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Research article

On weak compactons to the nonlinearly dispersive mK(m,n,a,b) equation

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Abstract: In this paper, we investigated the precise existence conditions for weak compacton solutions of the nonlinearly dispersive mK(m,n,a,b) equation. By employing cutoff conditions, we derived a complete classification of parameters for compacton solutions. Specifically, Theorem 3.1 established four sets of sufficient conditions (Rt1–Rt4) under which a traveling wave solution possessed compact support. For the special parameters $a = b - 1$, $m = 1$, and $n = 2b$, we derived the corresponding traveling wave ordinary differential equation and present explicit conserved integrals for mass, momentum, and energy. Furthermore, in Theorem 4.1, we provided general quadrature formulae for the compacton solutions. The results extend other studies on the mK(m,n,a,b) systems and offer an analytical framework for compacton existence in nonlinear dispersive equations.

Keywords: mK(m,n,a,b) equation; solitary waves; weak compactons
Mathematics Subject Classification: 35C07, 37K10, 37K40

1. Introduction

As is well known, since the solitary wave phenomenon was first observed by Russell [1], the search for solitary wave solutions to nonlinear models has been a central theme in mathematical physics and soliton theory. From the KdV equation describing shallow water waves to more complex models, such as the mKdV, Boussinesq, Zakharov-Kuznetsov (ZK), and Kadomtsev-Petviashvili (KP) equations, the search for solitary wave solutions exhibiting particle-like behavior has remained a central objective in this field. A significant conceptual breakthrough occurred in 1993 when Rosenau and Hyman introduced the K(m,n) equation [2]. This model departed from the weakly nonlinear regime of the classical KdV equation and remarkably yielded compacton solutions-solitary waves with compact support that exhibit nearly elastic recovery upon interaction. This discovery opened a new research direction on "compact solitons", prompting extensive investigations into compacton solutions across nonlinear dispersive models. As an important extension along this line, Yan and Bluman [3] systematically studied compacton solutions and their dynamical properties for the B(m,n) equation,

$$u_{tt} - u_{xx} - a(u^n)_{xx} + b(u^m)_{xxxx} = 0, \tag{1.1}$$

employing four direct ansatz methods to explore special exact solutions, and obtaining novel compacton solutions for the particular B(2,2) case. In a separate study, Mustafa [4] examined modified nonlinear Klein-Gordon-type equations, specifically the mKG(m,n,k) equation, by applying several ansatz techniques

$$(u^m)_{tt} + a(u^n)_{xx} + b(u^k)_{xxxx} = 0, \tag{1.2}$$

and the author obtained new compact solutions and solitary pattern solutions. Inspired by these works, we focus on the nonlinearly dispersive mK(m,n,a,b) equation

$$u^{m-1}u_t + \alpha(u^n)_x + \beta(u^a(u^b)_{xx})_x = 0, \tag{1.3}$$

with $m, n \neq 0$. It is widely recognized that several notable nonlinear evolution equations are intimately connected to the mK(m,n,a,b) equation (1.3), including the nonlinearly dispersive K(m,n) equation, the KP(m,n) equation, and the R(m,n) equation, among others [5, 6]. Advances have further extended compacton theory through analytical and numerical approaches. These equations have been analyzed in numerous special cases, especially concerning the physical properties of their compacton structures [7–9]. These developments underscore the enduring relevance of solitary wave studies and motivate our systematic investigation of compacton existence conditions for the general mK(m,n,a,b) equation.

Despite considerable progress on the mK(m,n,a,b) equation and its reductions, a systematic characterization of parameter regimes that support weak compacton solutions has not been established in the literature. Most studies rely on ad hoc ansatz-based constructions to obtain isolated solitary solutions, leaving a fundamental question unresolved: Under what parameter conditions must a traveling wave solution necessarily possess compact support? Furthermore, given that Eq (1.3) admits a conservation-law structure and that methods for deriving conservation laws have been developed [10–12], we investigate the conservation laws of Eq (1.3) and the existence conditions of its compacton solutions. By directly applying cutoff conditions, we derive precise existence criteria and obtain a complete classification of admissible parameter ranges (Rt1–Rt4). These findings offer a new theoretical perspective on the compacton structures of this model.

The organization of this paper is as follows: In Section 2, we address conservation laws and the corresponding travelling wave ordinary differential equation. In Section 3, existence conditions for weak compactons are presented through cutoff conditions. In Section 4, we provide the general compacton solution formulas for equation (1.3). Finally, the conclusions are outlined in Section 5.

2. Traveling wave ordinary differential equations and conservation laws

In this section, we provide a concise overview of traveling wave ordinary differential equations and conservation laws for the mK(m, n, a, b) equation, focusing on the specific parameter conditions $a = b-1$, $m = 1$, and $n = 2b$. Prior to a detailed analysis, we will establish some necessary foundational concepts.

Definition 2.1. *[13] A function $u(t,x)$ is termed a weak solution of the generalized mK(m, n, a, b) equation (1.3) if it satisfies the integral equation*

$$\int\int_{\mathbb{R}\times\mathbb{R}}\left(\frac{1}{m}u^m\varphi_t+\alpha u^n\varphi_x+\beta(u^a(u^b)_{xx})\varphi_x\right)dtdx=0, \tag{2.1}$$

for all test functions $\varphi(t,x)$ that are taken from the space $C_c^\infty(\mathbb{R}\times\mathbb{R})$ of infinitely differentiable functions with compact support.

Lemma 2.1. *[14] Let $\varphi(\xi)$ be any test function and $U(\xi)\in C^N$ with $N\geq 1$. The following identity holds:*

$$\int_{\mathbb{R}}\varphi'(\xi)U(\xi),d\xi=\varphi(L)U(L)-\varphi(-L)U(-L)-\int_{\mathbb{R}}\varphi(\xi)U'(\xi)H(L\pm\xi)d\xi. \tag{2.2}$$

Here, $H(\cdot)$ denotes the Heaviside step function, defined by

$$H(s)=\begin{cases}1, & s\geq 0,\\ 0, & s<0.\end{cases}$$

The notation $H(L-\xi)$ is used near the right boundary $\xi=L$, while $H(L+\xi)$ is used near the left boundary $\xi=-L$. In both cases, the argument vanishes at the respective cutoff point.

Proof. The proof of Lemma 2.1 follows from standard analytical techniques (see, e.g., [14]). For brevity, the detailed steps are omitted here.

Considering Eq (1.3), we introduce the traveling wave transformation $u=U(\xi)$, $\xi=k(x-ct)$ and substituting this into the equation yields

$$-cU^{m-1}U_\xi+\alpha(U^n)_\xi+k^2\beta(U^a(U^b)_{\xi\xi})_\xi=0. \tag{2.3}$$

Now let $V=U^b$, and the derivatives transform as follows:

$$U_\xi=\frac{1}{b}V^{\frac{1}{b}-1}V_\xi,\qquad U^n=V^{\frac{n}{b}},\qquad U^a(U^b)_{\xi\xi}=U^aV_{\xi\xi}=V^{\frac{a}{b}}V_{\xi\xi}.$$

Substituting these into (2.3) gives

$$-\frac{c}{b}V^{\frac{m}{b}-1}V_\xi+\frac{\alpha n}{b}V^{\frac{n}{b}-1}V_\xi+k^2\beta(V^{\frac{a}{b}}V_{\xi\xi})_\xi=0.$$

Integrating once with respect to ξ, we have

$$-\frac{c}{m}V^{\frac{m-a}{b}}+\alpha V^{\frac{n-a}{b}}+k^2\beta V''=C_1V^{\frac{-a}{b}}, \tag{2.4}$$

where C_1 is an arbitrary constant.

Multiplying (2.4) by V_ξ and integrating once more leads to a nonlinear oscillator equation of the form:

$$\frac{1}{2}(V')^2-\frac{bC_1}{k^2\beta(b-a)}V^{\frac{b-a}{b}}-\frac{bc}{k^2\beta m(m+b-a)}V^{\frac{m+b-a}{b}}+\frac{\alpha b}{k^2\beta m(n+b-a)}V^{\frac{n+b-a}{b}}=E, \tag{2.5}$$

with $E = \frac{C_2}{k^2\beta}$, where C_2 is an arbitrary constant. The solution to this equation is then given implicitly by the integral:

$$\int \frac{dV}{\sqrt{2(E + \frac{bC_1}{k^2\beta(b-a)}V^{\frac{b-a}{b}} + \frac{bc}{k^2\beta m(m+b-a)}V^{\frac{m+b-a}{b}} - \frac{\alpha b}{k^2\beta m(n+b-a)}V^{\frac{n+b-a}{b}})}} = \pm\xi + C_3, \tag{2.6}$$

with C_3 denoting another arbitrary constant.

For the special parameters $a = b - 1$, $m = 1$, and $n = 2b$, Eq (1.3) admits a local conservation law of the form

$$D_t\rho + D_x\Phi = 0, \tag{2.7}$$

for any solution $u(t, x)$ of Eq (1.3), where ρ is the conserved density and Φ represents the corresponding flux. Set $u_i = \frac{\partial^i u}{\partial x^i}$, and substitute the formulas

$$\rho = \sum_{t_0,t_1,\dots t_p} b_i u_0^{t_0} u_1^{t_1} \dots u_p^{t_p}, \Phi = \sum_{t_0,t_1,\dots t_p} c_j u_0^{t_0} u_1^{t_1} \dots u_q^{t_q}$$

into Eq (2.7) with t_i as the power of u_i, b_i, c_j, which are the coefficients corresponding to u_i. Since all terms in the given equation must have the same rank, we have the first three densities and fluxes, as follows:

$$\begin{aligned}
&\rho_1 = u,\ \Phi_1 = \alpha u^n + \beta u^a (u^b)_{xx}, \\
&\rho_2 = \frac{1}{2}u^2,\ \Phi_2 = \alpha\frac{2b}{2b+1}u^{2b+1} + \beta b u^{2b-1}u_{xx} + \beta b(b - \frac{3}{2})u^{2b-2}(u_x)^2, \\
&\rho_3 = \frac{u^{2b+1}}{2b+1} - -\frac{\beta b}{4\alpha(b-1)}u^{2b-2}u_x^2, \\
&\Phi_3 = \frac{\alpha b}{2b+1}u^{4b} + \beta b u^{3b-1}u_{xx} - \frac{\beta b(2b-1)}{4(b-1)}u^{3b-2}u_x^2 \\
&\quad + \frac{\beta^2 b^2}{4\alpha(b-1)}\left[u^{2b-2}(u_{xx})^2 - \frac{2(b-2)}{2b-3}u_x^2 u_{xx} + \frac{(b-2)(2b-3)}{3(b-1)^2}u^{2b-4}u_x^4\right].
\end{aligned} \tag{2.8}$$

Noting that when $u = U(\xi)$, $\xi = k(x - ct)$, combining the density function with the nonlinear oscillator form of the traveling wave ordinary differential equation (ODE), for a solution that exhibits a single peak U_- above a background level U_+, the associated quadrature can be evaluated without requiring the explicit solution of Eq (1.3). Specifically,

$$\int_{\mathbb{R}} \rho dx = \int_{\mathbb{R}} \rho \mid_{u=U(\xi)} d\xi = \int_{U_-}^{U^+} \frac{\sqrt{2}\rho \mid_{U(\xi)}}{\sqrt{E - \Pi(U)}} dU, \tag{2.9}$$

with

$$\Pi(U) = -\frac{bC_1}{k^2\beta(b-a)}V^{\frac{b-a}{b}} - \frac{bc}{k^2\beta m(m+b-a)}V^{\frac{m+b-a}{b}} + \frac{\alpha b}{k^2\beta m(n+b-a)}V^{\frac{n+b-a}{b}},$$

so we can obtain the corresponding conserved integrals is given by

$$\Omega = \int_{\mathbb{R}} (\rho \mid_{u=U(\xi)} - \rho \mid_{u=U^+}) dx = \int_{U_-}^{U^+} \frac{\sqrt{2}(\rho \mid_{u=U(\xi)} - \rho \mid_{u=U^+})}{\sqrt{E - \Pi(U)}} dU.$$

Correspondingly, we obtain the corresponding conservations for mass, momentum, and energy as

$$M_1[u] = \int_{U_-}^{U^+} \frac{\sqrt{2}(U - U^+)}{\sqrt{E - \Pi(U)}} dU, \tag{2.10}$$

$$M_2[u] = \int_{U_-}^{U^+} \frac{U^2 - (U^+)^2}{\sqrt{2(E - \Pi(U))}} dU, \tag{2.11}$$

$$M_3[u] = \int_{U_-}^{U^+} \frac{U^{2b+1} - (U^+)^{2b+1}}{(2b+1)\sqrt{E - \Pi(U)}} dU + \int_{U_-}^{U^+} \frac{\beta b(U^{2b-2}U_x^2 - (U^+)^{2b-2}(U_x^+)^2}{4\alpha(b-1)\sqrt{E - \Pi(U)}} dU. \tag{2.12}$$

Remark 2.1. *When $b = 1$, the term $\frac{1}{b-1}$ in the second integral of (2.12) becomes singular. In this case, the expression is understood in the limiting sense $b \to 1$, which yields a finite and well-defined result. The same conclusion follows by directly setting $b = 1$ in the original traveling-wave ODE (2.3) before integration.*

3. Existence conditions of compacton solutions

In this section, we outline the criteria under which a traveling wave solution $U(\xi)$ gives rise to a compacton solution $U_c(\xi) = U(\xi)H(L \pm \xi)$, where $H(L \pm \xi)$ denotes the step function. The subsequent theorem is then derived.

Theorem 3.1. *Assume that a compacton profile $u = U_c(\xi) = U(\xi)H(L \pm \xi)$ exists, with $U(\xi) \in C^3(\mathbb{R})$, $k \neq 0$ and boundary asymptotics given by*

$$U(\xi) \sim U_0(L \pm \xi)^\gamma = U_0 H(L \pm \xi)^\gamma, as\ \xi \sim \pm L.$$

Then $U_c(\xi) = U(\xi)H(L \pm \xi)$ constitutes a compacton solution of Eq (1.3), provided that the parameters $a, b, m, n, \gamma,\ and U_0$ ($m, n > 0, m \neq n$) satisfy one of the following conditions:
(Rt1) $\gamma > \frac{2}{b}$;
(Rt2) $\gamma > \max(\frac{1}{n}, \frac{1}{m}, \frac{3}{a+b})$;
(Rt3) $\gamma = \frac{2}{a+b-n}$, $2m + n > a + b > 3n$, $U_0 = [-\frac{an}{\gamma^2\beta b(b\gamma-1)(a\gamma+b\gamma-2)}]^{\frac{1}{a+b-n}}$;
(Rt4) $\gamma = \frac{2}{a+b-m}$, $2n + m > a + b > 3m$, $U_0 = [\frac{c}{\gamma^2\beta b(b\gamma-1)(a\gamma+b\gamma-2)}]^{\frac{1}{a+b-m}}$.

Proof. To determine the precise exponent conditions under which the distributional identity (3.1) yields compact support, we first note that the standard regularity and boundary assumptions for compacton analysis require $U(\xi) \in C^3(R)$ and $U(\xi) \sim U_0(L \pm \xi)^\gamma$ [2, 3]. It is also essential for determining the exponent conditions under which A_0–A_3 vanish. With these assumptions, it follows from Definition 2.1 that

$$\begin{aligned} & -cU_c^{m-1}(U_c)_\xi + \alpha(U_c^n)_\xi + \gamma^2\beta(U_c^a(U_c^b)_{\xi\xi})_\xi \\ = \ & -\left[\frac{c}{m}(U^m)_\xi - \alpha(U^n)_\xi - \gamma^2\beta[U^a(U^b)_{\xi\xi}]_\xi\right] H - \left[\frac{c}{m}U^m - \alpha U^n - \gamma^2\beta[U^a(U^b)_{\xi\xi}]\right]\delta \\ & + \Big[[U^a(U^b)_\xi + (U^{a+b})_\xi]\delta\Big]_\xi + \Big[(U^{a+b})_\xi\delta\Big]_{\xi\xi}. \end{aligned} \tag{3.1}$$

In view of Lemma 2.1, consider a function $U(\xi)$ that behaves asymptotically as $U_0(L-|\xi|)^\gamma = U_0H(L\pm\xi)^\gamma$ when $\xi\sim\pm L$. Under this assumption, Eq (3.1) becomes

$$-cU_c^{m-1}(U_c)_\xi+\alpha(U_c^n)_\xi+\gamma^2\beta(U_c^a(U_c^b)_{\xi\xi})_\xi = A_0H+A_1\delta+(A_2\delta)_\xi+(A_3\delta)_{\xi\xi}, \tag{3.2}$$

with

$$\begin{aligned}
A_0 &= -\frac{c}{m}(U^m)_\xi+\alpha(U^n)_\xi+\gamma^2\beta[U^a(U^b)_{\xi\xi}]_\xi\\
&\sim -c\gamma U_0^m(L\pm\xi)^{m\gamma-1}+\alpha n\gamma U_0^n(L\pm\xi)^{n\gamma-1}+\gamma^2\beta b\gamma(b\gamma-1)(a\gamma+b\gamma-2)U_0^{a+b}(L\pm\xi)^{a\gamma+b\gamma-3};\\
A_1 &= -\frac{c}{m}U^m+\alpha U^n+\gamma^2\beta U^a(U^b)_{\xi\xi}\\
&\sim -\frac{c}{m}U_0^m(L\pm\xi)^{m\gamma}+\alpha U_0^n(L\pm\xi)^{n\gamma}+\gamma^2\beta b\gamma(b\gamma-1)U_0^{a+b}(L\pm\xi)^{b\gamma-2};\\
A_2 &= U^a(U^b)_\xi+(U^{a+b})_\xi\\
&\sim b\gamma U_0^{a+b}(L\pm\xi)^{b\gamma-1}+(a+b)\gamma U_0^{a+b}(L\pm\xi)^{(a+b)\gamma-1};\\
A_3 &= (U^{a+b})_\xi\sim(a+b)\gamma U_0^{a+b}(L\pm\xi)^{(a+b)\gamma-1}.
\end{aligned}$$

For $U(\xi)\in C^3(\mathbb{R})$, the behavior of weak compactons $U(\xi)$ near $\xi=\pm L$ is determined by the dominant terms from A_0, A_1, A_2, and A_3.

Case 1. If $A_3=0$, it requires that

$$A_3\sim(a+b)\gamma U_0^{a+b}(L\pm\xi)^{(a+b)\gamma-1}\longrightarrow 0,$$

near $\xi=\pm L$, which implies that $\gamma>\frac{1}{a+b}$.

Case 2. When $A_2=0$, it needs that

$$A_2\sim b\gamma U_0^{a+b}(L\pm\xi)^{b\gamma-1}+(a+b)\gamma U_0^{a+b}(L\pm\xi)^{(a+b)\gamma-1}\longrightarrow 0,$$

near $\xi=\pm L$. One has $\gamma>\max(\frac{1}{b},\frac{1}{a+b})$, that is $\gamma>\frac{1}{b}$.

Case 3. Let $A_1=0$, which implies that

$$A_1\sim-\frac{c}{m}U_0^m(L\pm\xi)^{m\gamma}+\alpha U_0^n(L\pm\xi)^{n\gamma}+\gamma^2\beta b\gamma(b\gamma-1)U_0^{a+b}(L\pm\xi)^{b\gamma-2}\longrightarrow 0,$$

near $\xi=\pm L$, that is $\gamma>\frac{2}{b}$.

Thus, the above three cases yield that $\gamma>\frac{2}{b}$.

Case 4. $A_0=0$, this is required that $-c\gamma U_0^m(L\pm\xi)^{m\gamma-1}$, $\alpha n\gamma U_0^n(L\pm\xi)^{n\gamma-1}$ and $\gamma^2\beta b\gamma(b\gamma-1)(a\gamma+b\gamma-2)U_0^{a+b}(L\pm\xi)^{a\gamma+b\gamma-3}$ go to 0, which yields

$$m\gamma-1>0,\ n\gamma-1>0,\ a\gamma+b\gamma-3>0.$$

Here, $\gamma=\frac{2}{a+b-n}$, and the exponents in A_0 become

$$m\gamma-1=\frac{2m}{a+b-n}-1,\ n\gamma-1=\frac{3n-a-b}{a+b-n},\ (a+b)\gamma-3=\frac{3n-a-b}{a+b-n}.$$

When $a+b>3n$, the latter two exponents are equal and negative. Their coefficients must cancel, yielding $\alpha n\gamma U_0^n+\gamma^2\beta b\gamma(b\gamma-1)(a\gamma+b\gamma-2)U_0^{a+b}=0$. Solving for U_0 gives

$$U_0=[-\frac{\alpha n}{\gamma^2\beta b(b\gamma-1)(a\gamma+b\gamma-2)}]^{\frac{1}{a+b-n}}.$$

This leads to Rt3.

When $\gamma = \frac{2}{a+b-m}$, $2n + m > a + b > 3m$, a similar cancellation occurs between the first and third terms in A_0. Then we have

$$-c\gamma U_0^m + \gamma^2\beta b\gamma(b\gamma - 1)(a\gamma + b\gamma - 2)U_0^{a+b} = 0,$$

which implies

$$U_0 = [\frac{c}{\gamma^2\beta b(b\gamma - 1)(a\gamma + b\gamma - 2)}]^{\frac{1}{a+b-m}}.$$

This matches Rt4.

Remark 3.1. (Physical background) We acknowledge that the mK(m,n,a,b) equation (1.3) is a generalized mathematical model that encompasses multiple equations with physical significance as special cases, rather than directly originating from a specific physical system. When $a = 0$ and $b = 1$, (1.3) reduces to the K(m,n) equation introduced by Rosenau and Hyman [2], which describes compacton propagation in nonlinear lattices. When $a = b - 1$, $m = 1$ and $n = 2b$, it corresponds to a generalized Boussinesq-type equation relevant to shallow-water wave dynamics [3]. Certain parameter choices also relate to models arising in plasma physics [5] and nonlinear optics [7]. Therefore, the existence conditions established in Theorem 3.1, when specialized to these cases, provide unified criteria for the occurrence of compacton solutions in these physical settings.

Thus, we complete the proof.

4. Compactons in the general case

In this section, we derive line compactons in the general case starting from the traveling wave solutions (2.6). To satisfy the endpoint condition $V(\pm L) = 0$ required by Theorem 3.1, $V(\xi)$ can be expressed in a more convenient form:

$$\int_0^V \frac{dV}{\sqrt{2E + AV^{\frac{b-a}{b}} + BV^{\frac{m+b-a}{b}} + CV^{\frac{n+b-a}{b}}}} = L \pm \xi, \tag{4.1}$$

with

$$L = C_4,\ E = \frac{2C_2}{\gamma^2\beta},\ A = \frac{2bC_1}{\gamma^2\beta(b-a)},\ B = \frac{2bc}{\gamma^2\beta m(m+b-a)},\ C = -\frac{2\alpha b}{\gamma^2\beta m(n+b-a)}.$$

From the preceding analysis, let

$$F(V) = 2E + AV^{\frac{b-a}{b}} + BV^{\frac{m+b-a}{b}} + CV^{\frac{n+b-a}{b}},$$

so that (4.1) is $\displaystyle\int_0^V \frac{dV}{\sqrt{F(V)}} = L \pm \xi$. For the solution to have compact support with $V(\pm L) = 0$, the integral must converge as $V \to 0^+$. The asymptotic behavior of $V(\xi)$ near the boundary is determined by the dominant term in $F(V)$ as $V \to 0$. Four cases:

Case $E \neq 0$: The asymptotics of $V(\xi)$ are dominated by E. Hence, as $\xi \sim \pm\xi$, it follows that $V \sim \sqrt{2E}(L \pm \xi)$. This forces the asymptotic profile $V(\xi) \sim V_0(L \pm \xi)^{b\gamma}$ to satisfy $b\gamma = 1$. In light of case Rt1 in Theorem 3.1, compact solutions are ruled out in this setting.

Case $E = 0,\ A \neq 0$: When the smallest exponent arises from $AV^{\frac{b-a}{b}}$, we obtain $V^{\frac{1}{2}(\frac{a}{b}+1)} \sim \frac{1}{2}(1 + \frac{a}{b})\sqrt{A}(L \pm \xi)$. This necessitates $A > 0$ and leads to $b\gamma = \frac{2b}{a+b}$. Following Theorem 3.1 and Cases Rt2–Rt4, compactons are excluded. Moreover, for compactons to be feasible, the equation $AV^{\frac{b-a}{b}} + BV^{\frac{m+b-a}{b}} + CV^{\frac{n+b-a}{b}} = 0$ must admit a minimal extremal solution.

Case $E = A = 0,\ B, C \neq 0$: Here, two subcases emerge depending on the relative sizes of m and n.

If $m > n$, we infer $V^{\frac{1}{2}(1-\frac{n-a}{b})} \sim \frac{1}{2}(1 - \frac{n-a}{b})\sqrt{C}(L \pm \xi)$, corresponding to $b\gamma = \frac{2b}{a+b-n}$. Therefore, under condition Rt3 for the existence of a weak compacton, we must have $2m + n > a + b > 3n$.

If $m < n$, we instead have $V^{\frac{1}{2}(1-\frac{m-a}{b})} \sim \frac{1}{2}(1 - \frac{m-a}{b})\sqrt{C}(L \pm \xi)$, with $b\gamma = \frac{2b}{a+b-m}$. Then, according to condition Rt4, a weak compacton exists, provided $2n + m > a + b > 3m$.

Case $E = A = B = 0,\ C \neq 0$: In this final case, the equation possesses no positive root, so compactons cannot occur.

This analysis establishes the conditions stated in Theorem 4.1.

Theorem 4.1. *Let $m, n > 0$ and $m \neq n$, then the line compacton solutions of Eq (1.3) can be obtained by the quadratures with $m > 0$, $n > 0$, and $b - a > 0$,*

$$\int_0^{u^b} \frac{bU^{b-1}dU}{\sqrt{AU^{b-a} + BU^{m+b-a} + CU^{n+b-a}}} = L \pm \xi, \tag{4.2}$$

$$m > n : B < 0,\ A > 0; C < 0,\ B = 0,\ A > 0;\ B > 0,\ C < 0,\ 0 < A < \frac{m-n}{mn}[\frac{(-Cn)^m}{(Bm)^n}]^{\frac{1}{m-n}};$$

$$m < n : C < 0,\ A > 0; C > 0,\ B < 0,\ 0 < A < \frac{n-m}{mn}[\frac{(Cn)^m}{(-Bm)^n}]^{\frac{1}{m-n}};$$

$$\int_0^{u^b} \frac{bU^{b-1}dU}{\sqrt{BU^{m+b-a} + CU^{n+b-a}}} = L \pm \xi, \tag{4.3}$$

$$m > 1,\ n > 1,\ a + b > 3;$$

$$m > n : n > 0,\ m > 1,\ 2m + n > a + b > 3n;$$

$$m < n : m > 0,\ n > 1,\ 2n + m > a + b > 3m.$$

Remark 4.1. *The conditions in Theorem 4.1 establish the existence of compacton solutions for many parameters. To illustrate these results concretely, we present below an explicit example where the quadrature (4.2) can be evaluated in closed form, together with graphical representations of the resulting compacton profiles.*

Example 4.1. Under specific parameter selections, the generalized integral (2.6) has multiple distribution types, as shown in Figure 1. When the integrability conditions Rt3 or Rt4 are satisfied, this integral (2.6) can generate explicit compact sub-distributions with compact support, thereby verifying the existence conditions derived in Theorem 3.1, as shown in Figure 2.

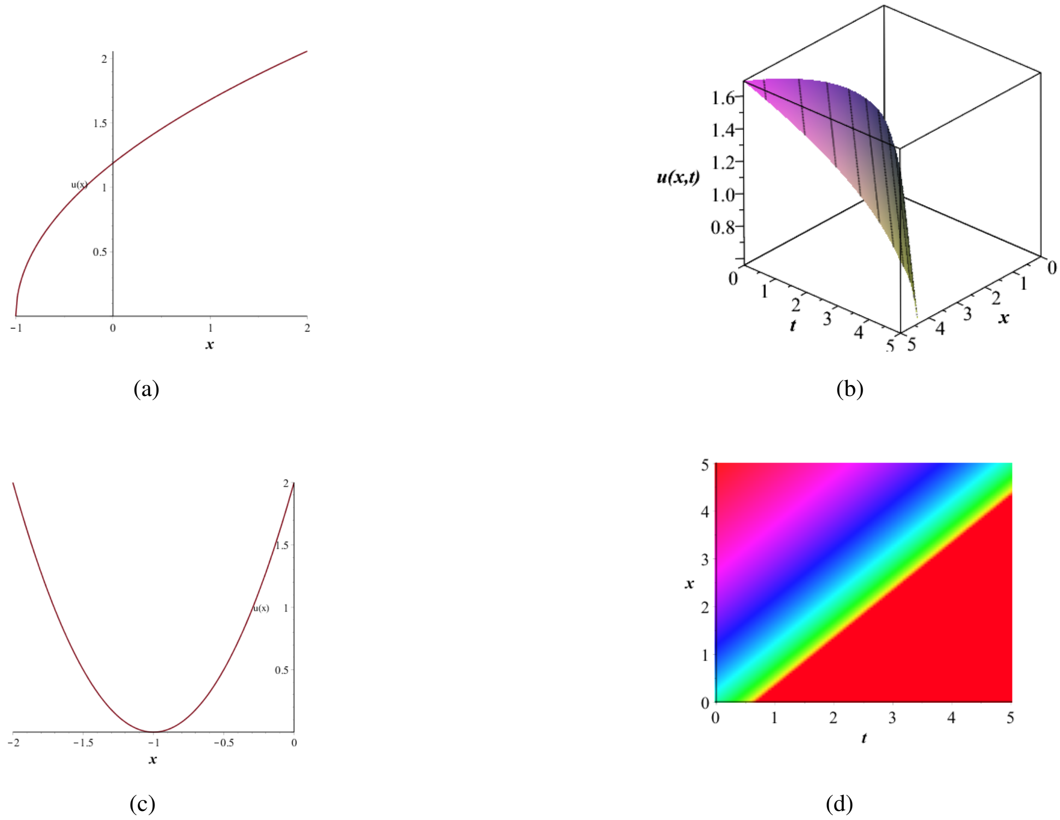

Figure 1. (a) Representative profiles (2.6), with $C_1 = c = \alpha = 0,\ E > 0$, (b) 3D surface plot of representative profiles (2.6), with $C_1 = c = \alpha = 0,\ E > 0$, (c) Representative profiles (2.6), with $C_1 = \alpha = 0,\ m = b,\ c \neq 0,\ E > 0$, and (d) 3D surface plot of representative profiles (2.6), with with $C_1 = \alpha = 0,\ m = b,\ c \neq 0,\ E > 0$.

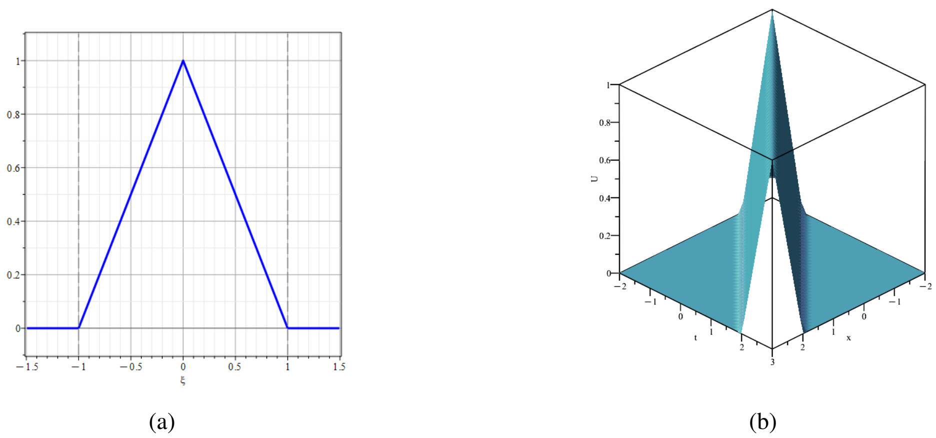

Figure 2. (a) Explicit compacton profile, with $m = 2,\ n = 1,\ a = 1,\ b = 2,\ \alpha = \beta = c = 1$, and (b) 3D surface plot of explicit compacton profile, with $m = 2,\ n = 1,\ a = 1,\ b = 2,\ \alpha = \beta = c = 1$.

5. Conclusions

In this work, we established the precise existence conditions for the weak compacton solutions of the mK(m,n,a,b) equation through truncation conditions, and provided the general integral formula for the corresponding compactons and concise analysis of the traveling wave ordinary differential equation and the conservation laws in special cases. Moreover, an important direction is the stability analysis of the weakly compact submanifold solutions. Although our existence conditions guarantee the formation of the compacton solutions, their dynamical stability under small perturbations needs to be studied separately. Furthermore, numerical verification and the extension of the cutoff method to other nonlinear dispersion equations are also important research directions that we will explore in the future.

Author contributions

Jingqun Wang: Methodology, software, writing–original draft, writing-review and editing; Jian-Gen Liu: Conceptualization, formal analysis, supervision, writing–review and editing; Chunyan Qin: ormal analysis, supervision, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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