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*Research article*

# Properties of H-Toeplitz operators on the bidisk Bergman space

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**Abstract:** This paper investigated H-Toeplitz operators on the bidisk Bergman space $L^2_a(\mathbb{D}^2)$. First, we provided the matrix-type representation for the H-Toeplitz operator $B_\phi$. Next, we presented the necessary and sufficient conditions for such operators to be co-isometries and partial isometries. Then, we proved that any diagonal H-Toeplitz operator with some pluriharmonic symbol must be the zero operator. Finally, we characterized several invariant subspaces and kernel spaces associated with H-Toeplitz operators.

**Keywords:** bidisk Bergman space; H-Toeplitz operators; partial isometry; invariant subspaces; kernel spaces
**Mathematics Subject Classification:** 47B35, 46E20

## 1. Introduction

Let $\mathbb{T}$ denote the boundary of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. The bidisk $\mathbb{D}^2$ and the torus $\mathbb{T}^2$ are subsets of $\mathbb{C}^2$, defined as the Cartesian products of two copies of $\mathbb{D}$ and $\mathbb{T}$, respectively. Hence, the bidisk $\mathbb{D}^2$ is given by

$$\mathbb{D}^2 = \{z = (z_1, z_2) : z_i \in \mathbb{C},\ |z_i| < 1,\ i = 1, 2\}.$$

Let $dV(z)$ be the normalized Lebesgue measure on $\mathbb{D}^2$, which is given by $dV(z) = dA(z_1)dA(z_2)$. Let $L^2(\mathbb{D}^2, dV)$ denote the space of all Lebesgue square-integrable functions on $\mathbb{D}^2$ with respect to the measure $dV$. It is endowed with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}^2} f(z)\overline{g(z)}\, dV(z) \quad \text{for all } f, g \in L^2(\mathbb{D}^2, dV).$$

The Bergman space over the bidisk $L^2_a(\mathbb{D}^2)$, which is the collection of all analytic functions on $\mathbb{D}^2$ contained in $L^2(\mathbb{D}^2, dV)$, is formally expressed as

$$L^2_a(\mathbb{D}^2) = \{f \in L^2(\mathbb{D}^2, dV) \mid f \text{ is analytic on } \mathbb{D}^2\}.$$

Thus, the Bergman space over the bidisk $L_a^2(\mathbb{D}^2)$ is a closed subspace of $L^2(\mathbb{D}^2, dV)$. For non-negative integers $n_1$ and $n_2$ , define

$$e_{n_1,n_2}(z) = \sqrt{n_1+1}\ \sqrt{n_2+1}\, z_1^{n_1} z_2^{n_2} \quad \text{for all } z = (z_1, z_2) \in \mathbb{D}^2.$$

Then $\{e_{n_1,n_2}\}_{n_1,n_2\geq 0}$ forms an orthonormal basis for $L_a^2(\mathbb{D}^2)$. Furthermore, $L_a^2(\mathbb{D}^2)$ is a reproducing kernel Hilbert space, whose reproducing kernel is expressed as

$$K_z(w) = \overline{K(z,w)} = \frac{1}{(1-\bar{z}_1 w_1)^2}\,\frac{1}{(1-\bar{z}_2 w_2)^2} \quad \text{for } z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{D}^2.$$

Accordingly, the normalized reproducing kernel $k_z(w)$ of $L_a^2(\mathbb{D}^2)$ takes the form

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{1-|z_1|^2}{(1-\bar{z}_1 w_1)^2}\,\frac{1-|z_2|^2}{(1-\bar{z}_2 w_2)^2}.$$

Let $P_{L_a^2} : L^2(\mathbb{D}^2, dV) \to L_a^2(\mathbb{D}^2)$ denote the Bergman orthogonal projection. Since $P_{L_a^2}(f) \in L_a^2(\mathbb{D}^2)$ for all $f \in L^2(\mathbb{D}^2, dV)$, by the reproducing kernel property, we have the following representation.

$$P_{L_a^2}(f)(z) = \langle P_{L_a^2} f, K_z\rangle = \langle f, K_z\rangle = \int_{\mathbb{D}^2} f(w)\overline{K_z(w)}\, dV(w) = \int_{\mathbb{D}^2} \frac{f(w)}{(1-\overline{w_1} z_1)^2(1-\overline{w_2} z_2)^2}\, dV(w).$$

In particular, for $f \in L_a^2(\mathbb{D}^2, dV)$, we have

$$f(z) = \int_{\mathbb{D}^2} \frac{f(w)}{(1-\overline{w_1} z_1)^2(1-\overline{w_2} z_2)^2}\, dV(w), \quad z = (z_1, z_2) \in \mathbb{D}^2.$$

$L^\infty(\mathbb{D}^2)$ denotes the Banach space consisting of all essentially bounded Lebesgue measurable functions $f$ on $\mathbb{D}^2$. Its norm is defined by

$$\|f\|_\infty = \operatorname{ess\,sup}\left\{|f(z)| : z = (z_1, z_2) \in \mathbb{D}^2\right\}.$$

Let $L^2_{\text{harm}}(\mathbb{D}^2)$ consist of all harmonic functions in $L^2(\mathbb{D}^2, dV)$. It can be expressed as

$$L^2_{\text{harm}}(\mathbb{D}^2) = \left\{ f \in L^2(\mathbb{D}^2, dV) \;\middle|\; \frac{\partial^2 f}{\partial z_1\, \partial \bar{z}_1} + \frac{\partial^2 f}{\partial z_2\, \partial \bar{z}_2} = 0 \right\},$$

with $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ being variables on the complex plane $\mathbb{C}$, and $f$ is a complex-valued function on $\mathbb{D}^2$. Let

$$P_{\text{harm}} \colon L^2(\mathbb{D}^2, dV) \to L^2_{\text{harm}}(\mathbb{D}^2)$$

be the orthogonal projection from the space $L^2(\mathbb{D}^2, dV)$ to the space $L^2_{\text{harm}}(\mathbb{D}^2)$. For $\phi \in L^\infty(\mathbb{D}^2)$, the multiplication operator $M_\phi \colon L^2(\mathbb{D}^2, dV) \to L^2(\mathbb{D}^2, dV)$ is defined by

$$M_\phi(f) = \phi f \quad \text{for all } f \in L^2(\mathbb{D}^2, dV).$$

$M_\phi$ is a bounded linear operator on $L^2(\mathbb{D}, dV)$ and $\|M_\phi\| = \|\phi\|_\infty$. Motivated by the research in [1], we first give the definitions of Toeplitz operators and Hankel-type operators on the bidisk.

**Definition 1.1.** *For $\phi \in L^\infty(\mathbb{D}^2)$, we define the Toeplitz operator $T_\phi\colon L_a^2(\mathbb{D}^2) \to L_a^2(\mathbb{D}^2)$ by*

$$T_\phi(f) = P_{L_a^2}(\phi f) \quad \text{for all } f \in L_a^2(\mathbb{D}^2).$$

**Proposition 1.2.** *For $\phi \in L^\infty(\mathbb{D}^2)$, $T_\phi$ is a bounded linear operator on $L_a^2(\mathbb{D}^2)$. In fact, we have $\|T_\phi\| \le \|\phi\|_\infty$.*

**Definition 1.3.** *For $\phi \in L^\infty(\mathbb{D}^2)$, the Hankel-type operators $H_\phi^{(i)}\colon L_a^2(\mathbb{D}^2) \to L_a^2(\mathbb{D}^2)$ satisfy*

$$H_\phi^{(i)}(f) = P_{L_a^2} M_\phi J^{(i)}(f), \quad \forall f \in L_a^2(\mathbb{D}^2),\ i \in \{1, 2, 3\},$$

*where the operator $J^{(i)}\colon L_a^2(\mathbb{D}^2) \to L^2(\mathbb{D}^2)$ is given by*

$$\begin{aligned} J^{(1)}(e_{n_1,n_2}) &= J(e_{n_1}) \cdot J(e_{n_2}), \\ J^{(2)}(e_{n_1,n_2}) &= J(e_{n_1}) \cdot e_{n_2}, \\ J^{(3)}(e_{n_1,n_2}) &= e_{n_1} \cdot J(e_{n_2}) \end{aligned}$$

*for all non-negative integers $n_1$, $n_2$. The operator $J\colon L_a^2(\mathbb{D}) \to \overline{L_a^2(\mathbb{D})}$ is given by $J(e_n) = \overline{e_{n+1}}$ for all non-negative integers $n$.*

**Proposition 1.4.** *For $\phi \in L^\infty(\mathbb{D}^2)$, $H_\phi^{(i)}$ is a bounded linear operator on $L_a^2(\mathbb{D}^2)$. In fact, we have*

$$\|H_\phi^{(i)}\| \le \|\phi\|_\infty \cdot \|J^{(i)}\|$$

*and $\|J^{(i)}\| = 1$ for each $i = 1, 2, 3$.*

In the fields of operator theory and functional analysis, Toeplitz and Hankel operators occupy a vital position as key research subjects. Motivated by the concepts of these operators, Arora and Paliwal [2] initially put forward and explored the concept of H-Toeplitz operators on the Hardy space. They characterized several important properties of H-Toeplitz operators, including partial isometry, co-isometry, the Hilbert-Schmidt property, and hyponormality. Subsequently, Gupta and Singh [1] extended the notion of H-Toeplitz operators to the Bergman space on the unit disk and systematically investigated fundamental issues such as matrix representations and invariant subspaces of such operators. Moreover, Kim and Lee [3, 4] studied the contractivity and expansivity of such operators. Two years later, they extended their fundamental properties to the weighted Bergman space. In recent years, scholars have also studied H-Toeplitz operators in the Dirichlet space [5] and the Fock space [6].

In addition, Liang et al. [7] examined commutative properties of H-Toeplitz operators whose symbols are quasi-homogeneous, while Ding and Chen [8] studied the product problem of H-Toeplitz operators and Toeplitz operators. Furthermore, Lu [9] first examined the commutativity of Toeplitz operators on the Bergman space of the bidisk. Thereafter, Lu and Zhang [10] explored the commutativity of Hankel operators and Toeplitz operators on the Hardy space over the bidisk, while Ding and Sun [11] inspected commuting Toeplitz operators on the same space. Meanwhile, Chen, Izuchi, and Lee [12] studied the kernel structure of Toeplitz operators. Using Berezin symbols and the Duhamel product, Garayev [13] investigated the invariant subspace problem for bounded linear operators on the unit disk. Subsequently, Li, Zhao, and Ding [14] investigated problems associated with invariant subspaces of Toeplitz operators on the Bergman space over the bidisk.

Motivated by the aforementioned studies, in this paper, we mainly investigate the properties of H-Toeplitz operators on $L^2_a(\mathbb{D}^2)$. In Section 2, we first give the matrix forms of Toeplitz operators and Hankel-type operators on $L^2_a(\mathbb{D}^2)$. Afterward, we introduce the definition of H-Toeplitz operators on $\mathbb{D}^2$ and further conduct a systematic study on the fundamental properties of such operators (e.g., co-isometry, partial isometry, diagonality). In Section 3, we study several invariant subspaces and kernel spaces associated with H-Toeplitz operators.

## 2. H-Toeplitz operators on the bidisk Bergman space $L^2_a(\mathbb{D}^2)$

For this section, we first present the following lemmas, which will be used in the derivation of subsequent conclusions.

**Lemma 2.1.** *We have $m_1, m_2, n_1, n_2 \in \mathbb{N} \cup \{0\}$. In the harmonic Bergman space on the bidisk $L^2_{harm}(\mathbb{D}^2)$, the following identities hold:*

$$P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}) = \begin{cases} \dfrac{(n_1-m_1+1)(n_2-m_2+1)}{(n_1+1)(n_2+1)}\, z_1^{n_1-m_1}z_2^{n_2-m_2}, & \text{if } n_1 \geq m_1 \text{ and } n_2 \geq m_2, \\ \dfrac{(m_1-n_1+1)(m_2-n_2+1)}{(m_1+1)(m_2+1)}\, \overline{z}_1^{m_1-n_1}\overline{z}_2^{m_2-n_2}, & \text{if } n_1 < m_1 \text{ and } n_2 < m_2, \\ \dfrac{(n_1-m_1+1)(m_2-n_2+1)}{(n_1+1)(m_2+1)}\, z_1^{n_1-m_1}\overline{z}_2^{m_2-n_2}, & \text{if } n_1 \geq m_1 \text{ and } n_2 < m_2, \\ \dfrac{(m_1-n_1+1)(n_2-m_2+1)}{(m_1+1)(n_2+1)}\, \overline{z}_1^{m_1-n_1}z_2^{n_2-m_2}, & \text{if } n_1 < m_1 \text{ and } n_2 \geq m_2. \end{cases}$$

*Proof.*
**Case (1):** When $n_1 \geqslant m_1$ and $n_2 \geqslant m_2$, we compute

$$\begin{aligned}
&\left\langle P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}),\, z_1^{k_1}z_2^{k_2}\right\rangle \\
&= \left\langle \overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2},\, z_1^{k_1}z_2^{k_2}\right\rangle \\
&= \int_{\mathbb{D}^2} \overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}\overline{z}_1^{k_1}\overline{z}_2^{k_2}\, dV(z_1,z_2) \\
&= \left(\int_{\mathbb{D}} \overline{z}_1^{m_1+k_1}z_1^{n_1}\, dA(z_1)\right)\cdot\left(\int_{\mathbb{D}} \overline{z}_2^{m_2+k_2}z_2^{n_2}\, dA(z_2)\right) \\
&= \begin{cases} \dfrac{1}{(n_1+1)(n_2+1)}, & \text{if } k_1 = n_1-m_1 \text{ and } k_2 = n_2-m_2, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{(k_1+1)(k_2+1)}{(n_1+1)(n_2+1)}\left\langle z_1^{k_1}z_2^{k_2},\, z_1^{k_1}z_2^{k_2}\right\rangle, & \text{if } k_1 = n_1-m_1 \text{ and } k_2 = n_2-m_2, \\ 0, & \text{otherwise,} \end{cases} \\
&= \left\langle \frac{(n_1-m_1+1)(n_2-m_2+1)}{(n_1+1)(n_2+1)} z_1^{n_1-m_1}z_2^{n_2-m_2},\, z_1^{k_1}z_2^{k_2}\right\rangle.
\end{aligned}$$

Furthermore, the inner product of $P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2})$ with $z_1^{k_1}z_2^{k_2}$, $\overline{z}_1^{k_1}z_2^{k_2}$, and $z_1^{k_1}\overline{z}_2^{k_2}$ equals zero. Thus, we have

$$P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}) = \frac{(n_1-m_1+1)(n_2-m_2+1)}{(n_1+1)(n_2+1)}\, z_1^{n_1-m_1}z_2^{n_2-m_2}.$$

**Case (2):** When $n_1 < m_1$ and $n_2 < m_2$, repeating the procedure in Case (1) gives us

$$\left\langle P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}),\, \overline{z}_1^{k_1}\overline{z}_2^{k_2}\right\rangle = \left\langle \frac{(m_1-n_1+1)(m_2-n_2+1)}{(m_1+1)(m_2+1)}\, \overline{z}_1^{m_1-n_1}\overline{z}_2^{m_2-n_2},\, \overline{z}_1^{k_1}\overline{z}_2^{k_2}\right\rangle.$$

Therefore, we can obtain that

$$P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}) = \frac{(m_1-n_1+1)(m_2-n_2+1)}{(m_1+1)(m_2+1)}\, \overline{z}_1^{m_1-n_1}\overline{z}_2^{m_2-n_2}.$$

**Case (3):** When $n_1 \geq m_1$ and $n_2 < m_2$, by the same method, we can obtain

$$P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}) = \frac{(n_1-m_1+1)(m_2-n_2+1)}{(n_1+1)(m_2+1)}\, z_1^{n_1-m_1}\overline{z}_2^{m_2-n_2}.$$

**Case (4):** When $m_1 > n_1$ and $m_2 \leq n_2$, we can obtain

$$P_{\text{harm}}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}) = \frac{(m_1-n_1+1)(n_2-m_2+1)}{(m_1+1)(n_2+1)}\, \overline{z}_1^{m_1-n_1}z_2^{n_2-m_2}.$$

□

For the analytic case, we obtain the lemma below.

**Lemma 2.2.** *We have $m_1, m_2, n_1, n_2 \in \mathbb{N}\cup\{0\}$. In the Bergman space on the bidisk $L_a^2(\mathbb{D}^2)$, the following identities hold:*

(a)

$$\left\langle z_1^{n_1}z_2^{n_2},\, z_1^{m_1}z_2^{m_2}\right\rangle = \begin{cases} \dfrac{1}{(n_1+1)(n_2+1)}, & \text{if } m_1 = n_1 \text{ and } m_2 = n_2,\\ 0, & \text{otherwise.}\end{cases}$$

(b)

$$P_{L_a^2}(\overline{z}_1^{m_1}\overline{z}_2^{m_2}z_1^{n_1}z_2^{n_2}) = \begin{cases} \dfrac{(n_1-m_1+1)(n_2-m_2+1)}{(n_1+1)(n_2+1)}\, z_1^{n_1-m_1}z_2^{n_2-m_2}, & \text{if } n_1 \geq m_1 \text{ and } n_2 \geq m_2,\\ 0, & \text{otherwise.}\end{cases}$$

Let $L^2_{\text{plurih}}(\mathbb{D}^2)$ denote the space of pluriharmonic functions in $L^2(\mathbb{D}^2, dV)$. Equivalently, it can be characterized as

$$L^2_{\text{plurih}}(\mathbb{D}^2) := \left\{ f \in L^2(\mathbb{D}^2, dV) \,\middle|\, f \in C^2(\mathbb{D}^2),\ f = h_1 + \overline{h_2},\ \text{and } h_2(0, z_2) \equiv 0,\ h_2(z_1, 0) \equiv 0 \right\}.$$

Moreover, any function $f$ in $L^2_{\text{plurih}}(\mathbb{D}^2)$ satisfies

$$\frac{\partial^2 f}{\partial z_j\, \partial \overline{z}_k} = 0$$

for all $j, k \in \{1, 2\}$. Next, we determine the matrix representations of the Toeplitz operator $T_\phi$ and Hankel-type operator $H_\phi$ on $L_a^2(\mathbb{D}^2)$ with some pluriharmonic symbol.

Let

$$\phi(z_1, z_2) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} b_{j_1,j_2} \overline{z}_1^{j_1} \overline{z}_2^{j_2} \in L^2_{\text{plurih}}(\mathbb{D}^2).$$

Subsequently, the $(m_1, m_2) - (n_1, n_2)$ entry of the matrix form of $T_\phi$ with respect to orthonormal basis $\{e_{n_1,n_2}\}_{n_1,n_2 \geq 0}$ of $L_a^2(\mathbb{D}^2)$ is given by

$$\begin{aligned}
\langle T_\phi e_{n_1,n_2},\, e_{m_1,m_2}\rangle &= \langle P_{L_a^2}(\phi \cdot e_{n_1,n_2}),\, e_{m_1,m_2}\rangle = \langle \phi \cdot e_{n_1,n_2},\, e_{m_1,m_2}\rangle \\
&= c_{n_1,n_2,m_1,m_2}\left\langle \Big(\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} b_{j_1,j_2} \overline{z}_1^{j_1}\overline{z}_2^{j_2}\Big) z_1^{n_1} z_2^{n_2},\, z_1^{m_1} z_2^{m_2}\right\rangle \\
&= c_{n_1,n_2,m_1,m_2}\left(\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a_{i_1,i_2}\langle z_1^{i_1+n_1} z_2^{i_2+n_2},\, z_1^{m_1} z_2^{m_2}\rangle + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} b_{j_1,j_2}\langle z_1^{n_1} z_2^{n_2},\, z_1^{m_1+j_1} z_2^{m_2+j_2}\rangle\right),
\end{aligned}$$

where

$$c_{n_1,n_2,m_1,m_2} = \sqrt{n_1+1}\,\sqrt{n_2+1}\,\sqrt{m_1+1}\,\sqrt{m_2+1}.$$

Therefore, we consider three cases separately:
**Case (1):** If $m_1 \geq n_1$ and $m_2 \geq n_2$, then we have

$$\begin{aligned}
\langle T_\phi e_{n_1,n_2},\, e_{m_1,m_2}\rangle &= c_{n_1,n_2,m_1,m_2}\Big(\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a_{i_1,i_2}\langle z_1^{i_1+n_1} z_2^{i_2+n_2},\, z_1^{m_1} z_2^{m_2}\rangle\Big) \\
&= c_{n_1,n_2,m_1,m_2} \cdot \frac{1}{(m_1+1)(m_2+1)}\, a_{m_1-n_1,m_2-n_2} \\
&= \frac{\sqrt{n_1+1}\,\sqrt{n_2+1}}{\sqrt{m_1+1}\,\sqrt{m_2+1}}\, a_{m_1-n_1,m_2-n_2}.
\end{aligned}$$

**Case (2):** If $m_1 < n_1$ and $m_2 < n_2$, then we also obtain

$$\langle T_\phi e_{n_1,n_2},\, e_{m_1,m_2}\rangle = \frac{\sqrt{m_1+1}\,\sqrt{m_2+1}}{\sqrt{n_1+1}\,\sqrt{n_2+1}}\, b_{n_1-m_1,n_2-m_2}.$$

**Case (3):** If $m_1 \geq n_1$ and $m_2 < n_2$, $m_1 < n_1$ and $m_2 \geq n_2$. Similarly, we can directly calculate

$$\langle T_\phi e_{n_1,n_2},\, e_{m_1,m_2}\rangle = 0.$$

Therefore, since $m_1$, $m_2$, $n_1$, and $n_2$ are non-negative intergers, we have

$$\langle T_\phi e_{n_1,n_2},\, e_{m_1,m_2}\rangle = \begin{cases} \dfrac{\sqrt{n_1+1}\,\sqrt{n_2+1}}{\sqrt{m_1+1}\,\sqrt{m_2+1}}\, a_{m_1-n_1,m_2-n_2}, & \text{for } m_1 \geq n_1 \text{ and } m_2 \geq n_2, \\ \dfrac{\sqrt{m_1+1}\,\sqrt{m_2+1}}{\sqrt{n_1+1}\,\sqrt{n_2+1}}\, b_{n_1-m_1,n_2-m_2}, & \text{for } m_1 < n_1 \text{ and } m_2 < n_2, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we partition the above inner product with two sets of indices: $m_1, n_1$ and $m_2, n_2$. We take $m_1, n_1$ as block indices and $m_2, n_2$ as intra-block indices, which correspond to the rows and columns of each two-dimensional subblock. Accordingly, we have

$$\mathrm{T}_\phi = \begin{bmatrix} T_{(0,0)} & T_{(0,1)} & T_{(0,2)} & \cdots \\ T_{(1,0)} & T_{(1,1)} & T_{(1,2)} & \cdots \\ T_{(2,0)} & T_{(2,1)} & T_{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where each subblock $T_{(m_1,n_1)}$ is a two-dimensional matrix indexed by $m_2$ and $n_2$. We then classify these subblocks into three types. When $m_1 = n_1$, the general formula of the diagonal blocks is given by:

$$[T_{m_1,n_1}]_{m_2,n_2} = \begin{cases} \dfrac{\sqrt{n_2+1}}{\sqrt{m_2+1}} a_{0,m_2-n_2}, & \text{for } m_2 \geq n_2, \\ 0, & \text{for } m_2 < n_2. \end{cases}$$

It follows that the diagonal elements are determined solely by the analytical coefficients of the symbols and are always lower-triangular matrices. Its matrix form can be expressed as:

$$T_{(m_1,n_1)} = \begin{bmatrix} a_{0,0} & 0 & 0 & \cdots \\ \sqrt{\dfrac{1}{2}} a_{0,1} & a_{0,0} & 0 & \cdots \\ \sqrt{\dfrac{1}{3}} a_{0,2} & \sqrt{\dfrac{2}{3}} a_{0,1} & a_{0,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

When $m_1 > n_1$, the general formula of the lower-triangular blocks can be expressed as:

$$[T_{m_1,n_1}]_{m_2,n_2} = \begin{cases} \dfrac{\sqrt{n_1+1}\,\sqrt{n_2+1}}{\sqrt{m_1+1}\,\sqrt{m_2+1}} a_{m_1-n_1,m_2-n_2}, & \text{for } m_2 \geq n_2, \\ 0, & \text{for } m_2 < n_2. \end{cases}$$

It can be seen that these subblocks are also generated by analytical coefficients and retain a lower-triangular matrix structure. Its matrix form can be expressed as:

$$T_{(m_1,n_1)} = \begin{bmatrix} \sqrt{\dfrac{n_1+1}{m_1+1}} a_{n_1-m_1,0} & 0 & 0 & \cdots \\ \dfrac{1}{\sqrt{2}} \sqrt{\dfrac{n_1+1}{m_1+1}} a_{n_1-m_1,1} & \sqrt{\dfrac{n_1+1}{m_1+1}} a_{n_1-m_1,0} & 0 & \cdots \\ \dfrac{1}{\sqrt{3}} \sqrt{\dfrac{n_1+1}{m_1+1}} a_{n_1-m_1,2} & \dfrac{\sqrt{2}}{\sqrt{3}} \sqrt{\dfrac{n_1+1}{m_1+1}} a_{n_1-m_1,1} & \sqrt{\dfrac{n_1+1}{m_1+1}} a_{n_1-m_1,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

When $m_1 < n_1$, the general formula of the upper-triangular blocks can be expressed as:

$$[T_{m_1,n_1}]_{m_2,n_2} = \begin{cases} \dfrac{\sqrt{m_1+1}\ \sqrt{m_2+1}}{\sqrt{n_1+1}\ \sqrt{n_2+1}}\, b_{n_1-m_1,n_2-m_2}, & \text{for } m_2 < n_2, \\ 0, & \text{for } m_2 \geq n_2. \end{cases}$$

Unlike the previous two types of subblocks, this type of subblock is uniquely determined by the anti-analytic coefficients and is a strictly upper-triangular matrix. Its matrix form can be expressed as:

$$\mathrm{T}_{(m_1,n_1)} = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{2}}\sqrt{\dfrac{m_1+1}{n_1+1}}\, b_{n_1-m_1,1} & \dfrac{1}{\sqrt{3}}\sqrt{\dfrac{m_1+1}{n_1+1}}\, b_{n_1-m_1,2} & \cdots \\ 0 & 0 & \dfrac{\sqrt{2}}{\sqrt{3}}\sqrt{\dfrac{m_1+1}{n_1+1}}\, b_{n_1-m_1,1} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Meanwhile, the adjoint matrix of $T_\phi$ is expressed as

$$T_\phi^* = \begin{bmatrix} T_{(0,0)}^* & T_{(1,0)}^* & T_{(2,0)}^* & \cdots \\ T_{(0,1)}^* & T_{(1,1)}^* & T_{(2,1)}^* & \cdots \\ T_{(0,2)}^* & T_{(1,2)}^* & T_{(2,2)}^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This corresponds exactly to the matrix of $T_{\overline{\phi}}$, hence $T_\phi^* = T_{\overline{\phi}}$.

**Example 2.3.** *Take the symbol function $\phi(z_1,z_2) = z_1 + \overline{z}_2$, whose coefficients satisfy $a_{1,0} = 1$ and $b_{0,1} = 1$, and all other coefficients are zero. Since the structure of the bidisk is relatively complex, to better reveal the properties of the matrix blocks, we perform a finite truncation with indices $m_1, n_1, m_2, n_2 \in \{0,1\}$ and obtain the truncated matrix. The operator $T_\phi$ is expressed in block form as*

$$T_\phi = \begin{bmatrix} T_{(0,0)} & T_{(0,1)} \\ T_{(1,0)} & T_{(1,1)} \end{bmatrix}.$$

*Through calculation, we find that the subblocks $T_{(0,0)}$, $T_{(0,1)}$, and $T_{(1,1)}$ are all zero matrices. Substituting $m_1 = 1$, $n_1 = 0$, and $a_{1,0} = 1$ into the formula, we obtain*

$$T_{(1,0)} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix}.$$

*Combining all subblocks, the complete truncated matrix is given by*

$$T_\phi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}.$$

Next, we derive the matrix forms of the three types of Hankel-type operators having some pluriharmonic symbol. We have

$$\phi(z_1, z_2) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \overline{z}_1^{j_1} \overline{z}_2^{j_2} \in L^2_{\text{plurih}}(\mathbb{D}^2).$$

Then, the $(m_1, m_2) - (n_1, n_2)$ entry of the matrix form of $H_\phi^{(1)}$ with respect to the orthonormal basis is given by:

$$\begin{aligned}
&\langle H_\phi^{(1)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle = \langle P_{L_a^2} M_\phi J^{(1)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle = \langle M_\phi J^{(1)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle \\
&= \left\langle \phi \cdot \sqrt{n_1+2}\,\sqrt{n_2+2}\,\overline{z}_1^{n_1+1} \overline{z}_2^{n_2+1},\ \sqrt{m_1+1}\,\sqrt{m_2+1}\, z_1^{m_1} z_2^{m_2} \right\rangle \\
&= d_{n_1,n_2,m_1,m_2} \left\langle \Big( \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \overline{z}_1^{j_1} \overline{z}_2^{j_2} \Big) \overline{z}_1^{n_1+1} \overline{z}_2^{n_2+1},\ z_1^{m_1} z_2^{m_2} \right\rangle \\
&= d_{n_1,n_2,m_1,m_2} \Big( \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} \langle z_1^{i_1} z_2^{i_2},\, z_1^{m_1+n_1+1} z_2^{m_2+n_2+1} \rangle + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \langle \overline{z}_1^{n_1+j_1+1} \overline{z}_2^{n_2+j_2+1},\, z_1^{m_1} z_2^{m_2} \rangle \Big) \\
&= \frac{d_{n_1,n_2,m_1,m_2}}{(m_1+n_1+2)(m_2+n_2+2)}\, a_{n_1+m_1+1,\, n_2+m_2+1},
\end{aligned}$$

where

$$d_{n_1,n_2,m_1,m_2} = \sqrt{n_1+2}\ \sqrt{n_2+2}\ \sqrt{m_1+1}\ \sqrt{m_2+1}.$$

Subsequently, using the same method, we compute the entries of the matrix forms of $H_\phi^{(2)}$ and $H_\phi^{(3)}$ with respect to the orthonormal basis.

$$\begin{aligned}
\langle H_\phi^{(2)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle &= \frac{l_{n_1,n_2,m_1,m_2}}{(m_1+n_1+2)(m_2+1)}\, a_{n_1+m_1+1,\, m_2-n_2}, \\
\langle H_\phi^{(3)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle &= \frac{r_{n_1,n_2,m_1,m_2}}{(m_1+1)(m_2+n_2+2)}\, a_{m_1-n_1,\, m_2+n_2+1},
\end{aligned}$$

where

$$\begin{aligned}
l_{n_1,n_2,m_1,m_2} &= \sqrt{m_1+1}\ \sqrt{m_2+1}\ \sqrt{n_1+2}\ \sqrt{n_2+1}, \\
r_{n_1,n_2,m_1,m_2} &= \sqrt{m_1+1}\ \sqrt{m_2+1}\ \sqrt{n_1+1}\ \sqrt{n_2+2}.
\end{aligned}$$

Similarly, the matrix form of $H_\phi^{(i)}(i = 1, 2, 3)$ on the bidisk is a four-dimensional infinite block matrix, where each block corresponds to a two-dimensional infinite matrix. So, we have

$$H_\phi^{(i)} = \begin{bmatrix} H_{(0,0)}^{(i)} & H_{(0,1)}^{(i)} & H_{(0,2)}^{(i)} & \cdots \\ H_{(1,0)}^{(i)} & H_{(1,1)}^{(i)} & H_{(1,2)}^{(i)} & \cdots \\ H_{(2,0)}^{(i)} & H_{(2,1)}^{(i)} & H_{(2,2)}^{(i)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (i = 1, 2, 3),$$

and the general formulas for $H_\phi^{(1)}$, $H_\phi^{(2)}$, and $H_\phi^{(3)}$ are given as follows:

$$\begin{aligned} [H_\phi^{(1)}]_{(m_1,m_2),(n_1,n_2)} &= \frac{d_{n_1,n_2,m_1,m_2}}{(m_1 + n_1 + 2)(m_2 + n_2 + 2)} \, a_{n_1+m_1+1,\, n_2+m_2+1}, \\ [H_\phi^{(2)}]_{(m_1,m_2),(n_1,n_2)} &= \frac{l_{n_1,n_2,m_1,m_2}}{(m_1 + n_1 + 2)(m_2 + 1)} \, a_{n_1+m_1+1,\, m_2-n_2}, \\ [H_\phi^{(3)}]_{(m_1,m_2),(n_1,n_2)} &= \frac{r_{n_1,n_2,m_1,m_2}}{(m_1 + 1)(m_2 + n_2 + 2)} \, a_{m_1-n_1,\, n_2+m_2+1}. \end{aligned}$$

From this, it can be seen that $[H_{(m_1,n_1)}^{(1)}]_{m_2,n_2}$ is a dense matrix. Its matrix form can be expressed as:

$$H_{(m_1,n_1)}^{(1)} = s \begin{bmatrix} \frac{\sqrt{2}}{2} a_{n_1+m_1+1,1} & \frac{\sqrt{3}}{3} a_{n_1+m_1+1,2} & \cdots & \\ \frac{2}{3} a_{n_1+m_1+1,2} & \frac{\sqrt{6}}{4} a_{n_1+m_1+1,3} & \cdots & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$s = \frac{\sqrt{m_1 + 1}\,\sqrt{n_1 + 2}}{m_1 + n_1 + 2}.$$

$[H_{(m_1,n_1)}^{(2)}]_{m_2,n_2}$ is a lower-triangular matrix. Its matrix form can be expressed as:

$$H_{(m_1,n_1)}^{(2)} = s \begin{bmatrix} a_{n_1+m_1+1,\,0} & 0 & 0 & \cdots \\ \frac{1}{\sqrt{2}} a_{n_1+m_1+1,\,1} & a_{n_1+m_1+1,\,0} & 0 & \cdots \\ \frac{1}{\sqrt{3}} a_{n_1+m_1+1,\,2} & \frac{\sqrt{2}}{\sqrt{3}} a_{n_1+m_1+1,\,1} & a_{n_1+m_1+1,\,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

$[H_{(m_1,n_1)}^{(3)}]_{m_2,n_2}$ is a dense matrix. Its matrix form can be expressed as:

$$H_{(m_1,n_1)}^{(3)} = t \begin{bmatrix} \frac{\sqrt{2}}{2} a_{m_1-n_1,\,1} & \frac{\sqrt{3}}{3} a_{m_1-n_1,\,2} & \cdots & \\ \frac{2}{3} a_{m_1-n_1,\,2} & \frac{\sqrt{6}}{4} a_{m_1-n_1,\,3} & \cdots & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$t = \frac{\sqrt{n_1 + 1}}{\sqrt{m_1 + 1}}.$$

**Example 2.4.** *Take the symbol function $\phi(z_1, z_2) = z_1^2 z_2^2$, whose coefficients satisfy $a_{2,2} = 1$ and $a_{i_1,i_2} = 0$ otherwise. From the non-zero condition, we have $n_1 + m_1 = 1$ and $n_2 + m_2 = 1$. Therefore, only the subblocks corresponding to $(m_1, n_1) = (0, 1)$ and $(1, 0)$ are non-zero, and all other outer subblocks are zero matrices. For the subblock $H^{(1)}_{(0,1)}$, its explicit form is computed as*

$$H^{(1)}_{(0,1)} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{2\sqrt{3}}{9} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

*Similarly, the subblock $H^{(1)}_{(1,0)}$ is given by*

$$H^{(1)}_{(1,0)} = \begin{bmatrix} 0 & \frac{2\sqrt{3}}{9} & 0 & 0 & \cdots \\ \frac{4}{9} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

*Consequently, the overall block matrix can be written as*

$$H^{(1)}_{\phi} = \begin{bmatrix} 0 & H^{(1)}_{(0,1)} & 0 & \cdots \\ H^{(1)}_{(1,0)} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

With a view to defining the notion of H-Toeplitz operators on $L^2_a(\mathbb{D}^2)$, we consider the operator $K\colon L^2_a(\mathbb{D}^2) \to L^2_{harm}(\mathbb{D}^2)$ satisfying the tensor product property

$$K(e_{m,n}(z_1, z_2)) = K(e_m(z_1)) \cdot K(e_n(z_2)),$$

which is defined by

$$\begin{aligned} K(e_{2m,2n}(z_1, z_2)) &= e_{m,n}(z_1, z_2) = \sqrt{m+1}\,\sqrt{n+1}\, z_1^m z_2^n, \\ K(e_{2m+1,2n}(z_1, z_2)) &= \overline{e_{m+1}(z_1)} \cdot e_n(z_2) = \sqrt{m+2}\,\sqrt{n+1}\, \overline{z}_1^{m+1} z_2^n, \\ K(e_{2m,2n+1}(z_1, z_2)) &= e_m(z_1) \cdot \overline{e_{n+1}(z_2)} = \sqrt{m+1}\,\sqrt{n+2}\, z_1^m \overline{z}_2^{n+1}, \\ K(e_{2m+1,2n+1}(z_1, z_2)) &= \overline{e_{m+1}(z_1)} \cdot \overline{e_{n+1}(z_2)} = \sqrt{m+2}\,\sqrt{n+2}\, \overline{z}_1^{m+1} \overline{z}_2^{n+1} \end{aligned}$$

for all $m, n \geq 0$ and $z_1, z_2 \in \mathbb{D}^2$. It is easy to check that the operator $K$ is bounded linear on $L_a^2(\mathbb{D}^2)$ with $\|K\| = 1$. Furthermore, we show the adjoint operator $K^*$ of operator $K$:

$$\begin{aligned}
K^*(e_{m,n}(z_1, z_2)) &= e_{2m,2n}(z_1, z_2) = \sqrt{2m+1}\,\sqrt{2n+1}\, z_1^{2m} z_2^{2n},\\
K^*(\overline{e_{m+1}(z_1)} \cdot e_n(z_2)) &= e_{2m+1,2n}(z_1, z_2) = \sqrt{2m+2}\,\sqrt{2n+1}\, z_1^{2m+1} z_2^{2n},\\
K^*(e_m(z_1) \cdot \overline{e_{n+1}(z_2)}) &= e_{2m,2n+1}(z_1, z_2) = \sqrt{2m+1}\,\sqrt{2n+2}\, z_1^{2m} z_2^{2n+1},\\
K^*(\overline{e_{m+1}(z_1)} \cdot \overline{e_{n+1}(z_2)}) &= e_{2m+1,2n+1}(z_1, z_2) = \sqrt{2m+2}\,\sqrt{2n+2}\, z_1^{2m+1} z_2^{2n+1}.
\end{aligned}$$

Thus, based on the definitions of $K$ and $K^*$, we have

$$\begin{aligned}
KK^*(e_{m,n}(z_1, z_2)) &= K(e_{2m,2n}(z_1, z_2)) = e_{m,n}(z_1, z_2),\\
KK^*(\overline{e_{m+1}(z_1)} \cdot e_n(z_2)) &= K(e_{2m+1,2n}(z_1, z_2)) = \overline{e_{m+1}(z_1)} \cdot e_n(z_2),\\
KK^*(e_m(z_1) \cdot \overline{e_{n+1}(z_2)}) &= K(e_{2m,2n+1}(z_1, z_2)) = e_m(z_1) \cdot \overline{e_{n+1}(z_2)},\\
KK^*(\overline{e_{m+1}(z_1)} \cdot \overline{e_{n+1}(z_2)}) &= K(e_{2m+1,2n+1}(z_1, z_2)) = \overline{e_{m+1}(z_1)} \cdot \overline{e_{n+1}(z_2)}.
\end{aligned}$$

Also,

$$\begin{aligned}
K^*K(e_{2m,2n}(z_1, z_2)) &= K^*(e_{m,n}(z_1, z_2)) = e_{2m,2n}(z_1, z_2),\\
K^*K(e_{2m+1,2n}(z_1, z_2)) &= K^*(\overline{e_{m+1}(z_1)} \cdot e_n(z_2)) = e_{2m+1,2n}(z_1, z_2),\\
K^*K(e_{2m,2n+1}(z_1, z_2)) &= K^*(e_m(z_1) \cdot \overline{e_{n+1}(z_2)}) = e_{2m,2n+1}(z_1, z_2),\\
K^*K(e_{2m+1,2n+1}(z_1, z_2)) &= K^*(\overline{e_{m+1}(z_1)} \cdot \overline{e_{n+1}(z_2)}) = e_{2m+1,2n+1}(z_1, z_2).
\end{aligned}$$

Therefore, we have $K^*K = I_{L_a^2(\mathbb{D}^2)}$ and $KK^* = I_{L^2_{\text{harm}}(\mathbb{D}^2)}$, and it follows that $K$ is a unitary operator. Next, we introduce the H-Toeplitz operator on $L_a^2(\mathbb{D}^2)$.

**Definition 2.5.** *For $\phi \in L^\infty(\mathbb{D}^2)$, we define the H-Toeplitz operator $B_\phi \colon L_a^2(\mathbb{D}^2) \to L_a^2(\mathbb{D}^2)$ by*

$$B_\phi(f) = P_{L_a^2} M_\phi K(f) \quad \text{for all } f \in L_a^2(\mathbb{D}^2).$$

The next proposition illustrates several fundamental properties of the H-Toeplitz operator $B_\phi$ on $L_a^2(\mathbb{D}^2)$.

**Proposition 2.6.** *For $\phi, \psi \in L^\infty(\mathbb{D}^2)$, the operator $B_\phi$ satisfies the following:*
(*i*) *$B_\phi$ is a bounded linear operator on $L_a^2(\mathbb{D}^2)$ with $\|B_\phi\| \leq \|\phi\|_\infty$.*
(*ii*) *For any scalar $\alpha$ and $\beta$, it holds that $B_{\alpha\phi+\beta\psi} = \alpha B_\phi + \beta B_\psi$.*
(*iii*) *The adjoint of H-Toeplitz operator $B_\phi$ is given by $B_\phi^* = K^* P_{harm} M_{\overline{\phi}}$.*

*Proof.*
(i):

$$\|B_\phi\| = \sup_{\|f\|=1} \left\| P_{L_a^2} M_\phi K(f) \right\| \leq \sup_{\|f\|=1} \|M_\phi\| \, \|K(f)\| = \|\phi\|_\infty \sup_{\|f\|=1} \|K(f)\| = \|\phi\|_\infty.$$

Therefore,

$$\|B_\phi\| \leq \|\phi\|_\infty.$$

(ii): For any $f \in L_a^2(\mathbb{D}^2)$, we have

$$B_{\alpha\phi+\beta\psi}(f) = P_{L_a^2}M_{\alpha\phi+\beta\psi}K(f) = \alpha P_{L_a^2}M_\phi K(f) + \beta P_{L_a^2}M_\psi K(f) = \alpha B_\phi(f) + \beta B_\psi(f).$$

Therefore,

$$B_{\alpha\phi+\beta\psi}(f) = \alpha B_\phi(f) + \beta B_\psi(f).$$

(iii):

$$\langle B_\phi f,\, g\rangle = \langle M_\phi K(f),\, P_{L_a^2}^* g\rangle = \langle K(f),\, M_\phi^* g\rangle = \langle K(f),\, M_{\overline{\phi}} g\rangle.$$

Since $K(f) \in L^2_{harm}(\mathbb{D}^2)$ and $P_{harm}: L^2(\mathbb{D}^2, dV) \to L^2_{harm}(\mathbb{D}^2, dV)$, it follows that

$$\langle K(f),\, M_{\overline{\phi}}(g)\rangle = \langle K(f),\, P_{harm}M_{\overline{\phi}}(g)\rangle = \langle f,\, K^* P_{harm}M_{\overline{\phi}}(g)\rangle.$$

Therefore,

$$\langle B_\phi f,\, g\rangle = \langle f,\, K^* P_{harm}M_{\overline{\phi}}(g)\rangle = \langle f,\, B_\phi^*(g)\rangle,$$

and we conclude that

$$B_\phi^* = K^* P_{harm} M_{\overline{\phi}}.$$

□

In general, $B_\phi B_\psi \neq B_{\phi\psi}$ for $\phi, \psi \in L^\infty(\mathbb{D}^2)$. We verify this by the subsequent example.

**Example 2.7.** *Take $\phi(z_1, z_2) = \psi(z_1, z_2) = z_1$. Then*

$$\begin{aligned} B_{z_1}B_{z_1}(1) &= B_{z_1}\big(P_{L_a^2}M_{z_1}K(1)\big) \\ &= B_{z_1}\big(P_{L_a^2}M_{z_1}K(e_{0,0})\big) \\ &= B_{z_1}\big(P_{L_a^2}M_{z_1}\cdot(1)\big) \\ &= B_{z_1}\big(P_{L_a^2}(z_1)\big) \\ &= P_{L_a^2}M_{z_1}K(z_1). \end{aligned}$$

*It can be calculated from the previous definition that:*

$$K(z_1) = K\left(\frac{e_{1,0}}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}K(e_{1,0}) = \frac{1}{\sqrt{2}}\cdot\sqrt{2}\,\overline{z}_1 = \overline{z}_1.$$

*Therefore, we can have*

$$B_{z_1}B_{z_1}(1) = P_{L_a^2}M_{z_1}(\overline{z}_1) = P_{L_a^2}(|z_1|^2) = \frac{1}{2},$$
$$B_{z_1^2}(1) = P_{L_a^2}M_{z_1^2}K(1) = P_{L_a^2}(z_1^2) = z_1^2.$$

*Thus, it follows that $B_{z_1}B_{z_1} \neq B_{z_1{}^2}$.*

**Example 2.8.** *Take $\phi(z_1, z_2) = z_1$ and $\psi(z_1, z_2) = \overline{z}_1$. We then obtain*

$$B_{z_1}B_{\overline{z}_1}(1) = B_{z_1}\big(P_{L_a^2}M_{\overline{z}_1}K(1)\big) = P_{L_a^2}M_{z_1}K\big(P_{L_a^2}(\overline{z}_1)\big) = 0,$$
$$B_{z_1\overline{z}_1}(1) = B_{|z_1|^2}(1) = P_{L_a^2}M_{|z_1|^2}K(1) = P_{L_a^2}(|z_1|^2) = \frac{1}{2}.$$

*Thus, $B_{z_1}B_{\overline{z}_1} \neq B_{z_1\overline{z}_1}$.*

Subsequently, we derive the matrix representation of H-Toeplitz operator $B_\phi$ with some pluriharmonic symbol on $L_a^2(\mathbb{D}^2)$. We take

$$\phi(z_1, z_2) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \bar{z}_1^{j_1} \bar{z}_2^{j_2} \in L^2_{\text{plurih}}(\mathbb{D}^2).$$

For $m, n \in \mathbb{N} \cup \{0\}$, we obtain

$$\begin{aligned}
B_\phi(e_{2m,2n}) &= P_{L_a^2} M_\phi K(e_{2m,2n}) = P_{L_a^2} M_\phi(e_{m,n}) = T_\phi(e_{m,n}),\\
B_\phi(e_{2m+1,2n+1}) &= P_{L_a^2} M_\phi K(e_{2m+1,2n+1}) = P_{L_a^2} M_\phi(\overline{e_{m+1,n+1}}) = P_{L_a^2} M_\phi J^{(1)}(e_{m,n}) = H^{(1)}(e_{m,n}),\\
B_\phi(e_{2m+1,2n}) &= P_{L_a^2} M_\phi K(e_{2m+1,2n}) = P_{L_a^2} M_\phi(\overline{e_{m+1}} \cdot e_n) = P_{L_a^2} M_\phi J^{(2)}(e_{m,n}) = H^{(2)}(e_{m,n}),\\
B_\phi(e_{2m,2n+1}) &= P_{L_a^2} M_\phi K(e_{2m,2n+1}) = P_{L_a^2} M_\phi(e_m \cdot \overline{e_{n+1}}) = P_{L_a^2} M_\phi J^{(3)}(e_{m,n}) = H^{(3)}(e_{m,n}).
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\langle B_\phi e_{2n_1,2n_2},\, e_{m_1,m_2} \rangle &= \langle T_\phi e_{n_1,n_2},\, e_{m_1,m_2} \rangle\\
&= \begin{cases}
\dfrac{\sqrt{n_1+1}\,\sqrt{n_2+1}}{\sqrt{m_1+1}\,\sqrt{m_2+1}}\, a_{m_1-n_1,\, m_2-n_2}, & \text{for } m_1 \geq n_1 \text{ and } m_2 \geq n_2,\\
\dfrac{\sqrt{m_1+1}\,\sqrt{m_2+1}}{\sqrt{n_1+1}\,\sqrt{n_2+1}}\, b_{n_1-m_1,\, n_2-m_2}, & \text{for } m_1 < n_1 \text{ and } m_2 < n_2,\\
0, & \text{otherwise},
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\langle B_\phi e_{2n_1+1,2n_2+1},\, e_{m_1,m_2} \rangle &= \langle H_\phi^{(1)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle = \frac{d_{n_1,n_2,m_1,m_2}}{(m_1+n_1+2)(m_2+n_2+2)}\, a_{n_1+m_1+1,\, n_2+m_2+1},\\
\langle B_\phi e_{2n_1+1,2n_2},\, e_{m_1,m_2} \rangle &= \langle H_\phi^{(2)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle = \frac{l_{n_1,n_2,m_1,m_2}}{(m_1+n_1+2)(m_2+1)}\, a_{n_1+m_1+1,\, m_2-n_2},\\
\langle B_\phi e_{2n_1,2n_2+1},\, e_{m_1,m_2} \rangle &= \langle H_\phi^{(3)} e_{n_1,n_2},\, e_{m_1,m_2} \rangle = \frac{r_{n_1,n_2,m_1,m_2}}{(m_1+1)(m_2+n_2+2)}\, a_{m_1-n_1,\, n_2+m_2+1},
\end{aligned}$$

where $m_1, m_2, n_1, n_2 \in \mathbb{N} \cup \{0\}$. Similarly, we take $m_1, n_1$ as block indices and $m_2, n_2$ as intra-block indices, corresponding, respectively, to the rows and columns of every two-dimensional subblock. The explicit expression for the matrix of $B_\phi$ under the orthonormal basis is presented below:

$$B_\phi = \begin{bmatrix} B_{(0,0)} & B_{(0,1)} & B_{(0,2)} & \cdots \\ B_{(1,0)} & B_{(1,1)} & B_{(1,2)} & \cdots \\ B_{(2,0)} & B_{(2,1)} & B_{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Moreover, from the above inner product, it follows that each matrix block $B$ is composed of one Toeplitz operator and three Hankel-type operators. So it can be written as follows:

$$B_{(x,y)} = \begin{bmatrix} T_{(x,y)} & H^{(2)}_{(x,y)} \\ H^{(3)}_{(x,y)} & H^{(1)}_{(x,y)} \end{bmatrix}.$$

The matrix of its adjiont is given by

$$B_\phi^* = \begin{bmatrix} B^*_{(0,0)} & B^*_{(1,0)} & B^*_{(2,0)} & \cdots \\ B^*_{(0,1)} & B^*_{(1,1)} & B^*_{(2,1)} & \cdots \\ B^*_{(0,2)} & B^*_{(1,2)} & B^*_{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$\left[B_{(x,y)}\right]^* = \begin{bmatrix} T^*_{(x,y)} & (H^{(3)}_{(x,y)})^* \\ (H^{(2)}_{(x,y)})^* & (H^{(1)}_{(x,y)})^* \end{bmatrix}.$$

**Example 2.9.** *Take the symbol function $\phi(z_1, z_2) = z_1^2 z_2^2$, whose coefficients satisfy $a_{2,2} = 1$ and all other coefficients are zero. Following the truncation rule from the previous section, take indices $x, y \in \{0, 1\}$, and we obtain the finite truncation block matrix*

$$B_\phi = \begin{bmatrix} B_{(0,0)} & B_{(0,1)} \\ B_{(1,0)} & B_{(1,1)} \end{bmatrix}.$$

*According to the previous analysis, under this symbol only the subblocks corresponding to $(x, y) = (0, 1)$ and $(1, 0)$ contain nonzero entries, hence $B_{(0,0)}$ and $B_{(1,1)}$ are both zero matrices.*

*Computation shows that $T_{(0,1)}$, $H^{(2)}_{(0,1)}$, and $H^{(3)}_{(0,1)}$ are all zero matrices, and only $H^{(1)}_{(0,1)}$ is a non-zero matrix. Thus,*

$$B_{(0,1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H^{(1)}_{(0,1)} \end{bmatrix}, \quad \text{where } H^{(1)}_{(0,1)} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2\sqrt{3}}{9} & 0 \end{bmatrix}.$$

*Similarly, we have*

$$B_{(1,0)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H^{(1)}_{(1,0)} \end{bmatrix}, \quad \text{where } H^{(1)}_{(1,0)} = \begin{bmatrix} 0 & \frac{2\sqrt{3}}{9} \\ \frac{4}{9} & 0 \end{bmatrix}.$$

*Finally, the overall truncation matrix can be written as*

$$B_\phi = \begin{bmatrix} \mathbf{0} & \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H^{(1)}_{(0,1)} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H^{(1)}_{(1,0)} \end{bmatrix} & \mathbf{0} \end{bmatrix}.$$

In the subsequent theorem, we give the necessary and sufficient condition such that the H-Toeplitz operator $B_\phi$ is a co-isometry on $L_a^2(\mathbb{D}^2)$. A function $f \in L^2(\mathbb{D}^2)$ is called co-analytic if $\overline{f}$ is analytic on $\mathbb{D}^2$. A bounded linear operator $T$ is co-isometric if and only if $TT^* = I_H$, where $I_H$ is the identity operator on $H$.

**Theorem 2.10.** *Let $\phi \in L^\infty(\mathbb{D}^2)$ be a co-analytic function. $B_\phi$ is a co-isometry on $L_a^2(\mathbb{D}^2)$ if and only if $|\phi| = 1$.*

*Proof.* First, we prove the sufficiency. Let $\phi \in L^\infty(\mathbb{D}^2)$ be a co-analytic function on $\mathbb{D}^2$. Then we have

$$\begin{aligned}
B_\phi B_\phi^*(z_1^{n_1} z_2^{n_2}) &= (P_{L_a^2} M_\phi K)(K^* P_{\text{harm}} M_{\overline{\phi}})(z_1^{n_1} z_2^{n_2}) \\
&= P_{L_a^2} M_\phi P_{\text{harm}} M_{\overline{\phi}}(z_1^{n_1} z_2^{n_2}) \\
&= P_{L_a^2} M_\phi P_{\text{harm}}(\overline{\phi}(z_1, z_2)\, z_1^{n_1} z_2^{n_2}) \\
&= P_{L_a^2} M_\phi(\overline{\phi}(z_1, z_2)\, z_1^{n_1} z_2^{n_2}) \\
&= P_{L_a^2}(\overline{\phi}(z_1 z_2)\phi(z_1, z_2)\, z_1^{n_1} z_2^{n_2}) \\
&= P_{L_a^2} M_{|\phi(z_1,z_2)|^2}(z_1^{n_1} z_2^{n_2}) \\
&= T_{|\phi(z_1,z_2)|^2}(z_1^{n_1} z_2^{n_2}),
\end{aligned}$$

where $n_1, n_2 \in \mathbb{N} \cup \{0\}$. As the polynomial set is dense in $L_a^2(\mathbb{D}^2)$, we can conclude that $B_\phi B_\phi^* = T_{|\phi|^2}$. Let $|\phi| = 1$, and then $B_\phi B_\phi^* = T_1 = I$. This implies that $B_\phi$ is a co-isometry on $L_a^2(\mathbb{D}^2)$.

Next, we prove necessity. If $B_\phi$ is a co-isometry on $L_a^2(\mathbb{D}^2)$, then it follows that $B_\phi B_\phi^* = I$. This is equivalent to $T_{1-|\phi|^2} = 0$, from which we deduce $|\phi| = 1$. □

The following theorem provides a necessary and sufficient condition for an H-Toeplitz operator to be a partial isometry on $L_a^2(\mathbb{D}^2)$. A bounded linear operator $T$ is a partial isometry if and only if $TT^*T = T$.

**Theorem 2.11.** *Let $\phi \in L^\infty(\mathbb{D}^2)$ be a co-analytic function. $B_\phi$ is a partial isometry on $L_a^2(\mathbb{D}^2)$ if and only if $\phi(z_1, z_2) = 0$ or $|\phi(z_1, z_2)| = 1$ for all $(z_1, z_2) \in \mathbb{D}^2$.*

*Proof.* First, we establish the sufficiency. By Theorem 2.10, if $\phi(z_1, z_2) = 0$ or $|\phi(z_1, z_2)| = 1$ for all $(z_1, z_2) \in \mathbb{D}^2$, then $B_\phi$ is a co-isometry. As we all know, every co-isometry is necessarily a partial isometry. Therefore, $B_\phi$ is a partial isometry.

For necessity, if $B_\phi$ is a partial isometry, then by [15], we have $B_\phi B_\phi^* B_\phi = B_\phi$. By substituting $B_\phi = P_{L_a^2} M_\phi K$ and $B_\phi^* = K^* P_{\text{harm}} M_{\overline{\phi}}$ into the above formula, we can obtain

$$\begin{aligned}
P_{L_a^2} M_\phi K &= (P_{L_a^2} M_\phi K) \cdot (K^* P_{\text{harm}} M_{\overline{\phi}}) \cdot (P_{L_a^2} M_\phi K) \\
&= P_{L_a^2} M_\phi \Big(P_{\text{harm}} M_{\overline{\phi}}(P_{L_a^2} M_\phi K)\Big) \\
&= P_{L_a^2} M_\phi(\overline{\phi}\,(P_{L_a^2} M_\phi K)) \\
&= P_{L_a^2}(\phi\,\overline{\phi}\,(P_{L_a^2} M_\phi K)) \\
&= P_{L_a^2} M_{|\phi|^2} P_{L_a^2} M_\phi K \\
&= T_{|\phi|^2} P_{L_a^2} M_\phi K.
\end{aligned}$$

So, we have $T_{|\phi|^2} B_\phi = B_\phi$, i.e., $(I - T_{|\phi|^2}) B_\phi = (I - T_{|\phi|^2}) T_\phi K = 0$. Especially, for $n_1, n_2 \in \mathbb{N} \cup \{0\}$, based on the parity combinations of $n_1$ and $n_2$, we discuss the following four cases:
**Case (1):** If $n_1 = 2p$ and $n_2 = 2q$, we can obtain

$$(I - T_{|\phi|^2}) T_\phi K(e_{2p,2q}) = (I - T_{|\phi|^2}) T_\phi(e_{p,q}) = 0.$$

**Case (2):** If $n_1 = 2p+1$ and $n_2 = 2q$, we can obtain

$$(I - T_{|\phi|^2})T_\phi K(e_{2p+1,2q}) = (I - T_{|\phi|^2})T_\phi(\overline{e_{p+1}} \cdot e_q) = 0.$$

**Case (3):** If $n_1 = 2p$ and $n_2 = 2q+1$, we can obtain

$$(I - T_{|\phi|^2})T_\phi K(e_{2p,2q+1}) = (I - T_{|\phi|^2})T_\phi(e_p \cdot \overline{e_{q+1}}) = 0.$$

**Case (4):** If $n_1 = 2p+1$ and $n_2 = 2q+1$, we can obtain

$$(I - T_{|\phi|^2})T_\phi K(e_{2p+1,2q+1}) = (I - T_{|\phi|^2})T_\phi(\overline{e_{p+1}} \cdot \overline{e_{q+1}}) = 0.$$

Thus, $T_{1-|\phi|^2}T_\phi = 0$, and by [16], we have $\phi(z_1, z_2) = 0$ or $|\phi(z_1, z_2)| = 1$ for all $(z_1, z_2) \in \mathbb{D}^2$. □

Next, we give the definition of a diagonal operator on the bidisk. On the Bergman space of the bidisk, an operator $T$ is a diagonal operator if and only if for any $(m,n), (p,q) \in N^2$, if $(m,n) \neq (p,q)$, then $\langle Te_{m,n}, e_{p,q}\rangle = 0$. Based on the foregoing analysis, we establish that a non-zero H-Toeplitz operator with some pluriharmonic symbol cannot be a diagonal operator acting on $L_a^2(\mathbb{D}^2)$.

**Theorem 2.12.** *Let $\phi$ be a pluriharmonic symbol and $\phi \in L^2_{plurih}(\mathbb{D}^2)$. $B_\phi$ is a diagonal operator if and only if $\phi \equiv 0$.*

*Proof.* The forward implication is trivial, so $B_\phi$ is a diagonal operator. Conversely, assume that $B_\phi$ is a diagonal operator if and only if for any $(m,n), (p,q) \in N^2$, when $(m,n) \neq (p,q)$, then $\langle Be_{m,n}, e_{p,q}\rangle = 0$. By substituting $B_\phi = P_{L_a^2}M_\phi K$ into the above formula, we can obtain

$$\left\langle P_{L_a^2}M_\phi K(e_{m,n}),\ e_{p,q}\right\rangle = \left\langle M_\phi K(e_{m,n}),\ e_{p,q}\right\rangle = \left\langle \phi K(e_{m,n}),\ e_{p,q}\right\rangle = 0.$$

If $m = 2k$, $n = 2l$, then

$$K(e_{2k,2l}) = e_{k,l} = \sqrt{k+1}\ \sqrt{l+1}\, z_1^k z_2^l.$$

Substituting into the above equation, and letting

$$g_{k,l,p,q} = \sqrt{k+1}\ \sqrt{l+1}\ \sqrt{p+1}\ \sqrt{q+1},$$

we can get:

$$\begin{aligned}
\langle \phi \cdot e_{k,l},\ e_{p,q}\rangle &= \left\langle \phi \cdot \sqrt{k+1}\,\sqrt{l+1}\, z_1^k z_2^l,\ \sqrt{p+1}\,\sqrt{q+1}\, z_1^p z_2^q\right\rangle \\
&= g_{k,l,p,q}\Big\langle\Big(\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} b_{j_1,j_2}\overline{z}_1^{j_1}\overline{z}_2^{j_2}\Big) z_1^k z_2^l,\ z_1^p z_2^q\Big\rangle \\
&= g_{k,l,p,q}\Big(\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a_{i_1,i_2}\langle z_1^{i_1+k} z_2^{i_2+l},\ z_1^p z_2^q\rangle + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} b_{j_1,j_2}\langle z_1^k z_2^l,\ z_1^{p+j_1} z_2^{q+j_2}\rangle\Big) \\
&= \begin{cases}
\dfrac{\sqrt{k+1}\ \sqrt{l+1}}{\sqrt{p+1}\ \sqrt{q+1}}\, a_{p-k,\,q-l}, & \text{if } p \geq k \text{ and } q \geq l, \\[2ex]
\dfrac{\sqrt{p+1}\ \sqrt{q+1}}{\sqrt{k+1}\ \sqrt{l+1}}\, b_{k-p,\,l-q}, & \text{if } p < k \text{ and } q < l, \\[2ex]
0, & \text{otherwise.}
\end{cases}
\end{aligned}$$

If $m = 2k+1$, $n = 2l$, then

$$K(e_{2k+1,2l}) = \overline{e_{k+1}} \cdot e_l = \sqrt{k+2}\ \sqrt{l+1}\, \bar{z}_1^{k+1} z_2^l.$$

Similarly, substitute into the above formula, and letting

$$h_{k,l,p,q} = \sqrt{k+2}\ \sqrt{l+1}\ \sqrt{p+1}\ \sqrt{q+1},$$

we can get:

$$\begin{aligned}
\langle \phi \cdot (\overline{e_{k+1}} \cdot e_l),\, e_{p,q} \rangle &= \left\langle \phi \cdot \sqrt{k+2}\sqrt{l+1}\, \bar{z}_1^{k+1} z_2^l,\ \sqrt{p+1}\sqrt{q+1}\, z_1^p z_2^q \right\rangle \\
&= h_{k,l,p,q}\left( \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} \langle z_1^{i_1} z_2^{i_2+l},\ z_1^{p+k+l} z_2^q \rangle + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \langle z_2^l,\ z_1^{p+j_1+k+1} z_2^{q+j_2} \rangle \right) \\
&= \frac{\sqrt{k+2}\ \sqrt{l+1}\ \sqrt{p+1}\ \sqrt{q+1}}{(p+k+2)(q+1)}\, a_{p+k+1,\, q-l}.
\end{aligned}$$

By the same method, if $m = 2k$, $n = 2l+1$, then

$$\langle \phi \cdot (e_k \cdot \overline{e_{l+1}}) \rangle = \frac{\sqrt{k+1}\ \sqrt{l+2}\ \sqrt{p+1}\ \sqrt{q+1}}{(q+l+2)(p+1)}\, a_{p-k,\, q+l+1}.$$

If $m = 2k+1$, $n = 2l+1$, then

$$\langle \phi \cdot \overline{e_{k+1,l+1}} \rangle = \frac{\sqrt{k+2}\ \sqrt{l+2}\ \sqrt{p+1}\ \sqrt{q+1}}{(q+l+2)(k+p+2)}\, a_{p+k+1,\, q+l+1}.$$

Since $B_\phi$ is a diagonal operator, then $\langle P_{L_a^2} M_\phi K e_{m,n},\, e_{p,q} \rangle = 0$. The above cases imply that $a_{i_1,i_2} = 0$ and $b_{j_1,j_2} = 0$ for all $i_1, i_2 \geq 0$ and $j_1, j_2 \geq 1$. Therefore, $\phi \equiv 0$. □

## 3. Invariant subspaces and kernel spaces of H-Toeplitz operators

In this section, we study several invariant subspaces and kernel spaces associated with H-Toeplitz operators on $L_a^2(\mathbb{D}^2)$. For two fixed non-negative integers $M_1$ and $M_2$, we define

$$H_{M_1,M_2} = \mathrm{Span}\{ z_1^{k_1} z_2^{k_2} \mid 0 \leq k_1 \leq 2M_1,\ 0 \leq k_2 \leq 2M_2 \}.$$

It is clear that $H_{M_1,M_2}$ forms a closed subspace of $L_a^2(\mathbb{D}^2)$. The orthogonal complement $H_{M_1,M_2}^{\perp}$ is defined as

$$H_{M_1,M_2}^{\perp} = \left\{ f \in L^2(\mathbb{D}^2, dV) \,\middle|\, \langle f,\, z_1^{k_1} z_2^{k_2} \rangle = 0,\ 0 \leq k_1 \leq 2M_1,\ 0 \leq k_2 \leq 2M_2 \right\}.$$

Before stating the theorem, we first present a lemma for the convenience of subsequent calculations.

**Lemma 3.1.** *We have $s, t \in \mathbb{N} \cup \{0\}$. In the bidisk Bergman space $L_a^2(\mathbb{D}^2)$, we compute:*

$$K(z_1^s z_2^t) = \begin{cases} \dfrac{\sqrt{p+1}\ \sqrt{q+1}}{\sqrt{2p+1}\ \sqrt{2q+1}}\, z_1^p z_2^q, & \text{if } s = 2p \text{ and } t = 2q, \\ \dfrac{\sqrt{p+1}\ \sqrt{q+2}}{\sqrt{2p+1}\ \sqrt{2q+2}}\, z_1^p \overline{z}_2^{q+1}, & \text{if } s = 2p \text{ and } t = 2q+1, \\ \dfrac{\sqrt{p+2}\ \sqrt{q+1}}{\sqrt{2p+2}\ \sqrt{2q+1}}\, \overline{z}_1^{p+1} z_2^q, & \text{if } s = 2p+1 \text{ and } t = 2q, \\ \dfrac{\sqrt{p+2}\ \sqrt{q+2}}{\sqrt{2p+2}\ \sqrt{2q+2}}\, \overline{z}_1^{p+1} \overline{z}_2^{q+1}, & \text{if } s = 2p+1 \text{ and } t = 2q+1. \end{cases}$$

*Proof.* If $s = 2p$ and $t = 2q$, using

$$z_1^{2p} z_2^{2q} = \frac{e_{2p,2q}}{\sqrt{2p+1}\ \sqrt{2q+1}} \quad \text{and} \quad K(e_{2p,2q}) = \sqrt{p+1}\ \sqrt{q+1}\, z_1^p z_2^q,$$

then

$$K(z_1^{2p} z_2^{2q}) = \frac{\sqrt{p+1}\ \sqrt{q+1}}{\sqrt{2p+1}\ \sqrt{2q+1}}\, z_1^p z_2^q.$$

If $s = 2p$ and $t = 2q+1$, using

$$z_1^{2p} z_2^{2q+1} = \frac{e_{2p,2q+1}}{\sqrt{2p+1}\ \sqrt{2q+2}} \quad \text{and} \quad K(e_{2p,2q+1}) = \sqrt{p+1}\ \sqrt{q+2}\, z_1^p \overline{z}_2^{q+1},$$

then

$$K(z_1^{2p} z_2^{2q+1}) = \frac{\sqrt{p+1}\ \sqrt{q+2}}{\sqrt{2p+1}\ \sqrt{2q+2}}\, z_1^p \overline{z}_2^{q+1}.$$

Following the same line of reasoning, we can derive that if $s = 2p+1$ and $t = 2q$, then

$$K(z_1^{2p+1} z_2^{2q}) = \frac{\sqrt{p+2}\ \sqrt{q+1}}{\sqrt{2p+2}\ \sqrt{2q+1}}\, \overline{z}_1^{p+1} z_2^q,$$

and if $s = 2p+1$ and $t = 2q+1$,

$$K(z_1^{2p+1} z_2^{2q+1}) = \frac{\sqrt{p+2}\ \sqrt{q+2}}{\sqrt{2p+2}\ \sqrt{2q+2}}\, \overline{z}_1^{p+1} \overline{z}_2^{q+1}.$$

□

In fact, the following theorem demonstrates that this property remains invariant for H-Toeplitz operators with co-analytic symbols.

**Theorem 3.2.** *If*

$$\psi(z_1, z_2) = \sum_{l_1=M_1}^{\infty} \sum_{l_2=M_2}^{\infty} a_{l_1,l_2}\, \overline{z}_1^{l_1} \overline{z}_2^{l_2} \in L^\infty(\mathbb{D}^2),$$

*the subspace $H_{M_1,M_2}$ of $L_a^2(\mathbb{D}^2)$ is invariant under $B_\psi$.*

*Proof.* By Lemma 3.1, for any non-negative integers $k_1$ and $k_2$, the operator $K$ defined on $L_a^2(\mathbb{D}^2)$ satisfies that:

$$K(z_1^{k_1}z_2^{k_2}) = \begin{cases} \dfrac{\sqrt{k_1+2}\ \sqrt{k_2+2}}{2\sqrt{k_1+1}\ \sqrt{k_2+1}}\, z_1^{\frac{k_1}{2}} z_2^{\frac{k_2}{2}}, & \text{if } k_1 \text{ is even and } k_2 \text{ is even,}\\ \dfrac{\sqrt{k_1+2}\ \sqrt{k_2+3}}{2\sqrt{k_1+1}\ \sqrt{k_2+1}}\, z_1^{\frac{k_1}{2}} \bar{z}_2^{\frac{k_2+1}{2}}, & \text{if } k_1 \text{ is even and } k_2 \text{ is odd,}\\ \dfrac{\sqrt{k_1+3}\ \sqrt{k_2+2}}{2\sqrt{k_1+1}\ \sqrt{k_2+1}}\, \bar{z}_1^{\frac{k_1+1}{2}} z_2^{\frac{k_2}{2}}, & \text{if } k_1 \text{ is odd and } k_2 \text{ is even,}\\ \dfrac{\sqrt{k_1+3}\ \sqrt{k_2+3}}{2\sqrt{k_1+1}\ \sqrt{k_2+1}}\, \bar{z}_1^{\frac{k_1+1}{2}} \bar{z}_2^{\frac{k_2+1}{2}}, & \text{if } k_1 \text{ is odd and } k_2 \text{ is odd.} \end{cases}$$

Next, we consider integers $i_1$, $i_2$ and $k_1$, $k_2$ such that $M_1 \le i_1 < \infty$, $M_2 \le i_2 < \infty$ and $0 \le k_1 \le 2M_1$, $0 \le k_2 \le 2M_2$. Subsequently, we analyze the problem by considering separate cases.

If $k_1$ is even and $k_2$ is even, then

$$\begin{aligned} B_{\bar{z}_1^{i_1}\bar{z}_2^{i_2}}(z_1^{k_1}z_2^{k_2}) &= P_{L_a^2} M_{\bar{z}_1^{i_1}\bar{z}_2^{i_2}}\left(\frac{\sqrt{(k_1+2)(k_2+2)}}{2\sqrt{(k_1+1)(k_2+1)}}\, z_1^{\frac{k_1}{2}} z_2^{\frac{k_2}{2}}\right)\\ &= P_{L_a^2}\left(\frac{\sqrt{(k_1+2)(k_2+2)}}{2\sqrt{(k_1+1)(k_2+1)}}\, z_1^{\frac{k_1}{2}}\bar{z}_1^{i_1}\, z_2^{\frac{k_2}{2}}\bar{z}_2^{i_2}\right)\\ &= \begin{cases} \dfrac{(k_1-2i_1+2)(k_2-2i_2+2)}{2\sqrt{(k_1+1)(k_2+1)(k_1+2)(k_2+2)}}\, z_1^{\frac{k_1}{2}-i_1} z_2^{\frac{k_2}{2}-i_2}, & \text{if } i_1 \le \frac{k_1}{2} \text{ and } i_2 \le \frac{k_2}{2},\\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

If $k_1$ is even and $k_2$ is odd, then

$$B_{\bar{z}_1^{i_1}\bar{z}_2^{i_2}}(z_1^{k_1}z_2^{k_2}) = P_{L_a^2}\left(\frac{\sqrt{(k_1+2)(k_2+3)}}{2\sqrt{(k_1+1)(k_2+1)}}\, z_1^{\frac{k_1}{2}}\, \bar{z}_1^{i_1}\, \bar{z}_2^{\frac{k_2+1}{2}+i_2}\right) = 0.$$

If $k_1$ is odd and $k_2$ is even, then

$$B_{\bar{z}_1^{i_1}\bar{z}_2^{i_2}}(z_1^{k_1}z_2^{k_2}) = P_{L_a^2}\left(\frac{\sqrt{(k_1+3)(k_2+2)}}{2\sqrt{(k_1+1)(k_2+1)}}\, \bar{z}_1^{\frac{k_1+1}{2}+i_1}\, z_2^{\frac{k_2}{2}}\, \bar{z}_2^{i_2}\right) = 0.$$

If $k_1$ is odd and $k_2$ is odd, then

$$B_{\bar{z}_1^{i_1}\bar{z}_2^{i_2}}(z_1^{k_1}z_2^{k_2}) = P_{L_a^2}\left(\frac{\sqrt{(k_1+3)(k_2+3)}}{2\sqrt{(k_1+1)(k_2+1)}}\, \bar{z}_1^{\frac{k_1+1}{2}+i_1}\, \bar{z}_2^{\frac{k_2+1}{2}+i_2}\right) = 0.$$

It is clear that the zero element must belong to the subspace $H_{M_1,M_2}$ and

$$\frac{k_1}{2} - i_1 \le \frac{k_1}{2} \le M_1, \qquad \frac{k_2}{2} - i_2 \le \frac{k_2}{2} \le M_2.$$

Therefore, for all $i_1, i_2$ with $M_1 \le i_1 < \infty$, $M_2 \le i_2 < \infty$ and for arbitrary $k_1, k_2$ satisfying $0 \le k_1 \le 2M_1$, $0 \le k_2 \le 2M_2$, it follows that

$$B_{\bar{z}_1^{i_1}\bar{z}_2^{i_2}}(z_1^{k_1}z_2^{k_2}) \in H_{M_1,M_2}.$$

If $k_1$ is even and $k_2$ is even, then

$$\begin{aligned}
B_\psi(z_1^{k_1}z_2^{k_2}) &= P_{L_a^2}M_\psi K(z_1^{k_1}z_2^{k_2}) \\
&= P_{L_a^2}M_\psi\left(\frac{\sqrt{(k_1+2)(k_2+2)}}{2\sqrt{(k_1+1)(k_2+1)}}\, z_1^{\frac{k_1}{2}} z_2^{\frac{k_2}{2}}\right) \\
&= P_{L_a^2}\left(\sum_{l_1=M_1}^{\infty}\sum_{l_2=M_2}^{\infty} a_{l_1,l_2}\frac{\sqrt{(k_1+2)(k_2+2)}}{2\sqrt{(k_1+1)(k_2+1)}}\, z_1^{\frac{k_1}{2}}\bar{z}_1^{l_1} z_2^{\frac{k_2}{2}}\bar{z}_2^{l_2}\right) \\
&= \sum_{l_1=M_1}^{\infty}\sum_{l_2=M_2}^{\infty} a_{l_1,l_2} B_{\bar{z}_1^{l_1}\bar{z}_2^{l_2}}(z_1^{k_1}z_2^{k_2}).
\end{aligned}$$

If $k_1$ is even and $k_2$ is odd, then

$$B_\psi(z_1^{k_1}z_2^{k_2}) = P_{L_a^2}M_\psi\left(\frac{\sqrt{(k_1+2)(k_2+3)}}{2\sqrt{(k_1+1)(k_2+1)}}\, z_1^{\frac{k_1}{2}}\, \bar{z}_2^{\frac{k_2+1}{2}}\right) = \sum_{l_1=M_1}^{\infty}\sum_{l_2=M_2}^{\infty} a_{l_1,l_2} B_{\bar{z}_1^{l_1}\bar{z}_2^{l_2}}(z_1^{k_1}z_2^{k_2}).$$

If $k_1$ is odd and $k_2$ is even, then

$$B_\psi(z_1^{k_1}z_2^{k_2}) = P_{L_a^2}M_\psi\left(\frac{\sqrt{(k_1+3)(k_2+2)}}{2\sqrt{(k_1+1)(k_2+1)}}\, \bar{z}_1^{\frac{k_1+1}{2}}\, z_2^{\frac{k_2}{2}}\right) = \sum_{l_1=M_1}^{\infty}\sum_{l_2=M_2}^{\infty} a_{l_1,l_2} B_{\bar{z}_1^{l_1}\bar{z}_2^{l_2}}(z_1^{k_1}z_2^{k_2}).$$

If $k_1$ is odd and $k_2$ is odd, then

$$B_\psi(z_1^{k_1}z_2^{k_2}) = P_{L_a^2}M_\psi\left(\frac{\sqrt{(k_1+3)(k_2+3)}}{2\sqrt{(k_1+1)(k_2+1)}}\, \bar{z}_1^{\frac{k_1+1}{2}}\, \bar{z}_2^{\frac{k_2+1}{2}}\right) = \sum_{l_1=M_1}^{\infty}\sum_{l_2=M_2}^{\infty} a_{l_1,l_2} B_{\bar{z}_1^{l_1}\bar{z}_2^{l_2}}(z_1^{k_1}z_2^{k_2}).$$

Thus, we get that

$$B_\psi(z_1^{k_1}z_2^{k_2}) = \sum_{l_1=M_1}^{\infty}\sum_{l_2=M_2}^{\infty} a_{l_1,l_2}\, B_{\bar{z}_1^{l_1}\bar{z}_2^{l_2}}(z_1^{k_1}z_2^{k_2}) \in H_{M_1,M_2},$$

for all $0 \le k_1 \le 2M_1, 0 \le k_2 \le 2M_2$. Therefore, for all $f \in H_{M_1,M_2}$, we have $B_\psi(f) \in H_{M_1,M_2}$. This implies that $H_{M_1,M_2}$ is invariant under $B_\psi$. □

**Theorem 3.3.** *If $\phi(z_1,z_2) = z_1^{M_1}z_2^{M_2}$, then the subspace $H_{M_1,M_2}$ in $L_a^2(\mathbb{D}^2)$ is a reducing subspace for the operator $B^*_{\phi(z_1,z_2)}$.*

*Proof.* To prove that the subspace $H_{M_1,M_2}$ is a reducing subspace of the operator $B^*_{\phi(z_1,z_2)}$, it suffices to show that both $H_{M_1,M_2}$ and $H^{\perp}_{M_1,M_2}$ are invariant subspaces of $B^*_{\phi(z_1,z_2)}$. First, we prove that $H_{M_1,M_2}$ is an invariant subspace of $B^*_{\phi(z_1,z_2)}$. Relying on Lemma 2.1, we get

$$\begin{aligned}
B^*_{\phi(z_1,z_2)}(z_1^{k_1}z_2^{k_2}) &= K^*M^*_{\phi}P_{L^2_a}(z_1^{k_1}z_2^{k_2}) \\
&= K^*P_{\text{harm}}M_{\overline{z}_1^{M_1}\overline{z}_2^{M_2}}(z_1^{k_1}z_2^{k_2}) \\
&= K^*P_{\text{harm}}\Big(\overline{z}_1^{M_1}\overline{z}_2^{M_2}z_1^{k_1}z_2^{k_2}\Big) \\
&= \begin{cases}
K^*\left(\dfrac{(k_1-M_1+1)(k_2-M_2+1)}{(k_1+1)(k_2+1)}\,z_1^{k_1-M_1}z_2^{k_2-M_2}\right), & \text{if } k_1 \ge M_1 \text{ and } k_2 \ge M_2, \\[2ex]
K^*\left(\dfrac{(M_1-k_1+1)(M_2-k_2+1)}{(M_1+1)(M_2+1)}\,\overline{z}_1^{M_1-k_1}\overline{z}_2^{M_2-k_2}\right), & \text{if } M_1 > k_1 \text{ and } M_2 > k_2, \\[2ex]
K^*\left(\dfrac{(k_1-M_1+1)(M_2-k_2+1)}{(k_1+1)(M_2+1)}\,z_1^{k_1-M_1}\overline{z}_2^{M_2-k_2}\right), & \text{if } k_1 \ge M_1 \text{ and } M_2 > k_2, \\[2ex]
K^*\left(\dfrac{(M_1-k_1+1)(k_2-M_2+1)}{(M_1+1)(k_2+1)}\,\overline{z}_1^{M_1-k_1}z_2^{k_2-M_2}\right), & \text{if } M_1 > k_1 \text{ and } k_2 \ge M_2.
\end{cases}
\end{aligned}$$

If $k_1 \ge M_1$ and $k_2 \ge M_2$, we can compute

$$\begin{aligned}
&K^*\Big(\frac{(k_1-M_1+1)(k_2-M_2+1)}{(k_1+1)(k_2+1)}\,z_1^{k_1-M_1}z_2^{k_2-M_2}\Big) \\
&= \frac{(k_1-M_1+1)(k_2-M_2+1)}{(k_1+1)(k_2+1)}\,K^*(z_1^{k_1-M_1}z_2^{k_2-M_2}) \\
&= \frac{\sqrt{k_1-M_1+1}\,\sqrt{k_2-M_2+1}}{(k_1+1)(k_2+1)}\,K^*(e_{k_1-M_1,\,k_2-M_2}) \\
&= \frac{\sqrt{k_1-M_1+1}\,\sqrt{k_2-M_2+1}}{(k_1+1)(k_2+1)}\,e_{2k_1-2M_1,\,2k_2-2M_2} \\
&= \frac{\sqrt{(k_1-M_1+1)(k_2-M_2+1)}\,\sqrt{(2k_1-2M_1+1)(2k_2-2M_2+1)}}{(k_1+1)(k_2+1)}\,z_1^{2k_1-2M_1}z_2^{2k_2-2M_2}.
\end{aligned}$$

Therefore, we get

$$B^*_{\phi(z_1,z_2)}(z_1^{k_1}z_2^{k_2}) = \frac{\sqrt{(k_1-M_1+1)(k_2-M_2+1)}\,\sqrt{(2k_1-2M_1+1)(2k_2-2M_2+1)}}{(k_1+1)(k_2+1)}\,z_1^{2k_1-2M_1}z_2^{2k_2-2M_2}.$$

If $M_1 > k_1$ and $M_2 > k_2$, we can obtain

$$B^*_{\phi(z_1,z_2)}(z_1^{k_1}z_2^{k_2}) = \frac{\sqrt{(M_1-k_1+1)(M_2-k_2+1)}\,\sqrt{(2M_1-2k_1)(2M_2-2k_2)}}{(M_1+1)(M_2+1)}\,z_1^{2M_1-2k_1-1}z_2^{2M_2-2k_2-1}.$$

If $k_1 \ge M_1$ and $M_2 > k_2$, we can obtain

$$B^*_{\phi(z_1,z_2)}(z_1^{k_1}z_2^{k_2}) = \frac{\sqrt{(k_1-M_1+1)(M_2-k_2+1)}\,\sqrt{(2k_1-2M_1+1)(2M_2-2k_2)}}{(k_1+1)(M_2+1)}\,z_1^{2k_1-2M_1}z_2^{2M_2-2k_2-1}.$$

If $M_1 > k_1$ and $k_2 \geq M_2$, we can obtain

$$B^*_{\phi(z_1,z_2)}(z_1^{k_1} z_2^{k_2}) = \frac{\sqrt{(M_1 - k_1 + 1)(k_2 - M_2 + 1)}\ \sqrt{(2M_1 - 2k_1)(2k_2 - 2M_2 + 1)}}{(M_1 + 1)(k_2 + 1)} z_1^{2M_1-2k_1-1} z_2^{2k_2-2M_2}.$$

We can observe that the exponents of $z_1$ and $z_2$ in both cases lie in $0 \sim 2M_1$ and $0 \sim 2M_2$, respectively. Thus, $B^*_{\phi(z_1,z_2)}(f) \in H_{M_1,M_2}$ for all $f \in H_{M_1,M_2}$. We conclude that the subspace $H_{M_1,M_2}$ is invariant under $B^*_{\phi(z_1,z_2)}$.

Next, we prove that $H^{\perp}_{M_1,M_2}$ is also an invariant subspace of $B^*_{\phi(z_1,z_2)}$. According to the definition of $H^{\perp}_{M_1,M_2}$, we have

$$\langle f(z_1, z_2),\ z_1^{k_1} z_2^{k_2} \rangle = 0 \quad \text{for all } 0 \leq k_1 \leq 2M_1,\ 0 \leq k_2 \leq 2M_2.$$

Then by computing the inner product of $f(z_1, z_2)$ with the monomial $z_1^{k_1} z_2^{k_2}$, we obtain, for any $f \in H^{\perp}_{M_1,M_2}$ , where $f$ can be expressed as the orthogonal decomposition:

$$\begin{aligned} f(z_1, z_2) = & \sum_{i_1=2M_1+1}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{i_1=0}^{2M_1} \sum_{i_2=2M_2+1}^{\infty} b_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{i_1=0}^{\infty} \sum_{j_2=1}^{\infty} c_{i_1,j_2} z_1^{i_1} \overline{z}_2^{j_2} \\ & + \sum_{j_1=1}^{\infty} \sum_{i_2=0}^{\infty} d_{j_1,i_2} \overline{z}_1^{j_1} z_2^{i_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} e_{j_1,j_2} \overline{z}_1^{j_1} \overline{z}_2^{j_2}. \end{aligned}$$

Moreover, by applying Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} & B^*_{\phi(z_1,z_2)}(f)(z_1, z_2) \\ & = K^* P_{\text{harm}} M_{\overline{\phi}(z_1,z_2)} \Bigg( \sum_{i_1=2M_1+1}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{i_1=0}^{2M_1} \sum_{i_2=2M_2+1}^{\infty} b_{i_1,i_2} z_1^{i_1} z_2^{i_2} + \sum_{i_1=0}^{\infty} \sum_{j_2=1}^{\infty} c_{i_1,j_2} z_1^{i_1} \overline{z}_2^{j_2} \\ & \quad + \sum_{j_1=1}^{\infty} \sum_{i_2=0}^{\infty} d_{j_1,i_2} \overline{z}_1^{j_1} z_2^{i_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} e_{j_1,j_2} \overline{z}_1^{j_1} \overline{z}_2^{j_2} \Bigg) \\ & = K^* P_{\text{harm}} \Bigg( \sum_{i_1=2M_1+1}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} \overline{z}_1^{M_1} \overline{z}_2^{M_2} + \sum_{i_1=0}^{2M_1} \sum_{i_2=2M_2+1}^{\infty} b_{i_1,i_2} z_1^{i_1} z_2^{i_2} \overline{z}_1^{M_1} \overline{z}_2^{M_2} \\ & \quad + \sum_{i_1=0}^{\infty} \sum_{j_2=1}^{\infty} c_{i_1,j_2} z_1^{i_1} \overline{z}_2^{j_2} \overline{z}_1^{M_1} \overline{z}_2^{M_2} + \sum_{j_1=1}^{\infty} \sum_{i_2=0}^{\infty} d_{j_1,i_2} \overline{z}_1^{j_1} z_2^{i_2} \overline{z}_1^{M_1} \overline{z}_2^{M_2} \\ & \quad + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} e_{j_1,j_2} \overline{z}_1^{j_1} \overline{z}_2^{j_2} \overline{z}_1^{M_1} \overline{z}_2^{M_2} \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_1=2M_1+1}^{\infty} \sum_{i_2=M_2}^{\infty} a_{i_1,i_2} \cdot A \cdot z_1^{2i_1-2M_1} z_2^{2i_2-2M_2} + \sum_{i_1=2M_1+1}^{\infty} \sum_{i_2=0}^{M_2-1} a_{i_1,i_2} \cdot B \cdot z_1^{2i_1-2M_1} z_2^{2M_2-2i_2-1} \\
&\quad + \sum_{i_1=M_1}^{2M_1} \sum_{i_2=2M_2+1}^{\infty} b_{i_1,i_2} \cdot A \cdot z_1^{2i_1-2M_1} z_2^{2i_2-2M_2} + \sum_{i_1=0}^{M_1-1} \sum_{i_2=2M_2+1}^{\infty} b_{i_1,i_2} \cdot C \cdot z_1^{2M_1-2i_1-1} z_2^{2i_2-2M_2} \\
&\quad + \sum_{i_1=0}^{M_1-1} \sum_{j_2=1}^{\infty} c_{i_1,j_2} \cdot D \cdot z_1^{2M_1-2i_1-1} z_2^{2M_2+2j_2-1} + \sum_{i_1=M_1}^{\infty} \sum_{j_2=1}^{\infty} c_{i_1,j_2} \cdot E \cdot z_1^{2i_1-2M_1} z_2^{2M_2+2j_2-1} \\
&\quad + \sum_{j_1=1}^{\infty} \sum_{i_2=0}^{M_2-1} d_{j_1,i_2} \cdot F \cdot z_1^{2M_1+2j_1-1} z_2^{2M_2-2i_2-1} + \sum_{j_1=1}^{\infty} \sum_{i_2=M_2}^{\infty} d_{j_1,i_2} \cdot G \cdot z_1^{2M_1+2j_1-1} z_2^{2i_2-2M_2} \\
&\quad + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} e_{j_1,j_2} \cdot H \cdot z_1^{2M_1+2j_1-1} z_2^{2M_2+2j_2-1},
\end{aligned}
$$

where $A$–$H$ are the corresponding coefficients. For the case when $i_1 > 2M_1$ and $i_2 \geq M_2$, we can verify that $\langle z_1^{2i_1-2M_1} z_2^{2i_2-2M_2}, z_1^{k_1} z_2^{k_2} \rangle = 0$. Similarly, the inner products corresponding to the remaining terms also vanish. It then follows that

$$\langle B^*_{\phi(z_1,z_2)}(f(z_1,z_2)), z_1^{k_1} z_2^{k_2} \rangle = 0 \quad \text{for all } 0 \leq k_1 \leq 2M_1,\ 0 \leq k_2 \leq 2M_2.$$

Therefore, $B^*_{\phi(z_1,z_2)}(f) \in H^{\perp}_{M_1,M_2}$. Since $H^{\perp}_{M_1,M_2}$ is orthogonal to $H_{M_1,M_2}$ and by the definition of $H^{\perp}_{M_1,M_2}$, the linear span of its basis elements is dense in $H^{\perp}_{M_1,M_2}$. Since $f$ is exactly a linear combination of these basis elements, we have $\overline{\text{Span}\{f\}} = H^{\perp}_{M_1,M_2}$, so the set of all such functions $f$ is dense in $H^{\perp}_{M_1,M_2}$. Thus, we have $B^*_{\phi(z_1,z_2)}(H^{\perp}_{M_1,M_2}) \subseteq H^{\perp}_{M_1,M_2}$. Hence, $H_{M_1,M_2}$ is a reducing subspace of $B^*_{\phi(z_1,z_2)}$. □

**Theorem 3.4.** *If*

$$\phi(z_1,z_2) = \sum_{m=N_1}^{\infty} \sum_{n=N_2}^{\infty} a_{m,n}\, \overline{z}_1^m \overline{z}_2^n \in L^{\infty}(\mathbb{D}^2),$$

*where $N_1, N_2 \geq 1$ are arbitrary integers, then*

$$\text{Ker}\, B_{\phi} = \text{Span}\left\{ z_1^{2p} z_2^{2q}\ (p < N_1 \text{ or } q < N_2),\ z_1^{2p+1} z_2^{2q},\ z_1^{2p} z_2^{2q+1},\ z_1^{2p+1} z_2^{2q+1}\ \middle|\ p,q \geq 0 \right\}.$$

*Proof.* First, we show that every element of the spanning set lies in $Ker B_{\phi}$. Let

$$f \in \text{Span}\left\{ z_1^{2p} z_2^{2q}\ (p < N_1 \text{ or } q < N_2),\ z_1^{2p+1} z_2^{2q},\ z_1^{2p} z_2^{2q+1},\ z_1^{2p+1} z_2^{2q+1}\ \middle|\ p,q \geq 0 \right\}.$$

Then, let $f = z_1^{2p} z_2^{2q} (p < N_1 \text{ or } q < N_2)$, and we compute that:

$$
\begin{aligned}
B_{\phi}(f) = P_{L_a^2} M_{\phi} K(z_1^{2p} z_2^{2q}) &= \frac{1}{\sqrt{2p+1}\,\sqrt{2q+1}} P_{L_a^2} M_{\phi}(z_1^p z_2^q) \\
&= \frac{1}{\sqrt{2p+1}\,\sqrt{2q+1}} P_{L_a^2}\Big[\Big( \sum_{m=N_1}^{\infty} \sum_{n=N_2}^{\infty} a_{m,n} \overline{z}_1^m \overline{z}_2^n \Big)(z_1^p z_2^q)\Big] = 0.
\end{aligned}
$$

Let $f = z_1^{2p+1}z_2^{2q}$, and we compute that:

$$
\begin{aligned}
B_\phi(f) = P_{L_a^2}M_\phi K(z_1^{2p+1}z_2^{2q}) &= \frac{1}{\sqrt{2p+2}\,\sqrt{2q+1}}\,P_{L_a^2}M_\phi(\overline{z}_1^{p+1}z_2^{q})\\
&= \frac{1}{\sqrt{2p+2}\,\sqrt{2q+1}}\,P_{L_a^2}\Big[\Big(\sum_{m=N_1}^{\infty}\sum_{n=N_2}^{\infty}a_{m,n}\overline{z}_1^{m}\overline{z}_2^{n}\Big)(\overline{z}_1^{p+1}z_2^{q})\Big] = 0.
\end{aligned}
$$

Similarly, for $f = z_1^{2p}z_2^{2q+1}$, $f = z_1^{2p+1}z_2^{2q+1}$, we have $B_\phi(f) = 0$. So, $f \in KerB_\phi$.

Conversely, we assume that $\forall f \in \operatorname{Ker} B_\phi$, we have $B_\phi(f) = 0$ and

$$
f = f_1 + f_2 + f_3 + f_4 \quad \text{for } f_1, f_2, f_3, f_4 \in L_a^2(\mathbb{D}^2),
$$

where

$$
\begin{aligned}
f_1 &= \sum_{p,q=0}^{\infty} C_{2p,2q}z_1^{2p}z_2^{2q},\\
f_2 &= \sum_{p,q=0}^{\infty} C_{2p+1,2q}z_1^{2p+1}z_2^{2q},\\
f_3 &= \sum_{p,q=0}^{\infty} C_{2p,2q+1}z_1^{2p}z_2^{2q+1},\\
f_4 &= \sum_{p,q=0}^{\infty} C_{2p+1,2q+1}z_1^{2p+1}z_2^{2q+1}.
\end{aligned}
$$

It then follows that

$$
P_{L_a^2(\mathbb{D}^2)}M_\phi K(f_1) + P_{L_a^2(\mathbb{D}^2)}M_\phi K(f_2) + P_{L_a^2(\mathbb{D}^2)}M_\phi K(f_3) + P_{L_a^2(\mathbb{D}^2)}M_\phi K(f_4) = 0.
$$

Subsequently, we consider the index

$$
P_{L_a^2(\mathbb{D}^2)}M_\phi K(f_2) = P_{L_a^2(\mathbb{D}^2)}M_\phi K\left(\sum_{p,q=0}^{\infty} C_{2p+1,2q}\,z_1^{2p+1}z_2^{2q}\right) = 0.
$$

Therefore, for all $i_1, j_1 \geq 0$, we have

$$
\left\langle P_{L_a^2}M_\phi K(f_2),\, z_1^{i}z_2^{j}\right\rangle = \sum_{m=N_1}^{\infty}\sum_{n=N_2}^{\infty}\frac{a_{m,n}C_{2p+1,2q}}{\sqrt{2p+2}\,\sqrt{2q+1}}\left\langle \overline{z}_1^{p+1}z_2^{q},\, z_1^{i+m}z_2^{j+n}\right\rangle = 0.
$$

Since the inner product contains the conjugate factor $\overline{z}_1^{p+1}$, the inner product vanishes for all $i, j, p, q \geq 0$. This implies that the coefficients $C_{2p+1,2q}$ are free of constraints. Consequently, $f_2$ is an arbitrary linear combination of these monomials, i.e.,

$$
f_2 \in \operatorname{Span}\{z_1^{2p+1}z_2^{2q} \mid p, q \geq 0\}.
$$

Similarly,

$$f_3 \in \operatorname{Span}\{z_1^{2p} z_2^{2q+1} \mid p, q \geq 0\}$$

and

$$f_4 \in \operatorname{Span}\{z_1^{2p+1} z_2^{2q+1} \mid p, q \geq 0\}.$$

We have $P_{L_a^2(\mathbb{D}^2)} M_\phi K(f_1) = 0$. Therefore, for all $i_1, j_1 \geq 0$, we have

$$\left\langle P_{L_a^2} M_\phi K(f_1),\, z_1^i z_2^j \right\rangle = \sum_{m=N_1}^{\infty} \sum_{n=N_2}^{\infty} \frac{a_{m,n} C_{2p,2q}}{\sqrt{2p+1}\,\sqrt{2q+1}} \left\langle z_1^p z_2^q,\, z_1^{i+m} z_2^{j+n} \right\rangle = 0.$$

By the orthogonality of monomials on the Bergman space, the inner product $\langle z_1^p z_2^q,\, z_1^{i+m} z_2^{j+n} \rangle \neq 0$ if and only if $p = i + m$ and $q = j + n$. For any $p \geq N_1$ and $q \geq N_2$, we can always find $i, j \geq 0$ such that the above equalities hold. This forces the corresponding coefficients to satisfy

$$C_{2p,2q} = 0, \quad \forall\, p \geq N_1,\ q \geq N_2.$$

In other words, all coefficients with $p \geq N_1$ and $q \geq N_2$ are required to vanish. Thus $f_1$ only consists of monomials $z_1^{2p} z_2^{2q}$ with $p < N_1$ or $q < N_2$, which yields

$$f_1 \in \operatorname{Span}\left\{ z_1^{2p} z_2^{2q} \mid p < N_1 \text{ or } q < N_2,\ p, q \geq 0 \right\}.$$

So,

$$f \in \operatorname{Span}\left\{ z_1^{2p} z_2^{2q}\ (p < N_1 \text{ or } q < N_2),\ z_1^{2p+1} z_2^{2q},\ z_1^{2p} z_2^{2q+1},\ z_1^{2p+1} z_2^{2q+1} \;\middle|\; p, q \geq 0 \right\}.$$

By the above proof, we obtain the mutual inclusion. Therefore,

$$\operatorname{Ker} B_\phi = \operatorname{Span}\left\{ z_1^{2p} z_2^{2q}\ (p < N_1 \text{ or } q < N_2),\ z_1^{2p+1} z_2^{2q},\ z_1^{2p} z_2^{2q+1},\ z_1^{2p+1} z_2^{2q+1} \;\middle|\; p, q \geq 0 \right\}.$$

□

## 4. Conclusions

In this research, we investigated H-Toeplitz operators on $L_a^2(\mathbb{D}^2)$. Specifically, co-isometry, partial isometry, diagonality, invariant subspaces, and kernel spaces of H-Toeplitz operators on $L_a^2(\mathbb{D}^2)$ were characterized.

### Author contributions

Jingwen Duan: Conceptualization, formal analysis, methodology, writing—original draft, and validation; Bo Zhang and Yinghui Liu: Writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

### Use of Generative-AI tools declaration

The authors declare they have not used Artiffcial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that there are no conflicts of interest.

## References

1. A. Gupta, S. K. Singh, H-Toeplitz operators on the Bergman space, *Bull. Korean Math. Soc.*, **58** (2021), 327–347. https://doi.org/10.4134/BKMS.b200260
2. S. C. Arora, S. Paliwal, On H-Toeplitz operators, *Bull. Pure Appl. Math.*, **1** (2007), 141–154.
3. S. Kim, J. Lee, Contractivity and expansivity of H-Toeplitz operators on the Bergman spaces, *AIMS Math.*, **7** (2022), 13927–13944. https://doi.org/10.3934/math.2022769
4. S. Kim, E. Ko, J. E. Lee, J. Lee, H-Toeplitz operators on the function spaces, *Monatsh. Math.*, **205** (2024), 757–786. https://doi.org/10.1007/s00605-024-01985-9
5. P. Y. Huang, Y. Y. Zhang, H-Toeplitz operators on the Dirichlet type space, *AIMS Math.*, **9** (2024), 17847–17870. https://doi.org/10.3934/math.2024868
6. A. Aggarwal, A. Gupta, H-Toeplitz operators on the Fock space, *Rend. Circ. Mat. Palermo (2)*, **74** (2025), 181. https://doi.org/10.1007/s12215-025-01298-2
7. J. J. Liang, L. L. Lai, Y. L. Zhao, Y. Chen, Commuting H-Toeplitz operators with quasihomogeneous symbols, *AIMS Math.*, **7** (2022), 7898–7908. https://doi.org/10.3934/math.2022442
8. Q. Ding, Y. Chen, Product of H-Toeplitz operator and Toeplitz operator on the Bergman space, *AIMS Math.*, **8** (2023), 20790–20801. https://doi.org/10.3934/math.20231059
9. Y. F. Lu, Commuting of Toeplitz operators on the Bergman spaces of the bidisc, *Bull. Aust. Math. Soc.*, **66** (2002), 345–351. https://doi.org/10.1017/S0004972700040181
10. Y. F. Lu, B. Zhang, Commuting Hankel and Toeplitz operators on the Hardy space of the bidisk, *J. Math. Res. Appl.*, **30** (2010), 205–216. https://doi.org/10.3770/j.issn:1000-341X.2010.02.002
11. X. Ding, S. Sun, D. Zheng, Commuting Toeplitz operators on the bidisk, *J. Funct. Anal.*, **263** (2012), 3333–3357. https://doi.org/10.1016/j.jfa.2012.08.005
12. Y. Chen, K. J. Izuchi, Y. J. Lee, Kernels of Toeplitz operators on the Hardy space over the bidisk, *J. Funct. Anal.*, **272** (2017), 3869–3903. https://doi.org/10.1016/j.jfa.2017.01.002
13. M. T. Garayev, Invariant subspaces of operators via Berezin symbols and Duhamel product, *Turk. J. Math.*, **47** (2023), 2139–2148. https://doi.org/10.55730/1300-0098.3485
14. Y. Li, Y. Zhao, X. Ding, The eigenvalues, numerical ranges, and invariant subspaces of the Bergman Toeplitz operators over the bidisk, *Ann. Funct. Anal.*, **15** (2024), 35. https://doi.org/10.1007/s43034-024-00336-x

15. G. J. Murphy, *$C^*$-algebras and operator theory*, Boston, MA: Academic Press, 1990. https://doi.org/10.1016/C2009-0-22289-6

16. P. Ahern, Ž. Čučković, Some examples related to the Brown-Halmos theorem for the Bergman space, *Acta Sci. Math.*, **70** (2004), 373–378.

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