Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 23404–23436.
DOI:10.3934/math.2026944
Received: 22 May 2026
Revised: 21 July 2026
Accepted: 27 July 2026
Published: 31 July 2026

Research article

Deferred weighted statistical and modular convergence generated by admissible lower triangular transformations: fractional q-difference applications

Gui-Ying Weng[1], Ömer Kişi[2], Mehmet Gürdal[3] and Qing-Bo Cai[4,*]

[1] Department of Mathematics, Yang-En University, Quanzhou 362014, China

[2] Department of Mathematics, Bartın University, Bartın, Turkey

[3] Department of Mathematics, Suleyman Demirel University, Isparta 32260, Turkey

[4] Fujian Provincial Key Laboratory of Data-Intensive Computing, Key Laboratory of Intelligent Computing and Information Processing, School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China

* **Correspondence:** Email: qbcai@126.com.

Abstract: We studied deferred weighted statistical and modular convergence generated by admissible lower triangular transformations. Let $A = (a_{nk})_{n,k\geq 0}$ be a triangle with nonzero diagonal entries, uniformly bounded absolute row sums, and null columns. Convergence is defined through the transformed sequence Ax along admissible deferred weighted windows. We proved that the column-null condition is equivalent to $A(\varphi) \subseteq c_0$, where φ denotes the space of finitely supported sequences, and use this characterization to obtain stability under finite modifications. We established uniqueness, linearity, strong-to-statistical implications, bounded converses, Cauchy characterizations, ideal and Musielak–Orlicz extensions, and window-comparison results. Analogous results are obtained for separable transformations of double sequences. As an application, we showed that the fractional q-difference triangle $Q^{(q,\xi)}$ has an ℓ_1 coefficient kernel, uniformly bounded absolute row sums, and null fixed columns. Integer orders yield banded triangles, whereas noninteger orders produce nonterminating but absolutely summable kernels.

Keywords: lower triangular transformation; deferred weighted statistical convergence; matrix summability; ideal convergence; Musielak–Orlicz modular convergence; fractional q-difference operator; double sequence; strong Cesàro convergence
Mathematics Subject Classification: 39A70, 40A35, 40B05, 40C05, 40G15, 46A80

1. Introduction

Statistical convergence extends ordinary convergence by allowing exceptional indices to form a set of asymptotically negligible size. The notion was introduced independently by Fast [6] and Steinhaus [19], and its relation to ordinary convergence, density, and strong Cesàro summability was subsequently clarified by Fridy [7] and Connor [3]. Since then, statistical convergence has become a useful tool in summability theory, approximation theory, and the study of sequence spaces.

Deferred summability provides a more flexible way to observe asymptotic behavior. Instead of considering only the initial segments $\{1, 2, \ldots, n\}$, one works with moving intervals whose left and right endpoints may vary independently. This idea originates in the deferred Cesàro means of Agnew [1] and was later incorporated into statistical convergence by Küçükaslan and Yılmaztürk [12]. The use of positive weights gives an additional degree of flexibility, since different parts of a window may contribute with different intensities. To retain a genuine statistical character, however, the window lengths must tend to infinity and no single weight should carry an asymptotically positive proportion of the total window mass.

Ideal convergence offers another extension of the classical theory. In the approach introduced by Kostyrko et al. [10], negligible sets are prescribed by an admissible ideal on $\mathbb{N}$. This makes it possible to treat several forms of convergence within a common set-theoretic setting. Musielak–Orlicz functions provide a complementary refinement by replacing fixed powers with position-dependent convex growth functions. The relevant background can be found in the works of Orlicz [17], Krasnosel'skii and Rutickii [11], and Musielak [16].

Matrix transformations form a natural setting in which these convergence methods can be studied. If $A = (a_{nk})_{n,k\geq 0}$ is a lower triangular matrix and $x = (x_k)$ is a scalar sequence, then the transformed sequence is given by $(Ax)_n = \sum_{k=0}^{n} a_{nk} x_k$. Classical matrix-domain theory usually fixes a sequence space X and studies the set of all x for which $Ax \in X$; see, for example, [2, 14]. A related problem is to define and compare convergence processes by testing the asymptotic behavior of Ax along prescribed families of windows.

Many elementary properties of such convergence processes depend only on the structural features of the transformation and not on the particular formula of its coefficients. For this reason, the present paper does not attach the general theory to a single difference operator. Instead, we identify a class of lower triangular transformations for which deferred weighted statistical, ideal, modular, and double-sequence convergence can be developed systematically. The fractional q-difference triangle is then examined as a concrete example of this class.

More precisely, we consider lower triangular transformations $A = (a_{nk})_{n,k\geq 0}$ with nonzero diagonal entries, uniformly bounded absolute row sums, and columns converging to zero. These assumptions serve different purposes. The nonzero diagonal entries guarantee triangular invertibility on the space of all sequences. The uniform boundedness of the absolute row sums provides control on bounded transformed data. The column-null condition, $\lim_{n\to\infty} a_{nr} = 0$ for every fixed $r \in \mathbb{N}_0$, ensures that every finitely supported perturbation is mapped into c_0.

One of the main structural results of the paper concerns finitely supported perturbations. We prove that $a_{nr} \to 0$ for every fixed $r \in \mathbb{N}_0$ if and only if $Ah \in c_0$ for every finitely supported sequence h. Thus the column-null condition precisely characterizes the lower triangular transformations that map every finitely supported sequence into a null sequence. We then use this property as a sufficient condition for showing that finite changes in the original sequence do not alter the statistical, ideal, or modular limit under the assumptions stated for the corresponding method. We do not claim that the column-null condition is necessary for finite-modification stability in every individual generalized convergence method.

Within this setting, we define the deferred weighted statistical convergence generated by A. We prove the uniqueness and linearity, obtain ordinary-to-statistical inclusions, and establish implications from strong to statistical convergence. We also prove the bounded converse results and a Cauchy characterization. The ideal and Musielak–Orlicz modular versions are then developed. Finally, we compare different deferred weighted windows under explicit domination assumptions. The proofs are written so that the hypotheses used in each result are visible, rather than hidden inside a particular coefficient formula.

The double-sequence theory is formulated through separable product transformations. If $A = (a_{mi})$ and $B = (b_{nj})$ are admissible lower triangular transformations, we consider

$$((A \otimes B)x)_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi} b_{nj} x_{ij}.$$

This form is compatible with rectangular deferred weighted windows and with Pringsheim convergence. It also permits independent control of the two coordinates. Corresponding uniqueness, linearity, strong convergence, and Cauchy results are obtained in the rectangular setting.

The general theory is applied to the fractional q-difference triangle. Difference sequence spaces were initiated by Kızmaz [9] and further developed in generalized form by Et and Çolak [5]. The basic tools of q-calculus, including q-numbers and q-binomial coefficients, were described by Kac and Cheung [8]. For $q \in (0, 1)$ and $\xi > 0$, the fractional q-difference coefficients define a lower triangular transformation $Q^{(q,\xi)}$. We prove that its coefficient kernel belongs to ℓ_1, that its absolute row sums are uniformly bounded, and that each fixed column converges to zero. Hence $Q^{(q,\xi)}$ satisfies the assumptions of the general theory.

The distinction between integer and noninteger orders is particularly clear in this example. If $\xi \in \mathbb{N}$, then $Q^{(q,\xi)}$ is banded and a finite perturbation produces only finitely many nonzero transformed terms. If $\xi \notin \mathbb{N}$, then the coefficient sequence does not terminate. A perturbation at a single coordinate therefore produces an infinite transformed tail. The absolute summability of the kernel ensures that this tail tends to zero and allows the finite-modification results of the general theory to be applied. Thus the fractional q-difference triangle illustrates both the finite-memory and infinite-memory cases within the same matrix setting.

The examples show that ordinary convergence of Ax is properly contained in the corresponding deferred weighted statistical convergence class. They also show that statistical convergence need not imply strong convergence unless the transformed error is bounded. Further examples illustrate sparse perturbations for fractional q-difference triangles and the corresponding behavior of double sequences.

The paper is organized as follows. Section 2 introduces admissible deferred weighted windows and admissible lower triangular transformations. Section 3 develops the statistical, strong, and Cauchy convergence generated by a triangle. Section 4 treats the ideal version, while Section 5 develops the Musielak–Orlicz modular theory. Section 6 concerns separable transformations of double sequences. Section 7 compares the convergence methods associated with different deferred weighted windows. Section 8 applies the general results to fractional q-difference triangles. Section 9 contains examples and strictness results. The final section summarizes the main conclusions and indicates possible directions for further study.

2. Admissible lower triangular transformations and deferred weighted windows

Admissible lower triangular transformations. Let ω denote the space of all complex-valued sequences indexed by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We write c_0 and ℓ_∞ for the spaces of null and bounded sequences, respectively, and $\varphi = \{h \in \omega : \text{supp}(h) \text{ is finite}\}$ for the space of finitely supported sequences.

An infinite matrix $A = (a_{nk})_{n,k\geq 0}$ is called a lower triangular transformation, or a triangle, if $a_{nk} = 0$ whenever $k > n$. For $x = (x_k) \in \omega$, define

$$(Ax)_n = \sum_{k=0}^{n} a_{nk} x_k, \quad n \in \mathbb{N}_0. \tag{2.1}$$

Since each sum is finite, $A : \omega \to \omega$ is well defined.

Definition 2.1. *A triangle $A = (a_{nk})_{n,k\geq 0}$ is called admissible if*

(T1) $a_{nn} \neq 0$ *for every $n \in \mathbb{N}_0$;*

(T2) $C_A := \sup_{n\geq 0} \sum_{k=0}^{n} |a_{nk}| < \infty$*;*

(T3) $a_{nr} \to 0$ *as $n \to \infty$ for every fixed $r \in \mathbb{N}_0$.*

Condition (T1) gives triangular invertibility on ω, Condition (T2) ensures the boundedness on ℓ_∞, and Condition (T3) ensures that finitely supported perturbations are mapped into c_0.

Lemma 2.2. *Let $A = (a_{nk})_{n,k\geq 0}$ be a triangle.*

(i) The mapping $A : \omega \to \omega$ is linear.

(ii) If $a_{nn} \neq 0$ for every n, then A has a unique lower triangular inverse on ω.

(iii) If $C_A < \infty$, then $A(\ell_\infty) \subseteq \ell_\infty$ and $\|Ax\|_\infty \leq C_A \|x\|_\infty$ for every $x \in \ell_\infty$.

Proof. Linearity follows from (2.1). If all diagonal entries are nonzero, the equation $Ax = y$ determines x recursively by

$$x_0 = \frac{y_0}{a_{00}}, \quad x_n = \frac{1}{a_{nn}} \left(y_n - \sum_{k=0}^{n-1} a_{nk} x_k \right) \quad (n \geq 1),$$

which gives a unique triangular inverse. Finally, for $x \in \ell_\infty$, we have

$$|(Ax)_n| \leq \sum_{k=0}^{n} |a_{nk}||x_k| \leq \|x\|_\infty \sum_{k=0}^{n} |a_{nk}| \leq C_A \|x\|_\infty.$$

Taking the supremum over n proves (iii). □

For $r \in \mathbb{N}_0$, let $e^{(r)}$ be the r-th unit sequence, that is, $e_k^{(r)} = \delta_{kr}$.

Theorem 2.3. *Let $A = (a_{nk})_{n,k\geq 0}$ be a triangle. The following assertions are equivalent:*

(i) $a_{nr} \to 0$ as $n \to \infty$ for every fixed $r \in \mathbb{N}_0$;

(ii) $Ae^{(r)} \in c_0$ for every $r \in \mathbb{N}_0$;

(iii) $Ah \in c_0$ for every $h \in \varphi$.

Proof. If (i) holds, then

$$(Ae^{(r)})_n = \begin{cases} 0, & n < r, \\ a_{nr}, & n \geq r, \end{cases}$$

so (ii) follows. If (ii) holds and $F = \mathrm{supp}(h)$, then $h = \sum_{r\in F} h_r e^{(r)}$ and $Ah = \sum_{r\in F} h_r Ae^{(r)} \in c_0$, proving (iii). Conversely, if (iii) is applied to $e^{(r)}$ gives $Ae^{(r)} \in c_0$, and hence $a_{nr} = (Ae^{(r)})_n \to 0$ for each fixed r. □

Corollary 2.4. *Let A be admissible and $h \in \varphi$. Then $(Ah)_n \to 0$. In particular, if $x, z \in \omega$ differ only on a finite set, then $\big(A(z-x)\big)_n \to 0$.*

Proof. This follows from Condition (T3) and Theorem 2.3. □

Admissible deferred weighted windows. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be sequences of non–negative integers such that $0 \leq \mu_n < \lambda_n$ and $\lambda_n \to \infty$. Set

$$I_n = (\mu_n, \lambda_n] \cap \mathbb{N}_0 \quad \text{and} \quad W_n = \sum_{k\in I_n} w_k,$$

where $w = (w_k)$ is a positive weight sequence.

Definition 2.5. *The triple $\mathcal{W} = (\lambda, \mu, w)$ is called an admissible deferred weighted window if*

$$|I_n| = \lambda_n - \mu_n \to \infty, \quad W_n \to \infty, \quad \vartheta_n(\mathcal{W}) := \max_{k\in I_n} \frac{w_k}{W_n} \to 0. \tag{2.2}$$

The first condition excludes bounded-width windows, the second makes fixed finite sets negligible, and the third prevents asymptotic concentration of the window's mass at one or finitely many indices. Standard and deferred Cesàro windows, together with positive weights bounded above and away from zero, provide basic examples.

Lemma 2.6. *Let $\mathcal{W}$ be admissible. If $E_n \subseteq I_n$ and $|E_n| \leq M$ for some fixed $M \in \mathbb{N}$, then $\frac{1}{W_n}\sum_{k\in E_n} w_k \to 0$.*

Proof. Since $w_k \leq \vartheta_n(\mathcal{W}) W_n$ for $k \in I_n$

$$0 \leq \frac{1}{W_n} \sum_{k\in E_n} w_k \leq |E_n| \vartheta_n(\mathcal{W}) \leq M\vartheta_n(\mathcal{W}) \to 0.$$

□

Remark 2.7. *Condition* (2.2) *excludes localized windows. For example, if $I_n = \{n\}$, then $\vartheta_n(\mathcal{W}) = 1$. More generally, Lemma* 2.6 *shows that a uniformly bounded number of moving indices cannot carry a non–negligible proportion of the total window mass.*

For $E \subseteq \mathbb{N}_0$, define its upper $\mathcal{W}$-density by

$$\overline{d}_{\mathcal{W}}(E) = \limsup_{n\to\infty} \frac{1}{W_n} \sum_{k\in I_n\cap E} w_k. \tag{2.3}$$

If the corresponding limit exists, it is denoted $d_{\mathcal{W}}(E)$.

Lemma 2.8. *Let $E, F \subseteq \mathbb{N}_0$. Then*

(i) $0 \le \overline{d}_{\mathcal{W}}(E) \le 1$, and $\overline{d}_{\mathcal{W}}(\mathbb{N}_0) = 1$;

(ii) if $E \subseteq F$, then $\overline{d}_{\mathcal{W}}(E) \le \overline{d}_{\mathcal{W}}(F)$;

(iii) $\overline{d}_{\mathcal{W}}(E \cup F) \le \overline{d}_{\mathcal{W}}(E) + \overline{d}_{\mathcal{W}}(F)$;

(iv) Every finite set has upper $\mathcal{W}$-density zero;

(v) $\overline{d}_{\mathcal{W}}(E) = 0$ if and only if $\frac{1}{W_n}\sum_{k\in I_n\cap E} w_k \to 0$.

Proof. Put

$$d_n^{\mathcal{W}}(E) = \frac{1}{W_n} \sum_{k\in I_n\cap E} w_k.$$

The positivity of the weights gives $0 \le d_n^{\mathcal{W}}(E) \le 1$ and $d_n^{\mathcal{W}}(\mathbb{N}_0) = 1$. Monotonicity and subadditivity follow from

$$I_n \cap E \subseteq I_n \cap F, \quad d_n^{\mathcal{W}}(E \cup F) \le d_n^{\mathcal{W}}(E) + d_n^{\mathcal{W}}(F).$$

If F is finite and $C_F = \sum_{k\in F} w_k$, then $0 \le d_n^{\mathcal{W}}(F) \le C_F/W_n \to 0$. Finally, $d_n^{\mathcal{W}}(E) \ge 0$, so its upper limit is zero exactly when the sequence itself converges to zero. □

3. Deferred weighted statistical, strong, and Cauchy convergence generated by a triangle

Throughout this section, $\mathcal{W} = (\lambda, \mu, w)$ denotes an admissible deferred weighted window and $A = (a_{nk})_{n,k\ge 0}$ denotes a lower triangular transformation. Additional assumptions on A are stated whenever they are required.

Statistical convergence generated by a triangle. For $x \in \omega$, $L \in \mathbb{C}$, and $\varepsilon > 0$, put

$$E_\varepsilon^A(x, L) = \{k \in \mathbb{N}_0 : |(Ax)_k - L| \ge \varepsilon\}.$$

Definition 3.1. *A sequence $x \in \omega$ is said to be deferred weighted statistically convergent to L generated by A if $\overline{d}_{\mathcal{W}}(E_\varepsilon^A(x, L)) = 0$ for every $\varepsilon > 0$. In this case, we write $S_{\mathcal{W}}(A)\text{-}\lim x = L$. The class of all such sequences is denoted by $S_{\mathcal{W}}(A)$; when $L = 0$, we write $x \in S_{\mathcal{W},0}(A)$.*

Theorem 3.2. *Let A be a lower triangular transformation.*

(i) The $S_{\mathcal{W}}(A)$-limit is unique.

(ii) If $S_{\mathcal{W}}(A)\text{-}\lim x = L$ and $S_{\mathcal{W}}(A)\text{-}\lim z = M$, then $S_{\mathcal{W}}(A)\text{-}\lim(\alpha x + \beta z) = \alpha L + \beta M$ for every $\alpha, \beta \in \mathbb{C}$.

(iii) The classes $S_{\mathcal{W}}(A)$ and $S_{\mathcal{W},0}(A)$ are linear spaces.

Proof. If we suppose that $L \neq M$ are two $S_{\mathcal{W}}(A)$-limits of x, and set $\varepsilon = |L - M|/2$. For $E_L = E_\varepsilon^A(x, L)$ and $E_M = E_\varepsilon^A(x, M)$, the triangle inequality gives $\mathbb{N}_0 = E_L \cup E_M$. Hence,

$$1 = \overline{d}_{\mathcal{W}}(\mathbb{N}_0) \leq \overline{d}_{\mathcal{W}}(E_L) + \overline{d}_{\mathcal{W}}(E_M) = 0,$$

a contradiction. Thus the limit is unique.

For (ii), the case $\alpha = \beta = 0$ is immediate. Otherwise, let $\varepsilon > 0$ and set $\delta = \varepsilon/[2(|\alpha| + |\beta|)]$. Define $E_x = E_\delta^A(x, L)$ and $E_z = E_\delta^A(z, M)$. By the linearity of A, we have

$$E_\varepsilon^A(\alpha x + \beta z, \alpha L + \beta M) \subseteq E_x \cup E_z.$$

Indeed, outside $E_x \cup E_z$, the transformed error is less than $(|\alpha| + |\beta|)\delta < \varepsilon$. Since both E_x and E_z have an upper $\mathcal{W}$-density of zero, so does their union. Assertion (iii) follows from (ii). □

For a classical sequence space X, its matrix domain under A is $X(A) = \{x \in \omega : Ax \in X\}$.

Theorem 3.3. *Let A be a lower triangular transformation. If $(Ax)_k \to L$, then $S_{\mathcal{W}}(A)\text{-}\lim x = L$. Consequently, $c(A) \subseteq S_{\mathcal{W}}(A)$ and $c_0(A) \subseteq S_{\mathcal{W},0}(A)$.*

Proof. For $\varepsilon > 0$, ordinary convergence gives a k_0 such that $|(Ax)_k - L| < \varepsilon$ for $k \geq k_0$. Hence,

$$E_\varepsilon^A(x, L) \subseteq \{0, 1, \ldots, k_0 - 1\},$$

and the latter set has an upper $\mathcal{W}$-density on zero. □

Corollary 3.4. *Suppose that $a_{nr} \to 0$ as $n \to \infty$ for every fixed $r \in \mathbb{N}_0$. If $x, z \in \omega$ differ only on a finite set, then $S_{\mathcal{W}}(A)\text{-}\lim x = L$ if and only if $S_{\mathcal{W}}(A)\text{-}\lim z = L$.*

Proof. Let $h = z - x$. Then $h \in \varphi$, and Theorem 2.3 gives $Ah \in c_0$. Hence, $S_{\mathcal{W}}(A)\text{-}\lim h = 0$ by Theorem 3.3. Since $z = x + h$, the conclusion follows from Theorem 3.2(ii); the converse is obtained by interchanging x and z. □

Remark 3.5. *Theorems 3.2 and 3.3 require only the linearity of the lower triangular transformation. By Theorem 2.3, the column-null condition is equivalent to the inclusion $A(\varphi) \subseteq c_0$. Corollary 3.4 uses this fact to show that changing finitely many terms of a sequence does not alter its generated statistical limit.*

Strong deferred weighted convergence.

Definition 3.6. *Let $p > 0$. A sequence $x \in \omega$ is said to be strongly deferred weighted convergent of order p to L generated by A if*

$$\frac{1}{W_n} \sum_{k \in I_n} w_k |(Ax)_k - L|^p \longrightarrow 0.$$

In this case, we write $N_{\mathcal{W}}^p(A)\text{-}\lim x = L$.

Theorem 3.7. *Let A be a lower triangular transformation.*

(i) For every $p > 0$, $N_{\mathcal{W}}^p(A)\text{-}\lim x = L$ implies $S_{\mathcal{W}}(A)\text{-}\lim x = L$.

(ii) If $S_{\mathcal{W}}(A)\text{-}\lim x = L$ and the transformed error sequence $((Ax)_k - L)$ is bounded, then $N^p_{\mathcal{W}}(A)\text{-}\lim x = L$ for every $p > 0$.

Proof. Let $p > 0$. For $\varepsilon > 0$ and $E_\varepsilon = E^A_\varepsilon(x, L)$, we have

$$\frac{1}{W_n} \sum_{k \in I_n \cap E_\varepsilon} w_k \le \frac{1}{\varepsilon^p W_n} \sum_{k \in I_n} w_k |(Ax)_k - L|^p.$$

Thus strong convergence implies statistical convergence.

Conversely, suppose that $|(Ax)_k - L| \le M$ for all k. For $\delta > 0$, set $E_\delta = E^A_\delta(x, L)$. Then

$$\frac{1}{W_n} \sum_{k \in I_n} w_k |(Ax)_k - L|^p \le M^p \frac{1}{W_n} \sum_{k \in I_n \cap E_\delta} w_k + \delta^p.$$

The first term tends to zero by statistical convergence. Therefore, the upper limit of the left-hand side is at most δ^p. Letting $\delta \downarrow 0$ proves (ii). □

Remark 3.8. *The boundedness assumption in Theorem* 3.7*(ii) concerns the transformed error* $((Ax)_k - L)$*, not necessarily the original sequence* x*. The examples in Section* 9 *show that this assumption cannot be omitted.*

A Cauchy characterization

Definition 3.9. *A sequence* $x \in \omega$ *is called* $S_{\mathcal{W}}(A)$*-Cauchy if, for every* $\varepsilon > 0$*,* $N \in \mathbb{N}_0$ *exists such that* $\overline{d}_{\mathcal{W}}(\{k \in \mathbb{N}_0 : |(Ax)_k - (Ax)_N| \ge \varepsilon\}) = 0$.

Lemma 3.10. *If* $E, F \subseteq \mathbb{N}_0$ *satisfy* $\overline{d}_{\mathcal{W}}(E^c) = \overline{d}_{\mathcal{W}}(F^c) = 0$*, then* $E \cap F \ne \emptyset$*.*

Proof. Otherwise, $\mathbb{N}_0 = E^c \cup F^c$, and therefore, we have

$$1 = \overline{d}_{\mathcal{W}}(\mathbb{N}_0) \le \overline{d}_{\mathcal{W}}(E^c) + \overline{d}_{\mathcal{W}}(F^c) = 0,$$

which is impossible. □

Theorem 3.11. *Let* A *be a lower triangular transformation. A sequence* $x \in \omega$ *is* $S_{\mathcal{W}}(A)$*-convergent if and only if it is* $S_{\mathcal{W}}(A)$*-Cauchy.*

Proof. Put $y = Ax$. Suppose first that $S_{\mathcal{W}}(A)\text{-}\lim x = L$. For $\varepsilon > 0$, let $G = \{k \in \mathbb{N}_0 : |y_k - L| < \varepsilon/2\}$. Since $\overline{d}_{\mathcal{W}}(G^c) = 0$, the set G is nonempty. Choose $N \in G$. Then

$$\{k : |y_k - y_N| \ge \varepsilon\} \subseteq \{k : |y_k - L| \ge \varepsilon/2\},$$

so x is $S_{\mathcal{W}}(A)$-Cauchy.

Conversely, assume that x is $S_{\mathcal{W}}(A)$-Cauchy. For each $m \in \mathbb{N}$, choose $N_m \in \mathbb{N}_0$ such that

$$\overline{d}_{\mathcal{W}}(\{k : |y_k - y_{N_m}| \ge 2^{-m}\}) = 0,$$

and put $G_m = \{k : |y_k - y_{N_m}| < 2^{-m}\}$. By Lemma 3.10, $G_m \cap G_r \ne \emptyset$ for all m, r. Choosing $k \in G_m \cap G_r$ gives $|y_{N_m} - y_{N_r}| < 2^{-m} + 2^{-r}$. Hence, (y_{N_m}) is Cauchy in $\mathbb{C}$ and converges to some $L \in \mathbb{C}$.

Let $\varepsilon > 0$. Choose m so that $2^{-m} < \varepsilon/2$ and $|y_{N_m} - L| < \varepsilon/2$. Then

$$\{k : |y_k - L| \ge \varepsilon\} \subseteq \{k : |y_k - y_{N_m}| \ge 2^{-m}\},$$

whose upper $\mathcal{W}$-density is zero. Thus $S_{\mathcal{W}}(A)\text{-}\lim x = L$. □

Remark 3.12. *The uniqueness, linearity, the strong convergence implications, and the Cauchy characterization hold for every lower triangular transformation. The column-null condition is not required for these results. It is used to ensure that finitely supported perturbations are mapped into c_0, and hence provides a sufficient condition for finite modification stability. Section 8 verifies this condition, together with the remaining admissibility assumptions, for the fractional q-difference triangle.*

4. Ideal convergence generated by a triangle

Throughout this section, $\mathcal{W} = (\lambda, \mu, w)$ is an admissible deferred weighted window, $A = (a_{nk})_{n,k\geq 0}$ is a lower triangular transformation, and $\mathcal{I}$ is a nontrivial admissible ideal of $\mathbb{N}$. Thus $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$, and is closed under subsets and finite unions, and $\mathbb{N} \notin \mathcal{I}$.

For a real sequence (r_n), we write $\mathcal{I}\text{-}\lim r_n = r$ if $\{n \in \mathbb{N} : |r_n - r| \geq \delta\} \in \mathcal{I}$ for every $\delta > 0$. For $E \subseteq \mathbb{N}_0$, put

$$D_n^{\mathcal{W}}(E) = \frac{1}{W_n} \sum_{k \in I_n \cap E} w_k \quad \text{and} \quad M_{n,p}^{A}(x, L) = \frac{1}{W_n} \sum_{k \in I_n} w_k |(Ax)_k - L|^p.$$

Lemma 4.1. *Let (a_n) and (b_n) be non-negative real sequences.*

(i) If $0 \leq a_n \leq b_n$ eventually and $\mathcal{I}\text{-}\lim b_n = 0$, then $\mathcal{I}\text{-}\lim a_n = 0$.

(ii) If $\mathcal{I}\text{-}\lim a_n = \mathcal{I}\text{-}\lim b_n = 0$, then $\mathcal{I}\text{-}\lim(a_n + b_n) = 0$.

(iii) Ordinary convergence $a_n \to 0$ implies $\mathcal{I}\text{-}\lim a_n = 0$.

(iv) If $\mathcal{I}\text{-}\lim a_n = 0$ and $c \geq 0$, then $\mathcal{I}\text{-}\lim c a_n = 0$.

Proof. For (i), if $a_n \leq b_n$ for $n \geq n_0$, then, for every $\delta > 0$, we have

$$\{n : a_n \geq \delta\} \subseteq \{1, \dots, n_0 - 1\} \cup \{n : b_n \geq \delta\},$$

and the right-hand side belongs to $\mathcal{I}$. For (ii), use

$$\{n : a_n + b_n \geq \delta\} \subseteq \{n : a_n \geq \delta/2\} \cup \{n : b_n \geq \delta/2\}.$$

Assertion (iii) follows because the corresponding exceptional set is finite. For (iv), the case $c = 0$ is immediate, while for $c > 0$, $\{n : c a_n \geq \delta\} = \{n : a_n \geq \delta/c\}$. □

Ideal deferred weighted statistical convergence

For $x \in \omega$, $L \in \mathbb{C}$, and $\varepsilon > 0$, let $E_\varepsilon^A(x, L) = \{k \in \mathbb{N}_0 : |(Ax)_k - L| \geq \varepsilon\}$.

Definition 4.2. *A sequence $x \in \omega$ is said to be $\mathcal{I}$-deferred weighted statistically convergent to L generated by A if $\mathcal{I}\text{-}\lim_{n\to\infty} D_n^{\mathcal{W}}(E_\varepsilon^A(x, L)) = 0$ for every $\varepsilon > 0$. We then write $S_{\mathcal{W}}^{\mathcal{I}}(A)\text{-}\lim x = L$ and denote the class of all such sequences as $S_{\mathcal{W}}^{\mathcal{I}}(A)$.*

Definition 4.3. *Let $p > 0$. A sequence $x \in \omega$ is said to be strongly $\mathcal{I}$-deferred weighted convergent of order p to L generated by A if $\mathcal{I}\text{-}\lim_{n\to\infty} M_{n,p}^{A}(x, L) = 0$. In this case, we write $N_{\mathcal{W}}^{p,\mathcal{I}}(A)\text{-}\lim x = L$.*

Theorem 4.4. *Let A be a lower triangular transformation.*

(i) The $S^{\mathcal{I}}_{\mathcal{W}}(A)$-limit is unique.

(ii) If $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$ and $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim z = M$, then $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim(\alpha x + \beta z) = \alpha L + \beta M$ for every $\alpha, \beta \in \mathbb{C}$.

(iii) The class $S^{\mathcal{I}}_{\mathcal{W}}(A)$ is a linear space.

Proof. Put $y = Ax$. Suppose that $L \neq M$ are two ideal statistical limits of x, and set $\varepsilon = |L - M|/2$. Define

$$u_n(L) = D^{\mathcal{W}}_n(\{k : |y_k - L| \geq \varepsilon\}), \quad u_n(M) = D^{\mathcal{W}}_n(\{k : |y_k - M| \geq \varepsilon\}).$$

For every k, at least one of the two defining inequalities holds; hence $1 \leq u_n(L) + u_n(M)$. Therefore

$$\mathbb{N} = \{n : u_n(L) \geq 1/2\} \cup \{n : u_n(M) \geq 1/2\}.$$

Both sets on the right belong to $\mathcal{I}$, which contradicts $\mathbb{N} \notin \mathcal{I}$. Thus the limit is unique.

For (ii), the case $\alpha = \beta = 0$ is immediate. Otherwise, let $\varepsilon > 0$ and set $\delta = \varepsilon/[2(|\alpha| + |\beta|)]$. With $E_1 = E^A_\delta(x, L)$ and $E_2 = E^A_\delta(z, M)$, the linearity of A gives

$$E^A_\varepsilon(\alpha x + \beta z, \alpha L + \beta M) \subseteq E_1 \cup E_2.$$

Consequently, we have

$$D^{\mathcal{W}}_n(E^A_\varepsilon(\alpha x + \beta z, \alpha L + \beta M)) \leq D^{\mathcal{W}}_n(E_1) + D^{\mathcal{W}}_n(E_2).$$

Lemma 4.1 yields the required ideal convergence. Part (iii) follows from (ii). □

Theorem 4.5. *Let A be a lower triangular transformation.*

(i) $S_{\mathcal{W}}(A)\text{-}\lim x = L$ *implies* $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$.

(ii) *For* $p > 0$, $N^p_{\mathcal{W}}(A)\text{-}\lim x = L$ *implies* $N^{p,\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$.

(iii) *If* $(Ax)_k \to L$, *then* $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$.

Proof. In (i), ordinary statistical convergence gives $D^{\mathcal{W}}_n(E^A_\varepsilon(x, L)) \to 0$ for every $\varepsilon > 0$. Lemma 4.1(iii) then gives the ideal limit. The same argument applied to $M^A_{n,p}(x, L)$ proves (ii). Part (iii) follows from Theorem 3.3 and Part (i). □

Theorem 4.6. *Suppose that $a_{nr} \to 0$ as $n \to \infty$ for every fixed $r \in \mathbb{N}_0$. If $x, z \in \omega$ differ only on a finite set, then $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$ if and only if $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim z = L$.*

Proof. Let $h = z - x$. Since $h \in \varphi$, Theorem 2.3 gives $Ah \in c_0$. Thus $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim h = 0$ by Theorem 4.5(iii). The conclusion follows from Theorem 4.4(ii), first with $z = x + h$ and then with $x = z - h$. □

Strong ideal convergence.

Theorem 4.7. *Let A be a lower triangular transformation and $p > 0$. If $N^{p,\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$, then $S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$.*

Proof. For every $\varepsilon > 0$, we have

$$D_n^{\mathcal{W}}(E_\varepsilon^A(x, L)) \leq \frac{1}{\varepsilon^p} M_{n,p}^A(x, L).$$

The conclusion follows from Lemma 4.1. □

Theorem 4.8. *Suppose that $S_{\mathcal{W}}^{\mathcal{I}}(A)$- $\lim x = L$ and that $((Ax)_k - L)$ is bounded. Then, for every $p > 0$, $N_{\mathcal{W}}^{p,\mathcal{I}}(A)$- $\lim x = L$.*

Proof. Choose $M > 0$ such that $|(Ax)_k - L| \leq M$ for every k. Fix $p > 0$ and $\rho > 0$, and choose $\delta > 0$ with $\delta^p < \rho/2$. Writing $E_\delta = E_\delta^A(x, L)$, we have

$$M_{n,p}^A(x, L) \leq M^p D_n^{\mathcal{W}}(E_\delta) + \delta^p.$$

Hence, we have

$$\{n : M_{n,p}^A(x, L) \geq \rho\} \subseteq \left\{n : D_n^{\mathcal{W}}(E_\delta) \geq \frac{\rho}{2M^p}\right\}.$$

The set on the right belongs to $\mathcal{I}$ by the assumed ideal statistical convergence. Since $\rho > 0$ was arbitrary, $\mathcal{I}$- $\lim M_{n,p}^A(x, L) = 0$. □

Corollary 4.9. *If the transformed error sequence $((Ax)_k - L)$ is bounded, then, for every $p > 0$, $S_{\mathcal{W}}^{\mathcal{I}}(A)$- $\lim x = L$ if and only if $N_{\mathcal{W}}^{p,\mathcal{I}}(A)$- $\lim x = L$.*

Proof. Apply Theorems 4.8 and 4.7. □

Remark 4.10. *For $\mathcal{I} =$ Fin, the ideal convergence of the quotient sequences is equivalent to ordinary convergence, so the methods of Section 3 are recovered. The algebraic and strong convergence results above require no special coefficient formula. The column-null condition is used in the finite-modification theorem through the inclusion $A(\varphi) \subseteq c_0$, exactly as in the nonideal setting.*

5. Musielak–Orlicz modular convergence generated by a triangle

Throughout this section, $\mathcal{W} = (\lambda, \mu, w)$ is an admissible deferred weighted window, $A = (a_{nk})_{n,k\geq 0}$ is a lower triangular transformation, and $\mathcal{M} = (M_k)_{k\geq 0}$ is a sequence of Orlicz functions. Thus each $M_k : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing, and convex, with $M_k(0) = 0$, $M_k(t) > 0$ for $t > 0$, and $M_k(t) \to \infty$ as $t \to \infty$.

We use the following uniform conditions:

$$m_{\mathcal{M}}(t) := \inf_{k\geq 0} M_k(t) > 0 \quad (t > 0), \tag{5.1}$$

$$M_{\mathcal{M}}(R) := \sup_{k\geq 0} M_k(R) < \infty \quad (R > 0), \tag{5.2}$$

$$\lim_{\delta\downarrow 0} M_{\mathcal{M}}(\delta) = 0. \tag{5.3}$$

Condition (5.1) prevents the degeneration of the modular family, (5.2) gives uniform boundedness on bounded intervals, and (5.3) gives uniform smallness at the origin.

For $\rho > 0$, define

$$\mathscr{M}_{n,\rho}^A(x, L) := \frac{1}{W_n} \sum_{k\in I_n} w_k M_k\left(\frac{|(Ax)_k - L|}{\rho}\right).$$

Definition 5.1. *A sequence $x \in \omega$ is said to be strongly $\mathcal{W}$-Musielak–Orlicz convergent to L generated by A if $\rho > 0$ exists such that*

$$\mathscr{M}^{A}_{n,\rho}(x, L) \longrightarrow 0. \tag{5.4}$$

In this case, we write $N^{\mathcal{M}}_{\mathcal{W}}(A)\text{-}\lim x = L$, and denote the class of all such sequences as $N^{\mathcal{M}}_{\mathcal{W}}(A)$.

Remark 5.2. *The parameter ρ in Definition* 5.1 *is a scale parameter and need not be unique. Whenever the lower condition* (5.1) *is assumed, Corollary* 5.4 *shows that the limit value L is unique; hence L is independent of the particular admissible value of ρ. If $M_k(t) = t^p$ for every k and some $p \geq 1$, then Definition* 5.1 *reduces to strong deferred weighted convergence of order p generated by A. The Musielak–Orlicz formulation therefore extends the fixed-power case by allowing the growth function to depend on the index.*

Relations with deferred weighted statistical convergence

Theorem 5.3. *Assume that $\mathcal{M}$ satisfies* (5.1)*. If $N^{\mathcal{M}}_{\mathcal{W}}(A)\text{-}\lim x = L$, then $S_{\mathcal{W}}(A)\text{-}\lim x = L$.*

Proof. Choose $\rho > 0$ such that $\mathscr{M}^{A}_{n,\rho}(x, L) \to 0$. For $\varepsilon > 0$, let $E_\varepsilon = E^A_\varepsilon(x, L)$. If $k \in E_\varepsilon$, then

$$M_k\left(\frac{|(Ax)_k - L|}{\rho}\right) \geq m_{\mathcal{M}}\left(\frac{\varepsilon}{\rho}\right) > 0.$$

Consequently

$$\frac{1}{W_n} \sum_{k \in I_n \cap E_\varepsilon} w_k \leq \frac{\mathscr{M}^{A}_{n,\rho}(x, L)}{m_{\mathcal{M}}(\varepsilon/\rho)} \longrightarrow 0.$$

Hence, $\overline{d}_{\mathcal{W}}(E_\varepsilon) = 0$ for every $\varepsilon > 0$. □

Corollary 5.4. *Under Condition* (5.1)*, the $N^{\mathcal{M}}_{\mathcal{W}}(A)$-limit is unique.*

Proof. Apply Theorem 5.3 and the uniqueness assertion in Theorem 3.2. □

Theorem 5.5. *Assume that $\mathcal{M}$ satisfies* (5.2) *and* (5.3)*. If $S_{\mathcal{W}}(A)\text{-}\lim x = L$ and the transformed error $((Ax)_k - L)$ is bounded, then $N^{\mathcal{M}}_{\mathcal{W}}(A)\text{-}\lim x = L$.*

Proof. Choose $B > 0$ such that $|(Ax)_k - L| \leq B$ for every k. For $\delta > 0$, put $E_\delta = E^A_\delta(x, L)$. Monotonicity of the functions M_k gives

$$\mathscr{M}^{A}_{n,1}(x, L) \leq M_{\mathcal{M}}(B) \frac{1}{W_n} \sum_{k \in I_n \cap E_\delta} w_k + M_{\mathcal{M}}(\delta).$$

The first term tends to zero by statistical convergence. Therefore, $\limsup_{n\to\infty} \mathscr{M}^{A}_{n,1}(x, L) \leq M_{\mathcal{M}}(\delta)$. Letting $\delta \downarrow 0$ and using (5.3) yields $\mathscr{M}^{A}_{n,1}(x, L) \to 0$. □

Corollary 5.6. *Assume that $\mathcal{M}$ satisfies* (5.1)*,* (5.2)*, and* (5.3)*. If $((Ax)_k - L)$ is bounded, then $S_{\mathcal{W}}(A)\text{-}\lim x = L$ if and only if $N^{\mathcal{M}}_{\mathcal{W}}(A)\text{-}\lim x = L$.*

Proof. Apply Theorems 5.5 and 5.3. □

Corollary 5.7. *Let $M_k(t) = t^{p_k}$, where $1 \le p_- \le p_k \le p_+ < \infty$ ($k \in \mathbb{N}_0$). Then Conditions* (5.1)–(5.3) *hold. Consequently, for a bounded transformed error, $S_{\mathcal{W}}(A)$- $\lim x = L$ if and only if $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim x = L$.*

Proof. Each $M_k(t) = t^{p_k}$ is an Orlicz function. If $0 < t < 1$, then $t^{p_+} \le t^{p_k} \le t^{p_-}$, whereas for $t \ge 1$, $t^{p_-} \le t^{p_k} \le t^{p_+}$. Thus

$$m_{\mathcal{M}}(t) \ge \begin{cases} t^{p_+}, & 0 < t < 1, \\ t^{p_-}, & t \ge 1, \end{cases}$$

so (5.1) holds. Similarly, we have

$$M_{\mathcal{M}}(R) \le \begin{cases} R^{p_-}, & 0 < R < 1, \\ R^{p_+}, & R \ge 1, \end{cases}$$

which proves (5.2). Finally, for $0 < \delta < 1$, $M_{\mathcal{M}}(\delta) \le \delta^{p_-} \to 0$, proving (5.3). The conclusion follows from Corollary 5.6. □

Linearity and finite modifications

Theorem 5.8. *Suppose that $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim x = L$ and $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim z = M$. Then, for every $\alpha, \beta \in \mathbb{C}$, $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim(\alpha x + \beta z) = \alpha L + \beta M$. Consequently, $N_{\mathcal{W}}^{\mathcal{M}}(A)$ is a linear space.*

Proof. Choose $\rho_1, \rho_2 > 0$ such that $\mathscr{M}_{n,\rho_1}^{A}(x, L) \to 0$ and $\mathscr{M}_{n,\rho_2}^{A}(z, M) \to 0$. The case $\alpha = \beta = 0$ is immediate. Otherwise, set

$$\rho = 2\max\{|\alpha|\rho_1, |\beta|\rho_2\} > 0.$$

By the linearity of A, monotonicity, and the convexity of M_k, we have

$$M_k\left(\frac{\left|(A(\alpha x + \beta z))_k - (\alpha L + \beta M)\right|}{\rho}\right) \le \frac{1}{2} M_k\left(\frac{|(Ax)_k - L|}{\rho_1}\right) + \frac{1}{2} M_k\left(\frac{|(Az)_k - M|}{\rho_2}\right).$$

Therefore

$$\mathscr{M}_{n,\rho}^{A}(\alpha x + \beta z, \alpha L + \beta M) \le \frac{1}{2}\mathscr{M}_{n,\rho_1}^{A}(x, L) + \frac{1}{2}\mathscr{M}_{n,\rho_2}^{A}(z, M),$$

and the right-hand side tends to zero. □

Lemma 5.9. *Assume that $\mathcal{M}$ satisfies* (5.2) *and* (5.3). *If $s_k \to 0$, then, for every $\rho > 0$, we have*

$$\frac{1}{W_n} \sum_{k \in I_n} w_k M_k\left(\frac{|s_k|}{\rho}\right) \longrightarrow 0.$$

Proof. Fix $\rho > 0$ and $\delta > 0$. Choose k_0 such that $|s_k|/\rho \le \delta$ for $k \ge k_0$, and set

$$C_0 = \sum_{k=0}^{k_0 - 1} w_k M_k\left(\frac{|s_k|}{\rho}\right).$$

Then

$$0 \le \frac{1}{W_n} \sum_{k \in I_n} w_k M_k\left(\frac{|s_k|}{\rho}\right) \le \frac{C_0}{W_n} + M_{\mathcal{M}}(\delta).$$

Taking the upper limit and then letting $\delta \downarrow 0$ proves the assertion. □

Corollary 5.10. *Assume that $\mathcal{M}$ satisfies* (5.2) *and* (5.3). *If $(Ax)_k \to L$, then $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim x = L$.*

Proof. Apply Lemma 5.9 to $s_k = (Ax)_k - L$. □

Theorem 5.11. *Assume that $\mathcal{M}$ satisfies* (5.2) *and* (5.3), *and that $a_{nr} \to 0$ as $n \to \infty$ for every fixed $r \in \mathbb{N}_0$. If $x, z \in \omega$ differ only on a finite set, then $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim x = L$ if and only if $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim z = L$.*

Proof. Let $h = z - x$. Since $h \in \varphi$, Theorem 2.3 gives $Ah \in c_0$. Corollary 5.10 therefore yields $N_{\mathcal{W}}^{\mathcal{M}}(A)$- $\lim h = 0$. The conclusion follows from Theorem 5.8, first with $z = x + h$ and then with $x = z - h$. □

Remark 5.12. *The modular to statistical implication is governed by the lower condition* (5.1). *The bounded converse and the treatment of ordinary null sequences require* (5.2) *and* (5.3). *The column-null condition on A is used only to obtain $A(\varphi) \subseteq c_0$, which is the step required in the finite modification result. Thus the assumptions on the modular family and those on the triangular transformation remain logically separate.*

6. Separable triangular transformations of double sequences

The classical convergence theory of double sequences goes back to Pringsheim [18]. Statistical convergence was later extended to double and multiple sequences by Mursaleen and Edely [15] and Móricz [13]. The deferred Cesàro mean and deferred statistical convergence of double sequences were studied by Dağadur and Sezgek [4]. Here, these ideas are combined with separable triangular transformations and rectangular deferred weighted windows.

Let $\Omega = \{x = (x_{mn})_{m,n\geq 0} : x_{mn} \in \mathbb{C}\}$, $\|x\|_\infty = \sup_{m,n\geq 0} |x_{mn}|$, and let $\ell_\infty^{(2)}$ denote the space of bounded elements of Ω. For lower triangular transformations $A = (a_{mi})_{m,i\geq 0}$ and $B = (b_{nj})_{n,j\geq 0}$, define

$$((A \otimes B)x)_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi} b_{nj} x_{ij}, \quad m, n \in \mathbb{N}_0. \tag{6.1}$$

Every sum is finite, so $A \otimes B : \Omega \to \Omega$ is well defined. The product structure treats the two coordinate directions independently; genuinely coupled transformations with mixed coefficients are not considered here.

Structural properties of product transformations

With respect to the product order $(i, j) \leq (m, n)$ if and only if $i \leq m$ and $j \leq n$, the transformation $A \otimes B$ is lower triangular.

Lemma 6.1. *Let A and B be lower triangular transformations.*

(i) The mapping $A \otimes B : \Omega \to \Omega$ is linear.

(ii) If $a_{mm} \neq 0$ and $b_{nn} \neq 0$ for all m, n, then $A \otimes B$ has a unique inverse on Ω, given by $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

(iii) If $C_A = \sup_{m\geq 0} \sum_{i=0}^{m} |a_{mi}| < \infty$ and $C_B = \sup_{n\geq 0} \sum_{j=0}^{n} |b_{nj}| < \infty$, then $A \otimes B$ maps $\ell_\infty^{(2)}$ continuously into itself and $\|(A \otimes B)x\|_\infty \leq C_A C_B \|x\|_\infty$.

Proof. Linearity follows from (6.1). Under the nonvanishing diagonal assumption, let $C = A^{-1} = (c_{mi})$ and $D = B^{-1} = (d_{nj})$. Finite rearrangement gives

$$((C \otimes D)(A \otimes B)x)_{mn} = \sum_{r=0}^{m} \sum_{s=0}^{n} \left(\sum_{i=r}^{m} c_{mi} a_{ir} \right) \left(\sum_{j=s}^{n} d_{nj} b_{js} \right) x_{rs} = x_{mn}.$$

The reverse composition is identical, proving (ii). Finally, we have

$$\left| ((A \otimes B)x)_{mn} \right| \leq \|x\|_\infty \left(\sum_{i=0}^{m} |a_{mi}| \right) \left(\sum_{j=0}^{n} |b_{nj}| \right) \leq C_A C_B \|x\|_\infty,$$

which proves (iii). □

The following proposition separates the roles of the assumptions on the two factors.

Proposition 6.2. *Let A and B be lower triangular transformations.*

(i) If $a_{mm} \neq 0$ and $b_{nn} \neq 0$ for every m, n, then the diagonal coefficient of $A \otimes B$ at (m, n) is $a_{mm} b_{nn} \neq 0$.

(ii) If $C_A, C_B < \infty$, then $\sup_{m,n \geq 0} \sum_{i=0}^{m} \sum_{j=0}^{n} |a_{mi} b_{nj}| \leq C_A C_B < \infty$.

(iii) If both A and B satisfy the column-null condition, then for every fixed $(i, j) \in \mathbb{N}_0^2$, $a_{mi} b_{nj} \longrightarrow 0$ in the Pringsheim sense.

(iv) Under the assumptions of (iii), if $h \in \Omega$ has finite support, then $((A \otimes B)h)_{mn} \longrightarrow 0$ in the Pringsheim sense.

Proof. Parts (i) and (ii) follow from

$$\sum_{i=0}^{m} \sum_{j=0}^{n} |a_{mi} b_{nj}| = \left(\sum_{i=0}^{m} |a_{mi}| \right) \left(\sum_{j=0}^{n} |b_{nj}| \right).$$

For (iii), fix (i, j) and $\varepsilon > 0$. The column-null assumptions give M, N such that $|a_{mi}| < \sqrt{\varepsilon}$ for $m \geq M$ and $|b_{nj}| < \sqrt{\varepsilon}$ for $n \geq N$. Hence $|a_{mi} b_{nj}| < \varepsilon$ whenever $m \geq M$ and $n \geq N$.

For (iv), if $H = \operatorname{supp}(h)$, then

$$((A \otimes B)h)_{mn} = \sum_{(i,j) \in H} a_{mi} b_{nj} h_{ij}.$$

The sum is finite and every summand tends to zero by (iii). □

In particular, if A and B are admissible, then their product inherits the corresponding nonzero-diagonal, bounded-row, and column-null properties.

Admissible rectangular deferred weighted windows

Let $\mathcal{W}_1 = (\lambda, \mu, u)$ and $\mathcal{W}_2 = (\theta, \kappa, v)$ be deferred weighted windows, and set

$$I_r = (\mu_r, \lambda_r] \cap \mathbb{N}_0, \quad U_r = \sum_{m \in I_r} u_m, \quad \vartheta_r^{(u)} = \max_{m \in I_r} \frac{u_m}{U_r},$$
$$J_s = (\kappa_s, \theta_s] \cap \mathbb{N}_0, \quad V_s = \sum_{n \in J_s} v_n, \quad \vartheta_s^{(v)} = \max_{n \in J_s} \frac{v_n}{V_s}.$$

Definition 6.3. *The product $\mathcal{P} = \mathcal{W}_1 \times \mathcal{W}_2$ is called an admissible rectangular deferred weighted window if*

$$\begin{aligned} |I_r| \to \infty, \quad U_r \to \infty, \quad \vartheta_r^{(u)} \to 0, \\ |J_s| \to \infty, \quad V_s \to \infty, \quad \vartheta_s^{(v)} \to 0. \end{aligned} \tag{6.2}$$

For a real double sequence (α_{rs}), its Pringsheim upper limit is $\limsup_{r,s\to\infty} \alpha_{rs} = \lim_{N\to\infty} \sup\{\alpha_{rs} : r, s \geq N\}$. For $E \subseteq \mathbb{N}_0^2$, define

$$Q_{rs}^{\mathcal{P}}(E) = \frac{1}{U_r V_s} \sum_{\substack{m\in I_r,\ n\in J_s \\ (m,n)\in E}} u_m v_n, \quad \overline{d}_{\mathcal{P}}(E) = \limsup_{r,s\to\infty} Q_{rs}^{\mathcal{P}}(E). \tag{6.3}$$

If the corresponding Pringsheim limit exists, it is denoted by $d_{\mathcal{P}}(E)$.

Lemma 6.4. *Let $\mathcal{P}$ be admissible. If $E_{rs} \subseteq I_r \times J_s$ and $|E_{rs}| \leq M$ for some fixed $M \in \mathbb{N}$, then*

$$\frac{1}{U_r V_s} \sum_{(m,n)\in E_{rs}} u_m v_n \longrightarrow 0$$

in the Pringsheim sense.

Proof. For $(m, n) \in I_r \times J_s$, $u_m v_n \leq \vartheta_r^{(u)} \vartheta_s^{(v)} U_r V_s$. Therefore,

$$0 \leq \frac{1}{U_r V_s} \sum_{(m,n)\in E_{rs}} u_m v_n \leq M \vartheta_r^{(u)} \vartheta_s^{(v)} \longrightarrow 0.$$

□

Remark 6.5. *The conditions in* (6.2) *exclude localized rectangles. For example, if $I_r = \{r\}$ or $J_s = \{s\}$, then the corresponding nonconcentration factor equals* 1. *Hence, an admissible rectangular window cannot reduce to a single moving row, column, or pair of indices.*

Lemma 6.6. *Let $E, F \subseteq \mathbb{N}_0^2$. We then have the following:*

(i) $0 \leq \overline{d}_{\mathcal{P}}(E) \leq 1$ and $\overline{d}_{\mathcal{P}}(\mathbb{N}_0^2) = 1$;

(ii) If $E \subseteq F$, then $\overline{d}_{\mathcal{P}}(E) \leq \overline{d}_{\mathcal{P}}(F)$;

(iii) $\overline{d}_{\mathcal{P}}(E \cup F) \leq \overline{d}_{\mathcal{P}}(E) + \overline{d}_{\mathcal{P}}(F)$;

(iv) If, for some $N \in \mathbb{N}_0$, $E \subseteq (\{0, \ldots, N\} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \{0, \ldots, N\})$, then $\overline{d}_{\mathcal{P}}(E) = 0$. In particular, every finite set has an upper rectangular density of zero;

(v) $\overline{d}_{\mathcal{P}}(E) = 0$ if and only if $Q_{rs}^{\mathcal{P}}(E) \to 0$ in the Pringsheim sense.

Proof. Parts (i)–(iii) follow directly from positivity of the weights and the definition of $Q_{rs}^{\mathcal{P}}$. For $V_N = \{0, \ldots, N\} \times \mathbb{N}_0$ and $H_N = \mathbb{N}_0 \times \{0, \ldots, N\}$

$$Q_{rs}^{\mathcal{P}}(V_N) \leq (N + 1)\vartheta_r^{(u)}, \quad Q_{rs}^{\mathcal{P}}(H_N) \leq (N + 1)\vartheta_s^{(v)}.$$

Both expressions tend to zero in the Pringsheim sense, and (iv) follows by subadditivity. Finally, $Q_{rs}^{\mathcal{P}}(E) \geq 0$, so its Pringsheim upper limit is zero exactly when it converges to zero. □

For the remainder of this section, $\mathcal{P}$ denotes an admissible rectangular deferred weighted window.

Rectangular statistical and strong convergence

For $x \in \Omega$, $L \in \mathbb{C}$, and $\varepsilon > 0$, put

$$E_\varepsilon^{A,B}(x, L) = \left\{(m, n) \in \mathbb{N}_0^2 : \left|((A \otimes B)x)_{mn} - L\right| \geq \varepsilon\right\},$$

and, for $p > 0$, define

$$\mathscr{R}_{rs,p}^{A,B}(x, L) = \frac{1}{U_r V_s} \sum_{m \in I_r} \sum_{n \in J_s} u_m v_n \left|((A \otimes B)x)_{mn} - L\right|^p .$$

Definition 6.7. *A double sequence $x \in \Omega$ is rectangular deferred weighted statistically convergent to L generated by $A \otimes B$ if $\overline{d}_{\mathcal{P}}(E_\varepsilon^{A,B}(x, L)) = 0$ for every $\varepsilon > 0$. We then write $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$.*

Definition 6.8. *Let $p > 0$. A double sequence $x \in \Omega$ is strongly rectangular deferred weighted convergent of order p to L generated by $A \otimes B$ if $\mathscr{R}_{rs,p}^{A,B}(x, L) \longrightarrow 0$ in the Pringsheim sense. We then write $N_{\mathcal{P}}^p(A \otimes B)\text{-}\lim x = L$.*

Theorem 6.9. *Let A and B be lower triangular transformations.*

(i) The $S_{\mathcal{P}}(A \otimes B)$-limit is unique.

(ii) If $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$ and $S_{\mathcal{P}}(A \otimes B)\text{-}\lim z = M$, then, for every $\alpha, \beta \in \mathbb{C}$, $S_{\mathcal{P}}(A \otimes B)\text{-}\lim(\alpha x + \beta z) = \alpha L + \beta M$.

Proof. Suppose that $L \neq M$ are two limits of x, and set $\varepsilon = |L - M|/2$. Then

$$\mathbb{N}_0^2 = E_\varepsilon^{A,B}(x, L) \cup E_\varepsilon^{A,B}(x, M),$$

which contradicts Lemma 6.6(i)–(iii). Hence the limit is unique.

For (ii), the case $\alpha = \beta = 0$ is immediate. Otherwise, take $\delta = \varepsilon/[2(|\alpha| + |\beta|)]$. Linearity gives

$$E_\varepsilon^{A,B}(\alpha x + \beta z, \alpha L + \beta M) \subseteq E_\delta^{A,B}(x, L) \cup E_\delta^{A,B}(z, M).$$

The conclusion follows from Lemma 6.6(ii)–(iii). □

Definition 6.10. *A double sequence $y = (y_{mn})$ is Pringsheim convergent to L if, for every $\varepsilon > 0$, $N \in \mathbb{N}$ exists such that $|y_{mn} - L| < \varepsilon$ whenever $m, n \geq N$. We write $\text{P-}\lim_{m,n\to\infty} y_{mn} = L$.*

Theorem 6.11. *Let A and B be lower triangular transformations. If $\text{P-}\lim_{m,n\to\infty}((A \otimes B)x)_{mn} = L$, then $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$.*

Proof. For every $\varepsilon > 0$, Pringsheim convergence gives N such that

$$E_\varepsilon^{A,B}(x, L) \subseteq (\{0, \ldots, N - 1\} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \{0, \ldots, N - 1\}).$$

The right-hand side has an upper rectangular density of zero by Lemma 6.6(iv). □

Theorem 6.12. *Suppose that A and B satisfy the column-null condition. If $x, z \in \Omega$ differ only on a finite subset of $\mathbb{N}_0^2$, then $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$ if and only if $S_{\mathcal{P}}(A \otimes B)\text{-}\lim z = L$.*

Proof. Let $h = z - x$. By Proposition 6.2(iv),

$$\operatorname*{P\text{-}lim}_{m,n\to\infty}((A \otimes B)h)_{mn} = 0.$$

Theorem 6.11 gives $S_{\mathcal{P}}(A \otimes B)\text{-}\lim h = 0$, and the result follows from Theorem 6.9(ii). □

Theorem 6.13. *Let A and B be lower triangular transformations and $p > 0$.*

(i) $N^p_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$ implies $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$.

(ii) If $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$ and the transformed error $(((A \otimes B)x)_{mn} - L)_{m,n\geq 0}$ is bounded, then $N^p_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$.

Proof. For every $\varepsilon > 0$,

$$Q^{\mathcal{P}}_{rs}(E^{A,B}_{\varepsilon}(x, L)) \leq \frac{1}{\varepsilon^p}\mathscr{R}^{A,B}_{rs,p}(x, L),$$

which proves (i).

For (ii), choose $M > 0$ such that $|((A \otimes B)x)_{mn} - L| \leq M$ for all m, n. For $\delta > 0$

$$\mathscr{R}^{A,B}_{rs,p}(x, L) \leq M^p Q^{\mathcal{P}}_{rs}(E^{A,B}_{\delta}(x, L)) + \delta^p.$$

The first term tends to zero by statistical convergence and Lemma 6.6(v). Hence, the Pringsheim upper limit of the left-hand side is at most δ^p. Letting $\delta \downarrow 0$ proves (ii). □

A rectangular Cauchy characterization

Definition 6.14. *A double sequence $x \in \Omega$ is called $S_{\mathcal{P}}(A\otimes B)$-Cauchy if, for every $\varepsilon > 0$, $(m_0, n_0) \in \mathbb{N}^2_0$ exists such that*

$$\overline{d}_{\mathcal{P}}\left(\left\{(m, n) : \left|((A \otimes B)x)_{mn} - ((A \otimes B)x)_{m_0n_0}\right| \geq \varepsilon\right\}\right) = 0.$$

Lemma 6.15. *If $E, F \subseteq \mathbb{N}^2_0$ satisfy $\overline{d}_{\mathcal{P}}(E^c) = \overline{d}_{\mathcal{P}}(F^c) = 0$, then $E \cap F \neq \emptyset$.*

Proof. Otherwise, $\mathbb{N}^2_0 = E^c \cup F^c$, contradicting Lemma 6.6(i)–(iii). □

Theorem 6.16. *Let A and B be lower triangular transformations. A double sequence $x \in \Omega$ is $S_{\mathcal{P}}(A \otimes B)$-convergent if and only if it is $S_{\mathcal{P}}(A \otimes B)$-Cauchy.*

Proof. Put $y = (A \otimes B)x$. Suppose first that $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$. For $\varepsilon > 0$, set

$$G = \{(m, n) : |y_{mn} - L| < \varepsilon/2\}.$$

Since $\overline{d}_{\mathcal{P}}(G^c) = 0$, choose $(m_0, n_0) \in G$. Then

$$\{(m, n) : |y_{mn} - y_{m_0n_0}| \geq \varepsilon\} \subseteq \{(m, n) : |y_{mn} - L| \geq \varepsilon/2\},$$

so x is $S_{\mathcal{P}}(A \otimes B)$-Cauchy.

Conversely, for each $\ell \in \mathbb{N}$, choose (m_ℓ, n_ℓ) such that

$$\overline{d}_{\mathcal{P}}(\{(m, n) : |y_{mn} - y_{m_\ell n_\ell}| \geq 2^{-\ell}\}) = 0,$$

and put $G_\ell = \{(m,n) : |y_{mn} - y_{m_\ell n_\ell}| < 2^{-\ell}\}$. Lemma 6.15 gives $G_\ell \cap G_j \neq \emptyset$, and therefore

$$|y_{m_\ell n_\ell} - y_{m_j n_j}| < 2^{-\ell} + 2^{-j}.$$

Thus $(y_{m_\ell n_\ell})$ is Cauchy in $\mathbb{C}$ and converges to some L.

Given $\varepsilon > 0$, choose ℓ so that $2^{-\ell} < \varepsilon/2$ and $|y_{m_\ell n_\ell} - L| < \varepsilon/2$. Then

$$\{(m,n) : |y_{mn} - L| \geq \varepsilon\} \subseteq \{(m,n) : |y_{mn} - y_{m_\ell n_\ell}| \geq 2^{-\ell}\},$$

whose upper rectangular density is zero. Hence $S_{\mathcal{P}}(A \otimes B)\text{-}\lim x = L$. □

Remark 6.17. *The structural conclusions concerning the product transformation use only the corresponding diagonal, row-sum, or column-null assumptions on its factors. The statistical, strong, and Cauchy results depend on the separable form of $A \otimes B$ and the rectangular geometry of $\mathcal{P}$, but not on a specific coefficient formula. More explicitly, the nonconcentration hypotheses on the rectangular windows enter through Lemmas* 6.4 *and* 6.6*, the column-null condition enters the finite modification result through Proposition* 6.2*(iv), and boundedness is used only in the bounded converse in Theorem* 6.13*(ii). Fractional q-difference products will later provide a concrete admissible example.*

7. Comparison of deferred weighted windows

In this section, the lower triangular transformation A is fixed, whereas the deferred weighted window is allowed to vary. Consequently, the results below concern only the geometry and weighted mass of the windows; no additional diagonal, row-sum, or column-null assumption on A is required.

Let $\mathcal{W} = (\lambda, \mu, w)$ and $\mathcal{V} = (\sigma, \tau, v)$ be admissible deferred weighted windows. Write

$$I_n^{\mathcal{W}} = (\mu_n, \lambda_n] \cap \mathbb{N}_0, \quad W_n = \sum_{k \in I_n^{\mathcal{W}}} w_k,$$
$$I_n^{\mathcal{V}} = (\tau_n, \sigma_n] \cap \mathbb{N}_0, \quad V_n = \sum_{k \in I_n^{\mathcal{V}}} v_k.$$

For $E \subseteq \mathbb{N}_0$ and a non–negative sequence $r = (r_k)_{k \geq 0}$, define

$$D_n^{\mathcal{W}}(E) = \frac{1}{W_n} \sum_{k \in I_n^{\mathcal{W}} \cap E} w_k, \quad \mathscr{A}_n^{\mathcal{W}}(r) = \frac{1}{W_n} \sum_{k \in I_n^{\mathcal{W}}} w_k r_k,$$
$$D_n^{\mathcal{V}}(E) = \frac{1}{V_n} \sum_{k \in I_n^{\mathcal{V}} \cap E} v_k, \quad \mathscr{A}_n^{\mathcal{V}}(r) = \frac{1}{V_n} \sum_{k \in I_n^{\mathcal{V}}} v_k r_k.$$

Thus $D_n^{\mathcal{W}}(E) = \mathscr{A}_n^{\mathcal{W}}(\chi_E)$ and $D_n^{\mathcal{V}}(E) = \mathscr{A}_n^{\mathcal{V}}(\chi_E)$, where χ_E is the characteristic sequence of E.

Definition 7.1. *We say that $\mathcal{V}$ is dominated by $\mathcal{W}$, and write $\mathcal{V} \preceq \mathcal{W}$, if $C > 0$ exists such that $\overline{d}_{\mathcal{V}}(E) \leq C\overline{d}_{\mathcal{W}}(E)$ for every $E \subseteq \mathbb{N}_0$.*

Thus $\mathcal{V} \preceq \mathcal{W}$ guarantees that every set of upper $\mathcal{W}$-density zero also has upper $\mathcal{V}$-density zero.

Definition 7.2. *We say that $\mathcal{V}$ is strongly dominated by $\mathcal{W}$, and write $\mathcal{V} \preceq_s \mathcal{W}$, if $C > 0$ exists and $n_0 \in \mathbb{N}$ such that $D_n^{\mathcal{V}}(E) \leq C D_n^{\mathcal{W}}(E)$ for every $E \subseteq \mathbb{N}_0$ and every $n \geq n_0$.*

Unlike ordinary domination, strong domination compares the quotient sequences themselves. This distinction is needed in the ideal setting, where convergence is imposed on the window index n.

Lemma 7.3. *If $\mathcal{V} \leq_s \mathcal{W}$, then $\mathcal{V} \leq \mathcal{W}$.*

Proof. Taking the upper limits in $D_n^{\mathcal{V}}(E) \leq CD_n^{\mathcal{W}}(E)$ gives $\overline{d}_{\mathcal{V}}(E) \leq C\overline{d}_{\mathcal{W}}(E)$ for every $E \subseteq \mathbb{N}_0$. □

Statistical comparison

Theorem 7.4. *If $\mathcal{V} \leq \mathcal{W}$, then $S_{\mathcal{W}}(A)\text{-}\lim x = L$ implies $S_{\mathcal{V}}(A)\text{-}\lim x = L$. Consequently, $S_{\mathcal{W}}(A) \subseteq S_{\mathcal{V}}(A)$.*

Proof. For $\varepsilon > 0$, let $E_\varepsilon = E_\varepsilon^A(x, L)$. If $S_{\mathcal{W}}(A)\text{-}\lim x = L$, then $\overline{d}_{\mathcal{W}}(E_\varepsilon) = 0$. Domination gives

$$0 \leq \overline{d}_{\mathcal{V}}(E_\varepsilon) \leq C\overline{d}_{\mathcal{W}}(E_\varepsilon) = 0,$$

and the conclusion follows. □

The following assumptions provide a direct sufficient criterion for comparison of all non–negative weighted averages.

Proposition 7.5. *Suppose that $K_1, K_2 > 0$ and $n_0 \in \mathbb{N}$ exists such that, for every $n \geq n_0$,*

$$I_n^{\mathcal{V}} \subseteq I_n^{\mathcal{W}}, \quad v_k \leq K_1 w_k \quad (k \in I_n^{\mathcal{V}}), \quad W_n \leq K_2 V_n.$$

Then, for every non–negative sequence $r = (r_k)$, we have

$$\mathscr{A}_n^{\mathcal{V}}(r) \leq K_1 K_2 \mathscr{A}_n^{\mathcal{W}}(r) \quad (n \geq n_0). \tag{7.1}$$

Proof. For $n \geq n_0$, the positivity of r and the stated assumptions give

$$\mathscr{A}_n^{\mathcal{V}}(r) = \frac{1}{V_n} \sum_{k \in I_n^{\mathcal{V}}} v_k r_k \leq \frac{K_1}{V_n} \sum_{k \in I_n^{\mathcal{V}}} w_k r_k \leq K_1 \frac{W_n}{V_n} \mathscr{A}_n^{\mathcal{W}}(r) \leq K_1 K_2 \mathscr{A}_n^{\mathcal{W}}(r).$$

□

Corollary 7.6. *Under the assumptions of Proposition 7.5, $\mathcal{V} \leq_s \mathcal{W}$ and hence $\mathcal{V} \leq \mathcal{W}$. In particular, $S_{\mathcal{W}}(A)\text{-}\lim x = L$ implies $S_{\mathcal{V}}(A)\text{-}\lim x = L$.*

Proof. Apply (7.1) to $r = \chi_E$ and then use Lemma 7.3 and Theorem 7.4. □

Strong, ideal, and modular comparisons

The next results are consequences of the same average estimate. They do not require further structural assumptions on the entries of A.

Theorem 7.7. *Assume the hypotheses of Proposition 7.5. If $p > 0$ and $N_{\mathcal{W}}^p(A)\text{-}\lim x = L$, then $N_{\mathcal{V}}^p(A)\text{-}\lim x = L$. Consequently, $N_{\mathcal{W}}^p(A) \subseteq N_{\mathcal{V}}^p(A)$.*

Proof. Apply (7.1) to $r_k = |(Ax)_k - L|^p$. □

Theorem 7.8. *Let $\mathcal{I}$ be a nontrivial admissible ideal of $\mathbb{N}$. If $\mathcal{V} \leq_s \mathcal{W}$, then*

$$S^{\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L \quad \Longrightarrow \quad S^{\mathcal{I}}_{\mathcal{V}}(A)\text{-}\lim x = L.$$

Consequently, $S^{\mathcal{I}}_{\mathcal{W}}(A) \subseteq S^{\mathcal{I}}_{\mathcal{V}}(A)$.

Proof. For $\varepsilon > 0$, set

$$a_n = D^{\mathcal{W}}_n(E^A_\varepsilon(x, L)), \quad b_n = D^{\mathcal{V}}_n(E^A_\varepsilon(x, L)).$$

Strong domination gives $0 \leq b_n \leq Ca_n$ for all sufficiently large values of n. Since $\mathcal{I}\text{-}\lim a_n = 0$, Lemma 4.1 yields $\mathcal{I}\text{-}\lim b_n = 0$. □

Theorem 7.9. *Let $\mathcal{I}$ be a nontrivial admissible ideal of $\mathbb{N}$ and assume the hypotheses of Proposition 7.5. If $p > 0$ and $N^{p,\mathcal{I}}_{\mathcal{W}}(A)\text{-}\lim x = L$, then $N^{p,\mathcal{I}}_{\mathcal{V}}(A)\text{-}\lim x = L$.*

Proof. Apply (7.1) to $r_k = |(Ax)_k - L|^p$ and then use Lemma 4.1. □

Theorem 7.10. *Assume the hypotheses of Proposition 7.5, and let $\mathcal{M} = (M_k)_{k\geq 0}$ be a sequence of Orlicz functions. If $N^{\mathcal{M}}_{\mathcal{W}}(A)\text{-}\lim x = L$, then $N^{\mathcal{M}}_{\mathcal{V}}(A)\text{-}\lim x = L$. Consequently, $N^{\mathcal{M}}_{\mathcal{W}}(A) \subseteq N^{\mathcal{M}}_{\mathcal{V}}(A)$.*

Proof. Choose $\rho > 0$ such that

$$\frac{1}{W_n} \sum_{k \in I^{\mathcal{W}}_n} w_k M_k \left(\frac{|(Ax)_k - L|}{\rho} \right) \longrightarrow 0.$$

Apply (7.1) to

$$r_k = M_k \left(\frac{|(Ax)_k - L|}{\rho} \right).$$

□

Mutually comparable windows

Corollary 7.11. *If $\mathcal{V} \leq \mathcal{W}$ and $\mathcal{W} \leq \mathcal{V}$, then $S_{\mathcal{W}}(A) = S_{\mathcal{V}}(A)$, and the corresponding limits agree.*

Proof. Apply Theorem 7.4 in both directions. Agreement of the limits follows from uniqueness. □

Corollary 7.12. *Suppose that the sufficient assumptions of Proposition 7.5 hold from $\mathcal{W}$ to $\mathcal{V}$ and also from $\mathcal{V}$ to $\mathcal{W}$, possibly with different constants. Then*

$$N^p_{\mathcal{W}}(A) = N^p_{\mathcal{V}}(A), \quad N^{p,\mathcal{I}}_{\mathcal{W}}(A) = N^{p,\mathcal{I}}_{\mathcal{V}}(A) \quad (p > 0),$$

and

$$N^{\mathcal{M}}_{\mathcal{W}}(A) = N^{\mathcal{M}}_{\mathcal{V}}(A), \quad S^{\mathcal{I}}_{\mathcal{W}}(A) = S^{\mathcal{I}}_{\mathcal{V}}(A).$$

Proof. Apply Theorems 7.7–7.10 in both directions. □

Remark 7.13. *The conditions in Proposition 7.5 are sufficient comparison conditions and are not claimed to be necessary. Throughout this section, A remains fixed, and the proofs use only non–negative averages of its transformed error sequence. In particular, they do not require invertibility, bounded absolute row sums, the column-null condition, or any specific coefficient formula.*

Analogous sufficient results hold for the rectangular product windows of Section 6. Under coordinatewise interval inclusions and weight comparisons, the one-dimensional domination constants multiply and the same argument applies to averages generated by $A \otimes B$.

8. Fractional q-difference triangles as admissible transformations

The preceding sections were developed for general lower triangular transformations. We now verify that the fractional q-difference operator defines an admissible triangle and identify the additional properties that follow from its explicit coefficient kernel. In particular, integer and noninteger orders are treated separately, since they correspond to finite-memory and infinite-memory transforms, respectively.

Throughout this section, let $0 < q < 1$ and $\xi > 0$. For $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$, set $[s]_q = \frac{1-q^s}{1-q}$, $[0]_q! = 1$ and $[n]_q! = \prod_{r=1}^{n}[r]_q$. The generalized q-binomial coefficient is defined by

$$\binom{\xi}{0}_q = 1, \quad \binom{\xi}{j}_q = \frac{\prod_{r=0}^{j-1}[\xi - r]_q}{[j]_q!} \quad (j \geq 1).$$

When $\xi = m \in \mathbb{N}$, this gives $\binom{m}{j}_q = 0$ for $j > m$. These conventions agree with the usual finite q-binomial coefficients; see [8].

Define

$$a_j^{(q,\xi)} = (-1)^j q^{j(j-1)/2}\binom{\xi}{j}_q, \tag{8.1}$$

$$b_j^{(q,\xi)} = \binom{\xi + j - 1}{j}_q, \quad j \in \mathbb{N}_0. \tag{8.2}$$

For completeness, let $(a;q)_0 = 1$, $(a;q)_j = \prod_{r=0}^{j-1}(1 - aq^r)$, and $(a;q)_\infty = \prod_{r=0}^{\infty}(1 - aq^r)$. The preceding coefficients can be written as

$$a_j^{(q,\xi)} = q^{j\xi}\frac{(q^{-\xi};q)_j}{(q;q)_j}, \quad b_j^{(q,\xi)} = \frac{(q^{\xi};q)_j}{(q;q)_j}.$$

The q-binomial theorem therefore gives, the following for $|z| < 1$:

$$\sum_{j=0}^{\infty} a_j^{(q,\xi)} z^j = \frac{(z;q)_\infty}{(q^\xi z;q)_\infty}, \quad \sum_{j=0}^{\infty} b_j^{(q,\xi)} z^j = \frac{(q^\xi z;q)_\infty}{(z;q)_\infty}.$$

Multiplication of these series yields

$$\sum_{j=0}^{m} a_j^{(q,\xi)} b_{m-j}^{(q,\xi)} = \begin{cases} 1, & m = 0, \\ 0, & m \geq 1. \end{cases} \tag{8.3}$$

The fractional q-difference triangle

Let

$$Q^{(q,\xi)} = (q_{nk}^{(q,\xi)})_{n,k\geq 0}, \qquad q_{nk}^{(q,\xi)} = \begin{cases} a_{n-k}^{(q,\xi)}, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases} \tag{8.4}$$

Then, for $x = (x_k) \in \omega$, we have

$$(Q^{(q,\xi)}x)_n = \sum_{k=0}^{n} a_{n-k}^{(q,\xi)} x_k = \sum_{j=0}^{n} a_j^{(q,\xi)} x_{n-j}. \tag{8.5}$$

We write $\nabla_q^{(\xi)} x = Q^{(q,\xi)} x$.

Similarly, define

$$R^{(q,\xi)} = (r_{nk}^{(q,\xi)})_{n,k\geq 0}, \qquad r_{nk}^{(q,\xi)} = \begin{cases} b_{n-k}^{(q,\xi)}, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases} \tag{8.6}$$

Proposition 8.1. *The matrix $Q^{(q,\xi)}$ is lower triangular with diagonal entries equal to* 1*, and $(Q^{(q,\xi)})^{-1} = R^{(q,\xi)}$ on ω. Equivalently, if $y = Q^{(q,\xi)} x$, then*

$$x_n = \sum_{j=0}^{n} b_j^{(q,\xi)} y_{n-j}, \quad n \in \mathbb{N}_0. \tag{8.7}$$

Proof. The triangular structure follows from (8.4), while $q_{nn}^{(q,\xi)} = a_0^{(q,\xi)} = 1$. If $x_n = \sum_{j=0}^{n} b_j^{(q,\xi)} y_{n-j}$, then finite rearrangement of the sums gives

$$(Q^{(q,\xi)} x)_n = \sum_{r=0}^{n} a_r^{(q,\xi)} \sum_{j=0}^{n-r} b_j^{(q,\xi)} y_{n-r-j} = \sum_{m=0}^{n} \left(\sum_{r=0}^{m} a_r^{(q,\xi)} b_{m-r}^{(q,\xi)} \right) y_{n-m} = y_n$$

by (8.3). The reverse composition follows from the same finite convolution identity. □

Remark 8.2. *Proposition* 8.1 *asserts an algebraic triangular inverse on ω. No boundedness of $R^{(q,\xi)}$ on ℓ_∞ is claimed or needed below.*

Absolute summability and admissibility

Lemma 8.3. *The coefficient sequence $(a_j^{(q,\xi)})_{j\geq 0}$ belongs to ℓ_1. Thus, we have*

$$A_{q,\xi} := \sum_{j=0}^{\infty} |a_j^{(q,\xi)}| < \infty, \tag{8.8}$$

and $a_j^{(q,\xi)} \to 0$.

Proof. If $\xi = m \in \mathbb{N}$, then $a_j^{(q,m)} = 0$ for $j > m$, so the assertion is immediate.

Suppose that $\xi \notin \mathbb{N}$, and set $c_j = |a_j^{(q,\xi)}|$. None of the factors in the generalized q-binomial coefficient vanishes, and

$$\frac{c_{j+1}}{c_j} = q^j \left| \frac{[\xi - j]_q}{[j+1]_q} \right| = \left| \frac{q^j - q^\xi}{1 - q^{j+1}} \right| \longrightarrow q^\xi.$$

Since $0 < q^\xi < 1$, the ratio test gives $\sum_{j\geq 0} c_j < \infty$. Absolute summability also implies $a_j^{(q,\xi)} \to 0$. □

Theorem 8.4. *For every $q \in (0,1)$ and $\xi > 0$, the triangle $Q^{(q,\xi)}$ is admissible in the sense of Definition* 2.1*. More precisely:*

(i) $q_{nn}^{(q,\xi)} = 1$ for every $n \in \mathbb{N}_0$;

(ii) $\sup_{n\geq 0} \sum_{k=0}^{n} |q_{nk}^{(q,\xi)}| \leq A_{q,\xi} < \infty$;

(iii) $q_{nr}^{(q,\xi)} \to 0$ as $n \to \infty$ for every fixed $r \in \mathbb{N}_0$.

Proof. Part (i) follows from $a_0^{(q,\xi)} = 1$. Moreover,

$$\sum_{k=0}^{n} |q_{nk}^{(q,\xi)}| = \sum_{j=0}^{n} |a_j^{(q,\xi)}| \le A_{q,\xi},$$

which proves (ii). For a fixed r and $n \ge r$, $q_{nr}^{(q,\xi)} = a_{n-r}^{(q,\xi)} \to 0$ by Lemma 8.3. □

Corollary 8.5. *The transformation $Q^{(q,\xi)}$ maps ℓ_∞ continuously into ℓ_∞, and*

$$\|Q^{(q,\xi)}x\|_\infty \le A_{q,\xi}\|x\|_\infty \quad (x \in \ell_\infty).$$

Proof. Apply Theorem 8.4 and Lemma 2.2. □

Finite and infinite q-memory

Proposition 8.6. *Let $Q^{(q,\xi)}$ be a fractional q-difference triangle.*

(i) If $\xi = m \in \mathbb{N}$, then $a_j^{(q,m)} = 0$ for $j > m$, and $Q^{(q,m)}$ is banded with a bandwidth $m + 1$.

(ii) If $\xi \notin \mathbb{N}$, then $a_j^{(q,\xi)} \ne 0$ for every $j \in \mathbb{N}_0$, and every column of $Q^{(q,\xi)}$ has infinitely many nonzero entries.

Proof. Part (i) follows from the integer-order q-binomial coefficient. If $\xi \notin \mathbb{N}$, none of $\xi, \xi - 1, \ldots, \xi - j + 1$ is zero. Since $[s]_q = 0$ exactly when $s = 0$, we have $\binom{\xi}{j}_q \ne 0$, and therefore, $a_j^{(q,\xi)} \ne 0$. The assertion about columns follows from $q_{nr}^{(q,\xi)} = a_{n-r}^{(q,\xi)}$ for $n \ge r$. □

For $r \in \mathbb{N}_0$, let $e^{(r)}$ denote the r-th unit sequence.

Proposition 8.7. *For every $r, n \in \mathbb{N}_0$, we have*

$$(Q^{(q,\xi)}e^{(r)})_n = \begin{cases} 0, & n < r, \\ a_{n-r}^{(q,\xi)}, & n \ge r. \end{cases}$$

Consequently, the following hold:

(i) if $\xi = m \in \mathbb{N}$, then $\operatorname{supp}(Q^{(q,m)}e^{(r)}) \subseteq \{r, r+1, \ldots, r+m\}$;

(ii) If $\xi \notin \mathbb{N}$, then $Q^{(q,\xi)}e^{(r)}$ has infinitely many nonzero coordinates but belongs to $\ell_1 \subset c_0$;

(iii) In either case, $\sum_{n=0}^{\infty} \left|(Q^{(q,\xi)}e^{(r)})_n\right| = A_{q,\xi}$.

Proof. In (8.5), the term $e_{n-j}^{(r)}$ is nonzero exactly when $j = n - r$. The stated formula follows. Parts (i) and (ii) follow from Proposition 8.6 and Lemma 8.3, while

$$\sum_{n=r}^{\infty} |a_{n-r}^{(q,\xi)}| = \sum_{j=0}^{\infty} |a_j^{(q,\xi)}| = A_{q,\xi}$$

proves (iii). □

Corollary 8.8. *If we let Let $h \in \varphi$, then $Q^{(q,\xi)}h \in c_0$. If $F = \mathrm{supp}(h)$, then $(Q^{(q,\xi)}h)_n = \sum_{\substack{r\in F\\ r\le n}} a^{(q,\xi)}_{n-r} h_r$. For $\xi \in \mathbb{N}$, the transformed sequence has finite support. For $\xi \notin \mathbb{N}$, it may have an infinite tail, but that tail converges to zero.*

Proof. The inclusion in c_0 follows from Theorem 8.4 and Theorem 2.3. The representation follows from the definition of $Q^{(q,\xi)}$. The final assertions follow from Propositions 8.6 and 8.7. □

Remark 8.9. *For integer order, the transformation has finite memory: A perturbation affects only finitely many subsequent transformed coordinates. For a noninteger order, a single perturbation affects infinitely many coordinates. For the fractional q-difference kernel considered here, Lemma* 8.3 *shows that the resulting tail is absolutely summable and hence converges to zero.*

The classical limit as $q \to 1^-$

Proposition 8.10. *For every fixed $\xi > 0$ and $j \in \mathbb{N}_0$, we have*

$$\lim_{q\to 1^-} a_j^{(q,\xi)} = (-1)^j \binom{\xi}{j}, \qquad \lim_{q\to 1^-} b_j^{(q,\xi)} = \binom{\xi + j - 1}{j}.$$

Hence, $Q^{(q,\xi)}$ converges entrywise to the classical fractional backward difference triangle.

Proof. For every fixed $s \in \mathbb{R}$, $[s]_q \to s$ as $q \to 1^-$, while $q^{j(j-1)/2} \to 1$. Substitution into (8.1) and (8.2) gives the stated limits. □

For $m \in \mathbb{N}$, let Δ^m denote the classical backward difference triangle

$$(\Delta^m x)_n = \sum_{j=0}^{\min\{m,n\}} (-1)^j \binom{m}{j} x_{n-j}.$$

Corollary 8.11. *For every $m \in \mathbb{N}$, $\lim_{q\to 1^-} \|Q^{(q,m)} - \Delta^m\|_{\ell_\infty\to\ell_\infty} = 0$.*

Proof. Both coefficient kernels vanish after m, and therefore

$$\|Q^{(q,m)} - \Delta^m\|_{\ell_\infty\to\ell_\infty} \le \sum_{j=0}^{m} \left| a_j^{(q,m)} - (-1)^j \binom{m}{j} \right|.$$

Every term in this finite sum tends to zero by Proposition 8.10. □

Remark 8.12. *For a noninteger ξ, Proposition* 8.10 *establishes only entrywise convergence. It does not by itself imply operator-norm convergence or the equality of the corresponding convergence classes; either conclusion would require the estimates to be uniform in both j and q. Such uniform estimates are not established in the present paper and remain open in this level of generality for the fractional q-difference kernel.*

Specialization of the general theory

Since $Q^{(q,\xi)}$ is admissible, set

$$\begin{aligned} &S_{\mathcal{W}}(\nabla_q^{(\xi)}) := S_{\mathcal{W}}(Q^{(q,\xi)}), \quad N^p_{\mathcal{W}}(\nabla_q^{(\xi)}) := N^p_{\mathcal{W}}(Q^{(q,\xi)}),\\ &S^{\mathcal{I}}_{\mathcal{W}}(\nabla_q^{(\xi)}) := S^{\mathcal{I}}_{\mathcal{W}}(Q^{(q,\xi)}), \quad N^{p,\mathcal{I}}_{\mathcal{W}}(\nabla_q^{(\xi)}) := N^{p,\mathcal{I}}_{\mathcal{W}}(Q^{(q,\xi)}),\\ &N^{\mathcal{M}}_{\mathcal{W}}(\nabla_q^{(\xi)}) := N^{\mathcal{M}}_{\mathcal{W}}(Q^{(q,\xi)}). \end{aligned}$$

Corollary 8.13. *Let $q \in (0,1)$ and $\xi > 0$. Every result of Sections 3, 4, 5, and 7 applies to $Q^{(q,\xi)}$ whenever the additional hypotheses of that result are satisfied. In particular, the generated limits are unique, the corresponding classes are linear, strong convergence implies statistical convergence, and finite modifications preserve the generated limits under the modular or ideal assumptions stated in the respective sections.*

Proof. Apply Theorem 8.4 to the results of the cited sections. □

The separable double fractional q-difference triangle

Let $q_1, q_2 \in (0,1)$ and $\xi, \eta > 0$. For a double sequence $x = (x_{ij})$, define

$$(\nabla^{(\xi,\eta)}_{q_1,q_2} x)_{mn} := \left(\left(Q^{(q_1,\xi)} \otimes Q^{(q_2,\eta)}\right) x\right)_{mn} = \sum_{r=0}^{m} \sum_{s=0}^{n} a_r^{(q_1,\xi)} a_s^{(q_2,\eta)} x_{m-r,n-s}. \tag{8.9}$$

Proposition 8.14. *The transformation $\nabla^{(\xi,\eta)}_{q_1,q_2} = Q^{(q_1,\xi)} \otimes Q^{(q_2,\eta)}$ has the following properties:*

(i) It is lower triangular in the product order and has diagonal entries equal to 1;

(ii) It is invertible on Ω, with inverse $R^{(q_1,\xi)} \otimes R^{(q_2,\eta)}$;

(iii) Its absolute row sums are bounded by $A_{q_1,\xi} A_{q_2,\eta}$;

(iv) Every fixed product column converges to zero in the Pringsheim sense;

(v) Every finitely supported double sequence is transformed into a Pringsheim-null double sequence.

Proof. Both factors are admissible by Theorem 8.4. The claims follow from Lemma 6.1 and Proposition 6.2, with the relevant assumption used separately in each assertion. □

Corollary 8.15. *Every rectangular statistical, strong, finite modification, and Cauchy result of Section 6 applies to $\nabla^{(\xi,\eta)}_{q_1,q_2}$ under the hypotheses stated in the corresponding result.*

Proof. Apply Proposition 8.14 and the results of Section 6. □

Remark 8.16. *Sections 3–7 contain the convergence results that hold for general lower triangular transformations. The purpose of the present section is to verify that the fractional q-difference triangle satisfies the assumptions of that theory. The explicit coefficient kernel yields the ratio limit q^{ξ}, geometric decay of the coefficients, finite support in the integer-order case, nonterminating but absolutely summable tails in the noninteger-order case, and the coefficientwise classical limit as $q \to 1^{-}$. These properties describe the role of the q-difference structure in the application without attributing the general convergence theorems themselves to q-calculus.*

9. Examples and strictness results

This section distinguishes the phenomena arising from the explicit fractional q-difference kernel from those belonging to the general triangular framework. In Examples 9.2 and 9.3, sparse impulses are placed in the original sequence, and their transformed values are produced by the nonterminating q-kernel. The later examples use triangular invertibility to exhibit strictness phenomena that do not depend on a particular coefficient formula.

Throughout the one-dimensional examples, let $q \in (0,1)$ and $\xi > 0$, unless an additional restriction is stated, and let $\mathcal{W}_0$ denote the standard Cesàro window

$$I_n = \{1, \ldots, n\}, \quad W_n = n.$$

Decay of the fractional kernel

Lemma 9.1. *Let $0 < q < 1$ and $\xi > 0$. For every τ with $q^\xi < \tau < 1$, the constants $C_\tau, D_\tau > 0$ exist such that*

$$|a_j^{(q,\xi)}| \le C_\tau \tau^j \quad (j \ge 0), \qquad \sum_{j=N}^{\infty} |a_j^{(q,\xi)}| \le D_\tau \tau^N \quad (N \ge 0).$$

Proof. If $\xi \in \mathbb{N}$, the coefficient sequence has finite support, and the assertion is immediate. Suppose that $\xi \notin \mathbb{N}$, and put $c_j = |a_j^{(q,\xi)}|$. The calculation in the proof of Lemma 8.3 gives

$$\frac{c_{j+1}}{c_j} \longrightarrow q^\xi < \tau.$$

Hence, $c_{j+1} \le \tau c_j$ for all sufficiently large values of j, which gives $c_j \le C_\tau \tau^j$ after enlarging C_τ to cover the finitely many initial terms. Therefore, we have

$$\sum_{j=N}^{\infty} |a_j^{(q,\xi)}| \le C_\tau \sum_{j=N}^{\infty} \tau^j = \frac{C_\tau}{1-\tau} \tau^N.$$

Take $D_\tau = C_\tau/(1-\tau)$. □

Proper inclusions based on the fractional q-difference kernel

Example 9.2. *Assume that $\xi \notin \mathbb{N}$ and define*

$$x_k = \begin{cases} 1, & k = m^2 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

For $y = Q^{(q,\xi)} x$

$$y_n = \sum_{m^2 \le n} a_{n-m^2}^{(q,\xi)}. \tag{9.1}$$

Thus y is the superposition of the q-kernel tails generated by the square impulses; it is not prescribed independently of x.

Let $\varepsilon > 0$. Choose $J \in \mathbb{N}_0$ such that $\sum_{j>J} |a_j^{(q,\xi)}| < \frac{\varepsilon}{2}$, and define $E_J = \bigcup_{j=0}^{J} \{m^2 + j : m \in \mathbb{N}\}$. Suppose that $n > J$ and $n \notin E_J$. Then $x_{n-j} = 0$ for every $j = 0, 1, \ldots, J$. Indeed, if $x_{n-j} = 1$ for some such j, then $n - j = m^2$ for some $m \in \mathbb{N}$, and hence $n = m^2 + j \in E_J$, which is a contradiction. Therefore

$$|y_n| = \left| \sum_{j=0}^{n} a_j^{(q,\xi)} x_{n-j} \right| \le \sum_{j>J} |a_j^{(q,\xi)}| < \frac{\varepsilon}{2}.$$

Consequently, $\{n \in \mathbb{N} : |y_n| \ge \varepsilon\} \subseteq E_J \cup \{1, \ldots, J\}$. Moreover, $|E_J \cap \{1, \ldots, N\}| \le (J+1)\lfloor \sqrt{N} \rfloor$, so E_J has a natural density of zero. Since the finite set $\{1, \ldots, J\}$ also has a density of zero, it follows that $S_{\mathcal{W}_0}(Q^{(q,\xi)})\text{-}\lim x = 0$.

At the square indices, we have

$$y_{m^2} = 1 + \sum_{\ell=1}^{m-1} a^{(q,\xi)}_{m^2-\ell^2}.$$

Since $m^2 - \ell^2 \geq 2m - 1$ for $\ell < m$

$$|y_{m^2} - 1| \leq \sum_{j=2m-1}^{\infty} |a^{(q,\xi)}_j| \longrightarrow 0.$$

Thus, $y_{m^2} \to 1$, whereas y is statistically convergent to 0. Consequently, y is not ordinarily convergent and $c(Q^{(q,\xi)}) \subsetneq S_{\mathcal{W}_0}(Q^{(q,\xi)})$.

Example 9.3. *Assume that $\xi \notin \mathbb{N}$, fix $p > 0$, and define*

$$x_k = \begin{cases} m^{2/p}, & k = m^2 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

For $y = Q^{(q,\xi)}x$, $y_n = \sum_{m^2 \leq n} a^{(q,\xi)}_{n-m^2} m^{2/p}$.

Choose $q^{\xi} < \tau < 1$ and then $\gamma > 0$ such that $\gamma|\log\tau| > 2/p + 2$. By Lemma 9.1, $|a^{(q,\xi)}_j| \leq C_\tau \tau^j$. Given $\varepsilon > 0$, choose $D > 0$ so that

$$C_\tau \tau^D \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} < \varepsilon,$$

and set

$$L_m = \lceil D + \gamma \log(m+1) \rceil, \quad E = \bigcup_{m\geq 1} \{m^2, \ldots, m^2 + L_m\}.$$

If $n \notin E$ and $m^2 \leq n$, then $n - m^2 > L_m$, and therefore

$$\tau^{n-m^2} m^{2/p} \leq \tau^D (m+1)^{\gamma \log\tau} m^{2/p} \leq \frac{\tau^D}{(m+1)^2}.$$

It follows that $|y_n| < \varepsilon$ for $n \notin E$. Since

$$|E \cap \{1, \ldots, N\}| \leq \sum_{m \leq \sqrt{N}} (L_m + 1) = O(\sqrt{N} \log N),$$

the set E has a natural density of zero. Hence, $S_{\mathcal{W}_0}(Q^{(q,\xi)})$- $\lim x = 0$.

At $n = m^2$,

$$\left|y_{m^2} - m^{2/p}\right| \leq m^{2/p} \sum_{j=2m-1}^{\infty} |a^{(q,\xi)}_j| = o(m^{2/p}).$$

Thus, $y_{m^2}/m^{2/p} \to 1$; consequently, $|y_{m^2}| \geq m^{2/p}/2$ for all sufficiently large values of m. Therefore

$$\frac{1}{N} \sum_{n=1}^{N} |y_n|^p \geq \frac{1}{2^p N} \sum_{\substack{m^2 \leq N \\ m \geq m_0}} m^2 \longrightarrow \infty.$$

Hence, $N^p_{\mathcal{W}_0}(Q^{(q,\xi)})$- $\lim x = 0$ does not hold. This proves that the boundedness assumption in Theorem 3.7(ii) is essential.

Remark 9.4. *In Examples* 9.2 *and* 9.3*, the squares specify only the locations of the impulses in the original sequence. The transformed values are generated by the full fractional q-difference kernel. These examples therefore depend on the explicit coefficient kernel rather than on transformed data prescribed in advance.*

Finite perturbations and classical limits

Example 9.5. *Let $z = x + ce^{(r)}$, where $r \in \mathbb{N}_0$ and $c \in \mathbb{C}$. Then*

$$(Q^{(q,\xi)}z)_n - (Q^{(q,\xi)}x)_n = \begin{cases} 0, & n < r, \\ c\, a_{n-r}^{(q,\xi)}, & n \geq r. \end{cases}$$

If $\xi = m \in \mathbb{N}$, the transformed perturbation vanishes for $n > r + m$. If $\xi \notin \mathbb{N}$, it has an infinite tail, but

$$\sum_{n=r}^{\infty} |c\, a_{n-r}^{(q,\xi)}| = |c| A_{q,\xi} < \infty.$$

Thus the tail converges to zero. Accordingly, x and z have the same generated statistical, ideal, and Musielak–Orlicz limits under the hypotheses of the corresponding finite modification theorems.

Example 9.6. *For $\xi = 1$, $a_0^{(q,1)} = 1$, $a_1^{(q,1)} = -1$, and $a_j^{(q,1)} = 0 \quad (j \geq 2)$, so*

$$(Q^{(q,1)}x)_n = x_n - x_{n-1} \quad (n \geq 1).$$

For $\xi = m \in \mathbb{N}$, the triangle is banded. For fixed $\xi > 0$ and $j \in \mathbb{N}_0$

$$a_j^{(q,\xi)} \longrightarrow (-1)^j \binom{\xi}{j} \qquad (q \to 1^-).$$

Thus the classical fractional backward difference triangle is recovered entrywise. When $\xi = m \in \mathbb{N}$, this improves to $\|Q^{(q,m)} - \Delta^m\|_{\ell_\infty \to \ell_\infty} \longrightarrow 0$. No operator-norm conclusion is asserted for noninteger ξ.

A variable Musielak–Orlicz example

Example 9.7. *Let*

$$M_k(t) = t^{p_k}, \quad p_k = \begin{cases} 1, & k \text{ even}, \\ 2, & k \text{ odd}. \end{cases}$$

Since $1 \leq p_k \leq 2$ for every $k \in \mathbb{N}_0$, Corollary 5.7 shows that the corresponding Musielak–Orlicz family satisfies the uniform conditions (5.1)–(5.3). This modular treats the two parity classes by different powers.

Define

$$y_k = \begin{cases} m, & k = 2m^3 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

and let x be the unique sequence satisfying $Q^{(q,\xi)}x = y$, whose existence follows from Proposition 8.1. With $\rho = 1$, we have

$$\frac{1}{n} \sum_{k=1}^{n} M_k(|y_k|) = \frac{1}{n} \sum_{2m^3 \leq n} m \longrightarrow 0,$$

whereas

$$\frac{1}{n}\sum_{k=1}^{n}|y_k|^2 = \frac{1}{n}\sum_{2m^3\le n} m^2 \longrightarrow \frac{1}{6}.$$

Hence, $N^{\mathcal{M}}_{\mathcal{W}_0}(Q^{(q,\xi)})$- $\lim x = 0$, but strong convergence of order 2 fails.

Conversely, define

$$u_k = \begin{cases} m, & k = 2m^3 + 1 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and let z satisfy $Q^{(q,\xi)}z = u$. Then

$$\frac{1}{n}\sum_{k=1}^{n}|u_k| \longrightarrow 0, \quad \frac{1}{n}\sum_{k=1}^{n} M_k(|u_k|) \longrightarrow \frac{1}{6}.$$

Thus $N^{1}_{\mathcal{W}_0}(Q^{(q,\xi)})$- $\lim z = 0$, but the Musielak–Orlicz modular convergence fails. Therefore, this variable modular is not equivalent to either of the fixed-power methods occurring in its definition.

Strictness for double sequences

Let $\mathcal{P}_0 = \mathcal{W}_0 \times \mathcal{W}_0$ be the standard rectangular window. Thus $I_r = \{1, \ldots, r\}$, $J_s = \{1, \ldots, s\}$, and $u_m = v_n = 1$.

Example 9.8. *Let $q_1, q_2 \in (0, 1)$ and $\xi, \eta > 0$. Define*

$$y_{mn} = \begin{cases} 1, & (m, n) = (a^2, b^2) \textit{ for some } a, b \in \mathbb{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

Its support $Q = \{(a^2, b^2) : a, b \in \mathbb{N}\}$ has a rectangular density of zero because

$$Q^{\mathcal{P}_0}_{rs}(Q) = \frac{\lfloor \sqrt{r}\rfloor \lfloor \sqrt{s}\rfloor}{rs} \longrightarrow 0$$

in the Pringsheim sense.

By Proposition 8.14(ii), there is a unique $x \in \Omega$ satisfying $(Q^{(q_1,\xi)} \otimes Q^{(q_2,\eta)})x = y$. Therefore

$$S_{\mathcal{P}_0}(Q^{(q_1,\xi)} \otimes Q^{(q_2,\eta)})\text{- } \lim x = 0.$$

However, y is not Pringsheim convergent to 0, since along the cofinal sequence $(m, n) = (a^2, a^2)$, one has $y_{a^2,a^2} = 1$. Hence, the inclusion from the Pringsheim-convergent matrix domain into the corresponding rectangular statistical class is proper.

Example 9.9. *Let $q_1, q_2 \in (0, 1)$, let $\xi, \eta > 0$, and fix $p > 0$. Define $Q = \{(a^2, b^2) : a, b \in \mathbb{N}\}$ and*

$$y_{mn} = \begin{cases} a^{2/p}b^{2/p}, & (m, n) = (a^2, b^2) \textit{ for some } a, b \in \mathbb{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

The support of y is Q, and

$$Q^{\mathcal{P}_0}_{rs}(Q) = \frac{\lfloor \sqrt{r}\rfloor \lfloor \sqrt{s}\rfloor}{rs} \longrightarrow 0$$

in the Pringsheim sense. Hence, Q has a rectangular density of zero.

By Proposition 8.14(ii), a unique $x \in \Omega$ exists such that $(Q^{(q_1,\xi)} \otimes Q^{(q_2,\eta)})x = y$. Therefore, $S_{\mathcal{P}_0}(Q^{(q_1,\xi)} \otimes Q^{(q_2,\eta)})\text{-}\lim x = 0$.

On the other hand

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}|y_{mn}|^p = \frac{1}{rs}\left(\sum_{a^2\le r} a^2\right)\left(\sum_{b^2\le s} b^2\right)$$
$$\sim \frac{1}{9}r^{1/2}s^{1/2} \longrightarrow \infty$$

in the Pringsheim sense. Thus $N^p_{\mathcal{P}_0}(Q^{(q_1,\xi)} \otimes Q^{(q_2,\eta)})\text{-}\lim x = 0$ does not hold. Hence, the boundedness hypothesis in Theorem 6.13(ii) cannot be omitted.

Corollary 9.10. *Let* $q \in (0,1)$ *and* $\xi \notin \mathbb{N}$.

(i) $c(Q^{(q,\xi)}) \subsetneq S_{\mathcal{W}_0}(Q^{(q,\xi)})$.

(ii) For every $p > 0$, $N^p_{\mathcal{W}_0}(Q^{(q,\xi)}) \subsetneq S_{\mathcal{W}_0}(Q^{(q,\xi)})$.

(iii) Let $q_1, q_2 \in (0,1)$ and $\xi_1, \xi_2 > 0$. Then the inclusion from the Pringsheim-convergent matrix domain into the corresponding rectangular statistical class is proper. Moreover, for every $p > 0$, we have

$$N^p_{\mathcal{P}_0}(Q^{(q_1,\xi_1)} \otimes Q^{(q_2,\xi_2)}) \subsetneq S_{\mathcal{P}_0}(Q^{(q_1,\xi_1)} \otimes Q^{(q_2,\xi_2)}).$$

Proof. Part (i) follows from Example 9.2. Part (ii) follows from Example 9.3 and Theorem 3.7(i). Part (iii) follows from Examples 9.8 and 9.9, together with Theorems 6.11 and 6.13(i). □

Remark 9.11. *Examples* 9.2 *and* 9.3 *use the explicit decay and nontermination of the fractional q-difference kernel. Examples* 9.7–9.9 *instead exploit triangular invertibility to prescribe transformed data. They therefore illustrate the strictness properties of the general triangular and product theories, rather than additional identities peculiar to the q-kernel.*

10. Conclusions

In this paper, we developed deferred weighted statistical, ideal, strong, and Musielak–Orlicz convergence methods generated by admissible lower triangular transformations. The general formulation separates the assumptions on the transformation from those on the deferred weighted window. In particular, we proved that the column-null condition is equivalent to the inclusion $A(\varphi) \subseteq c_0$. This equivalence precisely characterizes the lower triangular transformations that map every finitely supported sequence into a null sequence. It also yields finite modification stability for the convergence methods considered, under the assumptions stated in the corresponding results.

Within this framework, we established the uniqueness, linearity, ordinary to statistical inclusion, strong convergence implications, bounded converse results, and Cauchy characterizations. Corresponding ideal and Musielak–Orlicz versions were obtained under explicit assumptions. The theory was also extended to double sequences through separable product transformations $A \otimes B$ and admissible rectangular windows. Comparison results showed that inclusions between the convergence classes are governed by domination relations between the windows and do not depend on a particular coefficient formula.

The fractional q-difference triangle $Q^{(q,\xi)}$ was then studied as a concrete application of the general theory. Its coefficient kernel belongs to ℓ_1, its absolute row sums are uniformly bounded, and each fixed column converges to zero. The explicit kernel also gives the ratio limit q^{ξ} and a geometric estimate for the coefficient tails. Integer orders produce banded transformations, whereas noninteger orders produce nonterminating but absolutely summable tails. We also obtained the coefficientwise classical limit as $q \to 1^-$ and operator-norm convergence in the integer-order case.

The examples establish proper inclusions among the ordinary, statistical, strong, and modular convergence classes and show that the boundedness assumptions in the converse theorems are essential. They also distinguish the properties arising from the general triangular framework from those that use the explicit fractional q-difference kernel. Further work may consider nonseparable transformations of double sequences, sharper criteria for comparing deferred weighted windows, and q-dependent relations between $Q^{(q,\xi)}$ and its classical limit.

Author contributions

Gui-Ying Weng: Conceptualization, methodology, formal analysis, writing original draft, writing review and editing; Ömer Kişi: Conceptualization, methodology, formal analysis, writing original draft, writing review and editing; Mehmet Gürdal: Conceptualization, methodology, formal analysis, writing original draft, writing review and editing; Qing-Bo Cai: Conceptualization, methodology, formal analysis, writing original draft, writing review and editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Fujian Provincial Natural Science Foundation of China (Grant No. 2024J01792).

Conflict of interest

The authors declare no conflicts of interest.

References

1. R. P. Agnew, On deferred Cesàro means, *Ann. Math.*, **33** (1932), 413–421. https://doi.org/10.2307/1968524
2. F. Başar, *Summability theory and its applications* 2 Eds., New York: Chapman and Hall/CRC, 2022. https://doi.org/10.1201/9781003294153
3. J. S. Connor, The statistical and strong p-Cesàro convergence of sequences, *Analysis*, **8** (1988), 47–64. https://doi.org/10.1524/anly.1988.8.12.47

4. İ. Dağadur, Ş. Sezgek, Deferred Cesàro mean and deferred statistical convergence of double sequences, *J. Inequal. Spec. Funct.*, **7** (2016), 118–136.
5. M. Et, R. Çolak, On some generalized difference sequence spaces, *Soochow J. Math.*, **21** (1995), 377–386.
6. H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241–244.
7. J. A. Fridy, On statistical convergence, *Analysis*, **5** (1985), 301–313. https://doi.org/10.1524/anly.1985.5.4.301
8. V. Kac, P. Cheung, *Quantum calculus*, New York: Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7
9. H. Kızmaz, On certain sequence spaces, *Can. Math. Bull.*, **24** (1981), 169–176. https://doi.org/10.4153/CMB-1981-027-5
10. P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}$-Convergence, *Real Anal. Exch.*, **26** (1999), 193–200.
11. M. A. Krasnosel'skii, Y. B. Rutickii, *Convex functions and Orlicz spaces*, New York: Gordon & Breach, 1961.
12. M. Küçükaslan, M. Yılmaztürk, On deferred statistical convergence of sequences, *Kyungpook Math. J.*, **56** (2016), 357–366. https://doi.org/10.5666/KMJ.2016.56.2.357
13. F. Móricz, Statistical convergence of multiple sequences, *Arch. Math.*, **81** (2003), 82–89. https://doi.org/10.1007/s00013-003-0506-9
14. M. Mursaleen, F. Başar, *Sequence spaces: topics in modern summability theory*, Boca Raton: CRC Press, 2020. https://doi.org/10.1201/9781003015116
15. M. Mursaleen, O. H. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288** (2003), 223–231. https://doi.org/10.1016/j.jmaa.2003.08.004
16. J. Musielak, *Orlicz spaces and modular spaces*, Berlin: Springer, 1983. https://doi.org/10.1007/BFb0072210
17. W. Orlicz, Über Räume (L^M), *Bull. Acad. Polonaise A*, **1936** (1936), 93–107.
18. A. Pringsheim, Zur theorie der zweifach unendlichen zahlenfolgen, *Math. Ann.*, **53** (1900), 289–321. https://doi.org/10.1007/BF01448977
19. H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.*, **2** (1951), 73–74.

AIMS Press