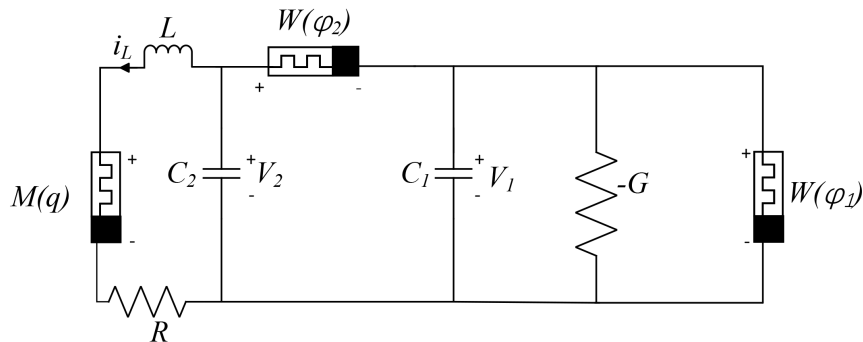


Figure 3. Circuit diagram of the six-dimensional hyperchaotic system.

Table 1. Comparison between the system in [20] and this work.

	System in [20]	This work
Number of memristors	Three	Three
Nonlinear coupling modes	Passive memristive nonlinear coupling based on KCL and KVL	Active-dissipative memristive nonlinear coupling based on KCL and KVL
Hyperchaotic characteristics	Phase portraits are shown in Figure 4 (a)–(c). The Lyapunov exponent spectrum is $LE = [0.2777, 0.0069, -0.0023, -0.0008, -0.0004, -6.3380]$	Phase portraits are shown in Figure 4 (d)–(f). The Lyapunov exponent spectrum is $LE = [0.5066, 0.0034, -0.0004, 0.0000, -0.0024, -11.8186]$
Spectral entropy (SE)	The Kolmogorov-Sinai entropy is $h_{KS} \approx 0.2846$ 0.532446	The Kolmogorov-Sinai entropy is $h_{KS} \approx 0.5100$ 0.605493
C0 complexity	0.182922	0.236474
Permutation entropy (PE)	0.659274	0.826386

Note: KCL and KVL denote Kirchhoff's current law and Kirchhoff's voltage law, respectively.

As shown in Table 1, the proposed system improves the original three-memristor circuit without increasing the number of memristors. The main advantage lies in the introduction of an active-dissipative modulation structure. The negative conductance branch provides energy compensation in the voltage equation, while the linear resistor branch introduces additional damping regulation in the inductor-current equation. This structure adjusts the energy exchange and dissipation balance of the circuit and enhances the effective nonlinear interaction among the state variables.

The phase portraits in Figure 4 further illustrate the differences in hyperchaotic behavior between the two systems. For the same simulation time, the attractors of the proposed system are more densely distributed in phase space. The maximum Lyapunov exponent increases from 0.2777 to 0.5066, while the Kolmogorov-Sinai entropy increases from 0.2846 to 0.5100. In addition, Table 1 shows that the proposed system achieves higher spectral entropy (SE), C0 complexity, and permutation entropy (PE) values than the system in [20]. These results demonstrate that the proposed system exhibits stronger hyperchaotic behavior and higher dynamic complexity under the selected parameters.

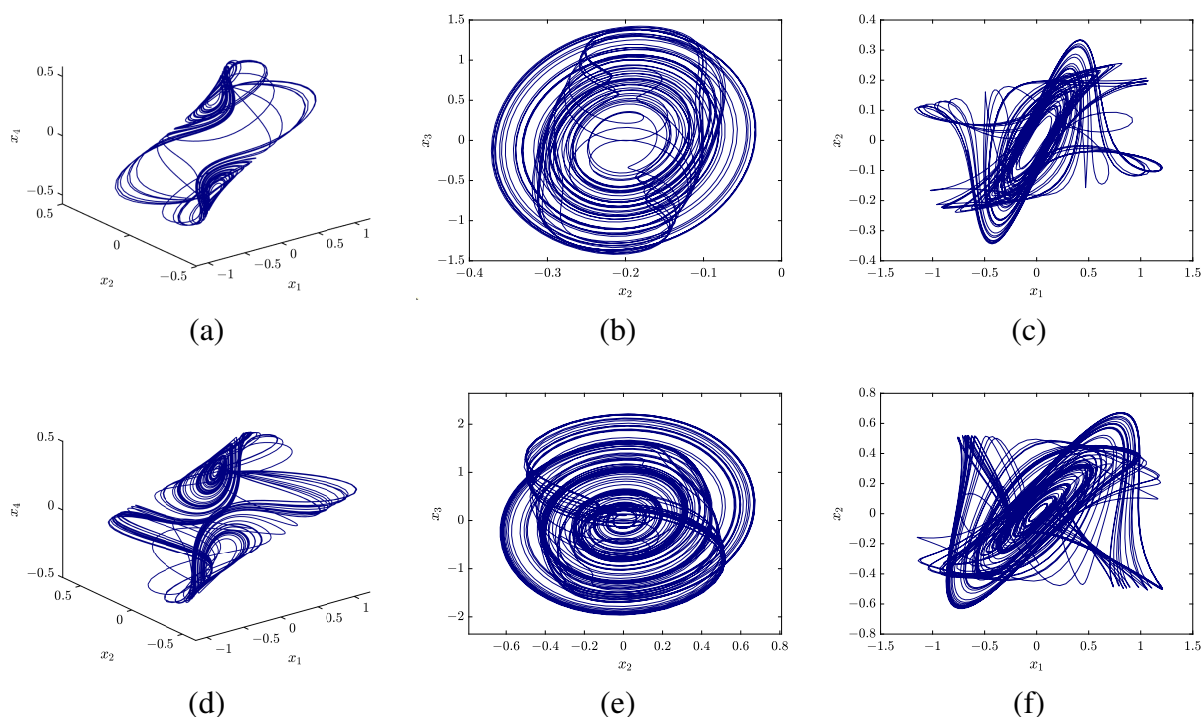


Figure 4. Comparison of phase portraits. (a)–(c) Phase portraits of the system in [20]; (d)–(f) phase portraits of the proposed system.

According to the voltage-current relationships of the charge-controlled and flux-controlled memristors given in (3) and (6), respectively, and based on Kirchhoff's current law and Kirchhoff's voltage law, the circuit can be described by (7).

$$\begin{cases} \dot{V}_1 = \frac{1}{C_1} (GV_1 - W_1(\varphi_1)V_1 + W_2(\varphi_2)(V_2 - V_1)), \\ \dot{V}_2 = \frac{1}{C_2} ((V_1 - V_2)W_2(\varphi_2) - i_L), \\ \dot{i}_L = \frac{1}{L} (V_2 - M(q)i_L - Ri_L), \\ \dot{\varphi}_1 = V_1, \\ \dot{\varphi}_2 = V_2 - V_1, \\ \dot{q} = i_L. \end{cases} \quad (7)$$

Substituting (2) and (5) into (7), and letting $x_1 = V_1$, $x_2 = V_2$, $x_3 = i_L$, $x_4 = \varphi_1$, $x_5 = \varphi_2$, $x_6 = q$, the system can be further expressed as (8).

$$\begin{cases} \dot{x}_1 = \frac{G - c_1 - c_2}{C_1}x_1 + \frac{c_2}{C_1}x_2 - \frac{3d_1}{C_1}x_1x_4^2 + \frac{3d_2}{C_1}x_2x_5^2 - \frac{3d_2}{C_1}x_1x_5^2, \\ \dot{x}_2 = \frac{c_2}{C_2}x_1 - \frac{c_2}{C_2}x_2 - \frac{1}{C_2}x_3 + \frac{3d_2}{C_2}x_1x_5^2 - \frac{3d_2}{C_2}x_2x_5^2, \\ \dot{x}_3 = \frac{1}{L}x_2 - \left(\frac{a+R}{L}\right)x_3 - \frac{3b}{L}x_3x_6^2, \\ \dot{x}_4 = x_1, \\ \dot{x}_5 = x_2 - x_1, \\ \dot{x}_6 = x_3. \end{cases} \quad (8)$$

where R , C_1 , C_2 , G , and L represent the resistance, capacitances, conductance, and inductance, respectively, while a , b , c_1 , d_1 , c_2 , and d_2 are the memristor parameters.

2.1. Equilibrium point and stability analysis

To determine the equilibrium points of the hyperchaotic system, the derivatives in (8) are set to zero. Then the equilibrium set of the hyperchaotic system can be obtained as follows:

$$S = \{(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = k_1, x_5 = k_2, x_6 = k_3) \mid (k_1, k_2, k_3) \in \mathbb{R}\}. \quad (9)$$

That is, any point in the three-dimensional subspace spanned by x_4 , x_5 , and x_6 is an equilibrium point of the hyperchaotic system. Linearizing the hyperchaotic system at an equilibrium point yields the following Jacobian matrix:

$$J = \begin{pmatrix} \frac{G-c_1-c_2}{C_1} - \frac{3d_1}{C_1}k_1^2 - \frac{3d_2}{C_1}k_2^2 & \frac{c_2}{C_1} + \frac{3d_2}{C_1}k_2^2 & 0 & 0 & 0 & 0 \\ \frac{c_2}{C_2} + \frac{3d_2}{C_2}k_2^2 & -\frac{c_2}{C_2} - \frac{3d_2}{C_2}k_2^2 & -\frac{1}{C_2} & 0 & 0 & 0 \\ 0 & \frac{1}{L} & -\frac{a+R}{L} - \frac{3b}{L}k_3^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (10)$$

The characteristic equation of the Jacobian matrix of system (8) at the equilibrium set S is given by

$$\lambda^3 (\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0) = 0, \quad (11)$$

where

$$\begin{aligned} p_0 &= 1058.4 k_1^2 + 3477.6 k_2^2 - 18 k_3^2 + 97.2 k_1^2 k_2^2 + 54 k_1^2 k_3^2 - 324 k_2^2 k_3^2 \\ &\quad + 972 k_1^2 k_2^2 k_3^2 - 157.8, \\ p_1 &= 56.7 k_1^2 - 311.4 k_2^2 - 2 k_3^2 + 972 k_1^2 k_2^2 + 27 k_1^2 k_3^2 \\ &\quad + 126 k_2^2 k_3^2 + 20.8, \end{aligned}$$

and

$$p_2 = 27 k_1^2 + 126 k_2^2 + k_3^2 - 1.9. \quad (12)$$

It can be seen from (11) that the Jacobian matrix has three zero eigenvalues, while the remaining three eigenvalues are determined by the cubic polynomial $\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0$.

According to the Routh-Hurwitz criterion, we found that

$$\begin{aligned} p_2 &> 0, \\ p_1p_2 - p_0 &> 0, \\ p_0(p_1p_2 - p_0) &> 0. \end{aligned} \tag{13}$$

When $k_1 = 0$ and $k_2 = 0$, we can get

$$p_2 = k_3^2 - 1.9, \quad p_1 = 20.8 - 2k_3^2, \quad p_0 = -18k_3^2 - 157.8.$$

Since $p_0 < 0$ for all $k_3 \in \mathbb{R}$, the Hurwitz conditions cannot be satisfied on the k_3 -axis.

When $k_1 = 0$ and $k_3 = 0$, the cubic part satisfies the Hurwitz condition on the k_2 -axis for

$$0.21 < |k_2| < 0.23. \tag{14}$$

When $k_2 = 0$ and $k_3 = 0$, the cubic part satisfies the Hurwitz condition on the k_1 -axis for

$$|k_1| > 0.39. \tag{15}$$

In both cases, due to the existence of zero eigenvalues, the equilibrium set can only be regarded as critically stable.

2.2. Phase diagrams and Lyapunov exponent analysis

Chaotic attractors are a special types of attractor in nonlinear dynamic systems. It is highly sensitive to the initial conditions, and even a slight perturbation can lead to exponential divergence of the trajectories. However, despite its irregular behavior, the trajectory remains confined within a bounded region. Let the system parameter $C_1 = 2/15$, $C_2 = 1/3$, $L = 1/13$, $a = 1/260$, $b = 1/39$, $c_1 = -1$, $d_1 = 6/5$, $c_2 = 2/3$, $d_2 = 4$, $G = 1/5$, and $R = 1/260$. In this case, the initial condition of the system is given by $[0.01, 0.01, 0.01, 0.01, 0.01, 0.01]^T$. The hyperchaotic attractors generated by System (8) are shown in Figure 4 (d)–(f).

The Lyapunov exponent is an important mathematical tool for measuring the sensitivity to the initial conditions in a dynamic system. Hyperchaotic systems were first proposed by the German physicist Rössler [43] in 1979, namely the Rössler system. A chaotic system is considered to be hyperchaotic if it has two or more positive Lyapunov exponents. Using the initial condition $\mathbf{x}(0) = [0.01, 0.01, 0.01, 0.01, 0.01, 0.01]^T$ and applying the Wolf method [44], the Lyapunov exponents are calculated; the result is shown in Figure 5.

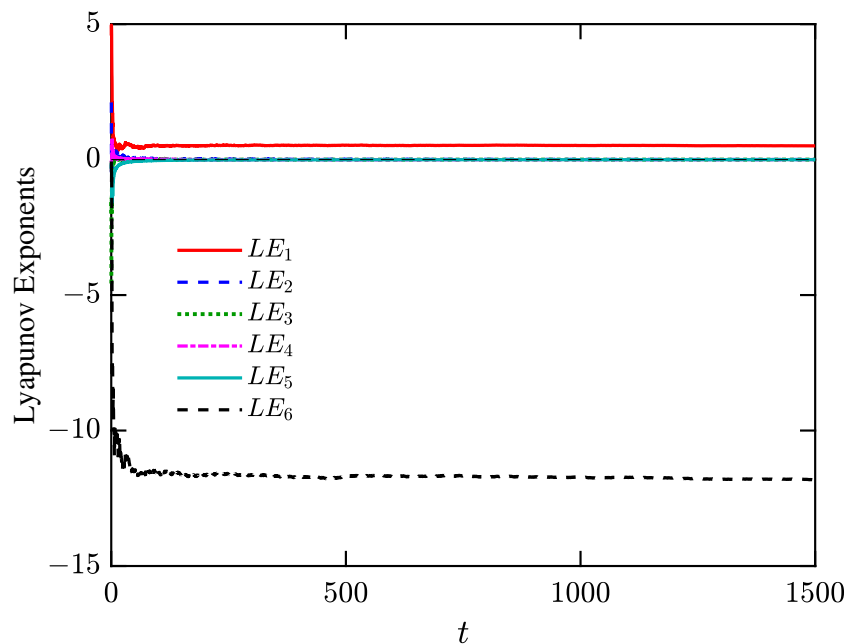


Figure 5. Lyapunov exponent spectrum of the hyperchaotic system.

When the system reaches stability, $LE_1 = 0.5066$, $LE_2 = 0.0034$, $LE_3 = -0.0004$, $LE_4 = 0.0000$, $LE_5 = -0.0024$, and $LE_6 = -11.8186$.

The Kaplan-Yorke dimension (D_{KY}) is calculated as follows:

$$\begin{aligned}
 D_{KY} &= \eta + \frac{\sum_{i=1}^{\eta} LE_i}{|LE_{\eta+1}|} \\
 &= 5 + \frac{0.5066 + 0.0034 - 0.0004 - 0.0000 - 0.0024}{|11.8186|} \\
 &\approx 5.0429.
 \end{aligned} \tag{16}$$

The calculated Kaplan-Yorke dimension is fractional, which further supports the existence of hyperchaotic behavior in the system.

2.3. Poincaré section and bifurcation diagram analysis

The Poincaré section provides an effective approach for analyzing dynamic systems. It simplifies the analysis by recording the intersection points between a trajectory and a selected section in the high-dimensional phase space. In this paper, the plane $x_6 = 0$ is selected as the Poincaré section, and only the points where the trajectory crosses this plane from the negative side to the positive side are recorded. To observe the distribution of the crossing points on the section, these points are projected onto the $x_1 - x_2$ and $x_1 - x_3$ planes, and the corresponding scatter plots are shown in Figure 6. The crossing points exhibit obvious scattering and folding characteristics, and no simple fixed-point or periodic structure can be observed. These results indicate that the proposed system has complex nonlinear dynamic behavior and further verify its hyperchaotic characteristics.

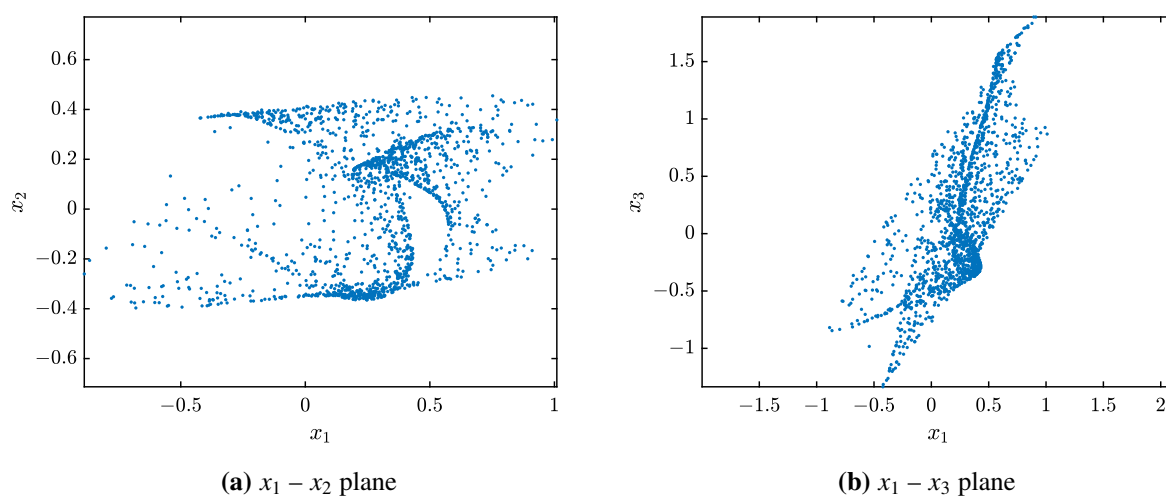


Figure 6. Poincaré section of the hyperchaotic system.

The bifurcation diagram is used to study the evolution of the stability structure or dynamic behavior of a system during parameter change, revealing the transformation process of the system from one dynamic state to another when a parameter is altered. In this paper, the system parameters G , a , b , c_1 , and d_1 are selected as bifurcation parameters to analyze the dynamic evolution of the proposed system. The corresponding bifurcation diagrams between these parameters and selected state variables are constructed, and the Lyapunov exponent spectra are also calculated for comparison.

The system parameters are set as $C_1 = 2/15$, $C_2 = 1/3$, $L = 1/13$, $b = 1/39$, $c_1 = -1$, $d_1 = 6/5$, $c_2 = 2/3$, $d_2 = 4$, $a = 1/260$, and $R = 1/260$, while the parameter G is varied in the range $G \in (0, 0.8)$. The bifurcation diagram of the state variable x_1 with respect to G is constructed, and the corresponding Lyapunov exponent spectrum is calculated, as shown in Figure 7. As can be observed from Figure 7, the system exhibits periodic behavior for $G \in (0, 0.12)$, and the Lyapunov exponents are $(0, -, -, -, -, -)$. For example, when $G = 0.08$, the Lyapunov exponents are $(0, -, -, -, -, -)$, indicating that the system is in a periodic state. As G increases from 0.12 to 0.45, the system transitions between periodic and hyperchaotic states. For example, when $G = 0.32$, the Lyapunov exponents are $(0.5770, 0.0223, -, -, -, -)$, indicating that the system is in a hyperchaotic state. When $G = 0.36$, the Lyapunov exponents are $(0, -, -, -, -, -)$, indicating that the system returns to a periodic state. As G increases from 0.46 to 0.8, the system remains in a periodic state.

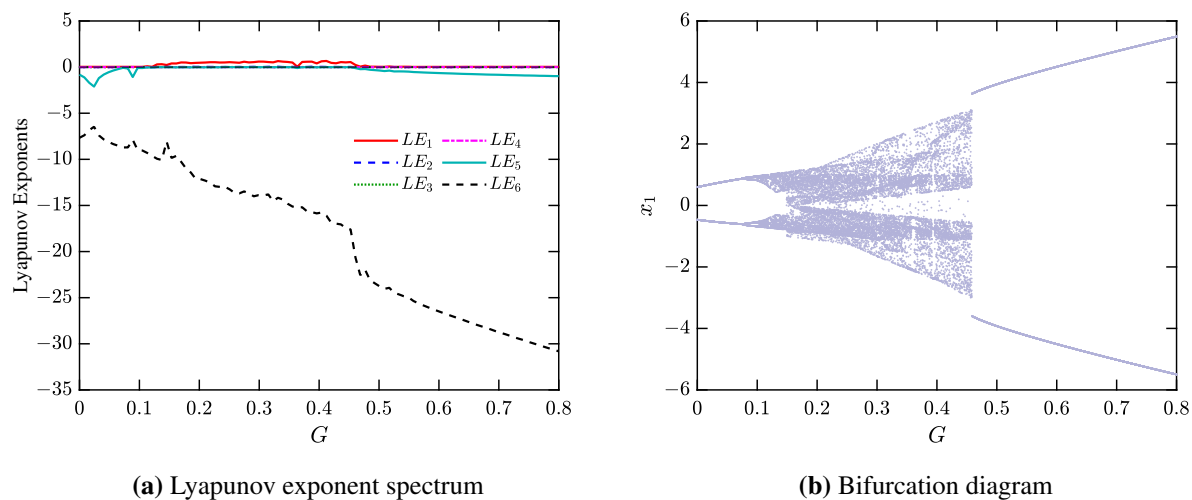


Figure 7. The Lyapunov exponent spectrum and bifurcation diagram for various values of the parameter G .

The system parameters are set as $C_1 = 2/15$, $C_2 = 1/3$, $L = 1/13$, $b = 1/39$, $c_1 = -1$, $d_1 = 6/5$, $c_2 = 2/3$, $d_2 = 4$, $G = 1/5$, and $R = 1/260$, while the parameter a is varied in the range $a \in (0, 0.02)$. The bifurcation diagram of the state variable x_2 with respect to a is plotted, and the corresponding Lyapunov exponent spectrum is also calculated, as illustrated in Figure 8. As shown in Figure 8, the system alternates among periodic, chaotic, and hyperchaotic behaviors as a varies. When $a \in (0, 0.0024) \cup (0.0028, 0.0077) \cup (0.0081, 0.0136) \cup (0.0139, 0.0188)$, the Lyapunov exponents are $(+, +, -, -, -, -)$, indicating that the system is in a hyperchaotic state. For example, when $a = 0.012$, the Lyapunov exponent spectrum is $(0.4436, 0.0218, 0, -, -, -)$, which further indicates hyperchaotic behavior. When $a \in (0.0025, 0.0027) \cup (0.0078, 0.0081) \cup (0.0137, 0.0138) \cup (0.0189, 0.0194)$, the Lyapunov exponents are $(0, -, -, -, -, -)$, indicating that the system enters a periodic state. For example, when $a = 0.008$, the Lyapunov exponents are $(0, -, -, -, -, -)$. When $a \in (0.0195, 0.02)$, the Lyapunov exponents are $(+, 0, -, -, -, -)$, indicating that the system enters a chaotic state. For example, when $a = 0.02$, the Lyapunov exponent spectrum is $(0.29, 0, -, -, -, -)$. These results show that variations in a can induce transitions among different dynamic states, leading to complex dynamic phenomena.

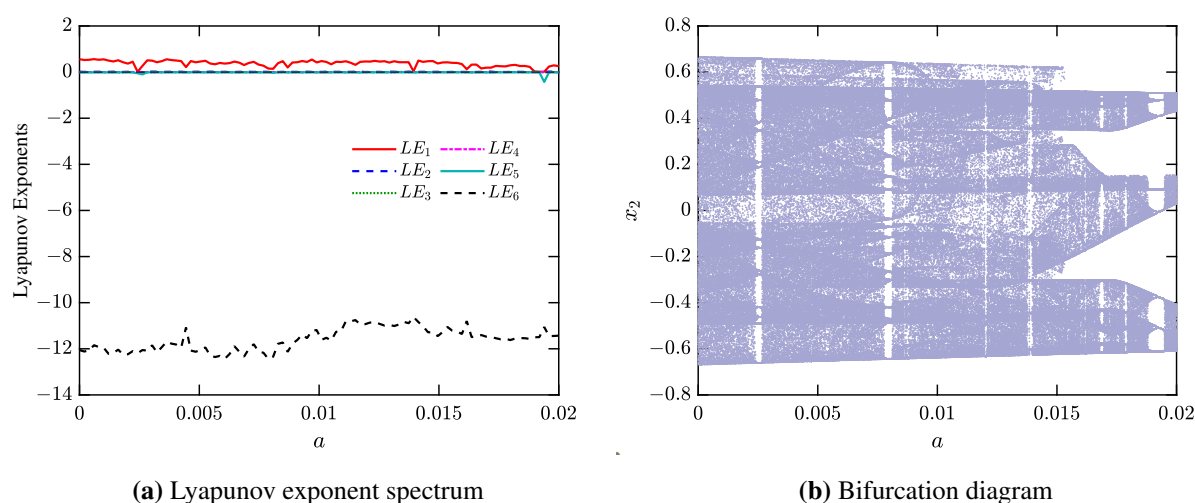


Figure 8. The Lyapunov exponent spectrum and bifurcation diagram for various parameters a .

The system parameters are set as $C_1 = 2/15$, $C_2 = 1/3$, $L = 1/13$, $a = 1/260$, $c_1 = -1$, $d_1 = 6/5$, $c_2 = 2/3$, $d_2 = 4$, $G = 1/5$, and $R = 1/260$, while the parameter b is varied in the range $b \in (0, 0.2)$. The bifurcation diagram between b and the state variable x_1 is plotted, and the corresponding Lyapunov exponent spectrum is also calculated, as illustrated in Figure 9. As shown in Figure 9, the system exhibits hyperchaotic behavior for $b \in (0, 0.06)$, and the Lyapunov exponents are $(+, +, -, -, -, -)$. For example, when $b = 0.02$, the Lyapunov exponents are $(0.5130, 0.0101, 0, -, -, -)$, indicating that the system is in a hyperchaotic state. As b increases from 0.07 to 0.14, the point set in the bifurcation diagram gradually evolves from a dense distribution to a regular branch structure, indicating that the system transforms from a hyperchaotic state to a periodic or quasi-periodic state. For example, when $b = 0.125$, the Lyapunov exponents are $(0, -, -, -, -, -)$, indicating that the system is in a periodic state. As b increases from 0.15 to 0.2, the bifurcation diagram converges to a stable branch, indicating that the system is in a stable state. For example, when $b = 0.18$, the Lyapunov exponents are $(-, -, -, -, -, -)$.

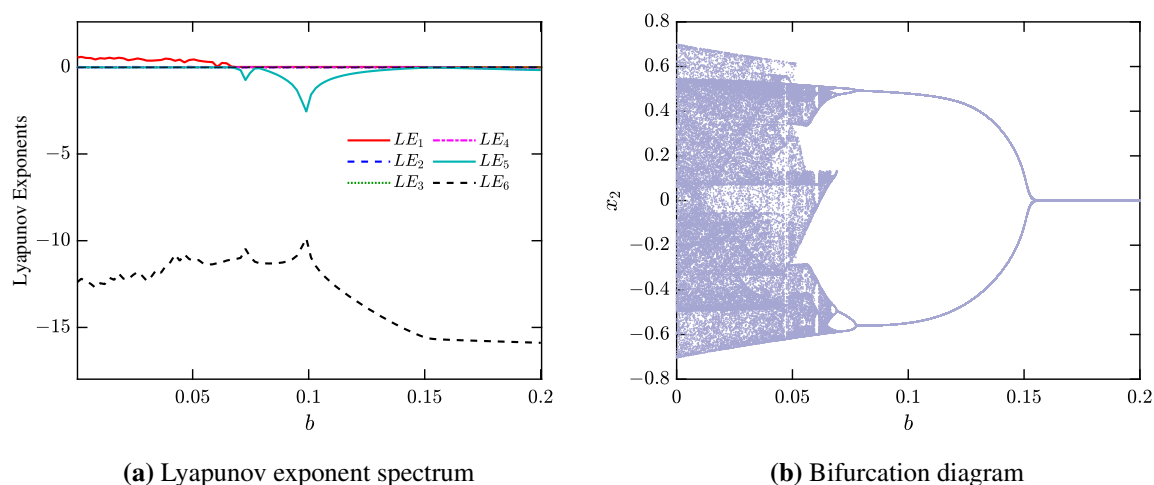


Figure 9. The Lyapunov exponent spectrum and bifurcation diagram for various parameters b .

The system parameters are set as $C_1 = 2/15$, $C_2 = 1/3$, $L = 1/13$, $a = 1/260$, $b = 1/39$, $d_1 = 6/5$, $c_2 = 2/3$, $d_2 = 4$, $G = 1/5$, and $R = 1/260$, while the parameter c_1 is varied in the range $c_1 \in (-2, 0)$. The bifurcation diagram between c_1 and the state variable x_1 is plotted, and the corresponding Lyapunov exponent spectrum is also calculated, as illustrated in Figure 10. As shown in Figure 10, the system exhibits periodic behavior for $c_1 \in (-2, -1.3)$, and the Lyapunov exponents are $(0, -, -, -, -, -)$. For example, when $c_1 = -1.7$, the Lyapunov exponents are $(0, -, -, -, -, -)$. As c_1 increases from -1.2 to -0.93 , the system transforms from periodic behavior to hyperchaotic behavior. For example, when $c_1 = -1$, the Lyapunov exponents are $(0.4640, 0.0058, -, -, -, -)$, indicating that the system is in a hyperchaotic state. As c_1 increases from -0.93 to -0.75 , the system enters a periodic state. For example, when $c_1 = -0.8$, the Lyapunov exponents are $(0, -, -, -, -, -)$. As c_1 further increases from -0.75 to 0 , the system becomes stable, and all Lyapunov exponents are negative, namely $(-, -, -, -, -, -)$.

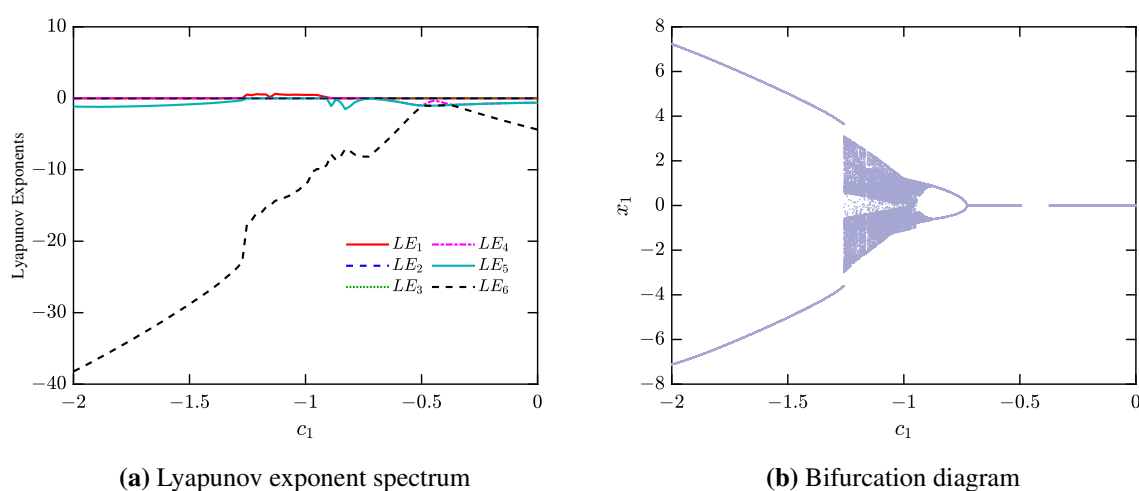


Figure 10. The Lyapunov exponent spectrum and bifurcation diagram for various parameters c_1 .

The system parameters are set as $C_1 = 2/15$, $C_2 = 1/3$, $L = 1/13$, $a = 1/260$, $b = 1/39$, $c_1 = -1$, $c_2 = 2/3$, $d_2 = 4$, $G = 1/5$, and $R = 1/260$, while the parameter d_1 is varied in the range $d_1 \in (0, 3)$. The bifurcation diagram between d_1 and the state variable x_1 is plotted, and the corresponding Lyapunov exponent spectrum is also calculated, as illustrated in Figure 11. As shown in Figure 11, the system exhibits periodic behavior for $d_1 \in (0, 0.4)$, and the Lyapunov exponents are $(0, -, -, -, -, -)$. For example, when $d_1 = 0.2$, the Lyapunov exponents are $(0, -, -, -, -, -)$. As d_1 increases from 0.5 to 1.4, the system transforms from a periodic state to a hyperchaotic state, and the Lyapunov exponents become $(+, +, -, -, -, -)$. For example, when $d_1 = 0.73$, the Lyapunov exponents are $(0.6284, 0.0147, -, -, -, -)$, indicating that the system is in a hyperchaotic state. As d_1 further increases from 1.5 to 3, the system undergoes a transition from hyperchaotic behavior to periodic behavior and finally reaches a stable state. For example, when $d_1 = 2$, the Lyapunov exponents are $(0, -, -, -, -, -)$, indicating periodic behavior. When $d_1 = 2.8$, the Lyapunov exponents are $(-, -, -, -, -, -)$, indicating that the system enters a stable state.

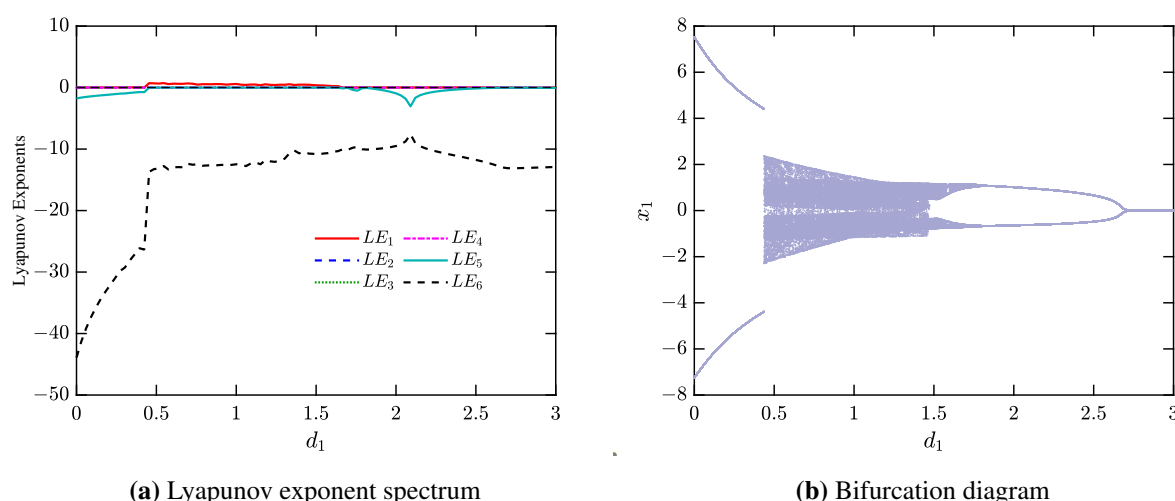


Figure 11. The Lyapunov exponent spectrum and bifurcation diagram for various parameters d_1 .

Overall, the bifurcation diagrams and Lyapunov exponent spectra with respect to different system parameters show that the proposed system is highly sensitive to parameter variations. As the parameters G , a , b , c_1 , and d_1 change, the system exhibits rich dynamic behaviors, including hyperchaotic states, chaotic states, periodic windows, and stable states. These results suggest that even slight changes in the system's parameters can lead to significant changes in the dynamic behavior of the system.

2.4. Complexity analysis

To evaluate the dynamic complexity of the proposed six-dimensional hyperchaotic system, SE, C0 complexity, and PE are used to analyze the output sequences of the system. SE is used to quantify the complexity of the frequency-domain energy distribution of the system's output sequence, C0 complexity measures the proportion of irregular components in the sequence, and PE characterizes the randomness of local ordinal patterns in a time series. In the numerical simulations, three two-dimensional parameter planes are considered. The grid size is set to 150×150 . The initial

condition is chosen as $(0.01, 0.01, 0.01, 0.01, 0.01, 0.01)^T$, and the remaining parameters are kept the same as those in the previous sections.

As shown in Figure 12 (a)–(c), the SE maps demonstrate that the complexity of the proposed hyperchaotic system varies significantly with different parameter combinations. In the $C_1 - c_2$ plane, high-complexity regions appear mainly in $C_1 \in (0.10, 0.15)$, $c_2 \in (0.5, 1.0)$, and $C_1 \in (0.05, 0.10)$, $c_2 \in (0, 0.5)$. The maximum SE value reaches 0.7282 near $(C_1, c_2) = (0.0701, 0.1094)$, indicating that the system has stronger spectral disorder in these regions. In contrast, the regions with low SE values suggest relatively simple dynamics, corresponding to periodic, quasi-periodic, or stable states. In the $C_2 - d_1$ plane, the SE values are generally high for $C_2 \in (0.1, 0.6)$ and $d_1 \in (0, 3)$, indicating that the system exhibits more complex dynamic behavior over a broad range of C_2 and d_1 combinations. In the $c_1 - d_1$ plane, the SE values change significantly with the variation of c_1 and d_1 . Relatively high SE values are mainly observed in the range of $c_1 \in (-1.8, -0.8)$ and $d_1 \in (0.5, 2.5)$, while relatively low SE values mainly appear when $c_1 \in (-0.8, 0)$ and $d_1 \in (0, 3)$. This indicates that the system complexity is sensitive to parameter variations.

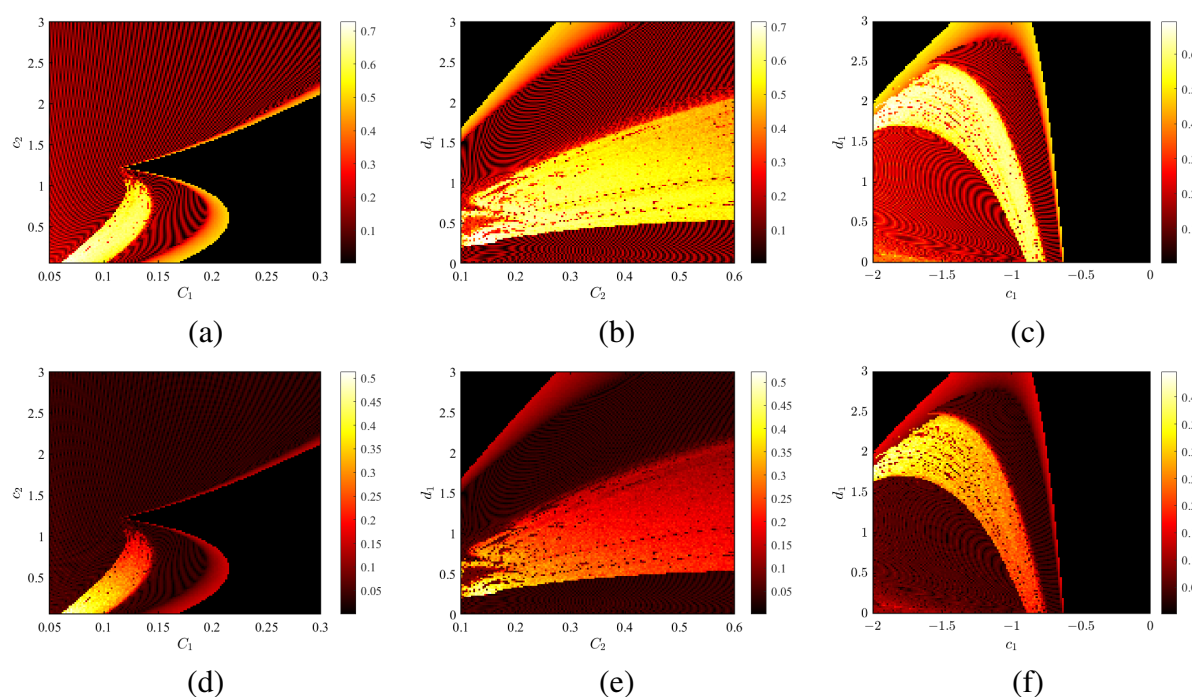


Figure 12. Complexity analysis of the proposed hyperchaotic system in different planes. (a)–(c) SE maps in the $C_1 - c_2$, $C_2 - d_1$, and $c_1 - d_1$ planes, respectively. (d)–(f) C0 complexity maps in the $C_1 - c_2$, $C_2 - d_1$, and $c_1 - d_1$ planes, respectively.

The C0 complexity maps in Figure 12 (d)–(f) show variation trends similar to those of the SE maps. In the $C_1 - c_2$ plane, relatively high C0 values are mainly observed in the ranges $C_1 \in (0.05, 0.15)$ and $c_2 \in (0, 1.0)$, indicating that the output sequence contains more irregular components under these parameter conditions. In the $C_2 - d_1$ plane, relatively high C0 values mainly appear for $C_2 \in (0.1, 0.6)$, $d_1 \in (0, 3)$. In the $c_1 - d_1$ plane, relatively high C0 values are mainly observed in the ranges $c_1 \in (-1.8, -0.8)$, $d_1 \in (0.5, 2.5)$. This indicates that the irregularity of the system output sequence is sensitive to parameter variations.

To further evaluate the dynamic complexity of the proposed hyperchaotic system, its complexity indices are compared with those of several existing chaotic systems. The comparison results are shown in Table 2.

Table 2. Complexity comparison with existing chaotic systems.

System	Parameters	SE	C0	PE
System in [20]	$\delta = 9, \beta = 1, \gamma = 12, a = 0.01, b = 0.03, c = -1.2, d = 1, e = 0.9, f = 0.9$	0.532446	0.182922	0.659274
System in [45]	$\alpha = 11.5, \beta = 3.8, a = 0.35$	0.261280	0.067347	0.692220
System in [7]	$a = 3.49, r = 2.89, k = 5.29$	0.840200	0.416900	0.675000
Proposed system	$C_1 = 2/15, C_2 = 1/3, L = 1/13, R = 1/260, G = 0.2, a = 1/260, b = 1/39, c_1 = -1, d_1 = 1.2, c_2 = 2/3, d_2 = 4$	0.605493	0.236474	0.826386

As shown in Table 2, under the selected parameter settings, the proposed system obtains a larger PE value than the selected comparison systems, indicating the strong local ordinal randomness of the generated sequence. Its SE value is also relatively high, suggesting a comparatively uniform spectral energy distribution. However, the C0 value is relatively small compared with some existing systems. These results indicate that the proposed system shows more evident complexity characteristics in PE and SE.

3. Analog circuit and FPGA implementation of the hyperchaotic system

3.1. Analog circuit simulation

To verify the correctness of the numerical simulation results of the hyperchaotic system, a circuit simulation is performed. According to the basic principles of analog circuit design, the circuit equations are derived from the inverting proportional circuits, inverting integrator circuits, and Kirchhoff's current law. For an inverting proportional circuit, the input V_{in} and output V_{out} voltages satisfy

$$V_{out} = -\frac{R_f}{R_i} V_{in}, \quad (17)$$

where R_f is the feedback resistor and R_i is the input resistor. For an inverting integrator circuit composed of an operational amplifier, a resistor, and a capacitor, the output voltage satisfies

$$V_{out} = -\frac{1}{R_i C} \int_0^t V_{in} dt, \quad (18)$$

and therefore

$$\frac{dV_{out}}{dt} = -\frac{1}{R_i C} V_{in}. \quad (19)$$

In the designed circuit, the output voltages of the six integrator circuits are defined as the state variables x_i ($i = 1, 2, \dots, 6$). Each term in (8) is introduced into the corresponding integrator through a resistor branch, and the nonlinear product terms are generated by analog multipliers. By applying Kirchhoff's current law at the summing node of each integrator, the derivative of each state variable can be expressed as the weighted sum of the corresponding input signals. The coefficient of each term is determined by the related resistor and capacitor in the form of $1/(C_i R_j)$. Based on the above circuit relationships, the circuit equations corresponding to (8) can be derived as follows:

$$\left\{ \begin{array}{l} \frac{dx_1}{d(\tau_0 t)} = \frac{1}{C_3 R_1} x_1 + \frac{1}{C_3 R_2} x_2 + \frac{1}{C_3 R_3} x_2 x_5^2 - \frac{1}{C_3 R_4} x_1 x_5^2 - \frac{1}{C_3 R_5} x_1 x_4^2, \\ \frac{dx_2}{d(\tau_0 t)} = \frac{1}{C_4 R_8} x_1 - \frac{1}{C_4 R_9} x_2 - \frac{1}{C_4 R_{12}} x_3 + \frac{1}{C_4 R_{10}} x_1 x_5^2 - \frac{1}{C_4 R_{11}} x_2 x_5^2, \\ \frac{dx_3}{d(\tau_0 t)} = \frac{1}{C_5 R_{15}} x_2 - \frac{1}{C_5 R_{16}} x_3 - \frac{1}{C_5 R_{17}} x_3 x_6^2, \\ \frac{dx_4}{d(\tau_0 t)} = \frac{1}{C_6 R_{20}} x_1, \\ \frac{dx_5}{d(\tau_0 t)} = -\frac{1}{C_7 R_{21}} x_1 + \frac{1}{C_7 R_{22}} x_2, \\ \frac{dx_6}{d(\tau_0 t)} = \frac{1}{C_8 R_{23}} x_3. \end{array} \right. \quad (20)$$

Remark 3.1. In (20), $t = \tau_0 t$ is used for the time-scale transformation, where $\tau_0 = 100$. This transformation is introduced to obtain suitable values of the resistance R_j ($j = 1, 2, \dots, 23$) and capacitance C_i ($i = 3, 4, \dots, 8$), as well as an appropriate operating time scale for the circuit.

By comparing the coefficients of (20) with System (8), we can obtain the following: $R_1 = 25 \text{ K}\Omega$, $R_2 = 20 \text{ K}\Omega$, $R_3 = 1.11 \text{ K}\Omega$, $R_4 = 1.11 \text{ K}\Omega$, $R_5 = 3.7 \text{ K}\Omega$, $R_8 = 50 \text{ K}\Omega$, $R_9 = 50 \text{ K}\Omega$, $R_{10} = 2.78 \text{ K}\Omega$, $R_{11} = 2.78 \text{ K}\Omega$, $R_{12} = 33.33 \text{ K}\Omega$, $R_{15} = 7.69 \text{ K}\Omega$, $R_{16} = 1000 \text{ K}\Omega$, $R_{17} = 100 \text{ K}\Omega$, $R_{20} = 100 \text{ K}\Omega$, $R_{21} = 100 \text{ K}\Omega$, $R_{22} = 100 \text{ K}\Omega$, $R_{23} = 100 \text{ K}\Omega$, $R_6 = 10 \text{ K}\Omega$, $R_7 = 10 \text{ K}\Omega$, $R_{13} = 10 \text{ K}\Omega$, $R_{14} = 10 \text{ K}\Omega$, $R_{18} = 10 \text{ K}\Omega$, $R_{19} = 10 \text{ K}\Omega$, $C_3 = 100 \text{ nF}$, $C_4 = 100 \text{ nF}$, $C_5 = 100 \text{ nF}$, $C_6 = 100 \text{ nF}$, $C_7 = 100 \text{ nF}$, and $C_8 = 100 \text{ nF}$.

According to the circuit equations in (20), the corresponding analog circuit is designed as shown in Figure 13. The circuit consists of operational amplifiers (TL084CN), capacitors, resistors, and multipliers. The operational amplifiers, capacitors, and resistors are mainly used to construct inverting integrator circuits and inverting proportional circuits, while the multipliers are used to realize the nonlinear coupling terms among the state variables. The oscilloscope output results of the circuit are shown in Figure 14. It can be observed that the circuit simulation results exhibit good consistency with the numerical simulation results, which further verifies the feasibility of the proposed system for practical circuit implementation.

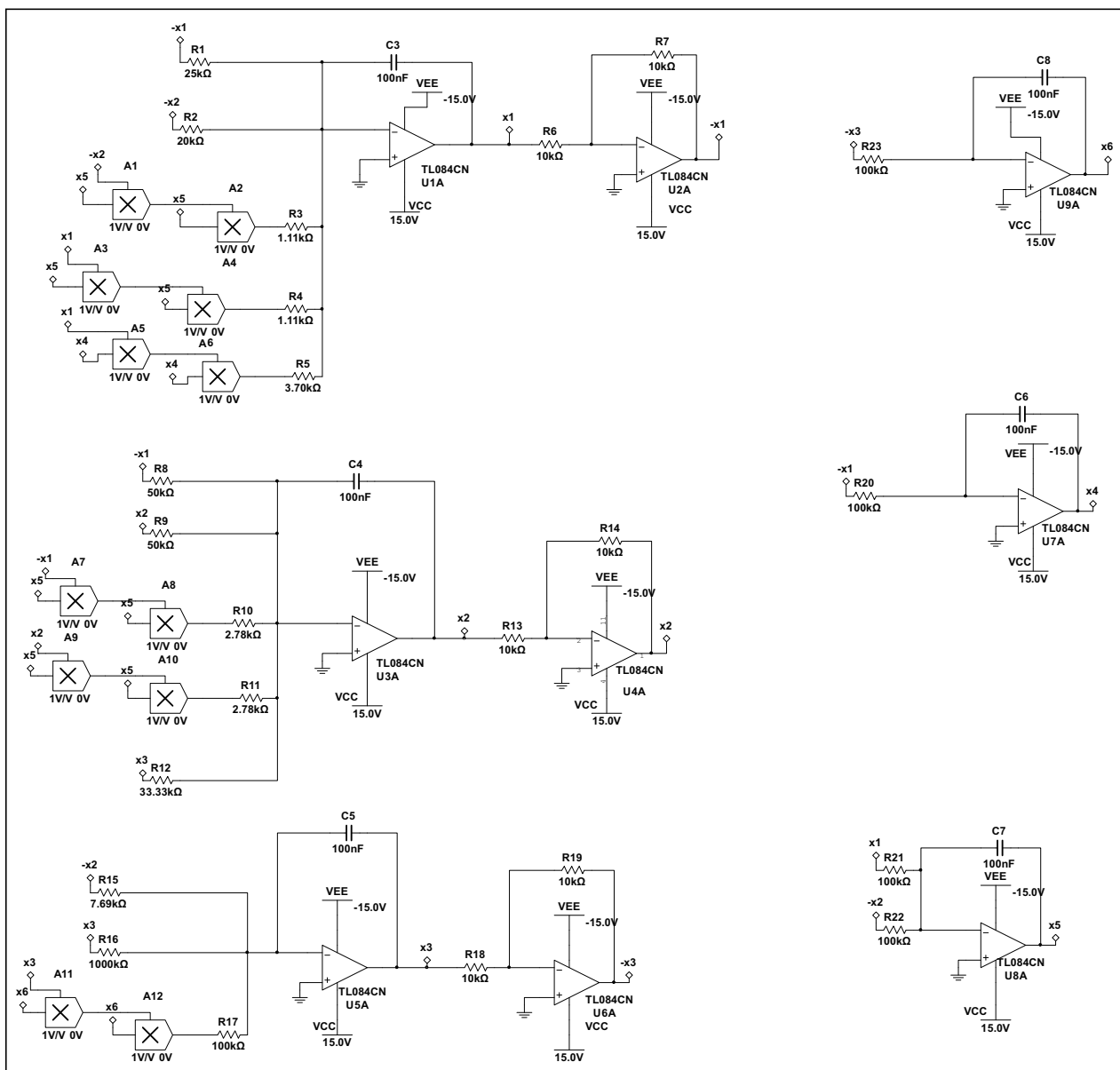


Figure 13. Circuit diagram of System (7).

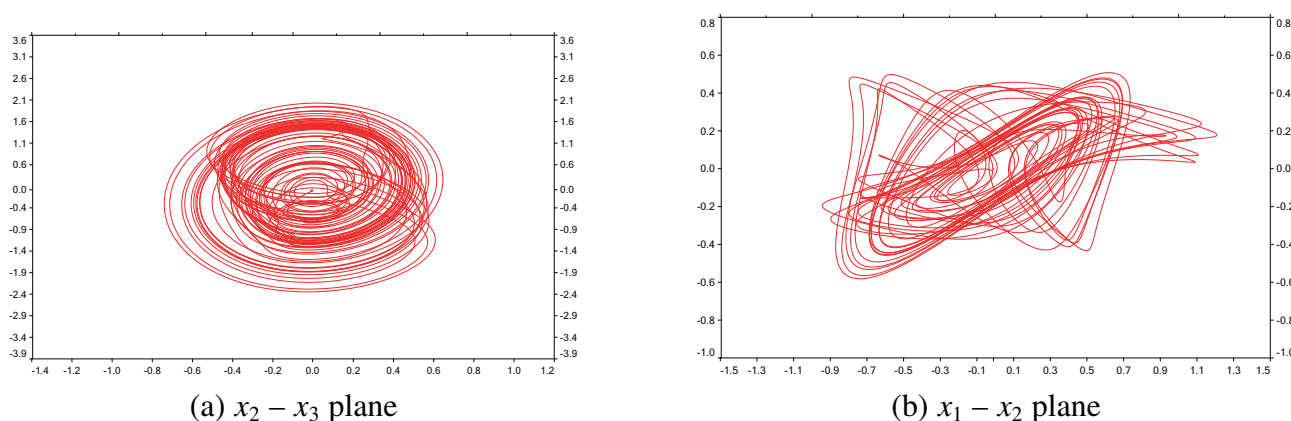


Figure 14. Attractor phase diagrams of the hyperchaotic system observed on the oscilloscope.

3.2. FPGA implementation

To further verify the hardware realizability of the proposed six-dimensional hyperchaotic system, the system is digitally implemented using an FPGA. Since FPGA devices operate on discrete digital signals, the proposed system is discretized using the improved Euler method. Suppose that the system can be expressed as

$$\dot{x} = F(x), \quad (21)$$

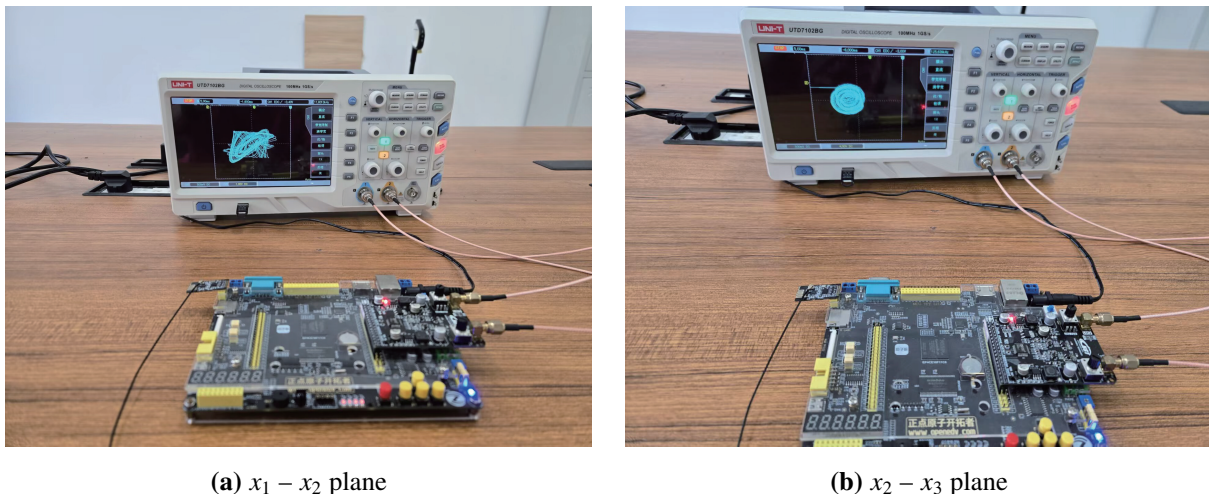
where $x = [x_1, x_2, x_3, x_4, x_5, x_6]^T$, and $F(x) = [F_1(x), F_2(x), F_3(x), F_4(x), F_5(x), F_6(x)]^T$ is the right-hand side function of the system. The prediction–correction format of the improved Euler method is given by

$$\tilde{x}_i(n+1) = x_i(n) + hF_i(x(n)), \quad (22)$$

$$x_i(n+1) = x_i(n) + \frac{h}{2} (F_i(x(n)) + F_i(\tilde{x}(n+1))), \quad i = 1, 2, \dots, 6, \quad (23)$$

where h is the discrete step size, $\tilde{x}_i(n+1)$ is the predicted value, and $x_i(n+1)$ is the corrected state value. During the FPGA digital implementation, a 32-bit signed fixed-point format is used to represent the system's state variables. The most significant bit is used as the sign bit, followed by 5 integer bits and 26 fractional bits, namely the Q5.26 fixed-point format. The initial states of the system are set as $x_1(0) = x_2(0) = x_3(0) = x_4(0) = x_5(0) = x_6(0) = 0.01$. In the Verilog program, the six state variables are implemented using Q5.26 fixed-point arithmetic. To ensure the consistency of the data format, the results of fixed-point multiplication are processed by sign-bit preservation, fractional-bit alignment, and bit-width truncation. Some coefficient multiplications are realized by shift-add operations to reduce hardware resource consumption.

In Quartus, the Verilog program is compiled and implemented to generate the FPGA bitstream file. ModelSim is then used to verify the state update logic and fixed-point operation process. After verification, the bitstream file is downloaded to the FPGA development board. The FPGA output signals are converted by a digital-to-analog conversion module and observed on an oscilloscope in X–Y mode. The obtained phase portraits are shown in Figure 15, which are basically consistent with the MATLAB numerical simulation results, further verifying the hardware realizability of the proposed system.

(a) $x_1 - x_2$ plane(b) $x_2 - x_3$ plane**Figure 15.** FPGA-based phase portraits of the proposed hyperchaotic system.

4. Model description and introduction of the control scheme

The system in (8) can be expressed in the following nonlinear form:

$$\dot{x}(t) = Ax(t) + f(x(t)), \quad (24)$$

where the state vector $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$, A is the constant matrix, and $f(x) = [f_1(x), f_2(x), \dots, f_n(x)]^T \in \mathbb{R}^n$ is a continuous nonlinear function.

Let System (24) be the drive system, according to the concept of chaos synchronization. The response system is described by

$$\dot{y}(t) = Ay(t) + f(y(t)) + u(t), \quad (25)$$

where the state vector $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T \in \mathbb{R}^n$, and $u(t)$ represents external control, given by the following equation:

$$u(t) = \begin{cases} H_k e(t_k) \delta(t - t_k), & t_k = t_0 + k\Delta t, k \in \mathbb{N}, t \leq T_1, \\ 0, & T_1 < t \leq T_2, \\ -De(t) - \rho \text{sign}(e(t))|e(t)|^\delta, & t > T_2, \end{cases} \quad (26)$$

where H_k is the impulsive control gain; H_k weakens as control progresses, following $H_k = H_{\min} + (H_{\max} - H_{\min})\alpha^{k-1}$; α satisfies $0 < \alpha < 1$ to ensure that the impulse gain decreases with the pulse index k ; $e(t_k)$ is the synchronization error at t_k ; $\delta(t)$ denotes the Dirac delta function; D is a diagonal matrix; ρ is a positive constant; $0 < \delta < 1$; and $\{t_1, t_2, \dots, t_k\}$ denotes the impulsive input instants for $t \leq T_1$. In addition, $\text{sign}(e(t)) \triangleq \text{diag}\{e_i(t)\} \in \mathbb{R}^{n \times n}$, and $|e(t)|^\delta \triangleq [|e_1(t)|^\delta, |e_2(t)|^\delta, \dots, |e_n(t)|^\delta]^T$.

The error system can be expressed in the following form:

$$\dot{e}(t) = \dot{y}(t) - \dot{x}(t) = Ae(t) + f(y(t)) - f(x(t)) + u(t), \quad (27)$$

where $e(t) = y(t) - x(t)$.

For $0 \leq t \leq T_1$, the response system can be expressed as follows:

$$\begin{cases} \dot{y}(t) = Ay(t) + f(y(t)), & t \neq t_k, \\ \Delta y(t_k) = H_k e(t_k), & k \in N, \end{cases} \quad (28)$$

where $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, $\Delta y(t_k)$ is a momentary jump at the pulse moment, and $y(t_k) = y(t_k^-)$. It follows from (24) and (28) that the synchronization error system can be given as follows:

$$\begin{cases} \dot{e}(t) = Ae(t) + f(y(t)) - f(x(t)), & t \neq t_k, \\ \Delta e(t_k) = H_k e(t_k), & k \in \mathbb{N}. \end{cases} \quad (29)$$

For $T_1 < t \leq T_2$, the response system can be expressed as follows:

$$\dot{y}(t) = Ay(t) + f(y(t)), \quad (30)$$

and the synchronization error system can be given as

$$\dot{e}(t) = Ae(t) + f(y(t)) - f(x(t)). \quad (31)$$

For $T_2 < t$, the response system can be expressed as follows:

$$\begin{cases} \dot{y}(t) = Ay(t) + f(y(t)) + u(t), \\ u(t) = -De(t) - \rho \text{sign}(e(t))|e(t)|^\delta, \end{cases} \quad (32)$$

and the synchronization error system can be given as

$$\dot{e}(t) = Ae(t) + f(y(t)) - f(x(t)) - De(t) - \rho \text{sign}(e(t))|e(t)|^\delta. \quad (33)$$

Assumption 4.1. Assume that the nonlinear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the Lipschitz condition, that is, a constant $L_1 > 0$ exists such that

$$\|f(x_1) - f(x_2)\| \leq L_1 \|x_1 - x_2\|. \quad (34)$$

In this paper, a finite-time controller consisting of impulsive control, free evolution, and continuous control is designed to realize the synchronization between the response system and the drive system within a finite settling time T , i.e., $\lim_{t \rightarrow T} e(t) = 0$.

5. Finite-time control of memristor-based hyperchaotic system

This section, based on Lyapunov stability theory, matrix inequality, and control-related theory, the condition for the stability of the system in (27) are given, and several related corollaries are derived.

Lemma 5.1. [33, 46] Suppose that a continuous, positive definite function $V(t)$ exists that satisfies the following inequality:

$$\dot{V}(t) \leq -lV^\eta(t) + hV(t), \quad \forall t \geq t_0, \quad V^{1-\eta}(t_0) \leq \frac{l}{h},$$

where $\eta \in (0, 1)$ and $l > 0, h > 0$. In this case, the settling time T_s satisfies $T_s \leq \frac{\ln\left(1 - \frac{h}{l} V^{1-\eta}(t_0)\right)}{h(\eta-1)}$.

Lemma 5.2. [47] For any $\varepsilon > 0$, $a, b \in \mathbb{R}^n$, the inequality $2a^T b \leq \varepsilon a^T a + \varepsilon^{-1} b^T b$ holds.

Lemma 5.3. [48] Suppose that $Q \in \mathbb{R}^{n \times n}$ is a real symmetric matrix, and $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ are its largest and smallest eigenvalues, respectively. Then, for any $x \in \mathbb{R}^n$, we have

$$\lambda_{\min}(Q)x^T x \leq x^T Qx \leq \lambda_{\max}(Q)x^T x. \quad (35)$$

Theorem 5.1. If Assumption 4.1 holds, and a constant $\xi > 1$ exists such that

$$\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}) + \ln(\lambda_k \xi) \leq 0, \quad (36)$$

$$\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} - 2\lambda_D > 0, \quad (37)$$

where $\lambda_A = \lambda_{\max}(A + A^T)$, $\lambda_D = \lambda_{\min}(D)$, $\lambda_k = \lambda_{\max}((H_k + I)^T(H_k + I))$. Then the system in (27) is finite-time stable under the controller in (26).

Proof. For $0 \leq t \leq T_1$, construct the Lyapunov function $V(e(t)) = e(t)^T e(t)$.

For $t \in (t_{k-1}, t_k]$, $k \in N$, we have

$$\begin{aligned} \dot{V}(e(t)) &= 2e^T(t)\dot{e}(t) \\ &= 2e^T(t)[Ae(t) + f(y(t)) - f(x(t))] \\ &= e^T(t)(A + A^T)e(t) + 2e^T(t)[f(y(t)) - f(x(t))]. \end{aligned} \quad (38)$$

According to Lemma 5.3, we can get

$$e^T(t)(A + A^T)e(t) \leq \lambda_A e^T(t)e(t), \quad (39)$$

where $\lambda_A = \lambda_{\max}(A + A^T)$. Moreover, by Lemma 5.2, for any $\varepsilon > 0$, we have

$$2e^T(t)(f(y(t)) - f(x(t))) \leq \varepsilon e^T(t)e(t) + \frac{1}{\varepsilon} \|f(y(t)) - f(x(t))\|^2. \quad (40)$$

Under Assumption 1, the nonlinear function $f(\cdot)$ satisfies

$$\|f(y(t)) - f(x(t))\| \leq L_1 \|y(t) - x(t)\| = L_1 \|e(t)\|. \quad (41)$$

Therefore,

$$2e^T(t)(f(y(t)) - f(x(t))) \leq \left(\varepsilon + \frac{L_1^2}{\varepsilon} \right) e^T(t)e(t), \quad (42)$$

it follows that

$$\dot{V}(e(t)) \leq \left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) V(e(t)). \quad (43)$$

We then get

$$V(e(t)) \leq V(e(t_{k-1}^+)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t - t_{k-1}) \right]. \quad (44)$$

When $t = t_k$, we have

$$V(e(t_k^+)) = e^T(t_k^+)e(t_k^+).$$

From (29), we can obtain

$$\begin{aligned} &= [H_k e(t_k) + e(t_k)]^T [H_k e(t_k) + e(t_k)] \\ &= [(H_k + I)e(t_k)]^T [(H_k + I)e(t_k)] \\ &\leq \lambda_k V(e(t_k)). \end{aligned} \quad (45)$$

where $\lambda_k = \lambda_{\max}((H_k + I)^T(H_k + I))$.

For $t \in [t_0, t_1]$, it follows from (44) and (45) that

$$V(e(t)) \leq V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t - t_0) \right]. \quad (46)$$

$$V(e(t_1)) \leq V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_1 - t_0) \right]. \quad (47)$$

$$V(e(t_1^+)) \leq \lambda_1 V(e(t_1)). \quad (48)$$

For $t \in (t_1, t_2]$, it follows from (44), (47), and (48) that

$$\begin{aligned} V(e(t)) &\leq V(e(t_1^+)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t - t_1) \right] \\ &\leq \lambda_1 V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t - t_0) \right]. \end{aligned} \quad (49)$$

By induction, for $t \in (t_{k-1}, t_k]$, we have

$$\begin{aligned} V(e(t)) &\leq V(e(t_{k-1}^+)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t - t_{k-1}) \right] \\ &\leq \lambda_1 \lambda_2 \cdots \lambda_{k-1} V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t - t_0) \right] \\ &\leq \lambda_1 \lambda_2 \cdots \lambda_{k-1} V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_0) \right] \\ &\leq V(e(t_0)) \lambda_1 \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_1 - t_0) \right] \\ &\quad \lambda_2 \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_2 - t_1) \right] \\ &\quad \cdots \lambda_{k-1} \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_{k-1} - t_{k-2}) \right] \end{aligned}$$

$$\exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}) \right]. \quad (50)$$

From Theorem 5.1, we can get

$$\begin{aligned} \ln(\xi \lambda_k) &\leq - \left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}), \\ \xi \lambda_k &\leq \exp \left[- \left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}) \right], \\ \lambda_k \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}) \right] &\leq \frac{1}{\xi}. \end{aligned} \quad (51)$$

Therefore

$$V(e(t)) \leq \frac{1}{\xi^{k-1}} V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}) \right]. \quad (52)$$

For convenience of analysis, let $t_k = T_1$, and we can get

$$V(e(T_1)) = \|e(T_1)\|^2 \leq \frac{1}{\xi^{k-1}} V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}) \right]. \quad (53)$$

Thus,

$$\|e(T_1)\| \leq \sqrt{\frac{1}{\xi^{k-1}} V(e(t_0)) \exp \left[\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t_k - t_{k-1}) \right]}. \quad (54)$$

For $T_1 < t \leq T_2$, we construct the Lyapunov function $V(e(t)) = e(t)^T e(t)$, and then we have

$$\begin{aligned} \dot{V}(e(t)) &= 2e^T(t) \dot{e}(t) \\ &= 2e^T(t) [Ae(t) + f(y(t)) - f(x(t))] \\ &\leq \left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) V(e(t)). \end{aligned} \quad (55)$$

According to (55), during the free evolution stage $T_1 < t \leq T_2$, one has

$$V(e(t)) \leq V(e(T_1)) \exp \left(\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (t - T_1) \right). \quad (56)$$

$$V(e(T_2)) \leq V(e(T_1)) \exp \left(\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) (T_2 - T_1) \right). \quad (57)$$

According to (53) and (57), we have

$$V(e(T_2)) \leq \frac{1}{\xi^{k-1}} V(e(t_0)) \exp \left(\left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} \right) ((t_k - t_{k-1}) + (T_2 - T_1)) \right). \quad (58)$$

For $t > T_2$, we construct the Lyapunov function $V(e(t)) = e(t)^T e(t)$. We then get

$$\begin{aligned}
 \dot{V}(e(t)) &= 2e^T(t)\dot{e}(t) \\
 &= 2e^T(t)[Ae(t) + f(y(t)) - f(x(t)) - De(t) - \rho \operatorname{sign}(e(t))|e(t)|^\delta] \\
 &= e^T(t)(A + A^T)e(t) + 2e^T(t)[f(y(t)) - f(x(t))] \\
 &\quad - 2e^T(t)De(t) - 2\rho e^T(t) \operatorname{sign}(e(t))|e(t)|^\delta \\
 &= e^T(t)(A + A^T)e(t) + 2e^T(t)[f(y(t)) - f(x(t))] \\
 &\quad - 2e^T(t)De(t) - 2\rho \sum_{i=1}^n |e_i(t)|^{1+\delta} \\
 &\leq \left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} - 2\lambda_D \right) V(e(t)) - 2\rho \sum_{i=1}^n |e_i(t)|^{1+\delta} \\
 &= -2\rho \left(\sum_{i=1}^n |e_i(t)|^2 \right)^{\frac{1+\delta}{2}} + \left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} - 2\lambda_D \right) V(e(t)) \\
 &= -2\rho V^{\frac{1+\delta}{2}}(e(t)) + \left(\lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} - 2\lambda_D \right) V(e(t)) \\
 &= -2\rho V^{\frac{1+\delta}{2}}(e(t)) + ZV(e(t)),
 \end{aligned} \tag{59}$$

where $Z = \lambda_A + \varepsilon + \frac{L_1^2}{\varepsilon} - 2\lambda_D$.

From Lemma 5.1, the drive-response system in (24) and (25) are synchronized under the controller (26), in a finite time

$$T \leq T_2 + T_s \leq T_2 + \frac{\ln\left(1 - \frac{Z}{2\rho} V(e(T_2))^{\frac{1-\delta}{2}}\right)}{Z\left(\frac{\delta-1}{2}\right)}. \tag{60}$$

□

6. Numerical simulation

In this section, numerical simulations are presented to verify the effectiveness of the proposed control scheme. The proposed controller is applied to the constructed hyperchaotic system, compared with existing methods, and further extended to a financial hyperchaotic system.

Example 6.1. Finite-time control and robustness analysis.

From System (8), we can obtain

$$A = \begin{bmatrix} \frac{G-c_1-c_2}{C_1} & \frac{c_2}{C_1} & 0 & 0 & 0 & 0 \\ \frac{c_2}{C_2} & -\frac{c_2}{C_2} & -\frac{1}{C_2} & 0 & 0 & 0 \\ 0 & \frac{1}{L} & -\frac{a+R}{L} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad f(x(t)) = \begin{bmatrix} -\frac{3d_1}{C_1}x_1(t)x_4^2(t) + \frac{3d_2}{C_1}x_2(t)x_5^2(t) - \frac{3d_2}{C_1}x_1(t)x_5^2(t) \\ \frac{3d_2}{C_2}x_1(t)x_5^2(t) - \frac{3d_2}{C_2}x_2(t)x_5^2(t) \\ -\frac{3b}{L}x_3(t)x_6^2(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{61}$$

To determine the Lipschitz constant L_1 of the nonlinear function $f(x(t))$, the boundedness of the system states is first analyzed, as shown in Figure 16. All state variables of the proposed system remain bounded during the evolution process. According to the numerical simulation, the maximum absolute values of the state variables are obtained as follows:

$$|x_1| \leq 1.2177, \quad |x_2| \leq 0.6571, \quad |x_3| \leq 2.1606, \quad |x_4| \leq 0.5104, \quad |x_5| \leq 0.3963, \quad |x_6| \leq 0.7303. \quad (62)$$

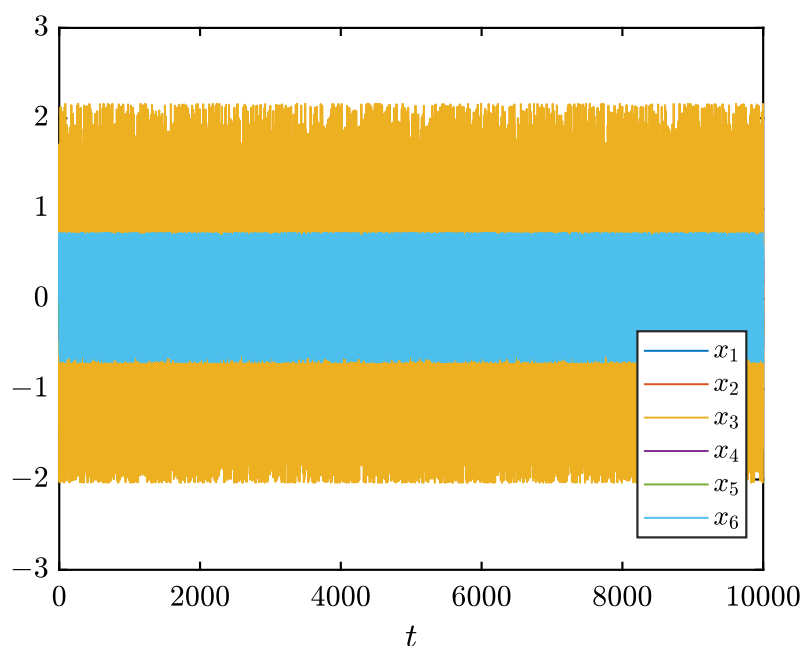


Figure 16. Time-domain waveforms of the six state variables.

After the parameters $C_1 = 2/15$, $C_2 = 1/3$, $L = 1/13$, $d_1 = 6/5$, $d_2 = 4$, and $b = 1/39$ are substituted into $f(x)$, the Jacobian matrix of $f(x)$ is obtained as

$$J_f(x) = \frac{\partial f(x)}{\partial x} = \begin{pmatrix} -27x_4^2 - 90x_5^2 & 90x_5^2 & 0 & -54x_1x_4 & 180(x_2 - x_1)x_5 & 0 \\ 36x_5^2 & -36x_5^2 & 0 & 0 & 72(x_1 - x_2)x_5 & 0 \\ 0 & 0 & -x_6^2 & 0 & 0 & -2x_3x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (63)$$

According to (62), the upper-bound matrix of $J_f(x)$ is given by

$$B = \begin{pmatrix} 21.1686 & 14.1349 & 0 & 33.5618 & 133.7370 & 0 \\ 5.6540 & 5.6540 & 0 & 0 & 53.4948 & 0 \\ 0 & 0 & 0.5334 & 0 & 0 & 3.1558 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (64)$$

According to the bounded-Jacobian criterion for the Lipschitz condition [49], the Lipschitz constant can be determined from an upper bound of the Jacobian matrix. Therefore, an upper bound of the Lipschitz constant can be obtained by $\|J_f(x)\|_2$.

Since $|J_f(x)| \leq B$, where the absolute value is taken componentwise, for any vector $z \in \mathbb{R}^6$, one has

$$|J_f(x)z| \leq |J_f(x)| |z| \leq B|z|. \quad (65)$$

Then,

$$\|J_f(x)z\|_2 \leq \|B|z|\|_2 \leq \|B\|_2 \|z\|_2 = \|B\|_2 \|z\|_2. \quad (66)$$

Thus,

$$\|J_f(x)\|_2 = \sup_{z \neq 0} \frac{\|J_f(x)z\|_2}{\|z\|_2} \leq \|B\|_2. \quad (67)$$

By calculating the spectral norm of B , we can get

$$\|B\|_2 = 149.7739. \quad (68)$$

Therefore,

$$\|J_f(x)\|_2 \leq 149.7739 < 150. \quad (69)$$

Hence, $f(x)$ satisfies the Lipschitz condition with $L_1 = 150$, namely

$$\|f(y) - f(x)\|_2 \leq 150\|y - x\|_2. \quad (70)$$

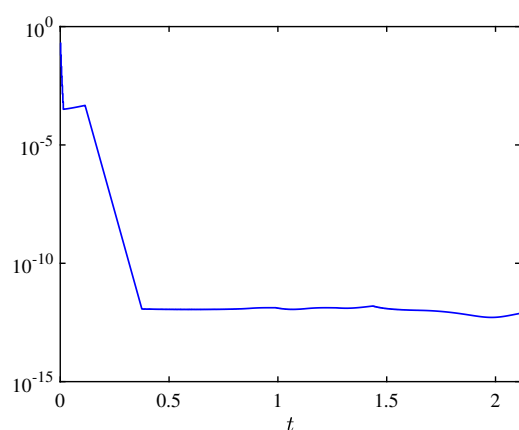
From Lemma 5.1, we set $\rho = 20$, $\delta = 0.7$, and $e(0) = y(0) - x(0)$. From Theorem 5.1, it can get

$$\Delta t \leq -\frac{\varepsilon \ln(\xi \lambda_k)}{\varepsilon(\lambda_A + \varepsilon) + L_1^2}, \quad (71)$$

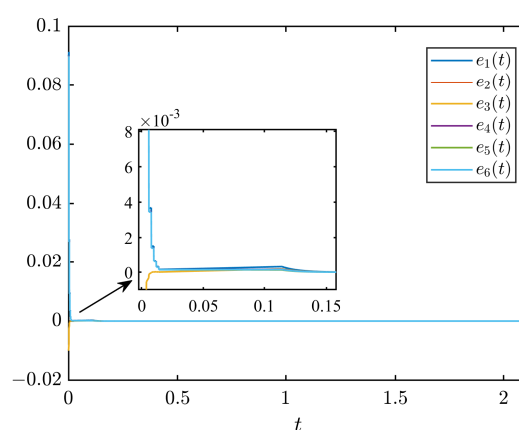
where $\lambda_k = \lambda_{\max}[(I + H_k)^T(I + H_k)]$ and $H_k = H_{\min} + (H_{\max} - H_{\min})\alpha^{k-1}$. Formula (71) gives a joint selection constraint for Δt , K , and α . Since λ_k depends on H_k and H_k is determined by α and the pulse index k , the pulse interval Δt should be selected together with K and α to satisfy $\xi \lambda_k < 1$ for all $k = 1, 2, \dots, K$.

In this paper, $\alpha = 0.9$, $K = 7$, and $\Delta t = 0.002$ s are chosen, which satisfy the condition above. Considering the stability, convergence performance, and control cost, the parameters are selected as $H_{\max} = -0.7I_6$, $H_{\min} = -0.3I_6$, and $\xi = 1.01$. Because $K = 7$, and the pulse interval is chosen as $\Delta t = 0.002$ s, which satisfies the theoretical upper bound. Therefore, the impulsive control stage ends at $T_1 = K\Delta t = 0.014$ s. After a free evolution duration of 0.1 s, the continuous finite-time control starts at $T_2 = 0.114$ s.

Furthermore, by taking $L_1 = 150$, $\rho = 20$, $\delta = 0.7$, $D = 150I_6$, $\varepsilon = \lambda_D = 150$, and $\lambda_A = 13.1648$, the theoretical settling time after T_2 is obtained as $T_s \leq 0.289$ s. The initial value of the response system is selected as $y(0) = [0.1, 0.1, 0, 0.1, 0.1, 0.1]^T$. Therefore, the total synchronization time is $T_3 = T_2 + T_s = 0.403$ s, and the synchronization results are shown in Figure 17.



(a) Synchronization error norm.



(b) Synchronization error trajectories.

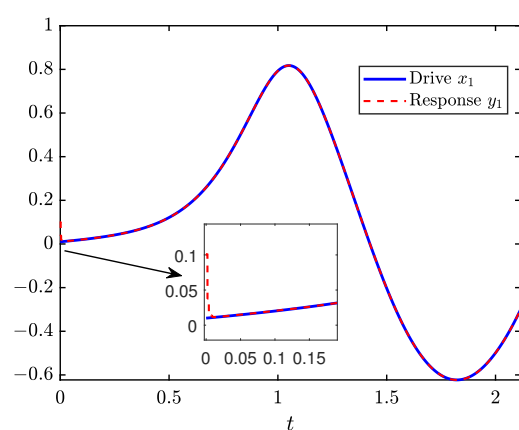
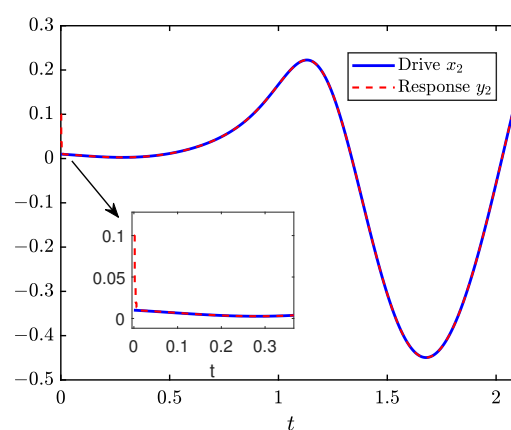
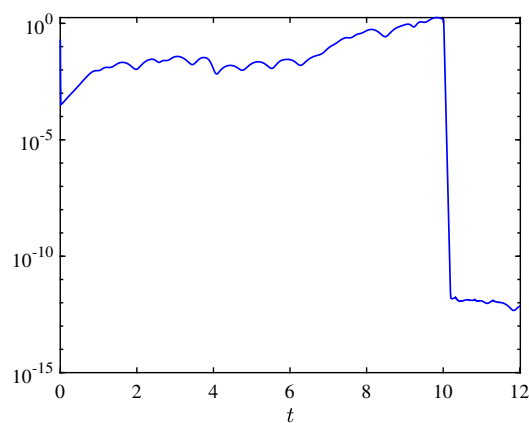
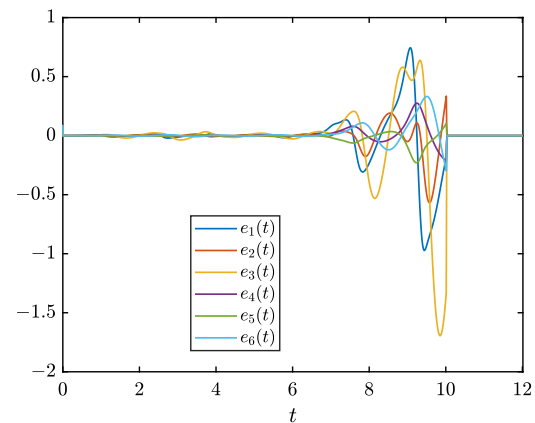
(c) Synchronization trajectories of $x_1(t)$ and $y_1(t)$.(d) Synchronization trajectories of $x_2(t)$ and $y_2(t)$.**Figure 17.** System synchronization performance analysis of $T_2 = 0.114$.

Figure 17 (a) shows the norm of the synchronization error. It can be seen from the figure that during the pulse control stage, the error decreases with the addition of pulses, and the error rebounds during the free evolution stage. In the continuous control stage, it converges to 10^{-12} and tends to be stable. Figure 17 (b) shows the trajectories of the synchronization error, and the error system reaches synchronization at 0.38 s, complying with the theoretical time. Figure 17 (c), (d) are the trajectories diagrams of $x_1 - y_1$ and $x_2 - y_2$, respectively. During the control stage, the trajectories match, which all indicate the feasibility of the control scheme proposed in this paper. It should be noted that the actual convergence time is 0.38 s, which is shorter than the theoretical upper bound of 0.403 s. Therefore, the derived synchronization conditions are only sufficient conditions.

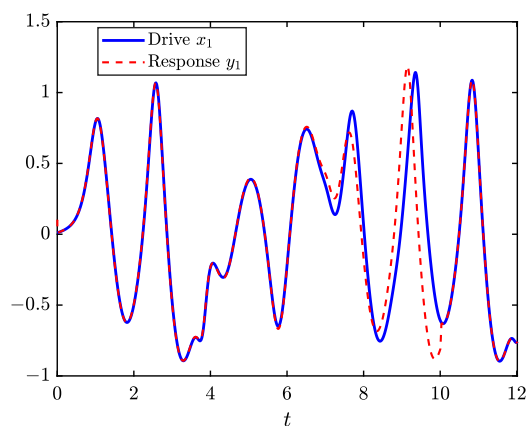
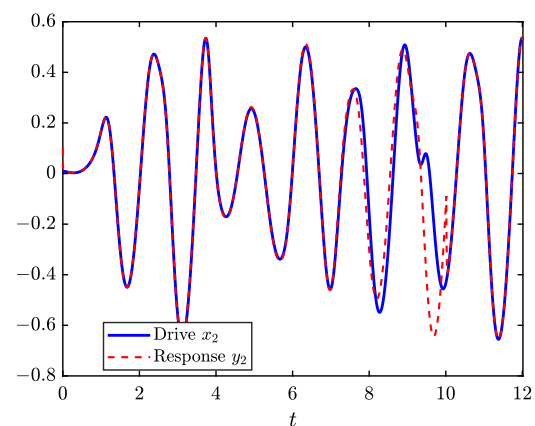
We further extended the free evolution stage to $T_2 = 10.014$ s; the results are shown in Figure 18. During the uncontrolled stage, the system gradually becomes unstable and the fluctuation range keeps increasing. The error system remained stable within $T = 10.303$ s. The system can still remain stable within a finite time, which indicates that this control scheme has a good control effect.



(a) Synchronization error norm.



(b) Synchronization error trajectories.

(c) Synchronization trajectories of $x_1(t)$ and $y_1(t)$.(d) Synchronization trajectories of $x_2(t)$ and $y_2(t)$.**Figure 18.** System synchronization performance analysis of $T_2 = 10.014$.

To verify the robustness of the proposed control scheme under an external disturbance, a bounded random disturbance is introduced into the response system. The disturbance is generated within the interval $[-0.2, 0.2]$, while the controller's parameters remain unchanged. The synchronization error trajectories under the disturbance are shown in Figure 19. It can be observed that all synchronization errors remain bounded without divergence. These results indicate that the proposed control scheme still has bounded synchronization errors in the presence of the external disturbance, demonstrating its robustness.

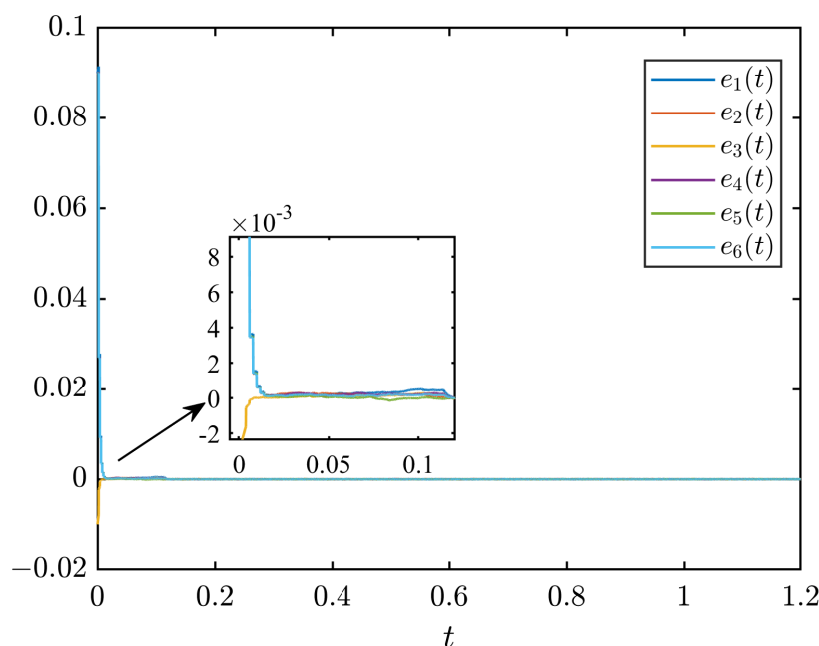


Figure 19. Synchronization error trajectories under random disturbance.

Example 6.2. Comparison with the adaptive synchronization control

The purpose of this example is to compare the finite-time control scheme proposed in this paper with the adaptive synchronization control method reported in [50]. In that study, an adaptive feedback controller was designed to synchronize two four-dimensional hyperchaotic hyperjerk systems, and the corresponding adaptive control laws are given as follows:

$$\begin{cases} u_1 = -e_2 - g_1 e_1, \\ u_2 = -e_3 - g_2 e_2, \\ u_3 = -e_4 - g_3 e_3, \\ u_4 = e_1 + e_2 + \hat{a}e_3 + \hat{b}(|y_2| - |x_2|) + \hat{c}(y_1^4 y_4 - x_1^4 x_4) - g_4 e_4, \end{cases} \quad (72)$$

where e_i ($i = 1, 2, 3, 4$) denote the synchronization errors; g_i ($i = 1, 2, 3, 4$) are positive gain constants; and $\hat{a}$, $\hat{b}$, and $\hat{c}$ denote the estimated system's parameters.

In the numerical simulation, the drive system, response system, system parameters, and adaptive feedback controller given in [50] are adopted for comparison in order to verify the applicability of the proposed control scheme, and the controller gains in the controller in (72) are set as $g_1 = g_2 = g_3 = g_4 = 150$. For the proposed control scheme, the parameters are selected as $\lambda_A = 3.6958$, $\varepsilon = L_1 = 230$. The remaining parameters are the same as those in Example 6.1. The result of the comparison is shown in Figure 20. It can be observed that both control methods can achieve synchronization between the drive system and the response system. However, since the proposed control scheme introduces impulsive control at the initial stage, the initial synchronization errors can be suppressed more rapidly, and the overall convergence time is about 0.01 s. In contrast, the adaptive synchronization method in [50] mainly relies on continuous feedback control and parameter estimation; therefore, its error convergence process is relatively smoother, and the convergence time is about 0.05 s. The results

indicate that the proposed control scheme can achieve faster synchronization.

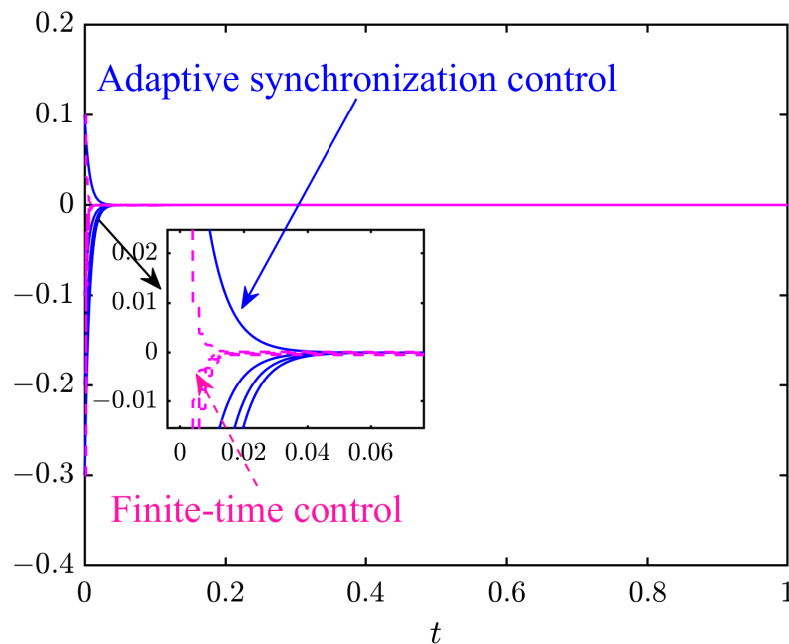


Figure 20. Comparison of synchronization error trajectories between the proposed finite-time control and adaptive synchronization control.

Example 6.3. Comparison with other finite-time control method

To verify the advantage of the proposed finite-time control scheme, a comparison is conducted with the synchronization control method reported in [51]. In that study, a finite-time synchronization strategy was proposed, and the corresponding controller is given as follows:

$$\begin{cases} u_5 = a_1 e_x + b_1 e_y + y_2 z_2 + \varepsilon_1 x_2 \cosh(w_2) - y_1 z_1 - \varepsilon_1 x_1 \cosh(w_1) + h_1 e_x, \\ u_6 = -e_x - e_y - x_2 z_2 + x_1 z_1 + h_1 e_y, \\ u_7 = c_3 e_z + y_1^2 - y_2^2 + d_3 x_1^2 + d_3 x_2^2 + h_1 e_z, \\ u_8 = -\varepsilon_1 x_2 \sinh(w_2) + \varepsilon_1 x_1 \sinh(w_1) + h_1 e_w, \end{cases} \quad (73)$$

where e_x , e_y , e_z , and e_w are the synchronization errors; (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) denote the state variables of the drive system and the response system, respectively; a_1 , b_1 , c_3 , d_3 , and ε_1 are the system's parameters; and h_1 is the feedback gain used to regulate the synchronization convergence rate.

System (8) is adopted as the synchronization system for the numerical simulation. The feedback gain of the controller in (73) is set to $h_1 = 150$; the remaining parameters are kept consistent with those in [51]. For the proposed controller, the continuous finite-time control is activated at $T_2 = 0.020$ s, while all other settings remain the same as in Example 6.1. The result, as shown in Figure 21, indicates that the synchronization control method in [51] can achieve synchronization between the drive system and the response system. However, this method exhibits a relatively slow error convergence process, with a synchronization time of 0.06 s. By comparison, the proposed control method achieves synchronization within 0.025 s, indicating that it can effectively shorten the synchronization time and improve the convergence speed.

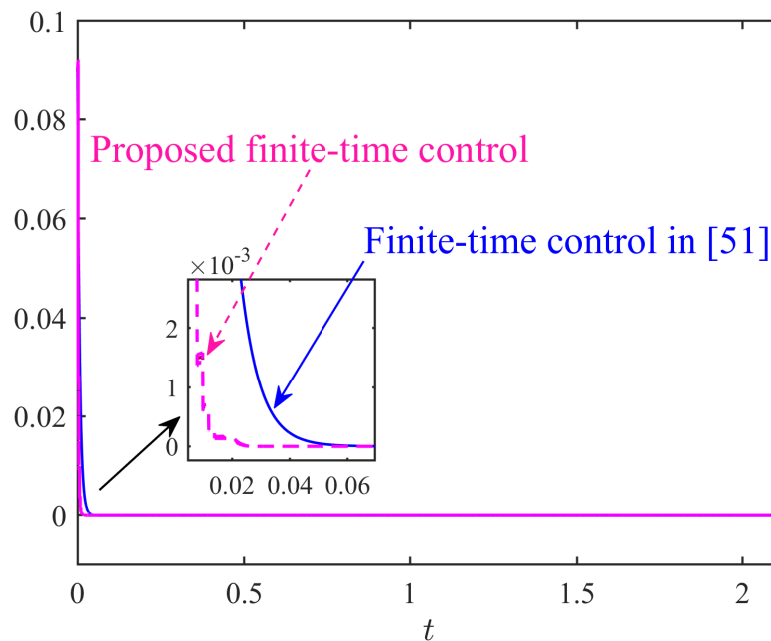


Figure 21. Comparison of synchronization error trajectories between the proposed finite-time control scheme and the finite-time control method in [51].

Example 6.4. Comparison with the active control method

To further verify the effectiveness of the proposed control scheme, a comparison is made with the active control method reported in [52]. In that study, an active controller was designed, and the corresponding control law is expressed as follows:

$$\begin{cases} \mu_a(t) = V_a(t), \\ \mu_b(t) = 2x_s z_s - 2x_m z_m + V_b(t), \\ \mu_c(t) = \chi(-x_s y_s + x_m y_m) + V_c(t), \\ \mu_d(t) = x_s y_s - x_m y_m + V_d(t), \end{cases} \quad (74)$$

where $\mu_a(t)$, $\mu_b(t)$, $\mu_c(t)$, and $\mu_d(t)$ are the active control inputs; χ is a positive parameter; and (x_m, y_m, z_m, w_m) and (x_s, y_s, z_s, w_s) denote the state variables of the master system and the slave system, respectively. The feedback control terms are selected as $V_a(t) = -\zeta_1 e_1$, $V_b(t) = -\zeta_2 e_2$, $V_c(t) = -\zeta_3 e_3$, and $V_d(t) = -\zeta_4 e_4$.

System (8) is adopted as the synchronization system for the numerical simulation. During the simulation, we set $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 150$; the remaining parameters are kept consistent with those in [52]. For the proposed controller, the continuous finite-time control is activated at $T_2 = 0.020$ s, while all other settings remain the same as in Example 6.1. The synchronization error trajectories under the two control methods are shown in Figure 22. It can be observed that the active control method in [52] achieves synchronization at 10.06 s, whereas the finite-time synchronization control method proposed in this paper achieves synchronization at 10.026 s. Since both controllers are activated at 10 s, the actual convergence times are 0.06 s and 0.026 s, respectively. This result indicates that the proposed method can reduce the synchronization time to some extent and provides a faster convergence process.

in this example.

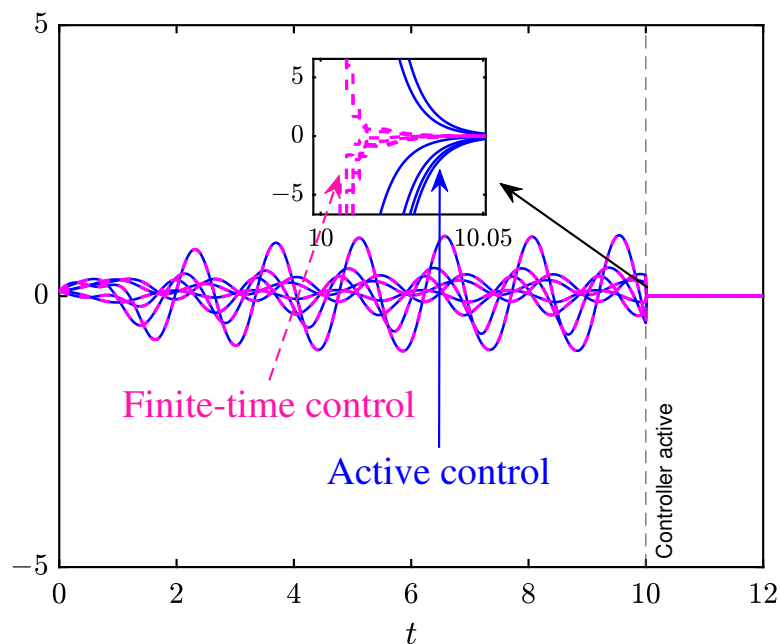


Figure 22. Comparison of synchronization error trajectories between the proposed finite-time control and active control.

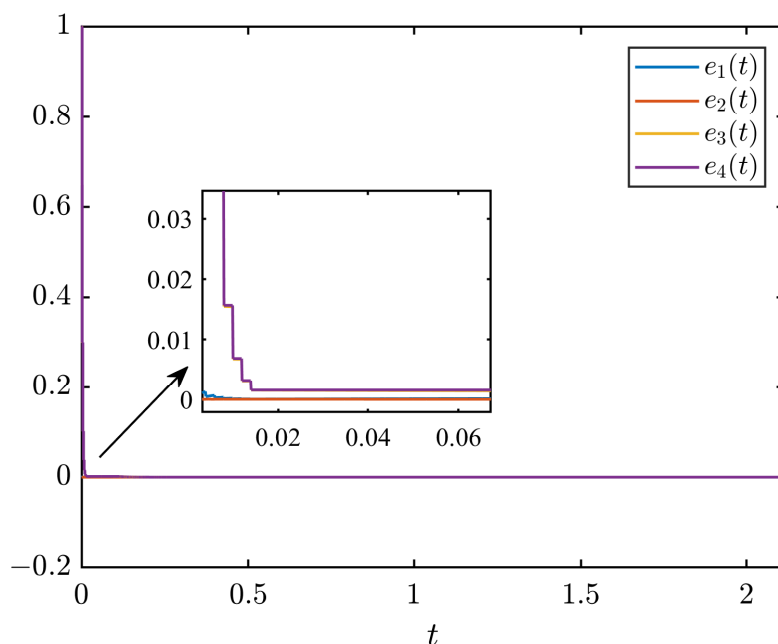


Figure 23. Synchronization error trajectories of the hyperchaotic financial system.

Example 6.5. Application of finite-time control in a financial hyperchaotic system

Hyperchaotic behavior plays an important role in financial systems, as it can simulate market uncertainty and reveal potential instability factors. Achieving control of a hyperchaotic financial system can help provide a new perspective for economic policy and risk prediction. In [53], a four-dimensional hyperchaotic financial system was constructed as follows:

$$\begin{cases} \dot{z}_1 = z_3 + (z_2 - \theta_1)z_1 + \theta_2 z_4, \\ \dot{z}_2 = 1 - \theta_3 z_2 - z_1^2, \\ \dot{z}_3 = -z_1 - \theta_4 z_3, \\ \dot{z}_4 = -\theta_5 z_1 z_2 - \theta_6 \sinh(z_1), \end{cases} \quad (75)$$

where z_1, z_2, z_3 , and z_4 are the interest rate, investment demand, price exponent, and average profit margin, respectively; the parameter θ_1 is the saving; θ_3 is the per investment cost; θ_4 signifies the elasticity of commercial demands; θ_2, θ_5 , and θ_6 are positive constants; Reference [53] sets $\theta_1 = 0.9$, $\theta_2 = 0.1$, $\theta_3 = 0.2$, $\theta_4 = 1.5$, $\theta_5 = 0.3$, and $\theta_6 = 0.15$, and thus the system in (75) is a hyperchaotic system.

The finite-time control scheme proposed in this paper is applied to synchronize the system in (75), for which the parameter values are the same as those used in Example 6.1. The simulation results are shown in Figure 23. It can be observed that the synchronization errors remain stable and gradually converge to zero, indicating the effectiveness of the proposed control scheme.

7. Conclusions

This paper proposed a memristor-based six-dimensional hyperchaotic system and investigated its dynamic characteristics. On the basis of the original three-memristor circuit, a linear resistor R and a negative conductance branch $-G$ are introduced. The resistor R provides an additional dissipation channel, while the negative conductance branch supplies active energy compensation. These modifications change the balance between energy dissipation and energy supply in the circuit and further enhance the nonlinear coupling among the voltage, current, flux, and charge variables. As a result, the proposed system exhibits more complex hyperchaotic behavior with a larger maximum Lyapunov exponent. The dynamic characteristics of the system are verified by equilibrium analysis, Lyapunov exponents, Poincaré sections, bifurcation diagrams, circuit simulation, and FPGA implementation.

Furthermore, a hybrid finite-time synchronization control scheme is developed by combining impulsive control, free evolution, and continuous control. In this scheme, impulsive control is used to rapidly suppress the initial synchronization error, the free-evolution stage reduces unnecessary control input, and continuous control ensures finite-time convergence. Based on Lyapunov stability theory and matrix inequalities, sufficient conditions for finite-time synchronization are derived. Numerical simulations, comparison experiments, application to a hyperchaotic financial system, and a robustness test under bounded random disturbance verified the effectiveness and applicability of the proposed scheme.

The results show that the proposed system has richer nonlinear dynamic characteristics, and the proposed control scheme provides an effective finite-time synchronization strategy for memristor-based hyperchaotic systems. In future work, the proposed hyperchaotic system will be further applied to

secure image and video encryption [54]. Inspired by nonlinear reconstruction methods [55], we will also investigate how such reconstruction strategies can be extended from discrete chaotic maps to continuous chaotic systems.

Author contributions

Zhongmeng Yuan: Writing—original draft, software, formal analysis, methodology, validation. Yuming Feng: Supervision, project administration, funding acquisition, writing—review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper..

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