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Research article

On some novel suprametric-interpolative contractions with applications

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Abstract: Fixed point theory in generalized metric-type spaces is useful for studying nonlinear problems where classical metric assumptions are too restrictive. The main contribution of this work is to establish new fixed point theorems for interpolative contraction mappings in strong extended s-suprametric spaces. These spaces are generalized distance frameworks that extend several known suprametric-type structures while preserving conditions suitable for convergence analysis. Interpolative contractions involve mixed-distance contractive conditions and therefore apply to mappings that may not satisfy the classical Banach contraction principle. Examples are included to demonstrate the efficacy of our findings. The existence of solutions to three categories of nonlinear problems, third-order boundary value problems, Fredholm integral equations, and dynamic programming problems, were demonstrated in the applications of the developed fixed point theory.

Keywords: suprametric spaces; interpolative contractions; fixed point; nonlinear problems
Mathematics Subject Classification: 34B15, 45G10, 47H10, 54E35, 54H25

1. Introduction

Fixed point theory is a cornerstone of modern mathematics with profound implications for theoretical analysis and applied sciences. Its origins trace back to Poincaré [22], whose pioneering work emphasized the role of fixed points in analysis and topology. A major advancement followed with Fréchet's [11] introduction of metric spaces, which provided a rigorous and generalized framework to study fixed points. The theory reached a decisive milestone with the Banach contraction principle [3], which established that every contraction in a complete metric space admits a unique fixed point. Due to its simplicity and broad applicability, Banach's theorem remains one of the fundamental results in nonlinear analysis.

Fixed point theory has been widely applied in solving nonlinear problems arising in applied mathematics and related disciplines. In particular, fixed point techniques provide powerful tools for establishing the existence and uniqueness of solutions for nonlinear integral equations, differential equations, and boundary value problems [27]. Several researchers have successfully utilized these methods to analyze nonlinear systems. Moreover, Berzig introduced generalized suprametric spaces which subsumes the JS-metric spaces and the b-suprametric spaces and established new fixed point theorems to map satisfying nonlinear contractions [6]. Aldosari et al. [2] introduced the concept of a robust extended s-suprametric space, a new extension that surpasses s-suprametric and extended suprametric spaces. It also examines the notions of existence and uniqueness using fundamental contractions, namely Banach and Kannan contractions.

Subsequent developments in metric fixed point theory have proceeded along two principal directions: One direction is the generalization of the underlying metric structure and extends classical contractive conditions. Numerous generalized metric spaces, including b-metric, partial metric, suprametric, s-suprametric, extended s-suprametric and interpolative metric spaces have been introduced to broaden the scope of fixed point results. Second, generalized contractions such as Kannan [14], Chatterjea [8], Reich [23], Hardy and Rogers [12], Geraghty, Meir and Keeler, F-contractions [28], Z-contractions, $F_{\omega e}$-Suzuki contractions [13], and interpolative contractions [16] have enriched the theory by providing flexible criteria for existence and uniqueness. Developments in fixed point theory highlight the importance of generalized metric spaces and new contractive conditions. The researchers in [19] defined generalized contraction on interpolative metric space and showed the advancement in both ways. In [20], the authors addressed an intrinsic relationship between some well-known contractions by proposing a novel metrizability method that converts some well-known contractions into a Banach contraction, preserving the completeness of the underlying metric space. The researchers in [21] developed two new classes of rational contraction mappings along with the corresponding fixed point theorems for these types of contractions in suprametric spaces. Moreover, authors [1, 9] applied fixed point principles to investigate the solvability of nonlinear integral equations and dynamic systems in generalized metric frameworks, demonstrating the effectiveness of these techniques in addressing complex mathematical models [2, 24]. In [25] the authors developed new fixed-point theorems in graphical extended b-metric spaces and demonstrated their applicability to elastic beam deformation problems and [26] introduced graphical rectangular b-metric spaces, established fixed-point results, and applied them to the vibration analysis of vertical heavy hanging cables. These contributions highlight the significant role of fixed point theory in theoretical developments and practical applications.

Alam et al. [30] studied fixed point results in ${M_v}^b$-metric spaces with applications to infectious disease models, while ϕ-interpolative contractions were applied to fixed figures and activation functions in neural networks [31]. Further progress was made through equiexpansive λ-set contractions and their applications to fractional differential systems [32].

Motivated by the need for more flexible distance structures in fixed point theory, we introduce new fixed point theorems for interpolative Reich–Rus–Ćirić-type and interpolative Kannan-type contraction in the framework of strong extended s-suprametric spaces, which extends several known suprametric-type frameworks while preserving suitable conditions for convergence analysis. The said interpolative contractive conditions are more general than classical Banach-type contractions because they involve mixed-distance terms and apply to mappings that may not satisfy a standard Lipschitz condition. The obtained results extend and generalize several fixed point theorems in metric and related generalized spaces. To demonstrate the usefulness of the proposed theory, we provide illustrative examples and apply the main results to prove the existence of solutions for three classes of nonlinear problems: third-order boundary value problems, Fredholm integral equations, and dynamic programming problems.

The manuscript is organized as follows: In Section 1, we provide a literature review on fixed point theory and Banach contraction principal in the first part. In subsequent paragraph, we discuss foundational and advancements in metric spaces and contractions. In the last paragraph, we present the work relevant to our findings in this manuscript. We also provide key definitions from semimetric space, suprametric space, b-suprametric space, and extended suprametric space. The definitions of strong extended s-suprametric spaces, their completeness, basis, open, and closed balls with some Prepositions 1 and 2, Remarks 1 and 2, and Examples 1–3 are also provided. In Section 2, we present Definitions 8 and 9 of interpolative contractions followed by the Theorems 1 and 2, respectively, which show the existence of fixed points of these contractions as well as illustrative Examples 4 and 5. Section 3 is based on the Applications of the established results. Section 4 Discussion. Sections 5 and 6 include conclusions and future scopes of the work, respectively.

Definition 1. *[10] Let ζ be a nonempty set and d be a mapping defined on $\zeta \times \zeta$ to $[0, \infty)$ that satisfies*

(i) $d(x, y) = 0 \iff x = y$,
(ii) $d(x, y) = d(y, x)$,

for all $x, y \in \zeta$. Then, (ζ, d) is referred to as a semimetric space.

Definition 2. *[5] Let (ζ, d) be a semimetric space that satisfies*

$$d(x, z) \leq d(x, y) + d(y, z) + \alpha d(x, y)d(y, z),$$

where $\alpha > 0$. Then the pair (ζ, d) is called a suprametric space.

Definition 3. *[7] If a semimetric space satisfies*

$$d(x, z) \leq b\left(d(x, y) + d(y, z)\right) + \alpha d(x, y)d(y, z),$$

where $\alpha > 0$ and $b > 1$, then the pair (ζ, d) is known as a b-suprametric space.

Example 1. *[7] Let $\zeta = C[0, 1]$, the space of continuous real-valued functions on $[0, 1]$. Define*

$$d(f, g) = \left(\int_0^1 |f(t) - g(t)|^p \, dt\right)^{\frac{1}{p}}, \qquad p \in (0, 1).$$

Then (ζ, d) is a b-suprametric space with $b = 2^{\frac{1}{p}}$ and any $\alpha > 0$.

Definition 4. *[18] If for all $x, y, z \in \zeta$, the inequality*

$$d(x, z) \leq d(x, y) + d(y, z) + \alpha(x, z)d(x, y)d(y, z)$$

holds, where $\alpha(x, z) > 0$, then the pair (ζ, d) is called an extended suprametric space.

Example 2. *Let $\zeta = [0, 1]$. Define*

$$d(x, y) = |x - y|^3, \qquad \forall x, y \in [0, 1].$$

Then (ζ, d) is an extended suprametric space with $\alpha(x, y) = 6 + xy$.

Definition 5. *[2] Let ζ be a nonempty set, $\alpha : \zeta \times \zeta \to [1, +\infty)$, $d : \zeta \times \zeta \to [0, +\infty)$ with $s > 1$, satisfying*

(i) $d(x, y) = 0 \Longleftrightarrow x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq sd(x, y) + d(y, z) + \alpha(x, z)d(x, y)d(y, z)$,
(iv) $d(x, z) \leq d(x, y) + sd(y, z) + \alpha(x, z)d(x, y)d(y, z)$,

for all $x, y, z \in \zeta$. Then (ζ, d) is called a strong extended s- suprametric space.

Example 3. *[2] Let $\alpha(x, y) = 2 + e^{x+y}$ and $d(x, y) = (x - y)^2$, where $\zeta \subseteq \mathbb{R}^+$. Then (ζ, d) is a strong extended s-suprametric space with $s \geq 10^2$.*

Example 4. *Let (ζ, d) be a metric space where $\zeta \in \mathbb{R}^+$ with the metric defined by $d(x, y) = |x - y|^p$ where $p \geq 2$ is an integer. This is a strong extended s-suprametric space for any $s \geq 10^p$ and $\alpha(x, y) = \sum_{n=1}^{p-1} {}^pC_n + e^{x+y}, \quad \forall x, y \in \zeta$.*

Example 5. *Let $\zeta = \{0, 1, 2, 3, 4\}$ be a non empty set. The function $\alpha : \zeta \times \zeta \to [1, \infty)$ is defined as $\alpha(x, y) = 1 + x + y$. If we define the metric d on ζ as follows*

$$d(x, y) = \begin{cases} 0, & if\, x = y, \\ \frac{1}{x+y}, & if\, x \neq y, \end{cases}$$

then we have Table 1.

Table 1. Metric values of $d(x, y)$.

d(x,y)	0	1	2	3	4
0	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
1	1	0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
2	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	0	$\frac{1}{7}$
4	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	0

From Table 1, it is obvious that the first two conditions of Definition 5 are satisfied by d. For conditions (iii) and (iv), let us consider some particular value of x, y and z.

If $x = 0, y = 1$ and $z = 4$, then from Table 1 $d(x, y) = 1, d(x, z) = \frac{1}{4}, d(y, z) = \frac{1}{5}$.

$$sd(x, y) + d(y, z) + \alpha(x, z)d(x, y)d(y, z) = 24.(1) + \frac{1}{5} + (1 + 0 + 4).1.\frac{1}{5} = 25.2,$$

hence,

$$d(x, z) \leq sd(x, y) + d(y, z) + \alpha(x, z)d(x, y)d(y, z).$$

again,

$$d(x, y) + sd(y, z) + \alpha(x, z)d(x, y)d(y, z) = (1) + 24.\frac{1}{5} + (1 + 0 + 4).1.\frac{1}{5} = 6.8.$$

hence,

$$d(x,z) \leq d(x,y) + sd(y,z) + \alpha(x,z)d(x,y)d(y,z).$$

Similarly, we can prove (iii) *and* (iv) *for other pair of values of* x, y *and* z*. Thus,* d *is a strong extended s-suprametric on* ζ *for* $s = 24$ *and* $\alpha(x,y) = 1 + x + y$, $\forall x, y \in \zeta$ *and* $x \neq y$.

Remark 1. *The strong extended s-suprametric space is an extended suprametric space [18] for* $s = 1$. *If* $s = 1$ *and* $\alpha(x,y) = \alpha > 0$*, then it will be a standard suprametric space [5].*

Definition 6. *[2] Let* (ζ, d) *be a strong extended s-suprametric space.*

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges to* $x \in \zeta$ *if for every* $\varepsilon > 0$*, there exists* $N_\varepsilon \in \mathbb{N}$ *such that* $d(x_n, x) < \varepsilon$ *for all* $n > N_\varepsilon$*, i.e.,* $\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ *is called Cauchy if for every* $\varepsilon > 0$*, there exists* $N_\varepsilon \in \mathbb{N}$ *such that* $d(x_n, x_m) < \varepsilon$ *for all* $m, n > N_\varepsilon$.
(iii) (ζ, d) *is complete if every Cauchy sequence converges in* ζ.

Remark 2. *[2] Let* (ζ, d) *be a strong extended s-suprametric space.*

(i) If d is continuous, then very convergent sequence has a unique limit.
(ii) In a complete space, every Cauchy sequence $\{x_n\}$ *has a limit* $x^* \in \zeta$ *such that* $\lim_{n\to\infty} d(x_n, x^*) = 0$*, and every subsequence* $\{x_{n_k}\}$ *converges to* x^*.

Definition 7. *[2] Let* (ζ, d) *be a strong extended s-suprametric space, then the open ball and closed ball with center* $a \in \zeta$ *and radius* $r > 0$ *are defined as follows:*

(i) The open ball : $\mathcal{B}(a, r) = \{x \in \zeta : d(a, x) < r\}$,
(ii) The closed ball : $\mathcal{B}[a, r] = \{x \in \zeta : d(a, x) \leq r\}$.

Proposition 1. *[2] In a strong extended s- suprametric space* (ζ, d)*, every open ball is an open set and every closed ball is a closed set.*

Proof. (i) Let $a \in \zeta$, $r > 0$, and suppose $y \in \mathcal{B}(a, r)$. Define

$$r_1 = \frac{r - d(y,a)}{s + \alpha(x,a)d(y,a)}.$$

For any $x \in \mathcal{B}(y, r_1)$,

$$d(x,a) \leq sd(x,y) + d(y,a) + \alpha(x,a)d(x,y)d(y,a) < sr_1 + d(y,a) + \alpha(x,a)r_1 d(y,a) < r.$$

Hence, $\mathcal{B}(y, r_1) \subseteq \mathcal{B}(a, r)$, so $\mathcal{B}(a, r)$ is open.
(ii) Let $\{x_n\} \subset \mathcal{B}[a, r]$, and $x_n \to x$. Then

$$d(a,x) \leq d(a,x_n) + sd(x_n,x) + \alpha(a,x)d(a,x_n)d(x_n,x).$$

Taking the limit as $n \to \infty$ gives $d(a, x) \leq r$, so $x \in \mathcal{B}[a, r]$. Thus, $\mathcal{B}[a, r]$ is closed.

□

Proposition 2. *[2] In a strong extended s-suprametric space (ζ, d), the family of open balls forms a basis for a topology on ζ.*

Proof. Suppose $u \in \mathcal{B}(x, \varepsilon) \cap \mathcal{B}(y, \varepsilon_0)$. Choose $r > 0$ such that

$$r + (s + r\alpha(x, v))d(x, u) < \varepsilon, \quad r + (s + r\alpha(y, v))d(y, u) < \varepsilon_0.$$

Then, for some $v \in \mathcal{B}(u, r)$, we have

$$d(x, v) < \varepsilon, \quad d(y, v) < \varepsilon_0.$$

Hence, $\mathcal{B}(u, r) \subseteq \mathcal{B}(x, \varepsilon) \cap \mathcal{B}(y, \varepsilon_0)$, showing that open balls form a base. □

1.1. Materials and methods

The main setting of the paper is a complete strong extended s- suprametric space (ζ, d). In this framework, the distance function satisfies the identity and symmetry properties together with the following properties for all $x, y, z \in (\zeta, d)$, $s > 1$ and $\alpha(x, y) : \zeta \times \zeta \to [1, \infty)$

(i) $d(x, z) \leq sd(x, y) + d(y, z) + \alpha(x, z)d(x, y)d(y, z)$,
(ii) $d(x, z) \leq d(x, y) + sd(y, z) + \alpha(x, z)d(x, y)d(y, z)$.

This setting is broader than the usual metric framework and is suitable for studying nonlinear mappings that may not satisfy a classical Lipschitz condition.

We also use interpolative contractions. These contractions are defined on the ground of strong extended s-suprametric spaces. The formulation of these contractions enable us to study the convergence of Picard iterations. The methodology used in the paper consists of the following steps:

(i) We mention the definitions of semi metric spaces, suprametic spaces, b-suprametric spaces, extended suprametic spaces, and strong extended s-suprametic spaces with examples. The completeness property of strong extended s-suprametric spaces and their topology are presented.
(ii) We introduce two interpolative contractions, named by Reich–Rus–Ciric type and Kannan type, in the setting of complete strong extended s- suprametric spaces.
(iii) We prove fixed point theorems for these contractions by constructing Picard sequences, establishing their Cauchy property, and then proving that their limits are fixed points.
(iv) In the first application we establish the existence of a solution to a third-order boundary value problem (BVP).
(v) As a second application, we find the solution of a nonlinear integral problem, and as a third application, the existence of a solution of a system of a functional equation in dynamic programming problem is determind.

2. Convergence results

In this section, we examine the existence of fixed points within the framework of strong extended s-suprametric spaces.

Definition 8. *In the framework of a strong extended s-suprametric space* (ζ, d)*, a mapping* $T : \zeta \to \zeta$ *is called an interpolative Reich–Rus–Ćirić type contraction if there exist constants* $\lambda \in [0, 1)$ *and* $\alpha, \beta \in (0, 1)$*, where* $\alpha + \beta < 1$ *such that*

$$d(T\xi, T\eta) \leq \lambda\, [d(\xi, \eta)]^{\beta} \cdot [d(\xi, T\xi)]^{\alpha} \cdot [d(\eta, T\eta)]^{1-\alpha-\beta} \tag{2.1}$$

for all $\xi, \eta \in \zeta \setminus \mathrm{Fix}(T)$*, where* $\mathrm{Fix}(T) = \{x \in \zeta : Tx = x\}$.

Theorem 1. *In the framework of a complete strong extended s-suprametric space* (ζ, d)*, if* $T : \zeta \to \zeta$ *is an interpolative Reich–Rus–Ćirić type contraction* (2.1)*, then* T *has a fixed point in* ζ.

Proof. Let us consider a sequence $u_n = Tu_{n-1}$ $\forall\, n \in \mathbb{N}$, starting from an arbitrary $u_0 \in \zeta$. Assume that the function α is bounded along the iterative sequence.

If $\lambda = 0$, the fixed point condition is trivially satisfied due to the continuity of T. Assume $0 < \lambda < 1$ and define the set $\Lambda = \{d(u, Tu) : u \in \zeta\}$, and let $r = \inf \Lambda$.

Suppose $r > 0$. Then there exists $u_p \in \zeta$ such that

$$d(u_p, Tu_p) < \frac{r}{\lambda}. \tag{2.2}$$

Applying the contraction condition (2.1), we obtain

$$\begin{aligned} d(Tu_p, T^2u_p) &\leq \lambda\, [d(u_p, Tu_p)]^{\beta} \cdot [d(u_p, Tu_p)]^{\alpha} \cdot [d(Tu_p, T^2u_p)]^{1-\alpha-\beta} \\ d(Tu_p, T^2u_p) &\leq \lambda\, [d(u_p, Tu_p)]^{\beta+\alpha} \cdot [d(Tu_p, T^2u_p)]^{1-\alpha-\beta} \\ d(Tu_p, T^2u_p)^{\alpha+\beta} &\leq \lambda\, [d(u_p, Tu_p)]^{\beta+\alpha} \\ [d(Tu_p, T^2u_p)]^{\alpha+\beta} &\leq \lambda \left[\frac{r}{\lambda}\right]^{\beta+\alpha}, \end{aligned} \tag{2.3}$$

from (2.2)

$$d(Tu_p, T^2u_p) \leq [\lambda^{\frac{1}{\alpha+\beta}-1} \cdot r] < r,$$

which contradicts the minimality of τ. Thus, $\tau = 0$, implying

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0.$$

Therefore, $\forall\, \varepsilon > 0$, $\exists\, m \in \mathbb{N}$ such that $\forall\, n \geq m$. Then, we have

$$d(u_n, u_{n+1}) < \varepsilon. \tag{2.4}$$

Now, we must show that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Using inequality (2.3), and for $a, b \in \mathbb{N}$ with $a > b > t$, and using the contraction definition (2.1), we get

$$d(u_a, u_{a+1}) \leq \lambda^{\frac{a-t}{\alpha+\beta}} d(u_t, u_{t+1}). \tag{2.5}$$

Again,

$$d(u_a, u_b) \leq d(u_a, u_{a+1}) + sd(u_{a+1}, u_b)$$

$$
\begin{aligned}
&+ \alpha(u_a, u_b) d(u_a, u_{a+1}) d(u_{a+1}, u_b) \\
&\leq \lambda^{\frac{a-t}{\alpha+\beta}} d(u_t, u_{t+1}) + s d(u_{a+1}, u_b) \\
&\quad + \alpha(u_a, u_b) \lambda^{\frac{a-t}{\alpha+\beta}} d(u_t, u_{t+1}) d(u_{a+1}, u_b) \\
&\leq \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon + s d(u_{a+1}, u_b) + \alpha(u_a, u_b) \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon d(u_{a+1}, u_b) \\
&\leq \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon + \left(s + \alpha(u_a, u_b) \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon\right) d(u_{a+1}, u_b). \qquad (2.6)
\end{aligned}
$$

Similarly, we achieve

$$
\begin{aligned}
d(u_{a+1}, u_b) &\leq d(u_{a+1}, u_{a+2}) + s d(u_{a+2}, u_b) \\
&\quad + \alpha(u_{a+1}, u_b) d(u_{a+1}, u_{a+2}) d(u_{a+2}, u_b) \\
&\leq \varepsilon \lambda^{\frac{a-t+1}{\alpha+\beta}} + \left(s + \alpha(u_{a+1}, u_b) \lambda^{\frac{a-t+1}{\alpha+\beta}} \varepsilon\right) d(u_{a+2}, u_b). \qquad (2.7)
\end{aligned}
$$

Substituting (2.7) into (2.6), we obtain

$$
\begin{aligned}
d(u_a, u_b) &\leq \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon + \left(s + \alpha(u_a, u_b) \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon\right) d(u_{a+1}, u_b) \\
&\leq \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon + \left(s + \alpha(u_a, u_b) \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon\right) \\
&\quad \times \left[\lambda^{\frac{a-t+1}{\alpha+\beta}} \varepsilon + \left(s + \alpha(u_{a+1}, u_b) \lambda^{\frac{a-t+1}{\alpha+\beta}} \varepsilon\right) d(u_{a+2}, u_b)\right] \\
&\leq \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon + \left(s + \alpha(u_a, u_b) \lambda^{\frac{a-t+1}{\alpha+\beta}} \varepsilon\right) \lambda^{\frac{a-t+1}{\alpha+\beta}} \varepsilon \\
&\quad + \left(s + \alpha(u_a, u_b) \lambda^{\frac{a-t}{\alpha+\beta}} \varepsilon\right) \left(s + \alpha(u_{a+1}, u_b) \lambda^{\frac{a-t+1}{\alpha+\beta}} \varepsilon\right) d(u_{a+2}, u_b).
\end{aligned}
$$

Continuing this process iteratively, we get

$$
d(u_a, u_b) \leq \varepsilon \lambda^{\frac{a-t}{\alpha+\beta}} \sum_{i=0}^{b-a-1} \lambda^{\frac{i}{\alpha+\beta}} \prod_{j=0}^{i-1} \left(s + \varepsilon \lambda^{\frac{a-t+j}{\alpha+\beta}} \alpha(u_{a+j}, u_b)\right),
$$

where the empty product $\prod_{j=0}^{-1}(\cdot)$ is taken as 1 by convention for $i = 0$.

Define

$$
Q_i = \lambda^{\frac{i}{\alpha+\beta}} \prod_{j=0}^{i-1} \left(s + \varepsilon \lambda^{\frac{a-t+j}{\alpha+\beta}} \alpha(u_{a+j}, u_b)\right). \qquad (2.8)
$$

Using the ratio test, since $\lambda \in [0, 1)$, we obtain

$$
\lim_{i \to \infty} \left| \frac{Q_{i+1}}{Q_i} \right| = \lambda^{\frac{1}{\alpha+\beta}} \left(s + \varepsilon \lambda^{\frac{a-t+1}{\alpha+\beta}} \alpha(u_{a+i}, u_b)\right) < 1.
$$

Therefore, the series converges, and as $a, b \to \infty$, we conclude

$$
\lim_{a,b \to \infty} d(u_a, u_b) = 0,
$$

so $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

By completeness of ζ, there exists $u^* \in \zeta$ such that

$$
\lim_{n \to \infty} u_n = u^*.
$$

Using Remark 2 and inequality (2.1), we get

$$d(Tu_{n_k}, Tu^*) \leq \lambda\,[d(u_{n_k}, u^*)]^{\beta} \cdot [d(u_{n_k}, Tu_{n_k})]^{\alpha} \cdot [d(u^*, Tu^*)]^{1-\alpha-\beta}.$$

Letting $k \to \infty$, we get $Tu^* = u^*$, i.e., u^* is a fixed point. □

Example 6. *Let $\zeta = [0, 1]$ be a complete strong extended s-suprametric space with metric d defined by $d(x, y) = (x - y)^2$, $s > 10^2$ and $\alpha(x, y) = 2 + e^{x+y}$. Let $T : \zeta \to \zeta$ be a self mapping defined by*

$$T(x) = \begin{cases} \frac{1}{5}, & x \in [0, \frac{1}{2}), \\ \frac{1}{4}, & x \in [\frac{1}{2}, 1]. \end{cases} \tag{2.9}$$

Then, T is an interpolative Reich–Rus–Ćirić type contraction for $\alpha = \beta = \frac{1}{4}$ and $\lambda = \frac{1}{2}$. From Theorem 1, it must have a fixed point in ζ, which is $x = \frac{1}{5}$.

Definition 9. *Let (ζ, d) be a strong extended s-suprametric space (ζ, d), and the mapping $T : \zeta \to \zeta$ is said to be an interpolative Kannan type contraction mapping if*

$$d(Tx, Ty) \leq \lambda[d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha}]. \tag{2.10}$$

For all $x, y \in \zeta \backslash \mathrm{Fix}(T)$, where $\mathrm{Fix}(T) = \{x \in \zeta : Tx = x\}$. Here, $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$.

Theorem 2. *In the framework of a complete strong extended s-suprametric space (ζ, d), if $T : \zeta \to \zeta$ is an interpolative Kannan type contraction* (2.10)*, then T has a fixed point in ζ.*

Proof. Form a sequence $\nu_n = T\nu_{n-1}\ \forall\ n \in \mathbb{N}$, starting from an arbitrary $\nu_0 \in \zeta$. Assume that the function α is bounded along the iterative sequence.

If $\lambda = 0$, the fixed point condition is trivially satisfied due to the continuity of T. Assume $0 < \lambda < 1$ and define the set $\Lambda = \{d(\nu, T\nu) : \nu \in \zeta\}$, and let $r = \inf \Lambda$.

Suppose $r > 0$. Then there exists $\nu_p \in \zeta$ such that

$$d(\nu_p, T\nu_p) < \frac{r}{\lambda}.$$

Then we obtain from (2.10)

$$d(T\nu_p, T^2\nu_p) \leq \lambda^{\frac{1}{\alpha}} d(\nu_p, T\nu_p). \tag{2.11}$$

Applying the contraction condition

$$\begin{aligned} d(T\nu_p, T^2\nu_p) &\leq \lambda\,[d(\nu_p, T\nu_p)]^{\alpha} \cdot [d(T\nu_p, T^2\nu_p)]^{1-\alpha} \\ d(T\nu_p, T^2\nu_p) &\leq \lambda^{\frac{1}{\alpha}}\,[d(\nu_p, T\nu_p)] \\ d(T\nu_p, T^2\nu_p) &\leq \lambda^{\frac{1}{\alpha}}\,\left[\frac{\tau}{\lambda}\right] \\ d(T\nu_p, T^2\nu_p) &< \lambda^{\frac{1}{\alpha}-1}\tau \end{aligned}$$

which contradicts the minimality of *r*. Thus, $r = 0$, implying

$$\lim_{n\to\infty} d(\nu_n, T\nu_n) = 0.$$

Hence, for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$d(\nu_n, \nu_{n+1}) < \varepsilon \quad \text{for all } n \geq m.$$

Now, we must show that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Using inequality (2.11), and for $a, b \in \mathbb{N}$ with $a > b > t$, we get using the contraction definition (2.10) ,

$$d(\nu_a, \nu_{a+1}) \leq \lambda^{\frac{a-t}{\alpha}} d(\nu_t, \nu_{t+1}). \tag{2.12}$$

Again consider,

$$\begin{aligned}
d(\nu_a, \nu_b) &\leq d(\nu_a, \nu_{a+1}) + sd(\nu_{a+1}, \nu_b) \\
&\quad + \alpha(\nu_a, \nu_b)d(\nu_a, \nu_{a+1})d(\nu_{a+1}, \nu_b) \\
&\leq \lambda^{\frac{a-t}{\alpha}} d(\nu_t, \nu_{t+1}) + sd(\nu_{a+1}, \nu_b) \\
&\quad + \alpha(\nu_a, \nu_b)\lambda^{\frac{a-t}{\alpha}} d(\nu_t, \nu_{t+1})d(\nu_{a+1}, \nu_b) \\
&\leq \lambda^{\frac{a-t}{\alpha}} \varepsilon + sd(\nu_{a+1}, \nu_b) + \alpha(\nu_a, \nu_b)\lambda^{\frac{a-t}{\alpha}} \varepsilon d(\nu_{a+1}, \nu_b) \\
&\leq \lambda^{\frac{a-t}{\alpha}} \varepsilon + \left(s + \alpha(\nu_a, \nu_b)\lambda^{\frac{a-t}{\alpha}} \varepsilon\right) d(\nu_{a+1}, \nu_b).
\end{aligned} \tag{2.13}$$

Similarly,

$$\begin{aligned}
d(\nu_{a+1}, \nu_b) &\leq d(\nu_{a+1}, \nu_{a+2}) + sd(\nu_{a+2}, \nu_b) \\
&\quad + \alpha(\nu_{a+1}, \nu_b)d(\nu_{a+1}, \nu_{a+2})d(\nu_{a+2}, \nu_b) \\
&\leq \varepsilon\lambda^{\frac{a-t+1}{\alpha}} + \left(s + \alpha(\nu_{a+1}, x_b)\lambda^{\frac{a-t+1}{\alpha}} \varepsilon\right) d(\nu_{a+2}, \nu_b).
\end{aligned} \tag{2.14}$$

Substituting (2.13) into (2.14) and we obtain

$$\begin{aligned}
d(\nu_a, \nu_b) &\leq \lambda^{\frac{a-t}{\alpha}} \varepsilon + \left(s + \alpha(\nu_a, \nu_b)\lambda^{\frac{a-t}{\alpha}} \varepsilon\right) \\
&\quad \times \left[\lambda^{\frac{a-t+1}{\alpha}} \varepsilon + \left(s + \alpha(\nu_{a+1}, \nu_b)\lambda^{\frac{a-t+1}{\alpha}} \varepsilon\right) d(\nu_{a+2}, \nu_b)\right] \\
&\leq \lambda^{\frac{a-t}{\alpha}} \varepsilon + \left(s + \alpha(\nu_a, \nu_b)\lambda^{\frac{a-t}{\alpha}} \varepsilon\right) \lambda^{\frac{a-t+1}{\alpha}} \varepsilon \\
&\quad + \left(s + \alpha(\nu_a, \nu_b)\lambda^{\frac{a-t}{\alpha}} \varepsilon\right) \left(s + \alpha(\nu_{a+1}, \nu_b)\lambda^{\frac{a-t+1}{\alpha}} \varepsilon\right) d(\nu_{a+2}, \nu_b).
\end{aligned}$$

Continuing this process iteratively, we get

$$d(\nu_a, \nu_b) \leq \varepsilon\lambda^{\frac{a-t}{\alpha}} \sum_{i=0}^{b-a-1} \lambda^{\frac{i}{\alpha}} \prod_{j=0}^{i-1} \left(s + \varepsilon\lambda^{\frac{a-t+j}{\alpha}} \alpha(\nu_{a+j}, \nu_b)\right), \tag{2.15}$$

where the empty product $\prod_{j=0}^{-1}(\cdot)$ is taken as 1 by convention for $i = 0$.

Define

$$Q_i = \lambda^{\frac{i}{\alpha}} \prod_{j=0}^{i-1} \left(s + \varepsilon\lambda^{\frac{a-t+j}{\alpha}} \alpha(\nu_{a+j}, \nu_b)\right). \tag{2.16}$$

Using the ratio test, since $\lambda \in [0, 1)$, we obtain

$$\lim_{i\to\infty} \left| \frac{Q_{i+1}}{Q_i} \right| = \lambda^{\frac{1}{\alpha}} \cdot \left(s + \varepsilon \lambda^{\frac{a-t+1}{\alpha}} \alpha(\nu_{a+i}, \nu_b) \right) < 1.$$

Therefore, the series converges, and as $a, b \to \infty$, we conclude

$$\lim_{a,b\to\infty} d(\nu_a, \nu_b) = 0,$$

so $\{\nu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

By completeness of ζ, there exists $\nu^* \in \zeta$ such that

$$\lim_{n\to\infty} \nu_n = \nu^*.$$

Using continuity of T and the contraction property

$$d(T\nu_{n_k}, T\nu^*) \leq \lambda\, d(\nu_{n_k}, T\nu_{n_k})^{\alpha}.d(T\nu^*, \nu^*)^{1-\alpha} \to 0.$$

Using Remark 2, we get $T\nu^* = \nu^*$, i.e., ν^* is a fixed point. □

Example 7. *Let (ζ, d) be a strong extended s-suprametric space define by 4. Here, $\zeta = \{0, 1, 2, 3, 4\}$ and d is defined by*

$$d(x, y) = \begin{cases} 0 & if x = y, \\ \frac{1}{x+y}, & if x \neq y. \end{cases}$$

Table 1 shows the range of metric d. Now let $T : \zeta \to \zeta$ be a self map defined by

$$T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 4 & 4 \end{pmatrix}.$$

Then it can be easily to seen that T is an interpolative Kannan type contraction for $\alpha = \frac{1}{2}$ and $\lambda = 0.9$. By Theorem 2, T has a fixed point in ζ, which is $x = 4$.

3. Applications

In the context of variational inequality theory, the deflection of a curved beam with a constant or varying cross-section, three-layer beams, rocket motion, heat conduction, subterranean water flow, thermo elasticity, plasma physics, electromagnetic waves, gravity-driven flows, boundary layer theory, the study of draining and coating flows, chemical engineering, control and optimization theory, and flow networks in biology, third-order boundary value problems (BVPs) are essential. In our first application, we establish the existence of a solution to a third-order boundary value problem (BVP). To achieve this, we begin by constructing the corresponding Green's function associated with the given BVP. Utilizing this Green's function, the solution is reformulated into an equivalent integral equation.

Subsequently, we define a suitable self-operator based on this integral representation. By employing our established fixed point Theorem 1, we demonstrate the existence of a fixed point of this operator, which corresponds to a solution of the original problem. Furthermore, we prove the uniqueness of this fixed point under the imposed conditions. Finally, we show that the Green's function-based Picard iterative sequence converges to this unique fixed point, thereby providing a constructive approach to approximating the solution.

3.1. Green's picard iterative procedure

To derive the Green's function embedded in Picard iterative procedure, we consider the following differential equation

$$L[x] + N[x] = f(t, x), \tag{3.1}$$

where $L[x]$ is a linear operator in x, $N[x]$ is a nonlinear operator in x, and $f(t, x)$ is a linear or nonlinear function in x.

We define the following linear integral operator $\mathcal{K}$, in terms of the Green's function

$$\mathcal{K}[x](t) = \int_a^b \mathcal{G}(t, s)L[x](s)\, ds, \tag{3.2}$$

where $\mathcal{G}$ is the Green's function associated with the linear differential operator $L[x]$.

Observe that x is a fixed point of $\mathcal{K}$ if and only if x is a solution of (3.1). Adding and subtracting $N[x] - f(s, x)$ from the integrand in (3.2) gives

$$\mathcal{K}[x](t) = \int_a^b \mathcal{G}(t, s)\left(L[x](s) + N[x](s) - f(s, x) + f(s, x) - N[x](s)\right)\, ds.$$

Thus,

$$\mathcal{K}[x](t) = x(t) + \int_a^b \mathcal{G}(t, s)\left(L[x](s) + N[x](s) - f(s, x)\right) ds.$$

We then apply the Picard's fixed point iteration formula, which reduces to

$$x_{n+1} = x_n + \int_a^b \mathcal{G}(t, s)(L[x_n] + N[x_n] - f(s, x_n))\, ds.$$

The initial guess x_0 is chosen as the solution to $L[x_0] = 0$, subject to the boundary conditions, taking into account that the Green's function satisfies the homogeneous part of the boundary conditions.

Consider the nonlinear boundary value problem

$$-y^{(3)}(t) = f(t, y, y', y''), \quad \text{subject to } y(1) = \mathbb{A}, \quad y'(1) = \mathbb{B}, \quad y(2) = \mathbb{C}. \tag{3.3}$$

The corresponding Green's function is

$$\mathcal{G}(t, s) = \begin{cases} -\frac{1}{2}s^2 + 2s - 2 + \left(\frac{1}{2}s^2 - 2s + 2\right)t, & 1 \le t \le s \le 2, \\ -s^2 + 2s - 2 + \left(\frac{1}{2}s^2 - s + 2\right)t - \frac{1}{2}t^2, & 1 \le s \le t \le 2. \end{cases}$$

The Green's Picard iterative procedure becomes

$$x_{n+1} = x_n + \int_1^2 \mathcal{G}(t, s)\left(x_n^{(3)}(s) - f(s, x_n, x_n', x_n'')\right) ds. \tag{3.4}$$

Choose x_0 satisfying $x_0^{(3)} = 0$ and

$$x_0(1) = \mathbb{A}, \quad x_0'(1) = \mathbb{B}, \quad x_0(2) = \mathbb{C}.$$

Define the operator $T_{\mathcal{G}} : C([1,2]) \to C([1,2])$ by

$$T_{\mathcal{G}}(x)(t) = x(t) + \int_1^2 \mathcal{G}(t,s)\left(x^{(3)}(s) - f(s,x,x',x'')\right) ds. \tag{3.5}$$

Using integration by parts and from [[17], Page no.3], the property

$$\int_1^2 \frac{\partial^3 \mathcal{G}}{\partial s^3}(t,s)x(s)\, ds = \int_1^2 \delta(t-s)x(s)\, ds = x(t),$$

where δ is the Kronecker Delta, we obtain

$$T_{\mathcal{G}}(x)(t) = (2-t)\mathbb{A} + \frac{1}{2}(t^2 - 3t + 2)\mathbb{B} + (t-1)\mathbb{C} - \int_1^2 \mathcal{G}(t,s) f(s,x,x',x'')\, ds.$$

Let the norm $\|x\| = \sup_{t\in[1,2]} |x(t)|^2$ and $d(x,y) = \sup_{x,y\in C[1,2]} |x-y|^2$.

From Example 4 $(C([1,2]), \|\cdot\|)$ will be a strong extended s-suprametric space, where $p = 2$ and $\alpha(x,y) = 2 + e^{x+y},\ \forall x,y \in C[1,2]$.

Theorem 3. *Suppose f satisfies the following condition*

$$|f(s,v,v',v'') - f(s,w,w',w'')| \le M_1|v-w|^\alpha + M_2|v - T_G(v)|^\beta + M_3|w - T_G(w)|^{1-\alpha-\beta},$$

for constants $M_1, M_2, M_3 > 0$, such that

$$\max\{M_1, M_2, M_3\} < 1.$$

Additionally,

$$\alpha, \beta \in \left(0, \frac{1}{2}\right).$$

Then, T_G is an interpolative Reich-Ruc-Ćirić type contraction on strong extended s-suprametric space $(C([1,2]), \|\cdot\|)$.

Proof. Using the given condition and the boundedness of $\mathcal{G}(t,s)$, consider the following for any $v, w \in (C[1,2], \|.\|)$

$$\begin{aligned}
&|T_G(v) - T_G(w)|^2 \\
&\quad = \left| \int_1^2 \mathcal{G}(t,s)\left[f(s,v,v',v'') - f(s,w,w',w'')\right] ds \right|^2 \\
&\quad \le \left| \sup_{t,s} \mathcal{G}(t,s) \right|^2 \left| \int_1^2 \Big(M_1|v-w|^\alpha + M_2|v - T_G(v)|^\beta \right. \\
&\qquad \left. + M_3|w - T_G(w)|^{1-\alpha-\beta} \Big) ds \right|^2
\end{aligned}$$

from [17] page number 8 in $1 \le s \le t \le 2$, $|\sup_{t,s} \mathcal{G}(t,s)| \le \mathcal{G}\left(\frac{3}{2}, 1\right)$. Thus,

$$|T_G(v) - T_G(w)|^2 \le \left(\mathcal{G}\left(\frac{3}{2}, 1\right)\right)^2 (\max\{M_1, M_2, M_3\})^2$$

$$
\begin{aligned}
&\times \int_1^2 |v-w|^{2\alpha}|v-T_G(v)|^{2\beta}|w-T_G(w)|^{2(1-\alpha-\beta)}\,ds\\
&\le \left(\frac{1}{8}\right)^2 (\max\{M_1, M_2, M_3\})^2\\
&\times \int_1^2 |v-w|^{2\alpha}|v-T_G(v)|^{2\beta}|w-T_G(w)|^{2(1-\alpha-\beta)}\,ds\\
\|T_G(v)-T_G(w)\| &< \lambda \int_1^2 \sup|v-w|^{2\alpha}\sup|v-T_G(v)|^{2\beta}\sup|w-T_G(w)|^{2(1-\alpha-\beta)}\,ds\\
\|T_G(v)-T_G(w)\| &< \lambda\|v-w\|^{\alpha}\|v-T_G(v)\|^{\beta}\|w-T_G(w)\|^{(1-\alpha-\beta)}.\\
&\textit{Here } \lambda = \frac{1}{64}(\max\{M_1, M_2, M_3\})^2 < 1.
\end{aligned}
$$

Therefore, T_G is an interpolative Reich–Rus– Ćirić contraction. Hence, from Theorem 1, T_G will have a fixed point in$(C([1,2]), \|\cdot\|)$. Now we will show that the Picard Sequence $(T(v_n))$, where $T(v_n) = v_{n+1}$ and v_{n+1} is defined by (3.4), will converge to the fixed point of T. □

Theorem 4. *Let T be the self-operator*

$$
T_{\mathcal{G}}(x)(t) = x(t) + \int_1^2 \mathcal{G}(t,s)\left(x^{(3)}(s) - f(s,x,x',x'')\right)ds. \tag{3.6}
$$

defined on strong extended s- suprametric space $T_G : C([1,2]) \to C([1,2])$ and follow the condition of Theorem 3. Then T is a Reich–Rus–Ćirić and have an unique fixed point. The picard sequence defined by (3.4) *will converge to the unique fixed point of T. Which will be the unique solution of the differential equation* (3.3).

Proof. Let T have two fixed point x^* and y^*. Then, from (3.5), we have

$$
\begin{aligned}
|Tx^* - Ty^*| &\le \left|\int_1^2 \mathcal{G}(t,s)\left[f(s,x^*,x^{*\prime},x^{*\prime\prime}) - f(s,y^*,y^{*\prime},y^{*\prime\prime})\right]ds\right|^2\\
&\le |\sup_{t,s}|\mathcal{G}(t,s)|^2\bigg|\int_1^2 \Big(M_1|x^*-y^*|^{\alpha}\\
&\quad + M_2|x^*-Tx^*|^{\beta} + M_3|y^*-Ty^*|^{1-\alpha-\beta}\Big)ds\bigg|^2 \qquad (3.7)\\
&\le {M_1}^2\int_1^2 |x^*-y^*|^{2\alpha}\,ds\\
\|Tx^*-Ty^*\| &\le {M_1}^2\|x^*-y^*\|^{\alpha}\\
\|x^*-y^*\| &\le {M_1}^2\|x^*-y^*\|^{\alpha}.
\end{aligned}
$$

Which is possible only when $x^* = y^*$ as $M_1 < 1$ and $\alpha < 1$.
Let $T(v_n) = v_{n+1}$ then consider

$$
|v_{n+1} - v_n|^2 = \left|\int_1^2 \mathcal{G}(t,s)\left[f(s,v_n,v_n',v_n'') - f(s,v_{n-1},v_{n-1}',v_{n-1}'')\right]ds\right|^2
$$

$$
\begin{aligned}
&\leq |\sup_{t,s} |\mathcal{G}(t,s)|^2| \times \left| \int_1^2 \Big(M_1|v_n - v_{n-1}|^\alpha + M_2|v_n - T_{\mathcal{G}}(v_n)|^\beta \right.\\
&\quad \left. + M_3|v_{n-1} - T(v_{n-1})|^{1-\alpha-\beta} \Big) ds \right|^2 \\
&\leq (\max\{M_1, M_2, M_3\})^2 \int_1^2 |v_n - v_{n-1}|^{2\alpha} |v_n - v_{n+1}|^{2\beta} \\
&\quad \times |v_n - v_{n-1}|^{2(1-\alpha-\beta)}\, ds \\
&\leq \lambda \int_1^2 |v_n - v_{n-1}|^{2(1-\beta)} |v_n - v_{n+1}|^{2\beta}\, ds
\end{aligned}
$$

here, $\lambda = (\max\{M_1, M_2, M_3\})^2 < 1$.

Now, taking suprimum,

$$
\begin{aligned}
\sup_{t,s} |v_{n+1} - v_n|^2 &< \lambda \int_1^2 \sup |v_n - v_{n-1}|^{2(1-\beta)} \sup |v_n - v_{n+1}|^{2\beta}\, ds \\
\|v_n - v_{n+1}\| &< \lambda^{\frac{1}{1-\beta}} \|v_n - v_{n-1}\| \\
\|v_n - v_{n+1}\| &< \lambda^{\frac{n}{1-\beta}} \|v_1 - v_0\|.
\end{aligned}
$$

Taking $n \to \infty$, we have $\|v_n - v_{n+1}\| \to 0$. Again, for some $a > b > t$ we, can have

$$
d(v_a, v_{a+1}) \leq \lambda^{\frac{a-t}{\alpha+\beta}} d(v_t, v_{t+1}).
$$

Following the logic of proof of Theorem 1, we can see that

$$
\lim_{a,b\to\infty} d(v_a, v_b) = 0.
$$

Hence, (v_n) is a Cauchy sequence. Now we will show that it will converge to the fixed point of T. Let v be the fixed point of T. Again, consider

$$
\begin{aligned}
|v_{n+1} - v|^2 &= \left| \int_1^2 \mathcal{G}(t,s) \left[f(s, v_n, v_n', v_n'') - f(s, v, v', v'') \right]\, ds \right|^2 \\
&\leq |\sup_{t,s} |\mathcal{G}(t,s)|^2 \left| \int_1^2 \Big(M_1|v_n - v|^\alpha + M_2|v - T_G(v)|^\beta + M_3|v_n - T(v_n)|^{1-\alpha-\beta} \Big)\, ds \right|^2 \\
&\leq (\max(M_1, M_2, M_3))^2 \int_1^2 \sup_{t,s} |v_n - v|^{2\alpha} \sup_{t,s} |v_n - v_{n+1}|^{2(1-\alpha-\beta)}\, ds \\
&\leq (\max(M_1, M_2, M_3))^2 \|v_n - v\|^\alpha \|v_n - v_{n+1}\|^{1-\alpha-\beta} \\
\|v_{n+1} - v\| &\leq \lambda \|v_n - v\|^\alpha\, \|v_n - v_{n+1}\|^{1-\alpha-\beta}.
\end{aligned}
$$

Taking $n \to \infty$, we get

$$
\|v_{n+1} - v\| \to 0.
$$

Hence, we have proved that

$$
\lim_{n\to\infty} v_{n+1} = v = \lim_{n\to\infty} v_n.
$$

□

3.2. Existence of a solution for a system of nonlinear integral equations

Our initial objective of this section is to find the solution of an integral problem. Let $[a, b] \subset \mathbb{R}$ and define

$$\zeta = C[a, b],$$

Define $d : \zeta \times \zeta \to \mathbb{R}^+$ by

$$d(\xi, \eta) = \max_{t \in [a,b]} |\xi(t) - \eta(t)|^p, \quad p > 1.$$

Then, from Example 4, (ζ, d) is a complete strong extended $s-$suprametric space for any $s > 1$ and $\alpha(\xi, \eta) = \sum_{n=1}^{p-1} {}^pC_n + e^{\xi+\eta}, \forall\, \xi, \eta \in \zeta$.

We establish the existence of a solution for the following system of nonlinear Fredholm integral equation

$$\eta(t) = f(t) + \mu \int_a^b D(t, r, \eta(r))\, dr, \tag{3.8}$$

where $\eta \in C[a, b]$, $\mu \in \mathbb{R}$, $t, r \in [a, b]$, $D : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$, and $f : [a, b] \to \mathbb{R}$ are continuous functions.

Define a mapping $F : \zeta \to \zeta$ by

$$F(\eta)(t) = f(t) + \mu \int_a^b D(t, r, \eta(r))\, dr. \tag{3.9}$$

Additionally, consider the following assumptions:

(i) $\exists$ a continuous function $\theta : [a, b] \times [a, b] \to \mathbb{R}^+$ such that

$$\max_{t \in [a,b]} \int_a^b \theta(t, r)\, dr \leq 1.$$

(ii) $\exists\, K \in (0, 1)$ such that $\forall t, r \in [a, b]$, $\eta \in \mathbb{R}$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$,

$$|D(t, r, \eta_1(r)) - D(t, r, \eta_2(r))|^p \leq \frac{K}{(b-a)^{p-1}} \theta(t, r)\, \Delta(\eta_1, \eta_2),$$
$$\text{where} \Delta(\eta_1, \eta_2) = |\eta_1(r) - \eta_2(r)|^{p\alpha} |\eta_1(r) - F\eta_1(r)|^{p\beta} |\eta_2(r) - F\eta_2(r)|^{p(1-\alpha-\beta)}.$$

(iii) $|\mu| \leq 1$.

Theorem 5. *Under assumptions* (i)–(iii), *the mappings* $F : \zeta \to \zeta$ *defined by* (3.9) *have a fixed point in* ζ. *Consequently, System* (3.8) *admits a solution.*

Proof. Let $\eta_1, \eta_2 \in \zeta$ and let $q > 1$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Using Hölder's inequality and assumptions (i)–(iii), we obtain

$$d(F\eta_1, F\eta_2) = \max_{t \in [a,b]} |F\eta_1(t) - F\eta_2(t)|^p$$

$$
\begin{aligned}
&= |\mu|^p \max_{t\in[a,b]} \left| \int_a^b (D(t,r,\eta_1(r)) - D(t,r,\eta_2(r)))\, dr \right|^p \\
&\le \left[|\mu| \max_{t\in[a,b]} \left(\int_a^b 1^q dr \right)^{\frac{1}{q}} \left(\int_a^b \left| (D(t,r,\eta_1(r)) - D(t,r,\eta_2(r))) \right|^p dr \right)^{\frac{1}{p}} \right]^p \\
&\le (b-a)^{p-1} \max_{t\in[a,b]} \int_a^b |D(t,r,\eta_1(r)) - D(t,r,\eta_2(r))|^p\, dr \\
&\le (b-a)^{p-1} \max_{t\in[a,b]} \int_a^b \frac{K}{(b-a)^{p-1}} \gamma(t,r) \Delta(\eta_1,\eta_2)\, dr \\
&\le K(\max_{t\in[a,b]} |\eta_1(r) - \eta_2(r)|^p)^\alpha (\max_{t\in[a,b]} |\eta_1(r) - F\eta_1(r)|^p)^\beta \\
&\quad \times (\max_{t\in[a,b]} |\eta_2(r) - F\eta_2(r)|^p)^{(1-\alpha-\beta)} \\
d(F\eta_1, F\eta_2) &\le \lambda\, d(\eta_1,\eta_2)^\alpha d(\eta_1, F\eta_1)^\beta d(\eta_2, F\eta_2)^{1-\alpha-\beta},
\end{aligned}
$$

where

$$\lambda = K \in (0,1).$$

Therefore, all the hypotheses of Theorem 1 are satisfied, and thus F admits a fixed point. Consequently, System (3.8) has a solution. □

3.3. Application to dynamic programming

Let $D \subseteq X_1$ be the decision space and $S \subseteq X_2$ be the state space, where X_1 and X_2 are Banach spaces. Denote by $\zeta(S)$ the Banach space of all bounded real-valued functions on S, endowed with the strong extended s- suprametric d

$$d(\xi,\eta) = \sup_{t\in S} |\xi(t) - \eta(t)|^p \quad \forall \xi,\eta \in \zeta(S),$$

where $p > 1$. The norm on $\zeta(S)$ is

$$\|F\| = \sup\{|F(t)|^p : t \in S\}.$$

Then from Example 4, (ζ, d) is a complete strong extended s-suprametric space for any $s \ge 10^p$ and $\alpha(\xi,\eta) = \sum_{n=1}^{p-1} {}^pC_n + e^{\xi+\eta}, \forall\, \xi,\eta \in \zeta$.

Some of the most noticeable advantages of dynamic programming are its adaptive nature and its computational flexibility. By the use of the functional equation of dynamic programming, we can ignore whatever happened in the past and consider only the present and future functions based on the current state of the problem. Furthermore, because of the computational flexibility, the functional equation technique can be applied in different ways, in combination with the classical techniques and separately, with different objectives in mind.

According to Bellman et al. [4], the functional equation in dynamic programming has the general form

$$f(\xi) = \mathrm{opt}_{\eta\in D} H(\xi,\eta, f(T(\xi,\eta))) \quad \xi \in S,$$

where T represents the process transformation, $f(\xi)$ denotes the optimal return function with initial state ξ, and opt stands for either sup or inf .

We consider the following system of functional equations

$$f(\nu_s) = \mathrm{opt}_{\nu_d \in D} \left(\eta(\nu_s, \nu_d) + \xi(\nu_s, \nu_d, f(\rho(\nu_s, \nu_d)))\right), \tag{3.10}$$

where $\nu_s \in S$ is the state vector, $\nu_d \in D$ is the decision vector, ρ is process transformations, and f represents optimal return functions.

Define $F : \zeta(S) \to \zeta(S)$ by

$$Ff(\nu_s) = \mathrm{opt}_{\nu_d \in D} \left(\eta(\nu_s, \nu_d) + \xi(\nu_s, \nu_d, f(\rho(\nu_s, \nu_d)))\right). \tag{3.11}$$

(Da) There exist constants $0 < h < 1$ and $0 < \alpha < 1$ such that for all $(\nu_s, \nu_d, f_1, f_2) \in S \times D \times \zeta(S) \times \zeta(S)$,

$$|\xi(\nu_s, \nu_{d_1}, f_1(\rho_1(\nu_s, \nu_{d_1}))) - \xi(\nu_s, \nu_{d_2}, f_2(\rho_2(\nu_s, \nu_{d_2})))| + |\eta_1(\nu_s, \nu_{d_1}) - \eta_2(\nu_s, \nu_{d_2})| \leq [hM(f_1, f_2)]^{\frac{1}{p}}, \tag{3.12}$$

where

$$M(f_1, f_2) = |f_1 - Ff_1|^{p\alpha} |f_2 - Ff_2|^{p(1-\alpha)}.$$

(Db) ρ and ξ are bounded.

Theorem 6. *If $F : \zeta(S) \to \zeta(S)$ are defined by* (3.11) *and satisfy conditions (Da) and (Db), then the System* (3.10) *admits a solution in $\zeta(S)$.*

Proof. Let $\nu_s \in S$ and $f_1, f_2 \in \zeta(S)$. Since ρ and ξ are bounded, there exists $L \geq 0$ such that

$$\sup\{\|\rho(\nu_s, \nu_d)\|, \|\xi(\nu_s, \nu_d, t)\| : (\nu_s, \nu_d, t) \in S \times D \times \mathbb{R}\} \leq L. \tag{3.13}$$

Hence, F is a self-mapping on $\zeta(S)$.

Proceeding as in the previous section and letting $\varepsilon \to 0$, we obtain

$$|Ff_1(\nu_s) - Ff_2(\nu_s)| \leq |\xi(\nu_s, \nu_{d_1}, f_1(\rho_1(\nu_s, \nu_{d_1}))) - \xi(\nu_s, \nu_{d_2}, f_2(\rho_2(\nu_s, \nu_{d_2})))| + |\eta_1(\nu_s, \nu_{d_1}) - \eta_2(\nu_s, \nu_{d_2})|.$$

Using assumption (Da),

$$|Ff_1(\nu_s) - Ff_2(\nu_s)| \leq [hM(f_1, f_2)]^{\frac{1}{p}}.$$

Taking supremum over $\nu_s \in S$, we get

$$\sup_{\nu_s \in S} |Ff_1(\nu_s) - Ff_2(\nu_s)|^p \leq h \sup_{\nu_s \in S} |f_1 - Ff_1|^{p\alpha} |f_2 - Ff_2|^{p(1-\alpha)}.$$

Therefore,

$$d(Ff_1, Ff_2) \leq h[d(f_1, Ff_1)]^{\alpha} [d(f_2, Ff_2)]^{1-\alpha}.$$

Hence, all the hypotheses of Theorem 2 are satisfied. Consequently, F admits a fixed point in $\zeta(S)$, which is the solution of System (3.11). □

4. Discussion

The results obtained in this paper provide a fixed point framework for studying nonlinear problems in complete extended s- suprametrtic spaces through interpolative contractions. The main theoretical contribution is given by Theorems 1 and 2, where two interpolative contractions, namely by Reich–Rus–Ciric type and Kannan contraction, are introduced and used to establish the existence of fixed points. These results extend the classical Banach contraction principle in two directions. First, the underlying space is not required to be a usual metric space; instead, the arguments are carried out in an extended s-suprametric space, where the triangle inequality is replaced by two conditions with weaker inequality. Second, the contraction conditions are not expressed only in terms of $d(x, y)$, but also involve the interpolative terms with distances between points and their images under the mapping with fractional powers. This makes the contractive framework more flexible than the standard Banach-type condition, while retaining a constructive Picard iteration scheme.

Our results are related to earlier developments in generalized metric and fixed point theory. Banach's theorem [3] gives a fundamental existence and uniqueness principle under a strict Lipschitz contraction in a complete metric space. It states that every contraction, in the complete metric space, admits a unique fixed point. This principle has been generalized by many ways (see [29]). On the contrary, Karapinar [16] introduced interpolative contractions and presented a method to obtain fixed points of such contractions. Karapinar, in subsequent papers, investigated Reich–Rus–Ciric-type interpolative contractions, Kannan type interpolative contractions [15] and to ensure the existence of fixed points in variant (generalized) metric spaces, while Berzig [5] developed the basic structure of suprametric spaces and showed that this setting can accommodate mappings that are not naturally handled by ordinary metric spaces. The theorems proved here combine these two lines of development: The interpolative contractions are used in the extended s- suprametric distance. In comparison with the classical metric setting, this framework enables the contractive behavior to depend not only on the distance between two points, but also on the interaction between the points and their images.

The examples following Theorems 1 and 2 illustrate how the abstract assumptions can be verified in extended s- suprametric spaces. These examples are not intended to replace the general proofs of the theorems; rather, they show that the hypotheses are nonempty and can be checked directly for concrete self-maps, which could be continuous or have a jump discontinuity. In our first application, we establish the existence of a solution to a third-order boundary value problem (BVP). Though the contraction conditions in Definitions 8 and 9 are for all $x, y \in \zeta$ such that $Tx \neq Ty$, we cannot claim that the existing fixed point of T is unique. However, in Theorems 3 and 4, we establish that with some conditions the Picard iterative sequence $Tx_n = x_{n+1}$ is converging to a unique fixed point.

In other two applications, we establish the existence of the solution of a non linear integral equation and functional equation of a dynamic programming problem. By giving three applications, we attempt to justify the versatility of our findings for applications in nonlinear analysis.

5. Conclusions

In this study, we develop fixed point results within the setting of strong extended s-suprametric spaces by employing interpolative contractions. The obtained theorems establish sufficient conditions for the existence of fixed points for these classes of contractions in the proposed framework. The results extend and enrich the literature by demonstrating that strong extended s-suprametric spaces provide an appropriate and more general environment for investigating such problems. To support the theoretical findings, illustrative examples are included, emphasizing the applicability and robustness of the established results. Furthermore, the developed theory is applied to certain nonlinear problems, confirming the practical relevance and potential of the proposed approach for addressing nonlinear phenomena.

6. Future research

In future studies, researchers may focus on extending the obtained results to other generalized contraction mappings, including Hardy–Rogers, Chatterjea, and Meir–Keeler type contractions in the setting of strong extended s-suprametric spaces. Furthermore, the development of common fixed point results, multivalued mappings, and applications to nonlinear integral and differential equations may provide promising directions for further investigation.

Author contributions

Deepali Patel: conceptualization, formal analysis, investigation, visualization, writingoriginal draft, writing-review and editing; Om Prakash Chauhan: conceptualization, formal analysis, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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