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Research article

On fixed points of mappings with contractive iterates and iterative Ulam–Hyers stability in strong b-metric spaces

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Abstract: In this paper, we study a class of self-mappings that are not necessarily contractions but admit a contractive iterate at a point. Working in the framework of strong b-metric spaces, we establish existence and uniqueness results for fixed points of such mappings. Moreover, we obtain an upper bound for the distance between an arbitrary initial point and the corresponding fixed point. Additionally, we investigate the behavior of fixed points under perturbations of the operator and prove the iterative Ulam–Hyers stability result for the associated fixed point problem.

Keywords: strong b-metric space; mappings with contractive iteration at a point; iterative Ulam-Hyers stable
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1. Introduction and preliminaries

The fixed point theory in metric and generalized metric spaces has been extensively studied due to its wide range of applications in nonlinear analyses and applied mathematics. Among various generalizations of metric spaces, the notion of a b-metric space relaxes the classical triangle inequality by allowing the following:

$$\tilde{d}(\tilde{x},\tilde{z}) \leq \hbar\,[\tilde{d}(\tilde{x},\tilde{y}) + \tilde{d}(\tilde{y},\tilde{z})], \quad \hbar \geq 1.$$

This framework has proven to be effective in establishing fixed point results and their applications; for example, see [13, 17, 19].

One of the important applications of the fixed-point theory is the analysis of ordinary and partial differential equations. In many cases, such problems can be reformulated as equivalent integral equations, thereby allowing Banach's contraction principle and its extensions to be employed in establishing the existence, uniqueness, and stability of solutions. Recent studies have successfully applied fixed-point methods to deterministic and stochastic differential equations, reaction–diffusion systems, epidemic models, and within-host virus dynamics [1, 3, 20]. These successful applications further highlight the importance of developing generalized fixed-point results in abstract spaces, where broader classes of contractive mappings can be studied.

In this context, we consider the framework of strong b-metric spaces. This class of spaces can be viewed as a natural generalization of metric spaces that still preserves many useful properties needed in the fixed point theory (see [12]). In particular, strong b-metric spaces refine the structure of b-metric spaces by imposing a stronger form of the triangle inequality, which leads to better regularity of the distance function and improves the behavior of convergent and Cauchy sequences. One important feature is that the distance function enjoys stronger continuity properties, which play a significant role in the study of convergence and the existence of fixed points.

Moreover, strong b-metric spaces provide a convenient setting to extend several fixed point results from metric spaces to more general spaces; for example, see [4, 8, 16]. Many well-known fixed point theorems, iterative methods, and stability results can be adapted to this framework. Because of these advantages, strong b-metric spaces have attracted increasing attention in recent studies in nonlinear analyses and the fixed point theory. Now, we recall the definition of a strong b-metric space.

Definition 1.1. *[12] Let $\tilde{X}$ be a non-empty set and $\hbar \geq 1$. Let $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ be a mapping that satisfies the following for each $\tilde{x}, \tilde{y}$ and $\tilde{z} \in \tilde{X}$:*
SbM1. $\tilde{d}(\tilde{x}, \tilde{y}) \geq 0$,
SbM2. $\tilde{d}(\tilde{x}, \tilde{y}) = 0 \Leftrightarrow \tilde{x} = \tilde{y}$,
SbM3. $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{y}, \tilde{x})$,
SbM4. $\tilde{d}(\tilde{x}, \tilde{z}) \leq \tilde{d}(\tilde{x}, \tilde{y}) + \hbar\, \tilde{d}(\tilde{y}, \tilde{z})$.

Then, the pair $(\tilde{X}, \tilde{d})$ is called a strong b-metric space ($SbMS$).

Example 1.1. *[2] Let $\tilde{X} = \mathbb{R}$, and $\tilde{d}(\tilde{x}, \tilde{y}) = \max\{|\tilde{x} - \tilde{y}|, 2|\tilde{x} - \tilde{y}| - 1\}$ for all $\tilde{x}, \tilde{y} \in \mathbb{R}$; then, $(\tilde{X}, \tilde{d})$ is an SbMS with $\hbar = 2$. $\tilde{d}$ is not a metric mapping on $\tilde{X}$ since*

$$\tilde{d}(2, 6) \nleq \tilde{d}(2, 2.5) + \tilde{d}(2.5, 6).$$

Fixed point results in metric and generalized metric spaces depend not only on the underlying space but also on the contractive nature of the mappings involved. In many situations, the mapping itself is not a contraction, although some of its iterates satisfy a contractive condition. Such phenomena naturally arise in generalized metric frameworks, including b-metric and strong b-metric spaces. In what follows, we focus on this type of mappings and present key results illustrating this behavior.

First, we mention some key results where the given map $\tilde{T}$ is not necessarily a contraction; however, for some n, the map $\tilde{T}^n$ becomes a contraction. To illustrate this fact, we give the following example.

Example 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(\tilde{x}) = \cos \tilde{x}$. Since*

$$f'(\tilde{x}) = -\sin \tilde{x}, \qquad \sup_{\tilde{x} \in \mathbb{R}} |f'(\tilde{x})| = 1,$$

the function f is not a contraction.

Consider the iterate $g(\tilde{x}) = f^2(\tilde{x}) = \cos(\cos\tilde{x})$*. Then,*

$$g'(\tilde{x}) = \sin(\cos\tilde{x})\,\sin\tilde{x}, \qquad |g'(\tilde{x})| \le \sin 1 < 1.$$

Hence, by the mean value Theorem,

$$|g(\tilde{x}) - g(\tilde{y})| \le (\sin 1)|\tilde{x} - \tilde{y}|, \quad \tilde{x}, \tilde{y} \in \mathbb{R}.$$

Thus, g is a contraction with the contraction constant $k = \sin 1 < 1$*. Consequently, f itself is not a contraction, but its iterate f^2 is.*

In [6], Bryant extended the Banach contraction principle by showing that the map $\tilde{T}$ itself need not be a contraction, provided that there exists a positive integer n such that the iterate $\tilde{T}^n$ is a contraction. Later, Sehgal [18] strengthened this result by allowing the contractive iterate to depend on the point, that is, for each $\tilde{x} \in \tilde{X}$, there exists a positive integer $n(\tilde{x})$ such that $\tilde{T}^{n(\tilde{x})}$ satisfies a contraction condition.

Theorem 1.1 ([18])**.** *Let $(\tilde{X}, \tilde{d})$ be a complete metric space and $\tilde{T} : \tilde{X} \to \tilde{X}$ be a continuous mapping. If for each $\tilde{x} \in \tilde{X}$, there exists a positive integer number $n(\tilde{x})$ such that for all $\tilde{y} \in \tilde{X}$, and for some* $0 < b < 1$,

$$\tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}) \quad \le \quad b\,\tilde{d}(\tilde{x}, \tilde{y}) \tag{1.1}$$

holds, then $\tilde{T}$ has a unique fixed point $\tilde{x}^* \in \tilde{X}$.

In [10], Guseman further refined Sehgal's work by eliminating the requirement for the map to be continuous. Subsequently, Ćirić [7] generalized Sehgal's findings.

Theorem 1.2 ([7])**.** *Let $(\tilde{X}, \tilde{d})$ be a complete metric space and $\tilde{T} : \tilde{X} \to \tilde{X}$ be a mapping. If for each $\tilde{x} \in \tilde{X}$, there exists a positive integer number $n(\tilde{x})$ such that for all $\tilde{y} \in \tilde{X}$, and for some* $0 < b < 1$,

$$\begin{aligned} \tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}) \quad \le \quad & b\,\max\{\tilde{d}(\tilde{x}, \tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^2\tilde{y}), \cdots, \\ & \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x})\} \end{aligned} \tag{1.2}$$

holds, then $\tilde{T}$ has a unique fixed point $\tilde{x}^* \in \tilde{X}$.

After that, Karapınar et al. [11] extended Ćirić's result to b-metric spaces.

Theorem 1.3 ([11])**.** *Let $(\tilde{X}, \tilde{d})$ be a complete b-metric space with $\hbar > 1$ and $\tilde{T} : \tilde{X} \to \tilde{X}$ be a mapping. If for each $\tilde{x} \in \tilde{X}$, there exists a positive integer number $n(\tilde{x})$ such that for all $\tilde{y} \in \tilde{X}$, and for some* $0 < b < \frac{1}{\hbar}$,

$$\begin{aligned} \tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}) \quad \le \quad & b\,\max\{\tilde{d}(\tilde{x}, \tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^2\tilde{y}), \cdots, \\ & \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x})\} \end{aligned} \tag{1.3}$$

holds, then $\tilde{T}$ has a unique fixed point $\tilde{x}^* \in \tilde{X}$.

In 2023, Dung [9] considered b-metric spaces satisfying the Fatou property. A b-metric $\tilde{d}$ is said to satisfy the Fatou property if, for all $\tilde{x}, \tilde{y} \in \tilde{X}$ and every sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty}\tilde{x}_n = \tilde{x}$, we have the following:

$$\tilde{d}(\tilde{x}, \tilde{y}) \le \liminf_{n\to+\infty}\tilde{d}(\tilde{x}_n, \tilde{y}).$$

It is known that continuity of a b-metric implies the Fatou property, whereas the converse does not hold in general. For further details, see Lu et al. [14], Examples 1 and 2, and Remark 1.

Under this assumption, Dung [9] extended the contraction constant considered in [11] to the case $b \in [0, 1)$ by imposing additional assumptions. The main result of Dung [9] can be stated as follows.

Theorem 1.4. *Let $(\tilde{X}, \tilde{d})$ be a complete b-metric space with constant $\hbar \geq 1$, and let $\tilde{T} : \tilde{X} \to \tilde{X}$ be a Ćirić-type contraction that satisfies Relation* (1.3) *with the contraction constant $b \in [0, 1)$. If one of the following conditions is satisfied:*

1) $\tilde{T}$ is continuous,
2) $\tilde{d}$ satisfies the Fatou property,
3) $b \in [0, \frac{1}{\hbar})$,

then $\tilde{T}$ admits a unique fixed point $\tilde{x}^* \in \tilde{X}$, and for every $\tilde{x} \in \tilde{X}$, the sequence $(\tilde{T}^n \tilde{x})_{n\in\mathbb{N}}$ converges to $\tilde{x}^*$.

Since strong b-metric spaces enjoy additional regularity properties, including the Fatou property, it is natural to investigate whether fixed point results for mappings with contractive iterates can be directly obtained within this framework. Moreover, providing clear and constructive proofs contributes to a deeper understanding of the underlying convergence mechanism. The aim of this paper is to establish a fixed point theorem for mappings whose iterates satisfy a contractive condition in strong b-metric spaces. In addition, we derive explicit estimates for the convergence of the associated iterative process and study the stability of fixed points under small perturbations of the operator in the sense of Ulam–Hyers.

2. New fixed point results for mappings with contractive iterates

In this section, we investigate mappings in strong b-metric spaces that possess a contractive iterate at a point, even though they are not contractions in the usual sense. We establish a fixed point theorem that ensures both the existence and uniqueness of the fixed point under these weaker conditions. This result can be viewed as an extension of Ćirić's fixed point theorem [7] to strong b-metric spaces.

Lemma 2.1. *Let $(\tilde{X}, \tilde{d})$ be a complete strong b-metric space with $\hbar \geq 1$ and $\tilde{T} : \tilde{X} \to \tilde{X}$ be a mapping. If for each $\tilde{x} \in \tilde{X}$, there exists a positive integer number $n(\tilde{x})$such that for all $\tilde{y} \in \tilde{X}$, and for some* $0 < k < 1$,

$$\tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}) \leq k \, \max\{\tilde{d}(\tilde{x}, \tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^2\tilde{y}), \cdots, \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x})\}$$

holds, then for each $\tilde{x} \in \tilde{X}$,

$$\sup\{\tilde{d}(\tilde{x}, \tilde{T}^q\tilde{x}) : q \in \mathbb{N}\} \leq \frac{\hbar}{1-k} \max\{\tilde{d}(\tilde{T}^p\tilde{x}, \tilde{x}),\ p = 1, 2, \cdots, n(\tilde{x})\}. \tag{2.1}$$

Proof. Let r be any, but fixed, positive integer and $m(m = m(\tilde{x}; r))$ a positive integer such that

$$\tilde{d}(\tilde{x}, \tilde{T}^m\tilde{x}) = \max\{\tilde{d}(\tilde{x}, \tilde{T}^q\tilde{x}) : 0 < q \leq r\}.$$

We assume that $r > n(\tilde{x})$ and $m > n(\tilde{x})$. Then, we have the following:

$$\tilde{d}(\tilde{x}, \tilde{T}^m\tilde{x}) \quad \leq \quad \hbar\, \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x}) + \tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{T}^{m-n(\tilde{x})}\tilde{x})$$

$$\begin{aligned} &\le \hbar\, \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x}) + k\ \max\{\tilde{d}(\tilde{x}, \tilde{T}^{m-n(\tilde{x})}\tilde{x}), \tilde{d}(\tilde{x}, \tilde{T}^{m+1-n(\tilde{x})}\tilde{x}), \cdots, \\ &\quad \tilde{d}(\tilde{x}, \tilde{T}^{m}\tilde{x}), \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x})\} \\ &\le \hbar\, \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x}) + k\ \max\{\tilde{d}(\tilde{x}, \tilde{T}^{q}\tilde{x}) : 0 < q \le r\}. \end{aligned}$$

Hence, $\tilde{d}(\tilde{x}, \tilde{T}^{m}\tilde{x}) \le \hbar\, \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x}) + k\, \tilde{d}(\tilde{x}, \tilde{T}^{m}\tilde{x})$; then,

$$\tilde{d}(\tilde{x}, \tilde{T}^{m}\tilde{x}) \le \hbar\, \frac{\tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x})}{1-k}.$$

By the arbitrariness of r, this obtains the following:

$$\sup_{r > n(\tilde{x})} \tilde{d}(\tilde{x}, \tilde{T}^{r}\tilde{x}) \le \tilde{d}(\tilde{x}, \tilde{T}^{m}\tilde{x}) \le \hbar\, \frac{\tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x})}{1-k}.$$

Since the set $\{\tilde{T}\tilde{x}, \ldots, \tilde{T}^{\,n(\tilde{x})}\tilde{x}\}$ is finite, it is bounded. By combining this with the above estimate for all $r > n(\tilde{x})$, Relation (2.1) follows. Consequently, for all $\tilde{x} \in \tilde{X}$, the orbit $(\tilde{T}^{r}\tilde{x})_{n\in\mathbb{N}}$ is bounded. □

The next result presents our main fixed point theorem for mappings with a contractive iterate. Additionally, we obtain an estimate that describes how the fixed point changes when the operator is slightly modified.

Theorem 2.1. *Let $(\tilde{X}, \tilde{d})$ be a complete strong b-metric space and $\tilde{T} : \tilde{X} \to \tilde{X}$ be a mapping. If for each $\tilde{x} \in \tilde{X}$, there exists a positive integer number $n(\tilde{x})$such that for all $\tilde{y} \in \tilde{X}$, and for some $0 < b < 1$,*

$$\begin{aligned} \tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}) \ &\le\ b\ \max\{\tilde{d}(\tilde{x}, \tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^{2}\tilde{y}), \cdots, \\ &\quad \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{x})\} \end{aligned} \tag{2.2}$$

holds, then the following hold:

1) *$\tilde{T}$ has a unique fixed point $\tilde{x}^* \in \tilde{X}$ and $\lim_{m\to+\infty} \tilde{T}^{m}(\tilde{x}) = \tilde{x}^*$ for each $\tilde{x} \in \tilde{X}$;*
2) *$\tilde{d}(\tilde{x}_0, \tilde{x}^*) \le \hbar\, \tilde{d}(\tilde{x}_0, \tilde{T}^{n(\tilde{x}_0)}\tilde{x}_0) + \hbar\, \frac{b}{1-b}\, a(\tilde{x}_0)$ for each $\tilde{x}_0 \in \tilde{X}$, where $a(\tilde{x}_0) = \sup\{\tilde{d}(\tilde{x}_0, \tilde{T}^{q}\tilde{x}_0) : q \in \mathbb{N}\}$;*
3) *Let $\tilde{g} : \tilde{X} \to \tilde{X}$ such that there exists $\eta > 0$ with*

$$\tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{g}(\tilde{x})) \le \eta,\ \forall \tilde{x} \in \tilde{X}.$$

Then,

$$\tilde{d}(\tilde{y}^*, \tilde{x}^*) \le \hbar\, \eta + \hbar\, \frac{b}{1-b}\, a(\tilde{y}^*),$$

for all $\tilde{y}^* \in Fix(\tilde{g})$.

Proof. 1) Let $\tilde{x}_0 \in \tilde{X}$ be arbitrary. Let $n_0 = n(\tilde{x}_0)$, $\tilde{x}_1 = \tilde{T}^{n_0}\tilde{x}_0$ and inductively $n_i = n(\tilde{x}_i)$, $\tilde{x}_{i+1} = \tilde{T}^{n_i}\tilde{x}_i$. Then, $(\tilde{x}_i)_{i\in\mathbb{N}}$ is a subsequence of $(\tilde{T}^{r}\tilde{x}_0)_{r\in\mathbb{N}}$. Let $\tilde{x}_k$ be an arbitrary fixed element in $(\tilde{x}_i)_{i\in\mathbb{N}}$ and let $\tilde{x}_p = \tilde{T}^{p}\tilde{x}_0$ and $\tilde{x}_q = \tilde{T}^{q}\tilde{x}_0$ be arbitrary two elements in $(\tilde{T}^{r}\tilde{x}_0)_{r\in\mathbb{N}}$ that are follower elements of $\tilde{x}_k$. Then, for some $u, v \in \mathbb{N}$, we have $\tilde{x}_p = \tilde{T}^{u}\tilde{x}_k$ and $\tilde{x}_q = \tilde{T}^{v}\tilde{x}_k$. Hence,

$$\begin{aligned} \tilde{d}(\tilde{x}_k, \tilde{x}_p) \ &=\ \tilde{d}(\tilde{x}_k, \tilde{T}^{u}\tilde{x}_k) \\ &=\ \tilde{d}(\tilde{T}^{n_{k-1}}\tilde{x}_{k-1}, \tilde{T}^{n_{k-1}}\tilde{T}^{u}\tilde{x}_{k-1}) \end{aligned}$$

$$
\begin{aligned}
&\leq b\,\max\{\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u}\tilde{x}_{k-1}),\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u+1}\tilde{x}_{k-1}),\cdots,\\
&\quad\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u+n_{k-1}}\tilde{x}_{k-1}),\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{n_{k-1}}\tilde{x}_{k-1})\}\\
&= b\,\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u_1}\tilde{x}_{k-1}),
\end{aligned}
$$

where $u_1 \in \{u, u+1, \cdots, u+n_{k-1}, n_{k-1}\}$ and such that

$$
\begin{aligned}
\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u_1}\tilde{x}_{k-1}) = \max\{&\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u}\tilde{x}_{k-1}),\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u+1}\tilde{x}_{k-1}),\cdots,\\
&\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u+n_{k-1}}\tilde{x}_{k-1}),\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{n_{k-1}}\tilde{x}_{k-1})\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u_1}\tilde{x}_{k-1}) &= \tilde{d}(\tilde{T}^{n_{k-2}}\tilde{x}_{k-2},\tilde{T}^{n_{k-2}}\tilde{T}^{u_1}\tilde{x}_{k-2})\\
&\leq b\,\max\{\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_1}\tilde{x}_{k-2}),\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_1+1}\tilde{x}_{k-2}),\cdots,\\
&\quad\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_1+n_{k-2}}\tilde{x}_{k-2}),\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{n_{k-2}}\tilde{x}_{k-2})\}\\
&= b\,\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_2}\tilde{x}_{k-2}),
\end{aligned}
$$

where $u_2 \in \{u_1, u_1+1, \cdots, u_1+n_{k-2}, n_{k-2}\}$ such that

$$
\begin{aligned}
\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_2}\tilde{x}_{k-2}) = \max\{&\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_1}\tilde{x}_{k-2}),\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_1+1}\tilde{x}_{k-2}),\cdots,\\
&\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_1+n_{k-2}}\tilde{x}_{k-2}),\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{n_{k-2}}\tilde{x}_{k-2})\}.
\end{aligned}
$$

Continuing in this way,

$$
\begin{aligned}
\tilde{d}(\tilde{x}_k,\tilde{x}_p) &\leq b\,\tilde{d}(\tilde{x}_{k-1},\tilde{T}^{u_1}\tilde{x}_{k-1})\\
&\leq b^2\,\tilde{d}(\tilde{x}_{k-2},\tilde{T}^{u_2}\tilde{x}_{k-2})\\
&\;\vdots\\
&\leq b^k\,\tilde{d}(\tilde{x}_0,\tilde{T}^{u_k}\tilde{x}_0),
\end{aligned}
$$

where $u_k \in \{u_{k-1}, u_{k-1}+1, \cdots, u_{k-1}+n_0, n_0\}$ such that

$$
\begin{aligned}
\tilde{d}(\tilde{x}_0,\tilde{T}^{u_k}\tilde{x}_0) = \max\{&\tilde{d}(\tilde{x}_0,\tilde{T}^{u_{k-1}}\tilde{x}_0),\tilde{d}(\tilde{x}_0,\tilde{T}^{u_{k-1}+1}\tilde{x}_0),\cdots,\\
&\tilde{d}(\tilde{x}_0,\tilde{T}^{u_{k-1}+n_0}\tilde{x}_0),\tilde{d}(\tilde{x}_0,\tilde{T}^{n_0}\tilde{x}_0)\}.
\end{aligned}
$$

Assume $a(\tilde{x}_0) = \sup\{\tilde{d}(\tilde{x}_0,\tilde{T}^{q}\tilde{x}_0) : q \in \mathbb{N}\}$; then,

$$
\tilde{d}(\tilde{x}_k,\tilde{x}_p) \leq b^k\,\tilde{d}(\tilde{x}_0,\tilde{T}^{u_k}\tilde{x}_0) \leq b^k\,a(\tilde{x}_0). \tag{2.3}
$$

Similarly, $\tilde{d}(\tilde{x}_k,\tilde{x}_q) \leq b^k\,a(\tilde{x}_0)$. Therefore,

$$
\tilde{d}(\tilde{x}_p,\tilde{x}_q) \leq \tilde{d}(\tilde{x}_k,\tilde{x}_p) + \hbar\,\tilde{d}(\tilde{x}_k,\tilde{x}_q) \leq b^k(1+\hbar)\,a(\tilde{x}_0)
$$

By Lemma 2.1, $a(\tilde{x}_0)$ is finite. Thus, the orbit sequence, $(\tilde{T}^r\tilde{x}_0)_{r\in\mathbb{N}}$, is a Cauchy sequence which converges to some $\tilde{x}^* \in \tilde{X}$, such that

$$
\lim_{n\to+\infty}\tilde{d}(\tilde{x}_n,\tilde{x}^*) = 0. \tag{2.4}
$$

To show $\tilde{x}^* = T\tilde{x}^*$, we show that $\tilde{x}^*$ is a fixed point of $\tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*$ at the first as follows:

$$\begin{array}{rcl} \tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*) & \leq & \hbar\, \tilde{d}(\tilde{x}^*, \tilde{T}^m\tilde{x}_0) + \tilde{d}(\tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{T}^{m-n(\tilde{x}^*)}\tilde{x}_0) \\ & \leq & \hbar\, \tilde{d}(\tilde{x}^*, \tilde{T}^m\tilde{x}_0) + b\, \max\{\tilde{d}(\tilde{x}^*, \tilde{T}^{m-n(\tilde{x}^*)}\tilde{x}_0), \cdots, \\ & & \tilde{d}(\tilde{x}^*, \tilde{T}^m\tilde{x}_0), \tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*)\}. \end{array}$$

When m tends to infinity, we have $\tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*) \leq b\, \tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*)$. Thus, $\tilde{x}^*$ is a fixed point of $\tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*$. Now, suppose $\tilde{x}^*$ is not a fixed point of $\tilde{T}$, and assume that

$$\tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*) = \max\{\tilde{d}(\tilde{x}^*, \tilde{T}^q\tilde{x}^*) : 0 < q \leq n(\tilde{x}^*)\}.$$

Then,

$$\begin{array}{rcl} \tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*) & = & \tilde{d}(\tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)+k}\tilde{x}^*) \\ & \leq & b\, \max\{\tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*), \tilde{d}(\tilde{x}^*, \tilde{T}^{k+1}\tilde{x}^*), \cdots, \\ & & \tilde{d}(\tilde{x}^*, \tilde{T}^{k+n(\tilde{x}^*)}\tilde{x}^*), \tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*)\}. \end{array}$$

Since $\tilde{T}^{n(\tilde{x}^*)}\tilde{x}^* = \tilde{x}^*$, for every $m \geq 1$, there exist integers $l \geq 0$ and $q \in \{1, \ldots, n(\tilde{x}^*)\}$ such that $m = l\,n(\tilde{x}^*) + q$. Hence, $\tilde{T}^m\tilde{x}^* = \tilde{T}^{l\,n(x^*)+q}\tilde{x}^* = \tilde{T}^q\tilde{x}^*$. Then, we have

$$\tilde{d}(\tilde{x}^*, \tilde{T}^m\tilde{x}^*) = \tilde{d}(\tilde{x}^*, \tilde{T}^q\tilde{x}^*) \leq \tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*),$$

and

$$\max\{\tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*), \ldots, \tilde{d}(\tilde{x}^*, \tilde{T}^{k+n(\tilde{x}^*)}\tilde{x}^*)\} = \tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*).$$

Therefore,

$$\tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*) \leq b\, \tilde{d}(\tilde{x}^*, \tilde{T}^k\tilde{x}^*),$$

which contradicts the assumption that $\tilde{x}^*$ is not a fixed point of $\tilde{T}$. Hence, $\tilde{T}\tilde{x}^* = \tilde{x}^*$. The uniqueness of a fixed point of $\tilde{T}$ immediately follows from (2.2).
We show that when m tends to infinity, $\tilde{T}^m(\tilde{x}) \to \tilde{x}^*$ for each $\tilde{x} \in \tilde{X}$. If m is sufficiently large, then $m = \alpha\, n(\tilde{x}^*) + \beta$, where $\alpha \in \mathbb{N}, \beta \in \mathbb{N} \cup \{0\}$ and $0 \leq \beta < n(\tilde{x}^*)$. Hence,

$$\begin{array}{rcl} \tilde{d}(\tilde{x}^*, \tilde{T}^m\tilde{x}) & = & \tilde{d}(\tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*, \tilde{T}^{\alpha\, n(\tilde{x}^*)+\beta}\tilde{x}) \\ & \leq & b\, \max\{\tilde{d}(\tilde{x}^*, \tilde{T}^{(\alpha-1)\, n(\tilde{x}^*)+\beta}\tilde{x}), \tilde{d}(\tilde{x}^*, \tilde{T}^{(\alpha-1)\, n(\tilde{x}^*)+\beta+1}\tilde{x}), \cdots, \\ & & \tilde{d}(\tilde{x}^*, \tilde{T}^{\alpha\, n(\tilde{x}^*)+\beta}\tilde{x}), \tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*)\}. \end{array}$$

Let us consider that

$$\begin{array}{rcl} \tilde{d}(\tilde{x}^*, \tilde{T}^{m_1}\tilde{x}) & = & \max\{\tilde{d}(\tilde{x}^*, \tilde{T}^{(\alpha-1)\, n(\tilde{x}^*)+\beta}\tilde{x}), \tilde{d}(\tilde{x}^*, \tilde{T}^{(\alpha-1)\, n(\tilde{x}^*)+\beta+1}\tilde{x}), \cdots, \\ & & \tilde{d}(\tilde{x}^*, \tilde{T}^{\alpha\, n(\tilde{x}^*)+\beta}\tilde{x}), \tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*)\}, \end{array}$$

where $m_1 \in \{(\alpha - 1)\, n(\tilde{x}^*) + \beta, (\alpha - 1)\, n(\tilde{x}^*) + \beta + 1, \cdots, \alpha\, n(\tilde{x}^*) + \beta, n(\tilde{x}^*)\}$.

Similarly,

$$\tilde{d}(\tilde{x}^*, \tilde{T}^{m_1}\tilde{x}) \quad \leq \quad b\, \max\{\tilde{d}(\tilde{x}^*, \tilde{T}^{(\alpha-2)\, n(\tilde{x}^*)+\beta}\tilde{x}), \tilde{d}(\tilde{x}^*, \tilde{T}^{(\alpha-2)\, n(\tilde{x}^*)+\beta+1}\tilde{x}), \cdots,$$

$$\tilde{d}(\tilde{x}^*, \tilde{T}^{(\alpha-1)\,n(\tilde{x}^*)+\beta}\tilde{x}), \tilde{d}(\tilde{x}^*, \tilde{T}^{n(\tilde{x}^*)}\tilde{x}^*)\} = b^2\tilde{d}(\tilde{x}^*, \tilde{T}^{m_2}\tilde{x}),$$

where $m_2 \in \{(\alpha-2)\,n(\tilde{x}^*)+\beta, (\alpha-2)\,n(\tilde{x}^*)+\beta+1, \cdots, (\alpha-1)\,n(\tilde{x}^*)+\beta, n(\tilde{x}^*)\}$.

Hence,

$$\tilde{d}(\tilde{x}^*, \tilde{T}^m\tilde{x}) \le b^2\tilde{d}(\tilde{x}^*, \tilde{T}^{m_2}\tilde{x}).$$

Continuing in this manner, we obtain the following:

$$\tilde{d}(\tilde{x}^*, \tilde{T}^m\tilde{x}) \le b^\alpha\tilde{d}(\tilde{x}^*, \tilde{T}^\beta\tilde{x}) \le b^\alpha r,$$

where $r = \max\{\tilde{d}(\tilde{x}^*, \tilde{T}^q\tilde{x}) : 0 \le q < n(\tilde{x}^*)\}$. When m tends to infinity, we get $\tilde{T}^m\tilde{x} \to \tilde{x}^*$ for each $\tilde{x} \in \tilde{X}$.

2) To prove the second statement, let $\tilde{x}_0 \in \tilde{X}$ be arbitrary. Let $n_0 = n(\tilde{x}_0)$, $\tilde{x}_1 = \tilde{T}^{n_0}\tilde{x}_0$ and inductively $n_i = n(\tilde{x}_i)$, $\tilde{x}_{i+1} = \tilde{T}^{n_i}\tilde{x}_i$. By simple calculation, we have the following:

$$\tilde{d}(\tilde{x}_k, \tilde{x}_{k+p}) \le \hbar\sum_{i=k}^{k+p-1}\tilde{d}(\tilde{x}_i, \tilde{x}_{i+1}).$$

Additionally, by arguing as in the proof of Relation (2.3), we obtain the following:

$$\tilde{d}(\tilde{x}_i, \tilde{x}_{i+1}) \le b^i\tilde{d}(\tilde{x}_0, \tilde{T}^{n(\tilde{x}_i)}\tilde{x}_0) \le b^i a(\tilde{x}_0).$$

Hence,

$$\tilde{d}(\tilde{x}_k, \tilde{x}_{k+p}) \le \hbar\, b^k\,(\frac{1-b^p}{1-b})\,a(\tilde{x}_0).$$

Letting $p \to +\infty$, and since the strong b-metric is continuous, we obtain the following:

$$\tilde{d}(\tilde{x}_k, \tilde{x}^*) \le \hbar\,\frac{b^k}{1-b}\,a(\tilde{x}_0).$$

For $k = 1$, we have $\tilde{d}(\tilde{x}^*, \tilde{x}_1) \le \hbar\,\frac{b}{1-b}\,a(\tilde{x}_0)$. Hence,

$$\begin{aligned}
\tilde{d}(\tilde{x}_0, \tilde{x}^*) &\le \hbar\,\tilde{d}(\tilde{x}_0, \tilde{x}_1) + \tilde{d}(\tilde{x}_1, \tilde{x}^*) \\
&\le \hbar\,\tilde{d}(\tilde{x}_0, \tilde{T}^{n(\tilde{x}_0)}\tilde{x}_0) + \hbar\,\frac{b}{1-b}\,a(\tilde{x}_0) \text{ where } \tilde{x}_1 = \tilde{T}^{n_0}\tilde{x}_0.
\end{aligned}$$

3) For the data dependence of the fixed points, using the result from (2) for $\tilde{x}_0 = \tilde{y}^*$, we have the following:

$$\begin{aligned}
\tilde{d}(\tilde{y}^*, \tilde{x}^*) &\le \hbar\,\tilde{d}(\tilde{y}^*, \tilde{T}^{n(\tilde{y}^*)}\tilde{y}^*) + \hbar\,\frac{b}{1-b}\,a(\tilde{y}^*) \\
&= \hbar\,\tilde{d}(\tilde{g}(\tilde{y}^*), \tilde{T}^{n(\tilde{y}^*)}\tilde{y}^*) + \hbar\,\frac{b}{1-b}\,a(\tilde{y}^*) \\
&\le \hbar\,\eta + \hbar\,\frac{b}{1-b}\,a(\tilde{y}^*).
\end{aligned}$$

This completes the proof of the theorem. □

Remark 2.1. *Since every strong b-metric space satisfies the Fatou property, the assumptions of Theorem 1.4 are fulfilled in our setting. However, our proof is directly obtained in strong b-metric spaces and does not rely on the Fatou property.*

Theorem 2.2. *Let $(\tilde{X}, \tilde{d})$ be a complete strong b-metric space and $\tilde{T} : \tilde{X} \to \tilde{X}$ be a mapping. If for each $\tilde{x} \in \tilde{X}$, there exists a positive integer number $n(\tilde{x})$such that for all $\tilde{y} \in \tilde{X}$, and for some $0 < b < 1$,*

$$\tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}) \leq b\,\tilde{d}(\tilde{x}, \tilde{y}) \tag{2.5}$$

holds, then $\tilde{T}$ has a unique fixed point $\tilde{x}^* \in X$ and $\tilde{T}^n(\tilde{x}) \to \tilde{x}^*$ for each $\tilde{x} \in \tilde{X}$.

Proof. Let Inequality (2.5) hold. Then,

$$\begin{aligned}\tilde{d}(\tilde{T}^{n(\tilde{x})}\tilde{x}, \tilde{T}^{n(\tilde{x})}\tilde{y}) \leq b\,\tilde{d}(\tilde{x}, \tilde{y}) \quad &\leq \quad b\, \max\{\tilde{d}(\tilde{x}, \tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^2\tilde{y}), \cdots,\\ &\qquad \tilde{d}(\tilde{x}, \tilde{T}^n\tilde{y}), \tilde{d}(\tilde{x}, \tilde{T}^n\tilde{x})\}.\end{aligned}$$

All hypothesis of Theorem 2.1 hold. Thus, $\tilde{T}$ has a unique fixed point $\tilde{x}^* \in \tilde{X}$ and $\tilde{T}^n(\tilde{x}) \to \tilde{x}^*$ for each $\tilde{x} \in \tilde{X}$. □

Remark 2.2. *Several classical fixed point results originally established in metric spaces were later extended to the setting of b-metric spaces. In particular, Sehgal's theorem [18, Theorem 2.1] was generalized to b-metric spaces by Bota [5, Theorem 3.2] and Mitrović [15, Theorem 2.1], while the Ćirić-type fixed point result [7, Theorem 1] was extended to b-metric spaces by Karapınar [11, Theorem 2.1] and Dung [9, Theorem 2.1]. The main results of the present paper complement these developments by establishing analogous conclusions in the framework of strong b-metric spaces.*

To conclude this section, we establish an iterative Ulam–Hyers stability result. We start by introducing the definition of iterative Ulam–Hyers stability for fixed point equations.

Definition 2.1. *Let $(\tilde{X}, \tilde{d})$ be a complete strong b-metric space with $\hbar \geq 1$, and let $\tilde{T} : \tilde{X} \to \tilde{X}$ be a mapping. The fixed point equation*

$$\tilde{x} = \tilde{T}\tilde{x} \tag{2.6}$$

is said to be iterative Ulam–Hyers stable if there exists a constant $c > 0$ such that for every $\varepsilon > 0$ and for every $\tilde{y}^* \in \tilde{X}$ satisfying

$$\tilde{d}(\tilde{y}^*, \tilde{T}^{n(\tilde{y}^*)}\tilde{y}^*) \leq \varepsilon$$

for some $n(\tilde{y}^*) \in \mathbb{N}$, there exists a solution $\tilde{x}^*$ of (2.6) such that

$$\tilde{d}(\tilde{y}^*, \tilde{x}^*) \leq c\,\varepsilon.$$

Theorem 2.3. *Let $(\tilde{X}, \tilde{d})$ be a complete strong b-metric space with $\hbar \geq 1$. Assume that all the hypotheses of Theorem 2.2 hold; then, the fixed point problem (2.6) is iterative Ulam-Hyers stable.*

Proof. Let us evaluate the following:

$$\begin{aligned}\tilde{d}(\tilde{y}^*, \tilde{x}^*) \quad &\leq \quad \hbar\,\tilde{d}(\tilde{y}^*, \tilde{T}^{n(\tilde{y}^*)}\tilde{y}^*) + \tilde{d}(\tilde{T}^{n(\tilde{y}^*)}\tilde{y}^*, \tilde{x}^*)\\ &= \quad \hbar\,\varepsilon + \tilde{d}(\tilde{T}^{n(\tilde{y}^*)}\tilde{y}^*, \tilde{T}^{n(\tilde{y}^*)}\tilde{x}^*)\end{aligned}$$

$$\leq \hbar\varepsilon + b\,\tilde{d}(\tilde{y}^*, \tilde{x}^*).$$

Hence,

$$\tilde{d}(\tilde{y}^*, \tilde{x}^*) \leq \frac{\hbar\varepsilon}{1-b}.$$

□

The contractive condition introduced in this paper involves a point-dependent iterate, which represents a distinctive feature of our approach. This naturally raises the question of whether such an iterative framework can be exploited in broader mathematical settings or practical applications. Motivated by this observation, we propose the following open problem.

Open problem. *An interesting direction for future research is to investigate applications of the point-dependent iterate $n(\tilde{x})$ that appears in our contractive condition. In particular, it is worthwhile to determine whether this iterative framework can be employed in the study of ordinary differential equations, partial differential equations, integral equations, and related nonlinear problems.*

3. Conclusions

The obtained results indicate that strong b-metric spaces constitute a natural setting to study iterative methods with contractive behaviors and they open the way for further extensions to other generalized metric structures.

Author contributions

The authors contributed equally to this work. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare that there are no conflicts of interest.

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