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Research article

Inverse problems for subdiffusion equations and approximation results

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Abstract: The aim of this paper is to investigate time-fractional inverse problems involving a self-adjoint operator A and their approximations. We prove existence and uniqueness results for the solution of the inverse problem consisting of the identification of a single constant conductivity, under overdeterminating conditions of energy type. Then we study their approximations in terms of suitable inverse problems via the Mosco convergence of the associated energies. Some applications to boundary value problems are also presented.

Keywords: inverse problems; Mosco convergence; fractional time derivatives; evolution equations in Hilbert spaces; resolvent families
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1. Introduction

The literature on inverse problems for fractional-in-time heat-type equations is not so extensive. As far as we know, a large part of the literature for inverse problems for fractional-in-time evolution equations deals with only one operator A, possibly non-autonomous, and investigates the identification of the source term and/or the fractional time derivative order; see, e.g., the monographs [23, 26]. Regarding the study of a fractional-in-time evolution equation in a Banach space, we cite the recent paper [16], where the authors, after proving well-posedness results for the direct problem, apply them to an inverse problem of determining an initial value and establishing the uniqueness.

Additional results on inverse source and backward problems associated with the Caputo fractional time derivative and the fractional Laplacian operator in space can be found in [37]. In the one-dimensional case, the determination of the fractional order α and the diffusion coefficient from boundary data are treated in [10]. Other inverse problems for one-dimensional fractional-in-time evolution equations are considered in [14, 21].

Direct and inverse problems for the nonhomogeneous time-fractional heat equation with a time-dependent diffusion coefficient for a positive operator are considered in [38] under linear overdeterminating conditions.

The case of inverse problems for time-fractional diffusion equations governed by operators in divergence form in the framework of homogenization problems is considered in [25]. We point out that, in [25], the identification of a diffusion coefficient for operators in divergence form in the one-dimensional case is considered, by assigning a point-wise overdeterminating condition. In particular, uniqueness and stability results for the solution of the inverse problem are proved.

In [11], we investigated inverse problems consisting of the identification of n constant coefficients for a fractional-in-time Cauchy problem governed by a finite sum of n positive self-adjoint conditions, coupled with overdeterminating conditions of energy type. We proved a conditioned existence result for the solution of the inverse problem, which is complemented with some numerical calculations; this was due to the presence of n different multiplicative constants.

In the present paper, we prove well-posedness results for the inverse problem consisting of the identification of a single constant conductivity for a fractional-in-time abstract Cauchy problem. This Cauchy problem is governed by a self-adjoint operator A, coupled with an overdeterminating condition quite feasible for industrial and physical applications. In view of future numerical approximation, we address the study of its approximation in terms of suitable inverse problems.

More precisely, we consider the direct time-fractional abstract Cauchy problem, formally stated as follows:

$$(P^\alpha)\begin{cases}\partial_t^\alpha u(t) + \lambda A u(t) = 0, \quad t > 0,\\ u(0) = u_0,\end{cases}$$

where $\alpha \in (0, 1]$, A is the generator of an associated coercive Dirichlet form E in a Hilbert space H, and the initial datum $u_0 \neq 0$ belongs to H.

Our goal is twofold. First, we investigate the well-posedness of the inverse problem P^α (defined in Section 3.1, see (3.5) and (3.6)), consisting of finding the solution (λ_*, u_*), where $u_*(t) \in H$ and $\lambda_* > 0$, with the additional condition

$$\|u_*(\overline{T})\| = \rho.$$

Here $\rho > 0$ is given, $\overline{T} > 0$ is a fixed measuring time instant, and $\|\cdot\|$ denotes the norm on H. We remark that the above requirement, in most of the physical and engineering applications, corresponds to measuring the energy of u at a fixed time. This leads to a quadratic functional of the solution, which in the relevant cases (such as, for instance, thermal subdiffusion devised in Section 4) is a feasible measurement.

Second, we address the question of whether, under suitable assumptions, the solution of the inverse problem can be seen as the limit of solutions of approximating problems.

To this aim, for every $n \in \mathbb{N}$, we introduce a sequence of problems (P_n^α), formally stated as

$$(P_n^\alpha) \begin{cases} \partial_t^\alpha u_n(t) + \lambda_n A_n u_n(t) = 0, & t > 0, \\ u_n(0) = u_0^{(n)}, \end{cases}$$

where $\{\lambda_n\}_n$ is a sequence of positive numbers, A_n is the generator of an associated coercive Dirichlet form E_n in H, and $u_0^{(n)} \neq 0$ belongs to H.

After proving the well-posedness results for problems (P^α) and (P_n^α) via operator theory [19], we consider the sequence of inverse problems P_n^α (defined in Section 3.1) associated to (P_n^α), for every $n \in \mathbb{N}$.

We show that, for every $n \in \mathbb{N}$, there exists a unique solution (λ_n, u_n) of P_n^α under the additional condition

$$\|u_n(\overline{T}_n)\| = \rho_n.$$

Here $\rho_n > 0$ is given and $\overline{T}_n > 0$ is a fixed measuring time instant. The well-posedness results of the inverse problems are proved in Section 3, where we extend and deepen the results proved in [31] (see also [32]).

Hence, we focus on the convergence of the sequence of solutions (λ_n, u_n), and we aim at proving that there exists a nontrivial limit which is indeed the solution of P^α.

Investigating the asymptotic behavior of the sequence of solutions (λ_n, u_n) relies on the notion of variational convergence. More precisely, by assuming that the sequence of energies E_n Mosco-converges to E in H (see Definition 2.3), we can prove that the sequence of the solutions of the direct problems (P_n^α) converges in H to the solution of (P^α) (see Theorem 3.4) and the sequence of the solutions of the corresponding approximating inverse problem P_n^α converges to the solution of the inverse problem P^α (see Theorem 3.5). This result extends in a natural way the result contained in [12] for the case $\alpha = 1$ to the fractional case $0 < \alpha < 1$. Indeed, in [12], we considered P^α for $\alpha = 1$, where $H = L^2(\Omega)$, Ω is an irregular domain possibly with a fractal boundary, A is the generator of a suitable Dirichlet form on $L^2(\Omega)$, and $0 \neq u_0 \in L^2(\Omega)$.

Our abstract result applies to different situations and different boundary value problems with various boundary conditions. Here, in Section 4, we present an application to fractional-in-time transmission problems across highly conductive fractal layers. These problems are modeled by a fractional heat equation with Wentzell-type transmission conditions on the interface. We note that these results can also be applied in the case of nonlocal (in space) operators such as those considered in Section 3.3 of [12]. A further application in Section 4 is the study of time-fractional differential equations in periodic structures with a small parameter $\varepsilon > 0$ and zero Dirichlet boundary conditions. Note that these periodic structures with a small parameter $\varepsilon > 0$ play an important role in the study of composite materials.

The paper is organized as follows.

In Section 2, we recall important results on Dirichlet forms, their associated operators, and semigroups. Moreover, we state general results on the fractional-in-time Caputo-type derivatives.

In Section 3, we prove the main results of the paper by providing the existence and uniqueness of the solutions of the direct and inverse problems at hand (Section 3.1) and proving, in addition to the convergence results for the direct problems, the convergence of the sequence of the solutions of the inverse problems P_n^α to the solution of the inverse problem P^α (see Theorem 3.5).

In Section 4, we present some meaningful applications.

2. Preliminaries

We set $\mathbb{R}_+ = (0, \infty)$ and $\overline{\mathbb{R}}_+ = [0, \infty)$. Let H be a (real) separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and its related norm $\| \cdot \|$. We denote by $\mathfrak{L}(H)$ the space of linear and continuous functionals on H, and we denote its norm by $\| \cdot \|_{\mathfrak{L}(H)}$.

We consider an energy form $E[\cdot]$ with effective domain $D(E) \subset H$ dense in H. We also consider a sequence of energy forms $E_n[\cdot]$ with effective domain $D(E_n) \subset H$ dense in H. We extend the forms E and E_n to the whole space H by defining

$$E[u] = +\infty \ \text{ for every } u \in H \setminus D(E)$$

and

$$E_n[u] = +\infty \ \text{ for every } u \in H \setminus D(E_n).$$

We equip $D(E)$ and $D(E_n)$ with the induced norms

$$\|u\|_{D(E)} := E[u] + \|u\|$$

and

$$\|u\|_{D(E_n)} := E_n[u] + \|u\|,$$

respectively. By $E(u, v)$ and $E_n(u, v)$, we denote the corresponding bilinear forms defined by polarization.

Definition 2.1. *(See Section 1.1 in [18]) E is a* symmetric form *if it is a non-negative definite symmetric bilinear form densely defined on H.*

We now recall the notion of *Dirichlet form* (for the main properties of Dirichlet forms, we refer to [18]).

We say that a symmetric form E is *closed* if for every Cauchy sequence $\{u_k\} \subseteq D(E)$, there exists $u \in D(E)$ such that

$$\|u_k - u\|_{D(E)} = E[u_k - u] + \|u_k - u\| \xrightarrow[k \to +\infty]{} 0.$$

Moreover, a closed symmetric form E is *Markovian* if for every $u \in D(E)$, it holds that

$$v := (u \vee 0) \wedge 1 \in D(E) \text{ and } E[v] \leq E[u].$$

Definition 2.2. *A symmetric form E on H is a Dirichlet form if it is closed and Markovian.*

From now on, we suppose that E and E_n are coercive and continuous Dirichlet forms on H.

Since $(E, D(E))$ is a closed bilinear form on H with domain $D(E)$ dense in H, from [24, Theorem 2.1, Chap. 6], there exists a unique self-adjoint non-negative operator A on H, with domain $D(A) \subseteq D(E)$ dense in H, such that

$$E(u, v) = \langle Au, v \rangle, \ u \in D(A), v \in D(E). \tag{2.1}$$

Moreover, in Theorem 1.3.1 of [18], it is proved that, to each closed symmetric form E, we can associate a family of linear operators $\{G_\beta, \beta > 0\}$ with the following property:

$$E(G_\beta u, v) + \beta \langle G_\beta u, v \rangle = \langle u, v \rangle, \ u \in H \text{ and } v \in D(E),$$

and this family is a strongly continuous resolvent with generator A, which also generates a strongly continuous semigroup $\{e^{-tA}\}_{t\geq 0}$ acting on H.

Proceeding as above, for every $n \in \mathbb{N}$, we denote by A_n the unique self-adjoint non-negative operator on H associated to E_n (which is supposed to be a symmetric form), with domain $D(A_n) \subseteq D(E_n)$ dense in H.

Moreover, we denote by $\{G^n_\beta, \beta > 0\}$ and $\{e^{-tA_n}\}_{t\geq 0}$ the resolvents and the semigroups, respectively, associated to E_n, for every $n \in \mathbb{N}$.

We recall here the main properties of the semigroups $\{e^{-tA}\}_{t\geq 0}$ and $\{e^{-tA_n}\}_{t\geq 0}$.

Proposition 2.1. *Let $\{e^{-tA}\}_{t\geq 0}$ and $\{e^{-tA_n}\}_{t\geq 0}$ be the strongly continuous semigroups generated by the operators $-A$ and $-A_n$, respectively. Then $\{e^{-tA}\}_{t\geq 0}$ and $\{e^{-tA_n}\}_{t\geq 0}$ are analytic contraction semigroups on H.*

The proof follows from [39, Theorem 6.2, Chapter 4], the Lumer-Phillips theorem (see Theorem 4.3, Chapter 1 in [36]), and Theorem 1.3.1 in [18].

We now recall the notion of M-convergence of energy forms adapted to our case, see [34, Definition 2.1.1] (cf. also [33]).

Definition 2.3. *Let H be a Hilbert space. A sequence of forms $\{E_n\}$ defined in H M-converges to a form E in H if the following conditions hold:*

a) *for every $\{v_n\} \in H$ weakly converging to $u \in H$,*

$$\varliminf_{n\to\infty} E_n[v_n] \geq E[u];$$

b) *for every $u \in H$, there exists a sequence $\{u_n\} \in H$ strongly converging to u in H, such that*

$$\varlimsup_{n\to\infty} E_n[u_n] \leq E[u].$$

Some important consequences of the M-convergence of energy forms hold.

Theorem 2.1. *(See Theorem 2.4.1 in [34]) If E_n M-converges to E, then the sequence of resolvents $\left\{G^n_\beta\right\}$ associated with the form E_n converges to the resolvent $\{G_\beta\}$ associated with the form E in the strong operator norm on H.*

Theorem 2.2. *(See Corollary 2.6.1 in [34]) If E_n M-converges to E, then the sequence of semigroups $\left\{e^{-tA_n}\right\}$ associated with the form E_n converges to the semigroup e^{-tA} associated with the form E in the operator norm uniformly on every interval $0 < t \leq T$.*

Definition 2.4. *(See Definition 2.3.1 in [34]) We say that a sequence of forms E_n is* asymptotically compact *in H if every sequence $\{u_n\}$ in H such that*

$$\varliminf_{n\to\infty} \left(E_n[u_n] + \|u_n\|^2\right) < +\infty$$

admits a subsequence that converges strongly in H.

Remark 2.1. *We remark that, if $\{E_n\}$ is asymptotically compact in H, then the notion of M-convergence is equivalent to the notion of Γ-convergence (see [33]).*

Theorem 2.3. *(See Theorem 2.6 in [27]) If E_n M-converges to E and E_n is asymptotically compact in H, then the limit set of the spectrum of A_n coincides with the spectrum of A.*

Theorem 2.4. *Let E_n be a sequence of asymptotically compact forms in H that M-converges to E. Assume also that E_n and E are coercive. Then, if μ_n^k denotes the k-th eigenvalue of A_n, for every k, it holds that* $\inf_n \mu_n^k > 0$.

Proof. Let us consider the sequence $\{\mu_n\}_n$ of minimal eigenvalues of A_n, and let us denote by $u_n \in D(A_n)$ the corresponding eigenfuction, i.e.,

$$A_n u_n = \mu_n u_n.$$

If we take the scalar product in H between both members of the above equation and $u_n \in D(A_n)$, using (2.1), we obtain

$$E_n[u_n] = \langle A_n u_n, u_n \rangle = \mu_n \|u_n\|^2. \tag{2.2}$$

Without loss of generality, up to normalization, we can suppose that $\|u_n\| = 1$. This implies that there exists a subsequence, which we still denote by $\{u_n\}$, which weakly converges to u in H.

Since E_n M-converges to E in H by hypothesis, this means that

$$\underline{\lim}_{n\to\infty} \left(E_n[u_n] + \|u_n\|^2 \right) \leq \overline{\lim_{n\to\infty}} \left(E_n[u_n] + 1 \right) < +\infty.$$

From the asymptotic compactness, it follows that actually u_n converges strongly to u in H. This implies that $\|u\| = 1$.

Hence, taking the lim inf in (2.2) and using again the M-convergence of the energies, we obtain that

$$\underline{\lim}_{n\to\infty} \mu_n = \underline{\lim}_{n\to\infty} E_n[u_n] \geq E[u],$$

and the right-hand side of the above inequality is strictly positive, since E is coercive and $u \neq 0$. □

Following [6, Theorem 7.7], we now recall some important properties of analytic semigroups generated by self-adjoint operators.

Proposition 2.2. *Let $\{e^{-tA}\}_{t\geq 0}$ be the semigroup generated by $-A$ on H. Then*

(1) for every $u_0 \in H$, the function $u(t) = e^{-tA}u_0$, $t > 0$, is the unique solution to the Cauchy problem

$$\begin{cases} \partial_t u(t) + Au(t) = 0, & t > 0, \\ u(0) = u_0, \end{cases}$$

with regularity

$$u \in C(\overline{\mathbb{R}}_+; H) \cap C^1(\mathbb{R}_+; H) \cap C\left(\mathbb{R}_+; D(A)\right);$$

(2) the identity

$$\frac{\partial}{\partial t}(e^{-tA}u_0) = -Ae^{-tA}u_0$$

holds for every $t > 0$ and every $u_0 \in H$;

(3) *if $\mu > 0$ denotes the smallest eigenvalue of A, the decay estimate*

$$\|e^{-tA}\|_{\mathfrak{L}(H)} \leq e^{-\mu t}$$

holds for every $t \geq 0$.

We now recall the main results on time-fractional derivatives. Following the notation in [19], for $\alpha \in (0, 1)$, we set

$$g_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where Γ denotes the usual Gamma function.

Definition 2.5. *Let Y be a Banach space, $T > 0$, and let $f \in C([0, T]; Y)$ be such that $g_{1-\alpha} * f \in W^{1,1}((0, T); Y)$.*

i) The Riemann-Liouville *fractional derivative of order $\alpha \in (0, 1)$ is defined as follows:*

$$D_t^\alpha f(t) := \frac{\mathrm{d}}{\mathrm{d}t}(g_{1-\alpha} * f)(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g_{1-\alpha}(t - \tau) f(\tau)\, \mathrm{d}\tau,$$

for a.e. $t \in (0, T]$.

ii) The Caputo-type *fractional derivative of order $\alpha \in (0, 1)$ is defined as follows:*

$$\partial_t^\alpha f(t) := D_t^\alpha (f(t) - f(0)),$$

for a.e. $t \in (0, T]$.

Remark 2.2. *We point out that Definition 2.5(ii) gives a weaker definition of the Caputo fractional derivative with respect to the original one in [9], since f is not assumed to be differentiable. Moreover, it holds that $\partial_t^\alpha(c) = 0$ for every constant $c \in \mathbb{R}$.*

We refer to [13] for more details on fractional derivatives.

Under the assumptions already introduced, we take $T > 0$, $u_0 \in H$, and $f : (0, T) \to H$ and consider the following non-homogeneous fractional Cauchy problem:

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) = f(t), & t \in (0, T), \\ u(0) = u_0. \end{cases} \tag{2.3}$$

Here we define the notion of *strong* solution for the problem (2.3).

Definition 2.6. *(See Definition 2.1.4 in [19]) We say that $u \in C([0, T); H)$ is a* strong solution *for* (2.3) *in* $[0, T]$ *if $u(0) = u_0$ and, for any $T_1, T_2 \in (0, T)$, $T_1 \leq T_2$,*

i) *$u(t) \in D(A)$ for all $t \in [T_1, T_2]$;*
ii) *$\partial_t^\alpha u \in C([T_1, T_2]; H)$;*
iii) *the differential equation $\partial_t^\alpha u(t) + Au(t) = f(t)$ is satisfied on $[T_1, T_2]$.*

We will now state a well-posedness result for (2.3), which will yield an explicit representation formula for its solution. We first recall the definition of the Wright-type function (see [20, Formula (28)]):

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)}, \quad 0 < \alpha < 1,\, z \in \mathbb{C}.$$

From [5, page 14], it holds that $\Phi_\alpha(t)$ is a probability density function, i.e.,

$$\Phi_\alpha(t) \geq 0 \quad \text{if } t > 0, \qquad \int_0^{+\infty} \Phi_\alpha(t)\,\mathrm{d}t = 1.$$

In addition, we have

$$\int_0^{\infty} t^p \Phi_\alpha(t)\ \mathrm{d}t = \frac{\Gamma(p+1)}{\Gamma(\alpha p + 1)}, \quad p > -1, \quad 0 < \alpha < 1$$

(see, e.g., [20]).

For more properties on the Wright function, we refer, e.g., to [5, 20, 40].

For $t \geq 0$ and $x \in H$, we define the (bounded) operators

$$S_\alpha(t)x := \int_0^{\infty} \Phi_\alpha(\tau) e^{-\tau t^\alpha A} x\,\mathrm{d}\tau \quad \text{and} \quad P_\alpha(t)x := \alpha t^{\alpha-1} \int_0^{\infty} \tau \Phi_\alpha(\tau) e^{-\tau t^\alpha A} x\,\mathrm{d}\tau.$$

Theorem 2.5. *Fix $T > 0$. Let $f : (0, T) \to H$ fulfill the following assumptions:*

(a) $f \in C^{0,\beta}((0, T); H)$ for some $\beta > 0$;
(b) there exists $r > 1/\alpha$ such that $\int_0^{T_0} \|f(t)\|^r\,\mathrm{d}t < \infty$ for some $T_0 \in (0, T)$.

Then, there exists a unique strong solution of (2.3) *(in the sense of Definition 2.6), given by*

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(s)\,\mathrm{d}s, \quad t \in (0, T).$$

Moreover, the following estimates hold:

$$\|S_\alpha(t)x\| \leq \|x\| \quad \text{and} \quad \|P_\alpha(t)x\| \leq \alpha t^{\alpha-1} \frac{\Gamma(2)}{\Gamma(1+\alpha)} \|x\|, \quad t > 0, \quad x \in H.$$

For the proof, we refer to [19, Theorem 2.1.7].

3. Time-fractional direct and inverse problems and their asymptotics

In this section, we state and prove the main abstract results in the paper. We take $\alpha \in (0, 1)$. We first prove the existence and uniqueness of the (strong) solution to the inverse conduction problem for a single constant in a family of abstract time-fractional homogeneous equations in H. Then, we prove a convergence result for the solutions of the inverse problems we will introduce.

3.1. Direct and inverse problems: existence and uniqueness results

Let $\lambda > 0$, $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers, and $T > 0$. Under the hypotheses and notations of Section 2, we consider the following abstract Cauchy problems:

$$(P^\alpha)\begin{cases}\partial_t^\alpha u + \lambda A u(t) = 0, & t \in (0,T),\\ u(0) = u_0,\end{cases}$$

and, for every $n \in \mathbb{N}$,

$$(P_n^\alpha)\begin{cases}\partial_t^\alpha u_n(t) + \lambda_n A_n\, u_n(t) = 0, & t \in (0,T),\\ u_n(0) = u_0^{(n)}.\end{cases}$$

From Theorem 2.5, given $u_0, u_0^{(n)} \in H$, problems (P^α) and (P_n^α), for every $n \in \mathbb{N}$, admit unique strong solutions given by the formulas

$$u(t) = S_{\alpha,\lambda}(t)u_0, \quad t \in (0,T), \tag{3.1}$$

and, for every $n \in \mathbb{N}$,

$$u_n(t) = S^{(n)}_{\alpha,\lambda_n}(t)\, u_0^{(n)}, \quad t \in (0,T), \tag{3.2}$$

respectively. Here, for every $t \in (0,T)$, we define

$$S_{\alpha,\lambda}(t)u_0 := \int_0^\infty \Phi_\alpha(\tau) e^{-\tau t^\alpha \lambda A} u_0 \,\mathrm{d}\tau \tag{3.3}$$

and, for every $n \in \mathbb{N}$,

$$S^{(n)}_{\alpha,\lambda_n}(t)u_0^{(n)} := \int_0^\infty \Phi_\alpha(\tau) e^{-\tau t^\alpha \lambda_n A_n} u_0^{(n)} \,\mathrm{d}\tau, \tag{3.4}$$

where $\left\{e^{-t\lambda A}\right\}_{t\geq 0}$ (resp., $\left\{e^{-t\lambda_n A_n}\right\}_{t\geq 0}$) is the analytic semigroup generated by the operator $-\lambda A$ (resp., the analytic semigroup generated by the operator $-\lambda_n A_n$) as in Section 2.

We now present the rigorous abstract formulation of the inverse problem that we will investigate.

Problem P^α**.** *Find a function $u_* : [0,T] \to H$ and a real constant $\lambda_* > 0$ satisfying the Cauchy problem*

$$\begin{cases}\partial_t^\alpha u(t) + \lambda A u(t) = 0, & t \in (0,T),\\ u(0) = u_0,\end{cases} \tag{3.5}$$

and the additional condition

$$\|u(\overline{T})\| = \rho, \tag{3.6}$$

where $0 \neq u_0 \in H$ and $\rho > 0$ are given, and $\overline{T} \in (0,T)$ is a fixed measuring time instant.

Theorem 3.1. *For a fixed $T > 0$, for every $0 \neq u_0 \in H$, and every $0 < \rho < \|u_0\|$, the inverse problem P^α admits a unique solution (λ_*, u_*) such that*

$$u_* \in C([T_1,T_2]; D(A)) \cap C([0,T); H), \quad \partial_t^\alpha u_* \in C([T_1,T_2]; H),$$

$\lambda_* \in \mathbb{R}_+$, $0 < T_1 \leq T_2 < T$, and $\overline{T} \in [T_1,T_2]$.

In order to prove Theorem 3.1, we first remark that, if we replace (3.1) in the overdeterminating condition (3.6) at the instant $t = \overline{T}$, the unknown constant λ must fulfill the following nonlinear equation:

$$f(\lambda) := \|S_{\alpha,\lambda}(\overline{T})u_0\|^2 = \rho^2, \quad \lambda > 0. \tag{3.7}$$

Hence, it is clear that the well-posedness for P^α is a consequence of the one for (3.7). This requires an investigation of the basic properties of the function f. Indeed, the following result holds.

Proposition 3.1. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be defined as in* (3.7). *Then:*

i) $f \in C^1(\mathbb{R}_+)$ *with* $f'(\lambda) < 0$ *for all* $\lambda > 0$*;*

ii) $\mathrm{Im} f = \left(0, \|u_0\|^2\right)$.

Proof. The regularity of f follows immediately by Proposition 2.2(1). Moreover, Proposition 2.2(2) implies, for every $T > 0$,

$$\frac{\partial}{\partial\lambda}(e^{-T\lambda A}u_0) = -TAe^{-T\lambda A}u_0$$

for all $\lambda > 0$. Then, by using the chain rule, the semigroup properties, and the coercivity of A, we get

$$\begin{aligned}
f'(\lambda) = \frac{\partial}{\partial\lambda}\|S_{\alpha,\lambda}(\overline{T})u_0\|^2 &= \frac{\partial}{\partial\lambda}\left\langle \int_0^\infty \Phi_\alpha(\tau)e^{-\tau\overline{T}^\alpha\lambda A}u_0\,\mathrm{d}\tau, \int_0^\infty \Phi_\alpha(\sigma)e^{-\sigma\overline{T}^\alpha\lambda A}u_0\,\mathrm{d}\sigma\right\rangle \\
&= \frac{\partial}{\partial\lambda}\int_0^\infty\int_0^\infty \Phi_\alpha(\tau)\Phi_\alpha(\sigma)\left\langle e^{-\frac{\tau+\sigma}{2}\overline{T}^\alpha\lambda A}u_0, e^{-\frac{\tau+\sigma}{2}\overline{T}^\alpha\lambda A}u_0\right\rangle \mathrm{d}\tau\,\mathrm{d}\sigma \\
&= -2\overline{T}^\alpha\int_0^\infty\int_0^\infty \Phi_\alpha(\tau)\Phi_\alpha(\sigma)\left(\frac{\tau+\sigma}{2}\right)\left\langle Ae^{-\frac{\tau+\sigma}{2}\overline{T}^\alpha\lambda A}u_0, e^{-\frac{\tau+\sigma}{2}\overline{T}^\alpha\lambda A}u_0\right\rangle \mathrm{d}\tau\,\mathrm{d}\sigma \\
&= -\overline{T}^\alpha\int_0^\infty\int_0^\infty \Phi_\alpha(\tau)\Phi_\alpha(\sigma)\,(\tau+\sigma)\left\|A^{\frac{1}{2}}e^{-\frac{\tau+\sigma}{2}\overline{T}^\alpha\lambda A}u_0\right\|^2 \mathrm{d}\tau\,\mathrm{d}\sigma < 0,
\end{aligned}$$

thus proving *i)*. We then notice that *ii)* is a direct consequence of the continuity of f and Proposition 2.2(3). □

Proposition 3.1 implies that, for every $0 < \rho < \|u_0\|$, Eq (3.7) admits a unique solution λ_*. As a byproduct, the H-valued function

$$u_*(t) = \int_0^\infty \Phi_\alpha(\tau)e^{-\tau t^\alpha\lambda_* A}u_0\,\mathrm{d}\tau, \quad t \in [T_1, T_2] \subset (0, T),$$

solves both the Cauchy problem (3.5) and the additional condition (3.6). This proves Theorem 3.1.

Now we investigate the following inverse problems depending on $n \in \mathbb{N}$.

Problem P_n^α. *Find a function $u_n : [0, T] \to H$ and a real constant $\lambda_n > 0$ satisfying the Cauchy problem*

$$\begin{cases} \partial_t^\alpha u_n(t) + \lambda_n A_n u_n(t) = 0, & t \in (0, T), \\ u_n(0) = u_0^{(n)}, \end{cases} \tag{3.8}$$

and the additional condition

$$\|u_n(\overline{T}_n)\| = \rho_n, \tag{3.9}$$

where $0 \neq u_0^{(n)} \in H$ and $\rho_n > 0$ are given, and $0 < \overline{T}_n < T$ is a measuring time instant.

The same argument used to prove the well-posedness of P^α immediately yields the following existence and uniqueness result for problem P^α_n.

Theorem 3.2. *Let $n \in \mathbb{N}$ and $T > 0$ be fixed. For every $0 \neq u_0^{(n)} \in H$ and every $0 < \rho_n < \|u_0^{(n)}\|$, problem P^α_n admits a unique solution (λ_n, u_n) such that*

$$u_n \in C([T_1, T_2]; D(A_n)) \cap C([0, T); H), \quad \partial_t^\alpha u_n \in C([T_1, T_2]; H),$$

where $0 < T_1 \leq T_2 < T$, $\overline{T}_n \in [T_1, T_2]$, and

$$u_n(t) = \int_0^\infty \Phi_\alpha(\tau) e^{-\tau t^\alpha \lambda_n A_n} u_0^{(n)} \, \mathrm{d}\tau, \quad t \in [T_1, T_2].$$

Moreover, $\lambda_n \in \mathbb{R}_+$ satisfies the equation

$$f_n(\lambda) := \left\| S^{(n)}_{\alpha,\lambda}(\overline{T}_n) \, u_0^{(n)} \right\|^2 = \rho_n^2, \quad \lambda > 0, \tag{3.10}$$

where we set

$$S^{(n)}_{\alpha,\lambda}(t) u_0^{(n)} := \int_0^\infty \Phi_\alpha(\tau) e^{-\tau t^\alpha \lambda A_n} u_0^{(n)} \, \mathrm{d}\tau. \tag{3.11}$$

We remark that, by adapting the proof of Theorem 3.6 in [11], we can prove the following result.

Proposition 3.2. *Let $u_0, u_0^{(n)} \in H$ be non-identically zero, $(\lambda_\alpha, u_\alpha)$ be the unique solution of problem P^α given by Theorem 3.1 and, for every $n \in \mathbb{N}$, let $(\lambda_{\alpha,n}, u_{\alpha,n})$ be the unique solution of problem P^α_n given by Theorem 3.2. Then, as $\alpha \to 1^-$,*

$$u_\alpha(t) \to u_1(t) \text{ in } H$$

for all $t \in (0, T)$, and

$$\lambda_\alpha \to \lambda_1 \text{ in } \mathbb{R}.$$

Analogously, for every $n \in \mathbb{N}$, as $\alpha \to 1^-$,

$$u_{\alpha,n}(t) \to u_{1,n}(t) \text{ in } H$$

for all $t \in (0, T)$, and

$$\lambda_{\alpha,n} \to \lambda_{1,n} \text{ in } \mathbb{R}.$$

Remark 3.1. *We observe that in [11], we chose an energy-type overdeterminating condition. This was due to the fact that we were dealing with the identifications of n constant coefficients for a time-fractional Cauchy problem governed by a finite sum of n positive self-adjoint operators. By proceeding as in the proof of Theorem 3.5 in [11], one can replace the overdeterminating condition considered in this paper by another energy-type one and prove a result similar to Proposition 3.1.*

3.2. Asymptotic results

In this subsection, we investigate the convergence of the solutions of the approximating problems (P^α_n) and P^α_n to the solutions of problems (P^α) and P^α, respectively.

From now on, we suppose that the sequence of energy forms $(E_n, D(E_n))$, associated with the sequence of operators A_n appearing in problem (P^α_n), M-converges to the form $(E, D(E))$, associated with the operator A appearing in problem (P^α) considered in Section 3.1.

We start with the direct problems. We first prove this auxiliary result.

Theorem 3.3. *For a fixed $T > 0$, let $S_{\alpha,\lambda}$ and $S^{(n)}_{\alpha,\lambda}$ be defined as in* (3.3) *and* (3.11)*, respectively. Then, under the hypotheses of Theorem 2.2, it holds that*

$$S^{(n)}_{\alpha,\lambda}(t) \xrightarrow[n\to+\infty]{} S_{\alpha,\lambda}(t)$$

in the operator norm uniformly on every interval $[T_1, T_2] \subset (0, T)$.

Proof. We sketch the proof for the readers' convenience. By using the definitions of the operators $S_{\alpha,\lambda}$ and $S^{(n)}_{\alpha,\lambda}$, for $0 \neq u_0 \in H$, we have that

$$\left\|S^{(n)}_{\alpha,\lambda}(t)u_0 - S_{\alpha,\lambda}(t)u_0\right\| = \left\|\int_0^\infty \Phi_\alpha(\tau)e^{-\tau t^\alpha \lambda A_n}u_0\,\mathrm{d}\tau - \int_0^\infty \Phi_\alpha(\tau)e^{-\tau t^\alpha \lambda A}u_0\,\mathrm{d}\tau\right\|$$

$$\leq \int_0^\infty \Phi_\alpha(\tau)\left\|\left(e^{-\tau t^\alpha \lambda A_n} - e^{-\tau t^\alpha \lambda A}\right)u_0\right\|\,\mathrm{d}\tau \leq \int_0^\infty \Phi_\alpha(\tau)\left\|e^{-\tau t^\alpha \lambda A_n} - e^{-\tau t^\alpha \lambda A}\right\|_{\mathfrak{L}(H)}\|u_0\|\,\mathrm{d}\tau.$$

Taking the supremum for $t \in [T_1, T_2] \subset (0, T)$, from Theorem 2.2, the first norm inside the last integral tends to zero as $n \to +\infty$ in the operator norm uniformly on every interval $[T_1, T_2] \subset (0, T)$. The thesis then follows from the dominated convergence theorem. □

Theorem 3.4. *Let $0 < T_1 \leq T_2 < T$. If $u^{(n)}_0 \xrightarrow[n\to+\infty]{} u_0$ strongly in H and $\lambda_n \xrightarrow[n\to+\infty]{} \lambda$, then for every $t \in [T_1, T_2]$, the solution $u_n(t)$ of problem (P^α_n) defined by* (3.2) *converges in H as $n \to +\infty$ to the solution $u(t)$ of problem (P^α) defined by* (3.1)*. Moreover, it holds that $u_n \xrightarrow[n\to+\infty]{} u$ in $L^2([T_1, T_2]; H)$.*

Proof. We sketch the proof for the readers' convenience. We first need to prove that, for every $t \in [T_1, T_2]$,

$$\|u_n(t) - u(t)\| = \left\|S^{(n)}_{\alpha,\lambda_n}(t)\,u^{(n)}_0 - S_{\alpha,\lambda}(t)\,u_0\right\| \xrightarrow[n\to+\infty]{} 0.$$

By adding and subtracting suitable terms, we get that

$$\begin{aligned}\|u_n(t) - u(t)\| &\leq \left\|S^{(n)}_{\alpha,\lambda_n}(t)\,u^{(n)}_0 - S^{(n)}_{\alpha,\lambda}(t)\,u^{(n)}_0\right\| + \left\|S^{(n)}_{\alpha,\lambda}(t)\,u^{(n)}_0 - S^{(n)}_{\alpha,\lambda}(t)\,u_0\right\| \\ &+ \left\|S^{(n)}_{\alpha,\lambda}(t)\,u_0 - S_{\alpha,\lambda}(t)\,u_0\right\| \leq \int_0^\infty \Phi_\alpha(\tau)\left\|e^{-\tau t^\alpha \lambda_n A_n} - e^{-\tau t^\alpha \lambda A_n}\right\|_{\mathfrak{L}(H)}\left\|u^{(n)}_0\right\|\,\mathrm{d}\tau \\ &+ \left\|S^{(n)}_{\alpha,\lambda}(t)\left(u^{(n)}_0 - u_0\right)\right\| + \left\|S^{(n)}_{\alpha,\lambda}(t) - S_{\alpha,\lambda}(t)\right\|_{\mathfrak{L}(H)}\|u_0\|.\end{aligned}$$

The third term on the right-hand side of the above chain of inequalities tends to zero as $n \to +\infty$ from Theorem 3.3. After using the estimates of Theorem 2.5, the second term tends to zero as $n \to +\infty$ since $u^{(n)}_0 \to u_0$ strongly in H. Moreover, since the semigroups generated by $-A$ and $-A_n$ are strongly continuous, analytic, and contractive, λ_n converges to λ, and $u^{(n)}_0$ strongly converges to u in H, then we can conclude that the first term also tends to zero as $n \to +\infty$.

The convergence in $L^2([T_1, T_2]; H)$ follows simply by integrating the above inequalities in $t \in [T_1, T_2]$ and by the dominated convergence theorem. □

We now turn to the main result of this paper. We prove the convergence of the sequence of solutions of the inverse problem P^α_n to the one of the inverse problem P^α.

The proof will rely on the following result (see [4, Theorem 1]).

Lemma 3.1. *Let $f_n : \mathrm{Dom}\, f_n \to \mathrm{Im}\, f_n$ be a sequence of continuous injective real functions defined on a common bounded interval $I \subseteq \bigcap_n \mathrm{Dom}\, f_n$, uniformly converging to a function f on I as $n \to +\infty$. If f is injective, then the sequence f_n^{-1} converges uniformly to f^{-1} on any bounded interval $J \subseteq \bigcap_n \mathrm{Im}\, f_n$.*

Theorem 3.5. *Under the assumptions of Theorems 3.1 and 3.2, let the couple (λ_*, u_*), with $u_* : [0, T] \to H$ and $\lambda_* > 0$, be the solution of the inverse problem P^α. In addition, let the couple (λ_n, u_n) be the solution of the inverse problem P_n^α. Assume that*

$$\begin{cases} \rho_n \to \rho > 0, \\ \overline{T}_n \to \overline{T} > 0, \\ u_0^{(n)} \to u_0 \text{ in } H, \\ \inf_n \mu_n^k > 0 \text{ for every } k \in \mathbb{N}, \end{cases}$$

where we denote by $\mu_n^k \in \mathbb{R}_+$ the k-th eigenvalue of A_n.

Then

$$\begin{cases} \lambda_n \to \lambda_*, \\ u_n(t) \to u_*(t) \text{ in } H \text{ uniformly on every bounded subset } [T_1, T_2] \subset (0, T). \end{cases}$$

Proof. We split the proof in six steps for the readers' convenience.

Step 1. We first prove the boundedness of the sequence $\{\lambda_n\}_n$. We argue by contradiction by assuming that, up to a subsequence, $\lambda_n \to +\infty$ as $n \to +\infty$. We denote by $\mu_n > 0$ the smallest eigenvalue of A_n. Then, using the estimate for $S_\alpha(t)$ in Theorem 2.5 and the estimate in Proposition 2.2(3), we see that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \rho_n = \left\| S^{(n)}_{\alpha,\lambda_n}(\overline{T}_n)\, u_0^{(n)} \right\| &\le \left\| S^{(n)}_{\alpha,\lambda_n}(\overline{T}_n)(u_0^{(n)} - u_0) \right\| + \left\| S^{(n)}_{\alpha,\lambda_n}(\overline{T}_n) u_0 \right\| \\ &\le \|u_0^{(n)} - u_0\| + \int_0^\infty \Phi_\alpha(\tau) \left\| e^{-\tau \overline{T}_n^\alpha \lambda_n A_n} \right\|_{\mathfrak{L}(H)} \|u_0\| \, \mathrm{d}\tau \\ &\le \|u_0^{(n)} - u_0\| + \|u_0\| \int_0^\infty \Phi_\alpha(\tau) e^{-\tau \overline{T}_n^\alpha \mu_n \lambda_n} \, \mathrm{d}\tau. \end{aligned}$$

Since $\inf_n \mu_n > 0$ by hypothesis, we deduce that $\overline{T}_n^\alpha \lambda_n \mu_n$ diverges as $n \to +\infty$. Moreover, $u_0^{(n)} \to u_0$ in H as $n \to +\infty$. Hence, the right-hand side of the above inequality vanishes as $n \to +\infty$. That is, $\rho_n \to \rho = 0$, which contradicts the assumption that $\rho > 0$.

Step 2. Due to the boundedness of the sequence $\{\lambda_n\}_n$ proved in Step 1, it is possible to choose a bounded interval $I \subset \mathbb{R}_+$ large enough so that $\lambda_n \in I$ for every $n \in \mathbb{N}$.

Step 3. We now introduce the function $f_n(\lambda)$, as in (3.10), which satisfies the assumptions of Lemma 3.1. As for the co-domains of f_n, we define the bounded interval

$$J = \bigcap_n \left(0, \|u_0^{(n)}\|^2\right) = \left(0, \inf_n \|u_0^{(n)}\|^2\right).$$

Step 4. We recall that from Theorem 3.3, we have that

$$S^{(n)}_{\alpha,\lambda}(t) \xrightarrow[n\to+\infty]{} S_{\alpha,\lambda}(t)$$

in the operator norm uniformly on every interval $[T_1, T_2] \subset (0, T)$. Then, it follows that

$$f_n(\lambda) \to f(\lambda) \text{ uniformly on } I \text{ as } n \to +\infty,$$

where I is the bounded interval chosen in Step 2. Since the limiting function f is injective, we can apply Lemma 3.1 to deduce

$$f_n^{-1}(\rho) \to f^{-1}(\rho) \text{ uniformly on } J \text{ as } n \to +\infty,$$

where J is the interval defined in Step 3.

Step 5. By uniform convergence, we immediately get

$$\lambda_n = f_n^{-1}(\rho_n) \xrightarrow[n\to+\infty]{} f^{-1}(\rho) = \lambda_*.$$

Step 6. Since λ_n tends to λ_*, again by Theorem 2.2, we deduce that

$$e^{-t\lambda_n A_n} u_0 \to e^{-t\lambda_* A} u_0 \text{ in } H \text{ uniformly on every bounded subset of } \overline{\mathbb{R}}_+.$$

This implies that

$$S^{(n)}_{\alpha,\lambda_n}(t)u_0 \to S_{\alpha,\lambda_*}(t)u_0 \text{ in } H \text{ uniformly on every bounded subset } [T_1, T_2] \subset (0, T). \tag{3.12}$$

Therefore, for every $[T_1, T_2] \subset (0, T)$, again using the estimate in Theorem 2.5 and (3.12), and taking into account (3.3) and (3.4), we have

$$\begin{aligned}
\sup_{t\in[T_1,T_2]} \|u_n(t) - u_*(t)\| &= \sup_{t\in[T_1,T_2]} \left\| S^{(n)}_{\alpha,\lambda_n}(t)u_0^{(n)} - S_{\alpha,\lambda_*}(t)u_0 \right\| \\
&\le \sup_{t\in[T_1,T_2]} \left\| S^{(n)}_{\alpha,\lambda_n}(t)(u_0^{(n)} - u_0) \right\| + \sup_{t\in[T_1,T_2]} \left\| \left(S^{(n)}_{\alpha,\lambda_n}(t) - S_{\alpha,\lambda_*}(t) \right) u_0 \right\| \\
&\le \left\| u_0^{(n)} - u_0 \right\| + \sup_{t\in[T_1,T_2]} \left\| \left(S^{(n)}_{\alpha,\lambda_n}(t) - S_{\alpha,\lambda_*}(t) \right) u_0 \right\| \to 0,
\end{aligned}$$

as $n \to +\infty$, thus concluding the proof of Theorem 3.5. □

4. Applications

In this section, we present some applications of the abstract results obtained in the previous section. In particular, in Section 4.1, we present a model of the fractional heat equation with Wentzell-type transmission conditions on the interface, while in Section 4.2, we apply our results to a two-dimensional time-fractional differential equation in periodic structures by using homogenization techniques.

We start by introducing the following functional spaces. Let $\mathcal{G}$ (resp., $\mathcal{S}$) be an open (resp., closed) subset of $\mathbb{R}^N$. By $L^p(\mathcal{G})$, for $p \ge 1$, we denote the Lebesgue space with respect to the Lebesgue measure $\mathrm{d}\mathcal{L}_N$, which will be left to the context whenever it does not create ambiguity. By $L^p(\partial\mathcal{G})$, we denote the Lebesgue space on $\partial\mathcal{G}$ with respect to a positive Borel measure μ supported on $\partial\mathcal{G}$. By $\mathcal{D}(\mathcal{G})$, we denote the space of infinitely differentiable functions with compact support in $\mathcal{G}$. By $W^{s,p}(\mathcal{G})$, where $s \in \mathbb{R}_+$, we denote the usual (possibly fractional) Sobolev spaces (see [1, 35]) and $W^{s,p}_0(\mathcal{G})$ denotes the closure of $\mathcal{D}(\mathcal{G})$ with respect to the $\|\cdot\|_{W^{s,p}}$-norm. By $C(\mathcal{S})$, we denote the space of continuous functions on $\mathcal{S}$, and by $C_0(\mathcal{S})$, we denote the space of continuous functions vanishing on $\partial\mathcal{S}$. We write $B(P, r) = \{P' \in \mathbb{R}^N : |P' - P| < r\}$, $P \in \mathbb{R}^N$, $r > 0$, for the Euclidean ball of radius r centered at P, and denote by $|A|$ the Lebesgue measure of a subset $A \subset \mathbb{R}^N$.

4.1. Time-fractional Wentzell problems on fractal domains

In this section, we study time-fractional transmission problems of Wentzell type across fractal structures.

We first introduce the notion of d-set (see [22]).

Definition 4.1. *A closed nonempty set $\mathcal{S} \subset \mathbb{R}^N$ is a d-set (for $0 < d \leq N$) if there exist a Borel measure μ with $\operatorname{supp} \mu = \mathcal{S}$ and two positive constants c_1 and c_2 such that*

$$c_1 r^d \leq \mu(\overline{B(x,r)} \cap \mathcal{S}) \leq c_2 r^d, \quad \forall\, x \in \mathcal{S}, \quad 0 < r \leq 1. \tag{4.1}$$

The measure μ is called a d-measure.

We now recall the definition of Besov space specialized to our case. For generalities on Besov spaces, we refer to [22].

Definition 4.2. *Let $\mathcal{S}$ be a d-set, $0 < \gamma < 1$, and $1 \leq p < \infty$. $B^{p,p}_\gamma(\mathcal{S})$ is the space of functions for which the following norm is finite:*

$$\|u\|^p_{B^{p,p}_\gamma(\mathcal{S})} = \|u\|^p_{L^p(\mathcal{S})} + \iint_{|x-y|<1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+p\gamma}} \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

The Koch curve

Following [15, 17], by the Koch curve K, we denote the uniquely determined self-similar closed bounded set with respect to a family Ψ of four contractions.

Let $A = (0,0)$, $B = (1,0)$, and K_0 denote the unit segment whose endpoints are A and B. We set $K_1 = \Psi(K_0)$ and $K_{n+1} = \bigcup_{M\in K_n} \Psi(M)$, where M denotes a segment of the n-th generation. K is the unique fixed point of the map of contractions Ψ, i.e., $K = \Psi(K)$. From now on, we assume that a clockwise orientation is given on K.

The Hausdorff dimension of the Koch snowflake is given by $d_f = \frac{\ln 4}{\ln 3}$. We denote by μ the normalized d_f-dimensional Hausdorff measure, restricted to K. For every $n \in \mathbb{N}$, K_n denotes the so-called pre-fractal (polygonal) curve approximating K at the n-th step.

The measure μ has the property that there exist two positive constants c_1 and c_2 such that

$$c_1 r^{d_f} \leq \mu(B(P,r) \cap K) \leq c_2 r^{d_f}, \quad \forall\, P \in K. \tag{4.2}$$

Hence, it follows that K is a d_f-set and that K_n is a 1-set.

From now on, Ω will be the polygonal open bounded rectangle $(0,1) \times (-1,1) \subset \mathbb{R}^2$, which is divided in two subdomains Ω_1 and Ω_2 by the Koch curve K, as it can be seen in Figure 1.

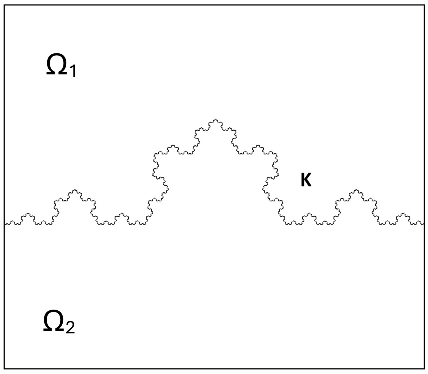

Figure 1. The open bounded set Ω.

We consider also the case when the rectangular domain $\Omega = (0, 1) \times (-1, 1)$ is divided in two subsets Ω_n^1 and Ω_n^2 by the n-th pre-fractal approximation of the Koch curve K_n, for $n \in \mathbb{N}$.

We now state a trace theorem for functions from Ω to K. For the proof, we refer to [22, Theorem 1, Chapter VII].

Proposition 4.1. *$B^{2,2}_{\frac{d_f}{2}}(K)$ is the trace space of $H^1(\Omega)$ in the following sense:*

(i) γ_0 is a continuous linear operator from $H^1(\Omega)$ to $B^{2,2}_{\frac{d_f}{2}}(K)$;

(ii) there exists a continuous linear operator Ext *from $B^{2,2}_{\frac{d_f}{2}}(K)$ to $H^1(\Omega)$ such that $\gamma_0 \circ$* Ext *is the identity operator in $B^{2,2}_{\frac{d_f}{2}}(K)$.*

We also state a trace theorem from Ω to the pre-fractal K_n, for $n \in \mathbb{N}$. We refer to Theorem 2.24 in [7].

Proposition 4.2. *For every $n \in \mathbb{N}$, $H^{\frac{1}{2}}(K_n)$ is the trace space of $H^1(\Omega)$ in the following sense:*

(i) γ_0 is a continuous linear operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(K_n)$;

(ii) there exists a continuous linear operator Ext *from $H^{\frac{1}{2}}(K_n)$ to $H^1(\Omega)$ such that $\gamma_0 \circ$* Ext *is the identity operator in $H^{\frac{1}{2}}(K_n)$.*

We introduce the spaces

$$H^{\frac{1}{2}}_{0,0}(K_n) = \{u \in L^2(K_n) \,:\, \exists\, v \in H^1_0(\Omega) \text{ s.t. } v = 0 \text{ on } \partial\Omega \text{ and } \gamma_0 v = u \text{ on } K_n\}$$

and

$$B^{2,2}_{\frac{d_f}{2},0}(K) = \{u \in L^2(K) \,:\, \exists\, v \in H^1_0(\Omega) \text{ s.t. } v = 0 \text{ on } \partial\Omega \text{ and } \gamma_0 v = u \text{ on } K\}.$$

By $(H^{\frac{1}{2}}_{0,0}(K_n))'$ and $(B^{2,2}_{\frac{d_f}{2},0}(K))'$, we denote their dual spaces.

For $i = 1, 2$, we set $u_i := u|_{\Omega_i}$ and we define the space

$$V(-\Delta, \Omega_i) := \{u \in H^1(\Omega) \,:\, -\Delta u_i \in L^2(\Omega_i) \text{ in the sense of distributions}\},$$

which is a Banach space equipped with the norm

$$\|u\|_{V(-\Delta,\Omega_i)} := \|u\|_{H^1(\Omega)} + \|\Delta u_i\|_{L^2(\Omega_i)}.$$

The following Green formula holds on Ω_i, $i = 1, 2$. For the proof, we refer to [28, Theorem 4.15].

Theorem 4.1. *(Green formula for fractal domains) For $u \in V(-\Delta, \Omega_i)$, $i = 1, 2$, there exists a bounded linear operator $\frac{\partial u_i}{\partial \nu_i}$ from $V(-\Delta, \Omega_i)$ to $(B^{2,2}_{\frac{d_f}{2},0}(K))'$ such that the following Green formula holds for every $u \in V(-\Delta, \Omega_i)$ and $v \in H^1_0(\Omega)$:*

$$\left\langle \frac{\partial u_i}{\partial \nu_i}, v|_K \right\rangle_{(B^{2,2}_{\frac{d_f}{2},0}(K))', B^{2,2}_{\frac{d_f}{2},0}(K)} = \int_{\Omega_i} \Delta u_i \, v \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_i} \nabla u_i \,\nabla v \,\mathrm{d}x\,\mathrm{d}y. \tag{4.3}$$

Analogously, for every $n \in \mathbb{N}$, for $i = 1, 2$, we set $u_{i,n} := u|_{\Omega_i^n}$ and we define the space $V(-\Delta, \Omega_i^n)$ on Ω_i^n as follows:

$$V(-\Delta, \Omega_i^n) := \{u \in H^1(\Omega) \,:\, -\Delta u_{i,n} \in L^2(\Omega_i^n) \text{ in the sense of distributions}\},$$

where the domains Ω_i^n are such that $\Omega = \Omega_1^n \cup \Omega_2^n$ and $\overline{\Omega_1^n} \cap \overline{\Omega_2^n} = K_n$, where K_n is the n-th approximation of K.

Theorem 4.2. *(Green formula for pre-fractal domains) For $u \in V(-\Delta, \Omega_i^n)$, $i = 1, 2$, there exists a bounded linear operator $\frac{\partial u_{i,n}}{\partial \nu_i}$ from $V(-\Delta, \Omega_i^n)$ to $(H^{\frac{1}{2}}_{0,0}(K_n))'$ such that the following Green formula holds for every $u \in V(-\Delta, \Omega_i^n)$ and $v \in H^1_0(\Omega)$:*

$$\left\langle \frac{\partial u_{i,n}}{\partial \nu_i}, v|_{K_n} \right\rangle_{(H^{\frac{1}{2}}_{0,0}(K_n))', H^{\frac{1}{2}}_{0,0}(K_n)} = \int_{\Omega_i^n} \Delta u_{i,n} \, v \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_i^n} \nabla u_{i,n} \,\nabla v \,\mathrm{d}x\,\mathrm{d}y. \tag{4.4}$$

For more details, we refer to [3].

Wentzell problems on fractal structures

Following [28], we introduce the energy form E_K on the Koch curve K associated with homogeneous Dirichlet boundary conditions with domain

$$D_0(K) := \{u : K \to \mathbb{R} \,:\, E_K[u] < \infty,\ u(A) = u(B) = 0\}$$

equipped with the norm

$$\|u\|_{D_0(K)} = (E_K[u])^{\frac{1}{2}}. \tag{4.5}$$

In the following, we denote by $E_K(u, v)$ the bilinear form, which is obtained from $E_K[u]$ by the polarization identity.

Proposition 4.3. *(See Definition 3.2 and Lemma 3.3 in [28]) The form E_K with domain $D_0(K)$ is a regular Dirichlet form in $L^2(K)$ and the space $D_0(K)$ is a Hilbert space under the intrinsic norm* (4.5).

We now define the Laplace operator on K. Since $(E_K, D_0(K))$ is a regular Dirichlet form on $L^2(K)$, with dense domain $D_0(K)$ in $L^2(K)$, there exists (see [24, Chap. 6, Theorem 2.1]) a unique self-adjoint, non-positive operator Δ_K on $L^2(K)$ (with domain $D(\Delta_K) \subseteq D_0(K)$ dense in $L^2(K)$) such that

$$E_K(u, v) = -\int_K \Delta_K u\, v\, \mathrm{d}\mu, \quad u \in D(\Delta_K), v \in D_0(K). \tag{4.6}$$

Let $(D_0(K))'$ denote the dual of the space $D_0(K)$. We now introduce the Laplace operator on the fractal K as a variational operator from $D_0(K) \to (D_0(K))'$ by

$$E_K(u, w) = -\langle \Delta_K z, w\rangle_{(D_0(K))', D_0(K)} \tag{4.7}$$

for every $z \in D_0(K)$ and $w \in D_0(K)$, where $\langle \cdot, \cdot\rangle_{(D_0(K))', D_0(K)}$ is the duality pairing between $(D_0(K))'$ and $D_0(K)$. We use the same symbol Δ_K to define the Laplace operator both as a self-adjoint operator in (4.6) and as a variational operator in (4.7). It will be clear from the context to which case we refer.

We now define the space

$$V(\Omega, K) = \left\{u \in H^1_0(\Omega) \;:\; u|_K \in D_0(K)\right\} \tag{4.8}$$

and we introduce the energy form

$$E[u] = \int_\Omega |\nabla u|^2\, \mathrm{d}\mathcal{L}_2 + E_K[u|_K] \tag{4.9}$$

defined on the domain $D(E) := V(\Omega, K)$. By $E(u, v)$, we denote the corresponding bilinear form.

According to [30, Proposition 4.2] and [28, Proposition 4.8], the form E defined in (4.9) is a continuous and coercive Dirichlet form in $L^2(\Omega)$ and the space $V(\Omega, K)$ is a Hilbert space equipped with the scalar product

$$(u, v)_{V(\Omega,K)} = (u, v)_{H^1_0(\Omega)} + E_K(u, v). \tag{4.10}$$

We denote by $\|u\|_{V(\Omega,K)}$ the norm in $V(\Omega, K)$ associated with (4.10), i.e.,

$$\|u\|_{V(\Omega,K)} = \left(\|u\|^2_{H^1_0(\Omega)} + \|u\|^2_{D_0(K)}\right)^{\frac{1}{2}}. \tag{4.11}$$

We now construct the sequence of energy forms E_{K_n} on the pre-fractal boundaries K_n, for every $n \in \mathbb{N}$. By $\mathrm{d}\ell$, we denote the one-dimensional measure given by the arc-length ℓ and by $M_n^{(j)}$, for $j = 1 \ldots, 4^n$, we denote the segments of K_n. We define $E_{K_n}[u]$ by setting

$$E_{K_n}[u] = \sum_{j=1}^{4^n} \left(\int_{M_n^{(j)}} \sigma_n |\nabla_\ell u|_{K_n}|^2\, \mathrm{d}\ell \right), \tag{4.12}$$

where σ_n is a positive constant and $u \in H^1_0(K_n)$, with $H^1_0(K_n)$ defined by (see [7])

$$H^1_0(K_n) := \left\{ v \in C_0(K_n) \;:\; u|_{\overset{\circ}{M}{}_n^{(j)}} \in H^1(\overset{\circ}{M}{}_n^{(j)}),\ j = 1, \ldots, 4^n \right\}.$$

We denote the corresponding bilinear form by $E_{K_n}(u, v)$.

We now introduce, for every $n \in \mathbb{N}$, the space $V(\Omega, K_n)$ of functions u defined on Ω for which the following norm is finite:

$$\|u\|^2_{V(\Omega,K_n)} = \|u\|^2_{H^1_0(\Omega)} + \|u\|^2_{H^1_0(K_n)} \tag{4.13}$$

and we define the energy form

$$E^{(n)}[u] = \int_\Omega |\nabla u|^2 \, d\mathcal{L}_2 + E_{K_n}[u|_{K_n}], \tag{4.14}$$

defined on the domain $D(E^{(n)}) := V(\Omega, K_n)$. By $E^{(n)}(u, v)$, we will denote the corresponding bilinear form.

According to [30, Proposition 4.1], for every $n \in \mathbb{N}$, the form $E^{(n)}$ defined in (4.14) with domain $V(\Omega, K_n)$ is a continuous and coercive Dirichlet form in $L^2(\Omega)$ and the space $V(\Omega, K_n)$ is a Hilbert space equipped with the norm

$$\|u\|_{V(\Omega,K_n)} = \left(\int_\Omega |\nabla u|^2 \, d\mathcal{L}_2 + E_{K_n}[u|_{K_n}] \right)^{\frac{1}{2}}. \tag{4.15}$$

We extend both E and $E^{(n)}$ to $L^2(\Omega)$ by setting them equal to $+\infty$ if $u \in L^2(\Omega) \setminus V(\Omega, K)$ and $u \in L^2(\Omega) \setminus V(\Omega, K_n)$, respectively.

By proceeding as in Section 2, we denote by A the non-negative operator associated to the energy form E and by $D(A) \subseteq V(\Omega, K)$ its domain, which is dense in $L^2(\Omega)$. We denote by $\{e^{-tA}\}_{t\geq 0}$ the strongly continuous analytic contraction semigroup generated by $-A$ as in Proposition 2.1, and by $\{G_\beta, \beta > 0\}$ the strongly continuous resolvent family with generator A.

Proceeding as above, we denote by A_n, $\{e^{-tA_n}\}_{t\geq 0}$ and $\{G^n_\beta, \beta > 0\}$ the generators, the semigroups, and the resolvents associated to $E^{(n)}$, for every $n \in \mathbb{N}$, respectively.

Moreover, since the sequence $E^{(n)}$ is asymptotically compact in $L^2(\Omega)$ (see [30, Proposition 4.5]), from Theorem 2.4, it follows that

$$\inf_n \mu^k_n > 0,$$

where we denote by μ^k_n the k-th eigenvalue of A_n for every $k \in \mathbb{N}$.

The following M-convergence result for the energies holds (see [30, Theorem 4.1]).

Theorem 4.3. *The sequence of forms $E^{(n)}$ defined in* (4.14) *converges in the sense of Definition 2.3 to the form E defined in* (4.9).

We now consider the fractional Cauchy problems

$$(P_W) \begin{cases} \partial^\alpha_t u(t) + \lambda A u(t) = 0, & t > 0, \\ u(0) = u_0 \end{cases}$$

and, for every $n \in \mathbb{N}$,

$$(P^{(n)}_W) \begin{cases} \partial^\alpha_t u_n(t) + \lambda_n A_n \, u_n(t) = 0, & t > 0, \\ u_n(0) = u^{(n)}_0, \end{cases}$$

where $\lambda > 0$, $\lambda_n > 0$, and $A\colon D(A) \subset L^2(\Omega) \to L^2(\Omega)$, $A_n\colon D(A_n) \subset L^2(\Omega) \to L^2(\Omega)$ are the generators associated with the energy form E and the energy forms E_n introduced in (4.9) and (4.14), respectively.

If u_0 and $u_0^{(n)}$ belong to $L^2(\Omega)$, the existence and uniqueness of solutions of problems (P_W) and $(P_W^{(n)})$ in terms of the solution operators $S_{\alpha,\lambda}(t)$ and $S^{(n)}_{\alpha,\lambda_n}(t)$ induced by $-A$ and $-A_n$, respectively, follow from Theorem 2.5.

We now give the strong formulation of problems (P_W) and $(P_W^{(n)})$.

Proposition 4.4. *Let u be the unique solution of problem (P_W) and $T > 0$. Then we have, for every fixed $t \in (0, T)$,*

$$(\overline{P}_W)\begin{cases} \partial_t^\alpha u_i(P,t) - \lambda\Delta u_i(P,t) = 0, & \text{a.e. in } \Omega_i,\ i = 1,2,\\ u(P,t) = 0, & \text{in } H^{\frac{1}{2}}(\partial\Omega),\\ [u(P,t)]_K = 0, & \text{in } B^{2,2}_{\frac{d_f}{2},0}(K),\\ \left\langle \left[\dfrac{\partial u}{\partial \nu}(P,t)\right]_K, v\right\rangle_{(D_0(K))',D_0(K)} + E_K(u,v) = 0, & \text{for every } v \in D_0(K),\\ u(P,0) = u^0(P), & \text{in } H^1_0(\Omega), \end{cases}$$

where, for $i = 1,2$, we set $u_i := u|_{\Omega_i}$, $[u]_K := u_1 - u_2$, and $\left[\dfrac{\partial u}{\partial \nu}\right]_K := \dfrac{\partial u_1}{\partial \nu_1} + \dfrac{\partial u_2}{\partial \nu_2}$. Here, for $i = 1,2$, ν_i denotes the "outward normal vector" to Ω_i and $\dfrac{\partial u_i}{\partial \nu_i}$ is the "normal derivative" in the sense of Theorem 4.1.

The proof can be achieved as in Theorem 3 in [8], and it relies on [28, Theorem 6.2] and [29, Theorem 6.1].

Proposition 4.5. *For every $n \in \mathbb{N}$, let u_n be the unique solution of problem $(P_W^{(n)})$ and $T > 0$. Then we have, for every fixed $t \in (0, T)$,*

$$(\overline{P}_W^{(n)})\begin{cases} \partial_t^\alpha u_{i,n}(P,t) - \lambda_n\Delta u_{i,n}(P,t) = 0, & \text{a.e. in } \Omega_i,\ i = 1,2,\\ u_n(P,t) = 0, & \text{in } H^{\frac{1}{2}}(\partial\Omega),\\ [u_n(P,t)]_{K_n} = 0, & \text{in } H^{\frac{1}{2}}_{0,0}(K_n),\\ \left[\dfrac{\partial u_n}{\partial \nu}(P,t)\right]_{K_n} - \Delta_{K_n} u = 0, & \text{in } L^2(K_n),\\ u_n(P,0) = u_0^{(n)}(P), & \text{in } H^1_0(\Omega), \end{cases}$$

where, for $i = 1,2$, we set $u_{i,n} := u|_{\Omega_i^n}$, $[u_n]_{K_n} := u_{1,n} - u_{2,n}$ and define

$$\left[\frac{\partial u_n}{\partial \nu}\right]_{K_n} := \frac{\partial u_{1,n}}{\partial \nu_1} + \frac{\partial u_{2,n}}{\partial \nu_2}.$$

Here, for $i = 1,2$, $\dfrac{\partial u_{i,n}}{\partial \nu_i}$ is the "normal derivative" in the sense of Theorem 4.2.

The proof can be achieved as in Theorem 4 in [8], by following [29, Theorem 6.3].

By combining Theorem 6 in [8] and Theorem 5.3 in [29], we obtain the convergence of the solutions of the direct problems, which is stronger than the one stated in the abstract setting in Theorem 3.4.

Proposition 4.6. *Let u and u_n be the solutions of problems (P_W) and $(P_W^{(n)})$, respectively, and $T > 0$. If $u_0^{(n)} \xrightarrow[n\to+\infty]{} u_0$ strongly in $L^2(\Omega)$ and $\lambda_n \xrightarrow[n\to+\infty]{} \lambda$, then $u_n(t)$ strongly converges to $u(t)$ in $H_0^1(\Omega)$ as $n \to +\infty$ for every $t \in [T_1, T_2] \subset (0, T)$.*

Finally, if we state the inverse problems $P^\alpha_{n,W}$ and P^α_W as in subsection 3.1, from Theorem 3.5 and Theorem 4.6, we obtain the following convergence result.

Proposition 4.7. *For $n \in \mathbb{N}$, $T > 0$, and $\overline{T}, \overline{T}_n \in (0, T)$, let $u_0^{(n)}, u_0 \in L^2(\Omega)$ be non-identically zero, $0 < \rho < \|u_0\|_{L^2(\Omega)}$, and $0 < \rho_n < \|u_0^{(n)}\|_{L^2(\Omega)}$. If*

$$\begin{cases} \rho_n \to \rho > 0, \\ \overline{T}_n \to \overline{T} > 0, \\ u_0^{(n)} \to u_0 \text{ in } L^2(\Omega), \end{cases}$$

as $n \to +\infty$, then, if we denote by (λ_n, u_n) the unique solution of problem $P^\alpha_{n,W}$ given by Theorem 3.2 and by (λ_*, u_*) the unique solution of problem P^α_W given by Theorem 3.1, we have

$$\lambda_n \xrightarrow[n\to+\infty]{} \lambda_*$$

and

$$u_n(t) \xrightarrow[n\to+\infty]{} u_*(t)$$

strongly in $H_0^1(\Omega)$ and uniformly on every bounded subset $[T_1, T_2] \subset (0, T)$.

4.2. Time-fractional homogenization problems

Let $\Omega \subset \mathbb{R}^2$ be an open bounded (sufficiently smooth) set with finitely many "small" (smooth) holes T_i, for $i = 1, \dots, N$. We set

$$\mathcal{T} := \bigcup_{i=1}^N T_i \quad \text{and} \quad \Omega^M := \Omega \setminus \mathcal{T}.$$

Ω^M is the so-called "material" and has a disconnected smooth boundary $\partial\Omega^M = \partial\mathcal{T} \cup \partial\Omega$.

Inspired by the ideas and results studied in [2] and [34], we now introduce a "microscopic" version of our domain. Note that this approach allows us to face some problems related to composite media, e.g., fiber, stratified, or porous materials, which play a relevant role in physical engineering. To this aim, let $\varepsilon > 0$ be sufficiently small and, for simplicity, assume that the structure is periodic.

For a fixed hole T_i, we set $T_i^\varepsilon = r_\varepsilon T_i + x_i^\varepsilon$, where $0 < r_\varepsilon < \varepsilon$ and x_i^ε is the center of the i-th cell. As before, we set

$$\mathcal{T}^\varepsilon := \bigcup_{i=1}^N T_i^\varepsilon.$$

We introduce the following sequence of problems:

$$(\overline{P}_H^\varepsilon) \begin{cases} \partial_t^\alpha u_\varepsilon(P, t) - \Delta u_\varepsilon(P, t) = 0, & \text{in } \Omega_\varepsilon \times (0, T), \\ u_\varepsilon(P, t) = 0, & \text{on } \partial\Omega_\varepsilon \times (0, T), \\ u_\varepsilon(P, 0) = u_0^\varepsilon(P), & \text{in } \Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \Omega \setminus \mathcal{T}^\varepsilon$.

We introduce the following sequence of energy functionals, for $\varepsilon > 0$:

$$F_\varepsilon[u] := \begin{cases} \displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, d\mathcal{L}_2, & \text{if } u \in \mathcal{K}^\varepsilon, \\ +\infty, & \text{if } u \in L^2(\Omega) \setminus \mathcal{K}^\varepsilon, \end{cases} \tag{4.16}$$

where the effective domain is the convex set $\mathcal{K}^\varepsilon := \{u \in H^1_0(\Omega) \,:\, u = 0 \text{ on } \mathcal{T}^\varepsilon\}$.

It can be easily seen that F_ε is a continuous and coercive Dirichlet form on $\mathcal{K}^\varepsilon$ and $\{F_\varepsilon\}_\varepsilon$ is asymptotically compact on $L^2(\Omega)$. Hence, proceeding as in Section 2, we denote by A_ε the non-negative operator associated to F_ε with domain $D(A_\varepsilon) \subset \mathcal{K}^\varepsilon$ dense in $L^2(\Omega)$.

As before, for $\varepsilon > 0$ and $\lambda_\varepsilon > 0$, we introduce the corresponding abstract Cauchy problem

$$(P^\varepsilon_H) \begin{cases} \partial^\alpha_t u_\varepsilon(t) + \lambda_\varepsilon A_\varepsilon \, u_\varepsilon(t) = 0, & t \in (0,T), \\ u_\varepsilon(0) = u^\varepsilon_0, \end{cases}$$

which, for $u^\varepsilon_0 \in L^2(\Omega)$, admits a unique strong solution. Moreover, proceeding as in the previous Subsection 4.1, we can prove that the strong solution of (P^ε_H) actually solves problem $(\overline{P}^\varepsilon_H)$.

From Theorem 1.27 in [2], if $0 < r_\varepsilon < \varepsilon^2$, we have that F_ε M-converges as $\varepsilon \to 0^+$ to the functional

$$F[u] := \begin{cases} \displaystyle\int_{\Omega} |\nabla u|^2 \, d\mathcal{L}_2, & \text{if } u \in H^1_0(\Omega), \\ +\infty, & \text{if } u \in L^2(\Omega) \setminus H^1_0(\Omega). \end{cases} \tag{4.17}$$

We now present the limit problem

$$(\overline{P}_H) \begin{cases} \partial^\alpha_t u(P,t) - \lambda \Delta u(P,t) = 0, & \text{in } \Omega \times (0,T), \\ u(P,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(P,0) = u_0(P), & \text{in } \Omega, \end{cases}$$

where $\lambda > 0$. The energy functional defined in (4.17) is associated to $(\overline{P}_H)$, and it can be proved that it is a continuous and coercive Dirichlet form on $L^2(\Omega)$. We construct as before an abstract Cauchy problem (P_H) involving the operator A associated with F, which, if $u_0 \in L^2(\Omega)$, admits a unique strong solution that also solves problem $(\overline{P}_H)$.

Now we state the convergence results.

Proposition 4.8. *Let u and u_ε be the solutions of problems (P_H) and (P^ε_H), respectively. If $u^\varepsilon_0 \xrightarrow[\varepsilon\to 0^+]{} u_0$ strongly in $L^2(\Omega)$ and $\lambda_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} \lambda$, then $u_\varepsilon(t)$ strongly converges to $u(t)$ in $H^1_0(\Omega)$ as $\varepsilon \to 0^+$ for every $t \in [T_1, T_2] \subset (0,T)$.*

The proof follows by combining Theorem 1.27 in [2] and Theorems 2.2 and 3.3.

Finally, if we state the inverse problems $P^\alpha_{\varepsilon,H}$ and P^α_H as in subsection 3.1, from Theorems 3.5 and 4.8, we obtain the following convergence result.

Proposition 4.9. *For $\varepsilon > 0$, $T > 0$, and $\overline{T}, \overline{T}_\varepsilon \in (0, T)$, let $u_0^\varepsilon, u_0 \in L^2(\Omega)$ be non-identically zero, $0 < \rho < \|u_0\|_{L^2(\Omega)}$, and $0 < \rho_\varepsilon < \|u_0^\varepsilon\|_{L^2(\Omega)}$. If*

$$\begin{cases} \rho_\varepsilon \to \rho > 0, \\ \overline{T}_\varepsilon \to \overline{T} > 0, \\ u_0^\varepsilon \to u_0 \text{ in } L^2(\Omega), \end{cases}$$

as $\varepsilon \to 0^+$, then, if we denote by $(\lambda_\varepsilon, u_\varepsilon)$ the unique solution of problem $P^\alpha_{\varepsilon,H}$ given by Theorem 3.2 and by (λ_*, u_*) the unique solution of problem P^α_H given by Theorem 3.1, we have

$$\lambda_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} \lambda_*$$

and

$$u_\varepsilon(t) \xrightarrow[\varepsilon\to 0^+]{} u_*(t)$$

strongly in $H^1_0(\Omega)$ and uniformly on every bounded subset $[T_1, T_2] \subset (0, T)$.

5. Conclusions

In this paper we investigated an abstract time-fractional (involving a Caputo-type fractional time derivative) inverse Cauchy problem, governed by a self-adjoint operator A, and its approximation. We proved existence and uniqueness results for the solution of the inverse problem consisting of the identification of a single constant conductivity, under overdeterminating conditions of energy type. Then we studied its approximation in terms of a sequence of suitable abstract time-fractional inverse Cauchy problems via the Mosco convergence of the associated energies. We concluded by presenting two applications to boundary value problems: the first one was a time-fractional transmission Wentzell-type problem across a fractal interface of Koch type, the second one was a time-fractional periodic homogenization problem.

Author contributions

All authors contributed equally to the conceptualization, methodology, formal analysis, and writing (original draft preparation, review, and editing) of the manuscript. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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